Competing Pre-match Investments Revisited: A Characterization of Monotone Bayes-Nash Equilibria in Large Markets

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Abstract

We solve an open problem in the literature that studies investment incentives provided by the competition in matching markets. We approximate continuum economies as in Peters and Siow's (2002) *competitive* model of competing pre-marital investments, with heterogeneous agents and frictionless matching under non-transferable utility, by pre-match investment games with finitely many agents and i.i.d. types. Our main result is a precise asymptotic characterization of side-symmetric, strictly monotone Bayes-Nash equilibrium (SSMBNE) behavior in large finite markets converging to an unbalanced limit economy. We show that SSMBNE investments *never* converge to the efficient investments predicted by competitive (hedonic) equilibrium. There is always inefficient overinvestment on both sides of the market. Our analysis relies on a completely novel way of using results about approximate distributions of order statistics that allows us to characterize equilibrium behavior asymptotically even though the limit strategies feature a discontinuity that is determined by a complex, two-sided interaction.
Keywords: Assortative matching, investment, large games, large contests.

1 Introduction

In many two-sided matching markets (labor, marriage, or education), participants on both sides make investments that are valued by their potential partners before entering. If the agents believed the outcome they will receive in the matching market to be independent of their ex ante, or *pre-match*, investments, the latter would suffer from a "hold-up" problem: ignoring the benefits created by their investment for their (future)

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partner, the agents would underinvest. Of course, competition provides additional incentives. Agents' market outcomes depend on their investments, and agents who make investments that are more valuable for their potential partners will generally get higher quality matches and earn higher returns.

An important question is then whether the anticipated competition for partners can provide *efficient* investment incentives for all market participants, at least under the ideal conditions of a *large and frictionless matching market* and a *deterministic investment technology*, i.e., a deterministic relationship between an agent's choice of investment and his or her match-relevant characteristics. According to *competitive* models of investment and one-to-one matching the answer is yes. For a two-sided continuum economy with agents who must first make investments and then form matches in a frictionless market, it is always possible to define an appropriate variant of *hedonic pricing equilibrium* (Rosen 1974), and to support any *pairwise efficient* allocation as an equilibrium allocation.[1] Remarkably, this is true regardless of whether utility is perfectly transferable (Cole, Mailath and Postlewaite 2001a, Dizdar 2018), imperfectly transferable (Nöldeke and Samuelson 2015) or non-transferable (Peters and Siow 2002) within pairs (after all agents have invested and entered the market). The other main competitive equilibrium concept that has been studied in the literature, *ex post (contracting)* equilibrium, allows for inefficient equilibria featuring coordination failures in investment activity, but efficient allocations can always be supported by an equilibrium (Cole, Mailath and Postlewaite 2001a, Nöldeke and Samuelson 2015, Dizdar 2018).

It has been noted, however, that these results rely heavily on assumptions about returns for *off-equilibrium* investments, and that there is a multitude of possible equilibrium concepts for the continuum economies, featuring different assumptions about such off-equilibrium returns. Relatedly, Peters (2007, 2011) and Felli and Roberts (2016) have identified cases where Nash or Bayes-Nash equilibrium investments in pre-match investment games with finitely many participants do not converge to efficient investments as the market grows large. Our understanding of the differences between noncooperative and competitive models of pre-match investment has remained very limited, however, and Nöldeke and Samuelson (2015) have recently concluded that making progress on this issue remains an important challenge for further work.

We make a major step in this direction, by resolving one of the main open questions in the literature. Motivated by the work of Peters (2007, 2011) and similar approaches in the literature on large double auctions (e.g., Rustichini, Satterthwaite and

[1]Pairwise efficiency is a refinement of Pareto efficiency that incorporates an "ex ante stability" requirement. It is satisfied by any allocation that may result when agents can *simultaneously* negotiate investments and matches in a frictionless market (see Nöldeke and Samuelson 2015).

Williams 1994) and large contests (Olszewski and Siegel 2016), we approximate a continuum economy as in Peters and Siow (2002), with ex ante heterogeneous agents, *non-transferable utility* (NTU) and positive assortative stable matching, by economies with finitely many agents whose types are drawn i.i.d. from two commonly known distributions. Following the most common terminology in the one-to-one matching literature, we refer to the agents as men and women. In our model, an agent's investment benefits his or her partner, these external benefits are increasing in the level of the investment, and higher types value increases in their partner's investment more than lower types.[2] We study the *side-symmetric, strictly monotone Bayes-Nash equilibria*[3] (SSMBNE) of the *pre-match investment game* where all men and women (who know their type and form beliefs about others' types according to the common prior) simultaneously choose investments and are then matched positive assortatively based on their investments, which corresponds to the stable outcome of the "post-investment" NTU matching market. Thus, as in the competitive equilibria for the continuum economy, the equilibrium matching in any SSMBNE is positive assortative, both with respect to agents' investments and with respect to their types.

Our objective is to characterize and understand SSMBNE behavior when the numbers of men (n) and women (k_n) grow large and the ratio $\frac{k_n}{n}$ converges to a constant $1 - r \in (0, 1)$. In this case, the empirical type distributions for the finite economies converge to the type distribution of a continuum economy that is not exactly balanced (the total measures of men and women are not exactly equal, as in Peters and Siow 2002). Without loss of generality, we assume that men are on the long side of the market.

Our main results show that if agents prefer a match with a partner who made only his or her *autarchy* investment (the optimal investment for the case that he or she remains unmatched) to staying unmatched,[4] SSMBNE allocations can *never* converge to the efficient allocation. There is always an *overinvestment* problem in large markets, regardless of any additional assumptions on preferences. In the finite economies, there are always some agents on the long side who face substantial uncertainty in equilibrium about whether they will be matched. It turns out that even though the measure of such types shrinks to zero as the market grows, their incentives to invest more aggressively to avoid "missing out" *always* have repercussions *on both sides* of the market that lead to significant overinvestment by a large number of agents.

[2]Our main results do not hinge on how exactly a strict single-crossing condition is build into the model. Analogous results apply if higher types have lower marginal costs of investment, as in Peters and Siow's (2002) original model, see Section 2.1 and Remark 4.

[3]All men (women) use the same, strictly increasing strategy.

[4]This assumption, made in Peters and Siow (2002) and most other models of pre-match investment competition, ensures "full matching," i.e., all agents on the short side get matched in equilibrium.

We now describe the problem and our results in a little more detail. Let $\underline{w}$ denote the lowest type of woman, and let m_r denote the *marginal* type of man (Peters 2011). This is the type who is matched to $\underline{w}$ in the assortative matching for the limit (continuum) economy. The limit economy's unique pairwise efficient allocation has the following features. First, types below m_r choose autarchy investments (anticipating that they will stay unmatched). Secondly, the types m_r and $\underline{w}$ make investments that are Pareto efficient for their relationship, and such that the type m_r is indifferent between this outcome and making his autarchy investment and staying unmatched. In particular, these Pareto efficient investments for the pair $(m_r, \underline{w})$ are strictly higher than the agents' autarchy investments, and also strictly higher than the "bilateral Nash" investments that the agents would make non-cooperatively (ignoring the external benefits for the partner) if they took for granted that they will be matched to each other. Thirdly, investments then increase continuously and are efficient for all pairs of higher, assortatively matched types. Hedonic equilibrium supports this allocation by imposing particular assumptions about the returns that agents expect to get for *off-equilibrium* investments, i.e., the investments between the autarchy investment and the efficient investment for the type m_r and below the efficient investment for the type $\underline{w}$, respectively, that deter deviations to these (less costly) investments (see Section 2.2).

SSMBNE strategies are continuous but, like hedonic equilibrium investments, limits of convergent subsequences of SSMBNE strategies must be discontinuous at m_r, and bounded away from the type $\underline{w}$'s autarchy investment for all types above $\underline{w}$. Indeed, it is not difficult to see that SSMBNE strategies in the large finite markets must increase rapidly, with maximal slopes of order $\sqrt{n}$, over intervals of types close to m_r (who face "non-negligible" uncertainty about whether they will get a partner) and $\underline{w}$.

The paper's central result, Theorem 2, provides an asymptotic characterization of the equilibrium behavior of such types close to the marginal type on the long side and close to the bottom on the short side, who face uncertain returns for their investment even in very large markets. Shifting and re-scaling men's type space so that men's *re-scaled strategies* (the strategies as functions of the re-scaled types, which correspond to original types close to m_r) satisfy a uniform Lipschitz bound, we show that any function that is the (locally uniform) limit of a subsequence of re-scaled SSMBNE strategies solves a particular fixed point (nonlinear integral) equation.

Obtaining this result requires a major methodological innovation. We introduce a novel way of using results about approximate distributions of order statistics to approximate monotone BNE of a discontinuous game for which the limits of equilibrium strategies are discontinuous, and for which the main object of interest, the size of the discontinuity, is determined by a complex, two-sided interaction. In particular, we systematically exploit that, due to the different concentration properties of central, intermediate and extreme order statistics, different types of agents face different amounts of uncertainty about their equilibrium outcome. These techniques are new to the literature on large Bayesian games, and should be useful for studying monotone BNE in other games where (as in this paper) agents' actions affect the payoffs of others directly, rather than "only" through positional externalities and/or effects on a market clearing price (as in contests with exogenously given prizes, or in double auctions).

Combining Theorem 2 with certain results from the theory of Wiener-Hopf equations, we then show that SSMBNE allocations can never converge to the efficient allocation for the limit economy. For any allocation that is a limit of SSMBNE allocations, types "just above" m_r and $\underline{w}$, and consequently also positive measures of types above m_r and $\underline{w}$ (potentially all), overinvest significantly (Theorem 3 and Theorem 4).

Thus, at least for matching under NTU, where the only possible "compensation" for a higher investment is a higher quality match, the fact that some agents compete not only to get a partner with a higher investment but also to get any match at all (combined with the intense competition in large markets) implies equilibrium behavior by these agents and their likely partners that "forces" a large share of market participants to overinvest.

The rest of the paper proceeds as follows. Section 1.1 discusses related work on pre-match investment competition and large contests. Section 2.1 presents the finite economy model. Section 2.2 explains the problems surrounding the choice of equilibrium concept for the limit economy. Section 2.3 defines notation and basic mathematical concepts. Section 2.4 establishes SSMBNE existence. In Section 3.1 we state and describe our main results about SSMBNE behavior in large markets (Theorems 2, 3 and 4). In particular, we provide an *intuitive* (but informal) explanation of the central fixed-point characterization, Theorem 2. In Section 3.2, we then present the formal results about approximate distributions of order statistics that we need for the rigorous proof of Theorem 2 (Section 3.2.1) and provide an overview of this proof, explaining its structure and the key ideas behind each main step in the argument (Section 3.2.2). Section 4 concludes with a brief discussion of open questions. The proofs of all formal results are given in the Appendix and the Online Appendix.

1.1 Related literature

By means of an ingenious argument, Peters (2007) showed for a model with *homogeneous* agents on each side and with assortative NTU matching (based on investments) that (mixed-strategy Nash) equilibria in large, unbalanced markets must feature overinvestment if certain sufficient conditions on the shapes of agents' indifference curves are satisfied. Peters (2011) adapted this argument for Bayesian pre-match investment games approximating a continuum economy with heterogeneous agents and heterogeneous equilibrium investments as in Peters and Siow (2002), i.e., for a model akin to the one studied in the present paper. However, understanding the equilibrium behavior of the types whose interaction determines whether overinvestment occurs, and hence what really causes the inefficiency (and whether the, rather strong, sufficient conditions for overinvestment are also necessary) seemed very much out of reach. Our results solve these open problems. They show that overinvestment is always an issue in unbalanced markets with assortative, NTU matching (without any additional assumptions about preferences), and they demonstrate precisely why this is the case. They also show how returns for off-equilibrium investments would have to be defined for an equilibrium concept for the limit model that is approximated by SSMBNE in large markets.

Cole, Mailath and Postlewaite (2001b) and Felli and Roberts (2016) studied models with finitely many buyers and sellers whose *types* (which determine investment costs in their models) are *commonly known* when all agents simultaneously invest, and with matching under *transferable utility* (TU). Their results show that full efficiency is unattainable unless the efficient investments satisfy certain "overlap" conditions, which happens only for *non-generic* instances of ex ante heterogeneity. Moreover, the (underinvestment) inefficiencies can remain significant even in large markets. In particular, this happens if the numbers of buyers and sellers are not equal, as the overlap condition is not even approximately satisfied in such cases. These complete information models are arguably more interesting for small markets, however, as they have the feature that a single agent's decision to deviate from the efficient investment can have a large impact on the expected utility of everybody else, regardless of the size of the market.

If women's (men's) investment strategies were exogenous in our model, the strategic interaction among men (women) would be equivalent to a standard all-pay contest with i.i.d. contestants and heterogeneous "prizes" (which correspond to matches with the various partners, characterized by their investments, see Sections 2.1 and 2.4). Olszewski and Siegel (2016) showed that the equilibrium outcomes of such all-pay contests with many agents and many prizes are always nicely approximated by the outcome of a unique tariff mechanism (for a single agent with a continuum types) that can be derived in a straightforward way from the utility function and the limits of the type and prize distributions.[5] These elegant results for *one-sided* contests (with exogenous prize distributions) cannot be applied to characterize equilibrium behavior in pre-match investment games with bilateral investments and external benefits, however. The reason is of course that the "prizes" that agents compete for are determined endogenously here, as a result of a complex interdependence between investment behavior on both sides of the market: the fundamental difficulty of understanding the equilibrium outcomes of

[5]Their results allow for ex ante asymmetric agents, and for incomplete or complete information.

large pre-match investments games is due to *two-sided* investment.[6]

Dizdar, Moldovanu and Szech (2019) studied a matching contest model with NTU matching, similar to the one of the present paper, where investments also serve as *signals* about agents' types. They focused on quantitative features of the feedback effect caused by the external benefits of investment in *small* markets, and on environments where investments/signals are partially wasteful. Most importantly, their techniques cannot be used to study SSMBNE behavior in large markets when investments are productive in the sense that hold-up would be an issue without competition. Thus, they do not allow addressing the questions at the heart of our study, which pertain to whether "competition can solve the hold-up problem," and to the difficulties of providing foundations for competitive models of pre-match investment.

Building on the tournament model of Lazear and Rosen (1981), Bhaskar and Hopkins (2016) analyzed a model where each agent's observable quality depends on his or her investment and on the realization of an idiosyncratic shock. The (NTU) stable matching is positive assortative with respect to agents' qualities. Assuming that the agents on each side are *homogeneous* prior to investing, they studied pure-strategy equilibria of a continuum model with balanced populations that assumes that an agent whose quality is below the support of the equilibrium distribution of qualities gets a match with the lowest-ranked partner with probability $1/2$ and stays unmatched otherwise. They also proved that their equilibria can be obtained as limits of equilibria of a finite model where the agents are uncertain about whether they are on the long side or the short side of a slightly imbalanced market. Their central finding is also an overinvestment result, albeit one that is very different from the results of the present paper. As all agents on either side make the same deterministic investment in equilibrium, agents compete solely to get a partner with a higher realization of the exogenous shock (rather than a partner with a higher investment). Bhaskar and Hopkins (2016) then showed that, as an intriguing but elementary implication of the Cauchy-Schwarz inequality, agents overinvest relative to any Pareto efficient levels (for the economy with shocks) unless men's shock distribution is an affine transformation of women's shock distribution.[7] They also provided an elegant analysis of how gender differences affect whether men (women) overinvest relative to utilitarian efficient levels.

[6]Given a sequence of large one-sided contests with a discontinuous limit prize distribution, understanding the equilibrium behavior of types likely to receive "prizes close to a discontinuity" is also not easy (Olszewski and Siegel (2016) left this as an open problem; our Theorem 2 implies a characterization of such behavior in the case of i.i.d. types). However, because the discontinuity is exogenous, this behavior does not affect the limit allocation (that can easily be derived from the continuum model).

[7]Another difference compared to our paper is that the pure-strategy equilibria studied in Bhaskar and Hopkins (2016) exist only if the utility from staying unmatched is sufficiently low, and that the equilibrium investments are then completely independent of the exact value of this utility.

2 Model and preliminaries

2.1 The finite economy model

Consider a two-sided, one-to-one matching market. To work with a consistent set of terms throughout, we refer to the agents as men and women. The numbers of men (n) and women (k) satisfy $2 \leq k < n$. Thus, the matching market is not exactly balanced, and men are on the long side. Each agent is characterized by a type. Types are independently distributed according to two commonly known, absolutely continuous cumulative distribution functions F (for men) and G (for women). The densities $f = F'$ and $g = G'$ are strictly positive and continuously differentiable on the measure supports $[\underline{m}, \overline{m}]$ and $[\underline{w}, \overline{w}]$, respectively, where $0 < \underline{m} < \overline{m} < \infty$ and $0 < \underline{w} < \overline{w} < \infty$.

Prior to matching, all agents make investments. If a man with type m and investment $b_M \in \mathbb{R}_+$ matches with a woman with type w and investment $b_W \in \mathbb{R}_+$, the man's utility is $U_M(b_M, b_W, m, w) = mv_M(b_W) - b_M$, and the woman's utility is $U_W(b_W, b_M, w, m) = wv_W(b_M) - b_W$, where the strictly increasing, non-negative functions v_M and v_W satisfy Assumption 1 below. A man who chooses investment b_M and remains unmatched receives utility $U_M^{\emptyset}(b_M, m) = -b_M$. Similarly, the utility of an unmatched woman with investment b_W is $U_W^{\emptyset}(b_W, w) = -b_W$. Consequently, agents' *autarchy investments* (Nöldeke and Samuelson 2014), the investments $b_M^{\emptyset}(m)$ and $b_W^{\emptyset}(w)$ maximizing $U_M^{\emptyset}(\cdot, m)$ and $U_W^{\emptyset}(\cdot, w)$, are equal to 0 for all types. We sometimes model an unmatched agent as being matched to a dummy type $\emptyset$, with dummy investment $b_\emptyset < 0$.

Assumption 1. *The functions $v_M : \mathbb{R}_+ \to \mathbb{R}_+$ and $v_W : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing, twice continuously differentiable and concave, and at least one of the two functions is strictly concave. Moreover, the following conditions are satisfied:*

$$v_M(0) > 0, \tag{1}$$

$$\underline{m}v_M'(0)\underline{w}v_W'(0) > 1, \tag{2}$$

$$\overline{m}\left(\lim_{b\to\infty} v_M'(b)\right)\overline{w}\left(\lim_{b\to\infty} v_W'(b)\right) < 1. \tag{3}$$

We will explain all these assumptions on preferences in detail throughout this section, and clarify in particular how they parallel assumptions in Peters and Siow (2002).

A matching of men and women is *positive assortative in investments* if all women are matched and if any man (woman) whose investment is strictly higher than that of another man (woman) is matched to a partner whose investment is weakly higher than the investment of the latter man's (woman's) partner.[8] Note that if all investments are

[8]For men one or both of these partners could be dummy types. We do not spell out the formal definition of a matching involving agents' identities (Roth and Sotomayor 1990), as we will not need it.

positive, all (complete information) stable matchings (e.g., Roth and Sotomayor 1990) for the nontransferable utility (NTU) matching market that results after everybody has invested are positive assortative in investments.[9] Furthermore, if there are no ties (i.e., there is no agent whose investment coincides with the investment of another agent from the same side of the market) the stable matching is unique. These properties hold of course because an agent's utility is independent of his or her partner's type, all agents prefer partners with higher investments, and $\underline{m}v_M(0) \geq 0$ and $\underline{w}v_W(0) \geq 0$.

The *pre-match investment game* $\Gamma(n, k)$ is the Bayesian game where all agents (who are privately informed about their types) simultaneously choose observable investments and utilities are determined as follows.[10] If there are no ties, agents receive utilities according to the unique stable matching. If there are ties, each agent's utility is equal to the expectation of a uniform lottery over her or his utilities in all stable matchings.[11]

As stable matchings do not depend on types, it is irrelevant (for interpreting the game) whether agents observe others' types (in addition to investments) in the "ex post" matching market. In particular, *signaling concerns* (which arise if types are unobservable ex post and values are interdependent) *play no role* here: investments are "purely productive," as in the existing competitive models of investment and matching.[12]

The asymmetric information at the investment stage implies of course that men and women face the kind of (interim) uncertainty about their competitors that is standard in Bayesian games with one-dimensional heterogeneity and ex ante symmetric agents. Moreover, they also face some uncertainty about the realized types, and hence the equilibrium investments, of their potential partners.

We study the *side-symmetric, strictly monotone Bayes-Nash equilibria* (SSMBNE) of the games $\Gamma(n, k)$, i.e., pure-strategy equilibria for which all men use the same, strictly increasing strategy and all women use the same, strictly increasing strategy. Note that, as higher types value any increase in their partner's investment more than lower types, agents' preferences satisfy a strict single crossing property that will allow us to obtain such monotone pure-strategy equilibria.[13] Formally, preferences are *separable* and sat-

[9] If $v_W(0) = 0$ and less than k men have made positive investments, there are additional stable matchings for which some women choose to remain unmatched. These stable matchings do not affect our results (and we could rule them out by assuming $v_W(0) > 0$ in addition to (1)).

[10] As the primitives F, G, v_M and v_W will always be fixed and arbitrary, subject to the assumptions made in this section, we suppress them in the notation.

[11] The choice of tie-breaking rule does not affect any results. Ties occur with probability zero in the side-symmetric equilibria we consider, regardless of the assumed tie-breaking rule.

[12] Remark 3 in the Online Appendix discusses the implications of our findings for settings in which investments have both a signaling and a productive function, as in Dizdar, Moldovanu and Szech (2019).

[13] Moreover, it follows from standard arguments that any side-symmetric BNE of a game $\Gamma(n, k)$ must be in strictly increasing, pure strategies. Compare also the discussion of condition (1) below.

isfy *strict outer single crossing*, as defined by Nöldeke and Samuelson (2014).[14] More specifically, agents' utilities are of a form that is commonly assumed in the literature on all-pay contests (Olszewski and Siegel 2016): each agent's utility is multiplicative in his or her type and in the "prize" that he or she gets, which is described here by an increasing function of the partner's investment. Moreover, this term is additively separated from the investment cost, which is type-independent. Assuming this functional form is convenient because, as is well known, the pure-strategy equilibrium of a (one-sided) all-pay contest with ex ante symmetric contestants has a closed form in this case. Our main results do not hinge on this assumption. Analogous characterizations of equilibrium behavior in large markets apply for models with type-dependent autarchy investments and utilities that are additively separable in b_M and b_W and satisfy strict outer single crossing (as in Peters and Siow 2002, Peters 2011, and Section 5.4 in Nöldeke and Samuelson 2014). Remark 4 in the Online Appendix provides a more detailed discussion of this point. Our techniques could also be used to study environments where utilities are strictly supermodular in the investments. However, it is clear that the details of the analysis would have to change very substantially to accommodate such cases, so we do not pursue this further in the paper.

Conditions (1)-(3) parallel assumptions in Peters and Siow (2002). Each agent weakly prefers a partner who made an autarchy investment to staying unmatched. By condition (1), this preference is strict for agents on the long side, which rules out the existence of a trivial BNE where all agents make autarchy investments, and also ensures *full matching*.[15] Condition (2) implies that without competition, investments would suffer from a "hold-up" problem. Indeed, following Peters and Siow (2002), let us call a pair of investments (b_M, b_W) such that b_M maximizes $U_M(\cdot, b_W, m, w)$ and b_W maximizes $U_W(\cdot, b_M, w, m)$ *bilateral Nash investments* for the pair (m, w). The interpretation is of course that these are investments that two types m and w would make (non-cooperatively) if they somehow took for granted that they will be matched to each other. As in Peters and Siow (2002) and Peters (2011), the bilateral Nash investments coincide here with the pair of autarchy investments, $(0, 0)$. Condition (2) implies that for any pair (m, w), the bilateral Nash investments are *too low compared to any Pareto efficient pair of investments*. Indeed, for a given (b_M, b_W), $mv'_M(b_W)$ $(wv'_W(b_M))$ is the

[14]Men's preferences are *separable* if there exist continuous, real-valued functions f_M, $f_M^{\emptyset}$ and $\hat{U}_M$ such that $U_M(b_M, b_W, m, w) = \hat{U}_M(f_M(b_M, b_W), b_M, m)$, $U_M^{\emptyset}(b_M, m) = \hat{U}_M(f_M^{\emptyset}(b_M), b_M, m)$ and $\hat{U}_M$ is strictly increasing in its first argument. This is obviously satisfied here, for $\hat{U}_M(x, b_M, m) = mx - b_M$, $f_M(b_M, b_W) = v_M(b_W)$ and $f_M^{\emptyset} \equiv 0$. An analogous observation applies for women. *Strict outer single crossing* is a standard strict single crossing property, defined for separable preferences. It requires that for all $m > m'$, $b > b'$ and all x and x', $\hat{U}_M(x, b, m') \geq \hat{U}_M(x', b', m')$ implies $\hat{U}_M(x, b, m) > \hat{U}_M(x', b', m)$, and that an analogous condition holds for women's preferences.

[15]All women are matched. Peters and Siow's assumptions correspond to $v_M(0) > 0$ and $v_W(0) > 0$.

marginal benefit for the man (woman) from a marginal increase in the woman's (man's) investment, and it is not difficult to see that if the product of these marginal external benefits is greater than the product of marginal costs (which is equal to 1 here), there is a pair of strictly higher investments that makes both partners better off. Finally, (3) ensures that the set of pairs of investments for which the partners obtain at least their *outside options* $U_M^{\emptyset}(b_M^{\emptyset}(m), m) = 0$ and $U_W^{\emptyset}(b_W^{\emptyset}(w), w) = 0$,

$$\mathcal{I}(m, w) := \{(b_M, b_W) \in \mathbb{R}_+^2 \,|\, mv_M(b_W) - b_M \geq 0 \text{ and } wv_W(b_M) - b_W \geq 0\}$$

is bounded, for all (m, w). We let $(\overline{b}_M(m, w), \overline{b}_W(m, w))$ denote the greatest element of $\mathcal{I}(m, w)$ in the usual (product) partial order on $\mathbb{R}_+^2$ and note that $(\overline{b}_M(m, w), \overline{b}_W(m, w))$ is the unique vector in $(0, \infty)^2$ satisfying

$$\overline{b}_M(m, w) = mv_M(\overline{b}_W(m, w)) \text{ and } \overline{b}_W(m, w) = wv_W(\overline{b}_M(m, w)). \tag{4}$$

We also define $\mathcal{I}_0^M(m) := \{(b_M, b_W) \in \mathbb{R}_+^2 \,|\, mv_M(b_W) - b_M = 0\}$ and

$$\begin{aligned}\mathcal{P}(m, w) &:= \{(b_M, b_W) \in \mathcal{I}(m, w) \,|\, (b_M, b_W) \text{ is Pareto efficient for } (m, w)\} \\ &= \{(b_M, b_W) \in \mathcal{I}(m, w) \,|\, mwv_M'(b_W)v_W'(b_M) = 1\},\end{aligned}$$

and we let $\left(b_M^e(m, w), b_W^e(m, w)\right)$ denote the Pareto efficient investments for which the man's utility is equal to his outside option. Figure 1 illustrates these definitions.

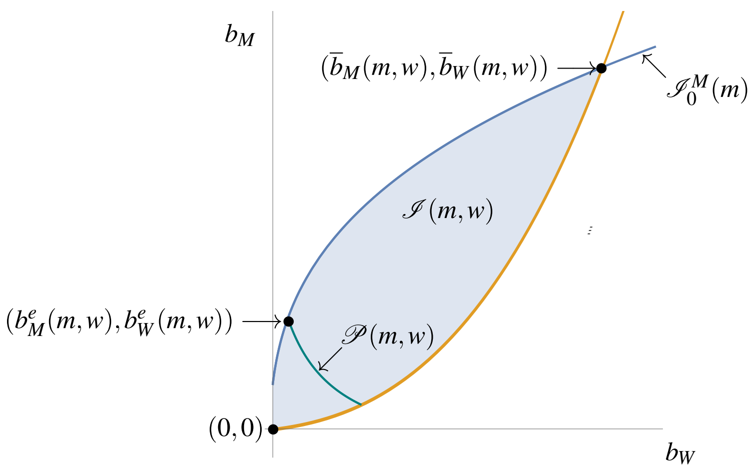

Figure 1: The zero utility indifference curves for two types m (the blue curve $\mathcal{I}_0^M(m)$) and w (orange curve), the sets $\mathcal{I}(m, w)$ and $\mathcal{P}(m, w)$, the bilateral Nash investments $(0, 0)$, and the pairs of investments $(b_M^e(m, w), b_W^e(m, w))$ and $(\overline{b}_M(m, w), \overline{b}_W(m, w))$, for a case where $v_W(0) = 0$.

2.2 The continuum economy, competitive models, and the importance of off-equilibrium returns

Here we briefly consider a continuum economy akin to those studied by competitive models of investment and matching. We maintain the assumptions about agents' preferences and about F and G. The population of women is described by a probability distribution with density g, and the population of men is described by a positive measure with density $f/(1-r)$, where $r \in [0, 1)$. The type distribution of this *r-continuum economy* is the limit of the empirical type distributions, normalized such that the total measure of women is equal to 1, for any sequence of games $\Gamma(n, k_n)$ satisfying $\lim_{n\to\infty} k_n/n = 1-r$.[16]

There are many possible equilibrium concepts for the continuum economy, with different assumptions about the returns (that agents expect to get) for off-equilibrium investments. We briefly explain the existing competitive equilibrium notions, and in particular how these support efficient behavior. We then explain some basic differences between the off-equilibrium returns assumed by these competitive models and features that off-equilibrium returns must have for a concept for which investments and returns are approximated by SSMBNE investments and (on-path) returns in large finite markets (see also Peters 2011). As our main results (presented in Section 3) pertain to investment behavior in large pre-match investment games approximating a continuum market that is *not exactly balanced*, we will assume throughout that $r > 0$.

We refer the reader to Nöldeke and Samuelson (2014) for a general and fully rigorous definition of *feasible allocations* for the kind of continuum economy considered here, formulated in the context of an "individualistic" model (with continuum populations of agents, each of which is characterized by his or her type).[17] Roughly speaking, a feasible allocation consists of a measure-preserving, one-to-one matching of men and women and of two functions describing investments, along with the resulting utilities for all agents. Given our assumptions on preferences, *equilibrium* allocations always feature the matching that is *positive assortative with respect to agents' types*, or are at least payoff equivalent to such an allocation. This holds for any of the equilibrium concepts that we will discuss below (and more generally for any "reasonable" equilibrium concept capturing frictionless matching ex post and a deterministic investment technology). The *positive assortative matching of types* is described by the function

$$\psi_r(m) = \begin{cases} \emptyset & \text{if } m < m_r \\ w_p & \text{if } m = m_{r+p(1-r)} \text{ for } p \in [0, 1], \end{cases}$$

[16] The convergence is uniform almost surely, by the Glivenko-Cantelli theorem.

[17] Greinecker and Kah (2021) clarify the relationship between "individualistic" and "distributional" continuum models of one-to-one matching markets and provide an elegant characterization of stability.

where $m_p = F^{-1}(p)$ and $w_p = G^{-1}(p)$ for all $p \in [0,1]$. Thus, types below the *marginal type* m_r remain unmatched, and the types in $[m_r, \overline{m}]$ and $[\underline{w}, \overline{w}]$ are matched positive assortatively. Correspondingly, equilibrium investments can always be described by nondecreasing functions $\gamma_M : [\underline{m}, \overline{m}] \to \mathbb{R}_+$ and $\gamma_W : [\underline{w}, \overline{w}] \to \mathbb{R}_+$.

For any $r > 0$, the r-continuum economy has a unique *pairwise efficient* allocation,[18] featuring the matching ψ_r and the following investments γ_M^e and γ_W^e. Unmatched types make autarchy investments, $\gamma_M^e(m) = 0$ for $m < m_r$, the types in the lowest matched pair, $(m_r, \underline{w})$, make the Pareto efficient investments for which the man receives his outside option utility, $\left(\gamma_M^e(m_r), \gamma_W^e(\underline{w})\right) = \left(b_M^e(m_r, \underline{w}), b_W^e(m_r, \underline{w})\right)$, and investments then increase (continuously) such that $\left(\gamma_M^e(m_{r+p(1-r)}), \gamma_W^e(w_p)\right) \in \mathcal{P}(m_{r+p(1-r)}, w_p)$ for all $p \in [0,1]$.

The first competitive equilibrium concept, due to Peters and Siow (2002), is a variant of hedonic pricing equilibrium (Rosen 1974) that is suitable for NTU matching, and which we will simply call *hedonic equilibrium*.[19] It requires that all agents choose utility-maximizing investments with respect to matching possibilities that are described by a single, strictly increasing and continuous *return function* $r_W : \mathbb{R}_+ \to \mathbb{R}_+$: for each investment $b_W \geq 0$, women expect a deterministic return corresponding to a match with a partner whose investment is $r_W(b_W)$ and, at the same time, men expect to get a partner with investment $r_W^{-1}(b_M)$ in return for any investment $b_M \geq r_W(0)$ (a man who chooses $b_M < r_W(0)$ expects to remain unmatched).[20] Of course, an equilibrium return function must also clear the market. The return function is the "functional equivalent of a complete set of prices" (Nöldeke and Samuelson 2014, p. 60), and hedonic equilibria are pairwise efficient. The intuition for this result is particularly clear for utilities that are concave in (b_M, b_W): the equilibrium indifference curves of any two matched agents, *including those of the bottom pair*, $(m_r, \underline{w})$, must be tangent to the graph of the return function, and hence to each other. In particular, hedonic equilibrium supports efficiency by "automatically" imposing returns for the off-equilibrium investments $b_M \in (0, b_M^e(m_r, \underline{w}))$ and $b_W \in [0, b_W^e(m_r, \underline{w}))$ that are low enough (and of a very specific form, corresponding to deterministic matches with investments that no potential partner actually makes) to deter deviations to these less costly investments.

Ex post equilibrium (Nöldeke and Samuelson 2014) assumes that a man (woman) with some investment, "on-path or off-path", can match with any "existing" partner, given her (his) sunk investment, as long as he (she) provides the partner with her (his)

[18]Pairwise efficiency is a refinement of Pareto efficiency that is satisfied by any *ex ante equilibrium* (see Nöldeke and Samuelson 2014).

[19]Peters and Siow (2002) called the concept rational expectations equilibrium, while Peters (2011) refers to an equivalent concept as hedonic equilibrium.

[20]The definition reflects the separability of preferences. In general, the return function would have to specify the investment and the type of the partner, depending on the agent's type and investment.

equilibrium utility, and that he (she) remains unmatched otherwise. Ex post equilibrium is less demanding than hedonic equilibrium, and the pairwise efficient allocation is one of many equilibrium allocations. For example, for every $(b_M, b_W) \in \mathcal{I}_0^M(m_r)$ satisfying $\underline{w}v_W(b_M) - b_W \geq \underline{w}v_W(0)$, the allocation with matching ψ_r and investments $\gamma_M(m) = 0$ for $m < m_r$, $\gamma_M(m) = b_M$ for $m \geq m_r$ and $\gamma_W(w) = b_W$ for all w is also an equilibrium.[21]

It should now be evident that there are many possible equilibrium concepts for the continuum model, featuring different assumptions about returns for off-equilibrium investments. For any SSMBNE of a pre-match investment game $\Gamma(n, k)$, the equilibrium strategies are continuous (see Section 2.4), and the types $\underline{m}$ and $\underline{w}$ make zero investments (their optimal investments knowing that they are the lowest-ranked agents with probability one). Moreover, by Helly's selection theorem, every subsequence of a given sequence of SSMBNE of games $\Gamma(n, k_n)$ satisfying $n \to \infty$ and $\lim_{n\to\infty} k_n/n = 1 - r$ contains a subsequence for which the equilibrium strategies converge to nondecreasing functions γ_M and γ_W, pointwise at all continuity points of γ_M and γ_W. In Section 3, we will see in particular that in large markets, SSMBNE strategies must increase rapidly close to m_r and $\underline{w}$, so that $\gamma_M^+(m_r) > \gamma_M^-(m_r) = 0$, where γ_M^+ and γ_M^- denote the usual one-sided limits.[22] As interim expected utilities are continuous for any SSMBNE, it follows easily that $(\gamma_M^+(m_r), \gamma_W^+(\underline{w})) \in \mathcal{I}_0^M(m_r)$ and that men's expected returns for investments $b_M \in (0, \gamma_M^+(m_r))$, which are off-equilibrium for the limit strategy but on-path in the large finite markets, must be such that (in the limit) the type m_r becomes indifferent between all $b_M \in [0, \gamma_M^+(m_r)]$. Similarly, the type $\underline{w}$ must become indifferent between all $b_W \in [0, \gamma_W^+(\underline{w})]$. We will see in Section 3 that the returns for investments in $(0, \gamma_M^+(m_r))$ and $[0, \gamma_W^+(\underline{w}))$ remain non-deterministic in the limit and converge to particular lotteries over partners with different investments (featuring positive probabilities of staying unmatched for men). Our main results characterize the form of these non-deterministic returns, and then show that all possible values of $(\gamma_M^+(m_r), \gamma_W^+(\underline{w}))$ are inefficiently high (exceed the investments $(b_M^e(m_r, \underline{w}), b_W^e(m_r, \underline{w}))$ by uniform constants).

2.3 Mathematical notation and definitions

Vectors, norms, probabilities, etc.

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the *floor* of x, the greatest integer less than or equal to x, and $\lceil x \rceil$ is the *ceiling* of x, the smallest integer greater than or equal to x. We let δ_{ij} denote

[21] A type $m \geq m_r$ has no incentive to deviate to some $b_M' < b_M$ because, according to the assumed off-equilibrium returns, he would stay unmatched (unable to provide any partner with her equilibrium utility). A deviation to some $b_M' > b_M$ is also unprofitable, as the agent does not get a higher return for this: there are no women with investments above b_W, and women cannot compensate the man ex post for his higher investment, due to NTU. The conditions for all other types are easily checked as well.

[22] For a function η defined in a neighborhood of $x \in \mathbb{R}$, $\eta^+(x) := \lim_{y\to x, y>x} \eta(y)$ and $\eta^-(x) := \lim_{y\to x, y<x} \eta(y)$ (if the limits exist, which is true for nondecreasing functions).

the *Kronecker delta* and write $\mathbf{1}_S$ for the *indicator function* of a set S. For $l \in \mathbb{N}$, we label the coordinates of a vector $u \in \mathbb{R}^l$ from 0 to $l-1$, i.e., $u = (u_i)_{i=0,\dots,l-1}$. We write $\|\eta\|_\infty = \sup\{|\eta(s)| : s \in S\}$ for the supremum norm of a bounded, real-valued function η with domain S, and $\|\eta\|_{\infty,T} = \sup\{|\eta(s)| : s \in T\}$ for the supremum norm of η on $T \subset S$. For two real-valued functions η_1 and η_2 defined on a set S, $\eta_1 \leq \eta_2$ means that $\eta_1(s) \leq \eta_2(s)$ for all $s \in S$. The set of real-valued, bounded and continuous functions on a topological space S is denoted $C_b(S)$. For a Lebesgue measurable set $S \subset \mathbb{R}$, we let $L^1(S)$ denote the space of functions on S that are integrable with respect to the restriction of Lebesgue measure, and we write $\|\cdot\|_{L^1(S)}$ for the corresponding norm. The *convolution* $\eta_1 * \eta_2 : \mathbb{R} \to \mathbb{R}$ of two functions $\eta_1 : \mathbb{R} \to \mathbb{R}$ and $\eta_2 : \mathbb{R} \to \mathbb{R}$ is defined as $(\eta_1 * \eta_2)(x) = \int_{\mathbb{R}} \eta_1(x-y)\eta_2(y)dy$, provided that the integral exists for all x.

$P[\cdot]$ denotes the probability of an event, and $E[\cdot]$, $E[\cdot|\cdot]$ and $Var[\cdot]$ designate expectations, conditional expectations, and variances of random variables. We write $U(0,1)$ for the uniform distribution on $(0,1)$, and φ_{t,σ^2} for the density of a normal distribution with mean $t \in \mathbb{R}$ and standard deviation σ, i.e., $\varphi_{t,\sigma^2}(y) = \frac{1}{\sigma}\varphi\left(\frac{y-t}{\sigma}\right)$, where φ is the density of the standard normal distribution, with c.d.f. Φ.

Definition 1. *Let H denote the c.d.f. of a probability measure on $\mathbb{R}$ that is absolutely continuous with respect to Lebesgue measure, has support $[\underline{x}, \overline{x}]$ for some $\underline{x}, \overline{x}$ satisfying $-\infty < \underline{x} < \overline{x} < \infty$, and whose density h is strictly positive on $[\underline{x}, \overline{x}]$.*

As H is continuous and strictly increasing on $[\underline{x}, \overline{x}]$, its quantile function is just the usual inverse H^{-1} of H (restricted to $[\underline{x}, \overline{x}]$). We define, for all $p \in [0,1]$,

$$x_p := H^{-1}(p). \tag{5}$$

We reserve the notation $m_p = F^{-1}(p)$ and $w_p = G^{-1}(p)$ for the quantiles of the distributions F and G, respectively.

Bachmann-Landau notation

We make extensive use of the following standard definitions. For two sequences (x_l) in $\mathbb{R}$ and (y_l) in $\mathbb{R}_+$, $x_l = o(y_l)$ if and only if $\lim_{l\to\infty} x_l/y_l = 0$, $x_l = O(y_l)$ if and only if $\limsup_{l\to\infty} |x_l|/y_l < \infty$, $x_l = \Omega(y_l)$ if and only if $\liminf_{l\to\infty} x_l/y_l > 0$,[23] and (if the elements of (x_l) are nonnegative) $x_l = \Theta(y_l)$ if and only if $x_l = O(y_l)$ and $x_l = \Omega(y_l)$.

We also define one non-standard piece of notation for sequences that decay exponentially with some positive power of l: for a sequence (x_l) in $\mathbb{R}$,

$$x_l = \mathcal{E}(l) \quad \text{if and only if there exists } \alpha > 0 \text{ such that } -\ln|x_l| = \Omega(l^\alpha). \tag{6}$$

[23] This is Knuth's definition of Ω.

Thus, $x_l = \mathcal{E}(l)$ if and only if there exist $\gamma > 0$ and $\alpha > 0$ such that $x_l = O\left(e^{-\gamma l^{\alpha}}\right)$.

Order Statistics

The *Bernstein basis polynomials* $B_{i,l} : [0,1] \to \mathbb{R}$ of degree $l \in \mathbb{N} \cup \{0\}$ are defined as $B_{i,l}(u) := \binom{l}{i} u^i (1-u)^{l-i}$ for $0 \le i \le l$. We let $H_{i:l}$ and $h_{i:l}$ denote the c.d.f. and density of $X_{i:l}$, where $X_{1:l} \le \ldots \le X_{l:l}$ are the *order statistics* of $l \in \mathbb{N}$ i.i.d. draws from H. Thus, we have for any $1 \le i \le l$ and $x \in \mathbb{R}$ (see for instance chapter 1.3 in Reiss 1989):

$$H_{i:l}(x) = \sum_{j=i}^{l} B_{j,l}(H(x)) \tag{7}$$

$$h_{i:l}(x) = l h(x) B_{i-1,l-1}(H(x)). \tag{8}$$

For notational purposes we also define $X_{0:l} \equiv \underline{x}$ and $H_{0:l} = \mathbf{1}_{[\underline{x},\infty)}$, for all $l \ge 0$.

Moreover, we always denote order statistics of i.i.d. draws from F, G and $U(0,1)$ by $M_{i:l}$, $W_{i:l}$ and $U_{i:l}$, respectively. The mean of $U_{i:l}$ is

$$\mu_{i,l} := E[U_{i:l}] = \frac{i}{l+1}. \tag{9}$$

We also define the following "approximate standard deviations," which are used to state the results about exponential bounds and about approximate distributions of order statistics that we borrow from the literature (Lemma 4 and Theorem 5 in Section 3.2.1).[24]

$$a_{i,l} := \left(\frac{i(l+1-i)}{(l+1)^2 l}\right)^{\frac{1}{2}} = \left(\frac{B_{1,2}(\mu_{i,l})}{2l}\right)^{\frac{1}{2}} = \left(\frac{l+2}{l} Var[U_{i:l}]\right)^{\frac{1}{2}}. \tag{10}$$

Note that $\mu_{i,l} = 1 - \mu_{l+1-i,l}$ and $a_{i,l} = a_{l+1-i,l}$, for all $1 \le i \le l$.

For a c.d.f. H, $l \in \mathbb{N}$, and a nondecreasing function $\eta : [\underline{x}, \overline{x}] \to \mathbb{R}$, we define the vector $d^{\eta}_{H,l} = \left(d^{\eta}_{H,l,i}\right)_{i=0,\ldots,l-1} \in \mathbb{R}^l$ as follows:

$$\text{For all } \ 0 \le i \le l-1, \quad d^{\eta}_{H,l,i} := E[\eta(X_{i+1:l}) - \eta(X_{i:l})].$$

Thus, if X is a random variable with c.d.f. H, the entries of $d^{\eta}_{H,l}$ are the *expectations* of the *spacings* (i.e., differences between adjacent order statistics) for l i.i.d. draws from the distribution of $\eta(X)$. For a c.d.f. H, $l \in \mathbb{N}$ and $1 \le i \le l$, we define $\xi_{H,l,i} : \mathbb{R} \to \mathbb{R}$ as

$$\xi_{H,l,i}(x) := (x - x_{\mu_{i,l}}) \frac{h(x_{\mu_{i,l}})}{a_{i,l}}. \tag{11}$$

Note that the affine transformation $\xi_{H,l,i}$ "approximately standardizes" the random vari-

[24]For the last identity in (10) see, for instance, chapter 1.7 in Reiss (1989).

able $X_{i:l}$.[25] We let $\check{h}_{i:l}$ denote the density of $\xi_{H,l,i}(X_{i:l})$, i.e.,

$$h_{i:l}(x) = \frac{h(x_{\mu_{i,l}})}{a_{i,l}} \check{h}_{i:l}\left(\xi_{H,l,i}(x)\right). \tag{12}$$

Finally, recall that $(X_{i_l:l})$ is a sequence of *central* order statistics if $i_l = \Omega(l)$ and $l - i_l = \Omega(l)$, a sequence of *lower* (*upper*) *extreme order statistics* if there is some constant $K \geq 0$ such that $i_l \leq K$ for all $l \in \mathbb{N}$ ($l - i_l \leq K$ for all $l \in \mathbb{N}$), and a sequence of *lower* (*upper*) *intermediate* order statistics if $i_l \to \infty$ and $i_l/l \to 0$ ($l - i_l \to \infty$ and $i_l/l \to 1$).

2.4 Equilibrium existence and basic characterization

Given a strictly increasing strategy $\beta_W : [\underline{w}, \overline{w}] \to \mathbb{R}_+$ for women, the strategic interaction among men in the game $\Gamma(n,k)$ is equivalent to an all-pay contest with incomplete information, n ex ante symmetric contestants and k heterogeneous prizes. Indeed, given the standard multiplicative form of the utility functions, the "prizes" that men are competing for are described by the expectations of the random variables $v_M(\beta_W(W_{i:k}))$. Moreover, the unique symmetric equilibrium of the all-pay contest is strictly monotone and has a closed form (Hoppe, Moldovanu and Sela 2009). An analogous observation applies for women, and we easily obtain the following basic relationship between men's and women's equilibrium strategies.

Lemma 1. *A pair* (β_M,β_W) *of strictly increasing functions* $\beta_M : [\underline{m}, \overline{m}] \to \mathbb{R}_+$ *and* $\beta_W : [\underline{w}, \overline{w}] \to \mathbb{R}_+$ *is a SSMBNE of the game* $\Gamma(n,k)$ *if and only if* β_M *and* β_W *satisfy the following system of equations. For all* $m \in [\underline{m}, \overline{m}]$ *and* $w \in [\underline{w}, \overline{w}]$,

$$\beta_M(m) = \int_{\underline{m}}^{m} s \sum_{i=0}^{k-1} f_{n-k+i:n-1}(s) \left(d_{G,k,i}^{v_M \circ \beta_W} + v_M(0)\delta_{i0}\right) ds \tag{13}$$

$$\beta_W(w) = \int_{\underline{w}}^{w} s \sum_{i=1}^{k-1} g_{i:k-1}(s) d_{F,n,n-k+i}^{v_W \circ \beta_M}. \tag{14}$$

To understand (13), note that if a man with type m matches with the $(i+1)$th lowest ranked partner rather than with the ith lowest one, where $0 \leq i \leq k-1$ and $i = 0$ corresponds to remaining unmatched, his expected (gross) utility increases by $mE[v_M(\beta_W(W_{i+1:k})) - v_M(\beta_W(W_{i:k}))] = md_{G,k,i}^{v_M \circ \beta_W}$ if $i \geq 1$, and by $mE[v_M(\beta_W(W_{1:k}))]$ if $i = 0$, which (using $W_{0:k} = \underline{w}$ and $\beta_W(\underline{w}) = 0$) is equal to $m\left(d_{G,k,0}^{v_M \circ \beta_W} + v_M(0)\right)$. In equilibrium, a man whose type is m receives the ith one of these utility increments with probability $F_{n-k+i:n-1}(m)$, the probability that at least $n-k+i$ of the other $n-1$ men have a type below m. The identity (13) then follows from a straightforward application of

[25] As h is continuous, $x_{\mu_{i,l}}$ and $a_{i,l}/h(x_{\mu_{i,l}})$ are good approximations of the mean and standard deviation of $X_{i:l}$ when l is large.

the standard first-order approach. An analogous argument for women (who get at least a match with the type $M_{n-k+1:n}$) yields (14).

The existence of a SSMBNE follows from an argument akin to those given in Peters (2007, 2011). First, it is easy to see that equilibrium investments must be bounded by the greatest element of $\mathcal{I}(\overline{m}, \overline{w})$:

Lemma 2. *For any SSMBNE (β_M, β_W) of a game $\Gamma(n,k)$, it holds that*

$$\beta_M(\overline{m}) \leq \bar{b}_M(\overline{m}, \overline{w}) \ \text{ and } \ \beta_W(\overline{w}) \leq \bar{b}_W(\overline{m}, \overline{w}). \tag{15}$$

The SSMBNE existence problem can then be formulated as a (finite-dimensional) fixed-point problem for the vector $d_{G,k}^{v_M \circ b_W}$, and a simple application of Brouwer's Theorem yields the desired result.

Theorem 1. *For any $n > k \geq 2$, the game $\Gamma(n,k)$ has a SSMBNE.*

We note that Theorem 1 does not claim that there is a unique SSMBNE, and that there is no reason to expect uniqueness.

3 Equilibrium behavior in large finite economies

To formulate our results, we fix F, G, v_M, v_W and $r \in (0,1)$ and consider an arbitrary sequence $(\Gamma(n, k_n))$ of pre-match investment games satisfying $\lim_{n\to\infty} \bar{k}_n/n = r$, where

$$\bar{k}_n := n - k_n$$

is the number of men who stay unmatched. As mentioned in Section 2.2, the empirical type distributions for the games $(\Gamma(n, k_n))$ then converge to the type distribution of the r-continuum economy. Throughout, $(\beta_{M,n,k_n}, \beta_{W,n,k_n})$ denotes a SSMBNE of $\Gamma(n, k_n)$. With slight abuse of notation, we will identify equilibrium strategies with the trivial extensions (defined on $\mathbb{R}$) obtained by setting $\beta_{M,n,k_n}(m) = 0$ for $m < \underline{m}$, $\beta_{M,n,k_n}(m) = \beta_{M,n,k_n}(\overline{m})$ for $m > \overline{m}$, $\beta_{W,n,k_n}(w) = 0$ for $w < \underline{w}$ and $\beta_{W,n,k_n}(w) = \beta_{W,n,k_n}(\overline{w})$ for $w > \overline{w}$.

3.1 The main results

In large markets, men with types from intervals $[m_{\bar{k}_n/n} - Cn^{-\frac{1}{2}}, m_{\bar{k}_n/n} + Cn^{-\frac{1}{2}}]$, where $C > 0$ is any constant, face non-negligible uncertainty about whether they will be matched: the (interim) probability of getting matched (in any SSMBNE of $\Gamma(n, k_n)$) is $F_{\bar{k}_n:n-1}(m_{\bar{k}_n/n} - Cn^{-\frac{1}{2}})$ for the type $m_{\bar{k}_n/n} - Cn^{-\frac{1}{2}}$, the (interim) probability of remaining unmatched is $1 - F_{\bar{k}_n:n-1}(m_{\bar{k}_n/n} + Cn^{-\frac{1}{2}})$ for the type $m_{\bar{k}_n/n} + Cn^{-\frac{1}{2}}$, and both of these terms are of order $\Omega(1)$. By contrast, for $\alpha > \frac{1}{2}$, the probabilities $F_{\bar{k}_n:n-1}(m_{\bar{k}_n/n} - n^{\alpha-1})$

and $1 - F_{\bar{k}_n:n-1}(m_{\bar{k}_n/n} + n^{\alpha-1})$ are of order $\mathcal{E}(n)$, i.e., they decay exponentially fast. Relatedly, for any $\alpha < 1$, the conditional (interim) distributions over ranks that types $m \in [m_{\bar{k}_n/n} - n^{\alpha-1}, m_{\bar{k}_n/n} + n^{\alpha-1}]$, which we shall also refer to as types of men that are *close to the margin* in the game $\Gamma(n, k_n)$, attain among men's realized types concentrate on integer intervals whose cardinality is of order "just above" $\Theta\left(n^{\frac{1}{2}}\right)$. These simple facts are due to basic concentration properties of *central* order statistics, which have standard deviations of order $\Theta\left(n^{-\frac{1}{2}}\right)$ (while the expected difference between two neighboring order statistics is of order $\Theta\left(n^{-1}\right)$).

Another simple but important observation is that the types of women who are likely to match with types of men close to the margin face much lower uncertainty about their rank (among women's realized types) than these men. The reason is that these types on the short side, very close to $\underline{w}$, are very likely among certain *lower intermediate or extreme* order statistics, which have much smaller standard deviations than central order statistics. Using these observations, it is not difficult to show that the derivatives β'_{M,n,k_n} and β'_{W,n,k_n} are of order $O\left(n^{\frac{1}{2}}\right)$ close to the margin and close to $\underline{w}$, and in fact of order $\Theta\left(n^{\frac{1}{2}}\right)$ on intervals of the form $[m_{\bar{k}_n/n} - Cn^{-\frac{1}{2}}, m_{\bar{k}_n/n} + Cn^{-\frac{1}{2}}]$ and $[\underline{w}, \underline{w} + Cn^{-\frac{1}{2}}]$ (see Lemma 6 and footnote 36 in Section 3.2.2). The central, much more difficult task is then to characterize the equilibrium behavior of types of men (very) close to the margin and types of women (very) close to $\underline{w}$, i.e., in the part of the type space where the equilibrium strategies become very steep, in a way that is endogenously determined by the strategic interaction induced by the bilateral external benefits of investment.

Maybe surprisingly, this is possible. We shift and re-scale men's type space by suitable affine transformations, so that men's *re-scaled equilibrium strategies* $\tilde{\beta}_{M,n,k_n}$ (the strategies as functions of the re-scaled types) satisfy a uniform Lipschitz bound. Consequently, sequences of re-scaled equilibrium strategies have locally uniformly convergent subsequences.[26] Our central result, Theorem 2, then shows that any function that is the limit of a subsequence of re-scaled equilibrium strategies solves a particular fixed point (nonlinear integral) equation.

The proof of Theorem 2, and in particular the proof of the underlying asymptotic bounds that we establish in Theorem 6 (see Section 3.2.2), constitutes the paper's main methodological contribution. Using advanced results about approximate distributions of order statistics in a novel way, we provide a precise asymptotic characterization of equilibrium behavior in a discontinuous game, for types close to a point where a discontinuity must develop in the limit, in a setting where the size of the discontinuity is the main object of interest and is determined by a complex, two-sided interaction.

[26]Recall that a sequence of functions on $\mathbb{R}$ (or any locally compact space) converges locally uniformly if and only if it converges uniformly on every compact set.

We will present the required tools in Section 3.2.1, and give an overview of the main steps in the proofs of Theorem 6 and Theorem 2 in Section 3.2.2. The proofs of all results presented in Section 3.2.2 are given in Appendix C and Online Appendix G. In the present section, we only state Theorem 2 and provide intuition for the result.

Based on Theorem 2, we will then proceed and establish our second main result, Theorem 3, which shows that types very close to (and above) $m_{\bar{k}_n/n}$ and $\underline{w}$ must inefficiently *overinvest* in sufficiently large markets. Types close to $m_{\bar{k}_n/n}$ compete not only to get a partner with a higher investment but also to avoid staying unmatched. Our analysis shows very clearly why this additional incentive must always lead to overinvestment when utility is nontransferable ex post and the limit economy is not exactly balanced (see below). Our final result, Theorem 4, which identifies non-empty intervals of types above m_r and $\underline{w}$ who overinvest will then be a simple corollary.

We use the affine functions $\xi_{F,n-1,\bar{k}_n}$ to define men's re-scaled strategies $\tilde{\beta}_{M,n,k_n}$. For all $z \in \mathbb{R}$,

$$\tilde{\beta}_{M,n,k_n}(z) := \beta_{M,n,k_n}\left(\xi^{-1}_{F,n-1,\bar{k}_n}(z)\right). \tag{16}$$

Recall that, by (9) and (11),

$$\xi_{F,n-1,\bar{k}_n}(m) = \left(m - m_{\bar{k}_n/n}\right)\frac{f(m_{\bar{k}_n/n})}{a_{\bar{k}_n,n-1}}.$$

Thus, in the game $\Gamma(n,k_n)$, the re-scaled type $z \in \mathbb{R}$ corresponds to the original type $m = m_{\bar{k}_n/n} + z\frac{a_{\bar{k}_n,n-1}}{f(m_{\bar{k}_n/n})}$. The factor $\frac{f(m_{\bar{k}_n/n})}{a_{\bar{k}_n,n-1}}$ is of order $\Theta\left(n^{\frac{1}{2}}\right)$, because $a_{\bar{k}_n,n-1} = \Theta\left(n^{-\frac{1}{2}}\right)$ (see Lemma 5 (iii)) and

$$0 < \min_{m\in[\underline{m},\overline{m}]} f(m) \le \|f\|_\infty < \infty. \tag{17}$$

Note also that if (ε_n) is a sequence satisfying $\varepsilon_n = \Theta\left(n^{\alpha-1}\right)$ for some $\alpha \in \left(\frac{1}{2},1\right)$, then

$$-\xi_{F,n-1,\bar{k}_n}(m_{\bar{k}_n/n} - \varepsilon_n) = \Theta\left(n^{\alpha-\frac{1}{2}}\right) \quad \text{and} \quad \xi_{F,n-1,\bar{k}_n}(m_{\bar{k}_n/n} + \varepsilon_n) = \Theta\left(n^{\alpha-\frac{1}{2}}\right). \tag{18}$$

Thus, the intervals $[m_{\bar{k}_n/n}-\varepsilon_n, m_{\bar{k}_n/n}+\varepsilon_n]$ correspond to intervals of re-scaled types whose boundaries tend to infinity like $n^{\alpha-\frac{1}{2}}$. The functions $\tilde{\beta}_{M,n,k_n}$ satisfy a common Lipschitz bound on such increasingly large intervals (see Corollary 3 in Section 3.2.2), so that the following lemma is a simple consequence of the Arzèla-Ascoli Theorem.

Lemma 3. *Any subsequence of a sequence of re-scaled strategies $(\tilde{\beta}_{M,n,k_n})$ has a subsequence that converges locally uniformly.*

We let $\mathcal{L}$ denote the set of all accumulation points (with respect to the topology of locally uniform convergence) of sequences of re-scaled equilibrium strategies:

$$\mathcal{L} := \left\{\tilde{\beta} \in C_b(\mathbb{R}) | \, \tilde{\beta} \text{ is an accumulation point of a sequence } (\tilde{\beta}_{M,n,k_n})\right\}. \tag{19}$$

We now define the operators that we need to state Theorem 2. Let

$$\mathcal{A}_1 := \left\{\tilde{\beta} \in C_b(\mathbb{R}) | \, \tilde{\beta} \text{ is nondecreasing, Lipschitz continuous, and } \lim_{y \to -\infty} \tilde{\beta}(y) = 0\right\},$$
$$\mathcal{A}_2 := \left\{\tilde{\zeta} \in C_b(\mathbb{R}_+) | \, \tilde{\zeta} \text{ is nondecreasing, Lipschitz continuous, and } \tilde{\zeta}(0) = 0\right\}.$$

We let $\Xi_1 : \mathcal{A}_1 \to \mathcal{A}_2$ and $\Xi_2 : \mathcal{A}_2 \to \mathcal{A}_1$ denote the following operators:

$$\text{For all } x \in \mathbb{R}_+ : \quad \Xi_1[\tilde{\beta}](x) = \underline{w}\left(\varphi * (v_W \circ \tilde{\beta})(x) - \varphi * (v_W \circ \tilde{\beta})(0)\right). \tag{20}$$
$$\text{For all } z \in \mathbb{R} : \quad \Xi_2[\tilde{\zeta}](z) = m_r \int_0^\infty \varphi(z - x)(v_M \circ \tilde{\zeta})(x) dx. \tag{21}$$

Using Assumption 1 and basic properties of the convolution, it is clear that Ξ_1 indeed maps $\mathcal{A}_1$ into $\mathcal{A}_2$, Ξ_2 maps $\mathcal{A}_2$ into $\mathcal{A}_1$, and that the functions $\Xi_1[\tilde{\beta}]$ and $\Xi_2[\tilde{\zeta}]$ are smooth, for any $\tilde{\beta} \in \mathcal{A}_1$ and any $\tilde{\zeta} \in \mathcal{A}_2$. Defining

$$\Psi_1 := \Xi_2 \circ \Xi_1 \quad \text{and} \quad \Psi_2 := \Xi_1 \circ \Xi_2,$$

we are ready to state our first main result.

Theorem 2. *Every $\tilde{\beta} \in \mathcal{L}$, i.e., every function that is the (locally uniform) limit of some subsequence of men's re-scaled equilibrium strategies solves the nonlinear integral equation*

$$\tilde{\beta} = \Psi_1[\tilde{\beta}]. \tag{22}$$

The fixed point equation (22) is of course equivalent to the fixed point equation

$$\tilde{\zeta} = \Psi_2[\tilde{\zeta}]. \tag{23}$$

The operators Ξ_1 and Ξ_2 may look complicated at first sight, but Theorem 2 has a very clear interpretation. To explain this, let us assume (counterfactually) that a given function $\tilde{\beta} \in \mathcal{A}_1$ describes men's re-scaled strategy in all games $\Gamma(n, k_n)$. In what follows, we describe the intuition for why similarly re-scaled symmetric equilibrium strategies for the resulting contests among women, expressed in types $x \geq 0$ of the form $x = (w - \underline{w})d_n$ where the d_n are suitable scaling factors of order $\Theta\left(n^{\frac{1}{2}}\right)$, must then converge to $\Xi_1[\tilde{\beta}]$. Similarly, we explain why for a given $\tilde{\zeta} \in \mathcal{A}_2$ describing women's re-scaled strategy in all markets, the re-scaled symmetric equilibrium strategies of the resulting contests among men must converge to $\Xi_2[\tilde{\zeta}]$.

Remark 1. *We "de-couple" the approximation arguments for men and women only for the intuitive explanation of the different parts of the fixed point operator. The proof of Theorem 6 uses only the actual equilibrium strategies, and shows how close the derivatives of men's re-scaled strategies are to solving the fixed point equation* $\tilde{\beta}' = \frac{m_{\bar{k}_n/n}}{m_r}\Psi_1[\tilde{\beta}]'$*. In particular, it will not be necessary to specify the factors* d_n *for the re-scaling of women's types explicitly (they would of course depend on* $\xi_{F,n-1,\bar{k}_n}$ *and* $g(\underline{w})$*).*

The formulas for Ξ_1 and Ξ_2 reflect three facts. First, the relationship between a man's re-scaled type and his "approximate rank," measured in multiples of the factor $na_{\bar{k}_n,n-1} = \Theta\left(n^{\frac{1}{2}}\right)$ above or below the rank $\bar{k}_n$ (from the bottom), remains uncertain as $n \to \infty$. Secondly, the relationship between a woman's re-scaled type and her "approximate rank" becomes deterministic in the limit: in large markets, the rank (from the bottom, among women's realized types) attained by a woman with re-scaled type x is very likely to be very close to $xna_{\bar{k}_n,n-1}$. This is a consequence of the above mentioned fact that the distributions of extreme and lower intermediate order statistics are much more concentrated than those of central order statistics. Thirdly, standardized central order statistics are asymptotically normal, which explains the occurrence of φ.

More precisely, if (i_n) is a sequence satisfying $i_n - xna_{\bar{k}_n,n-1} = o\left(n^{\frac{1}{2}}\right)$ for some $x \in \mathbb{R}_+$, then the distributions of the re-scaled order statistics $\xi_{F,n-1,\bar{k}_n}(M_{\bar{k}_n+i_n:n})$ converge to a normal distribution with mean x and variance 1. Thus, given some $\tilde{\beta} \in \mathcal{A}_1$, the expectation of $v_W \circ \tilde{\beta}$ with respect to the conditional distribution over the re-scaled types of partners for a woman with re-scaled type x, for whom the probability of matching with a partner whose rank is not within an $o\left(n^{\frac{1}{2}}\right)$ range around $\bar{k}_n + xna_{\bar{k}_n,n-1}$ decays exponentially fast, converges to $\int_{\mathbb{R}} \varphi(x-y)v_W(\tilde{\beta}(y))dy = \varphi * (v_W \circ \tilde{\beta})(x)$. In the limit, any given re-scaled type x must become indifferent between her equilibrium investment and the zero investment (which is the equilibrium investment for $x = 0$), because the corresponding original types converge to $\underline{w}$. This explains the formula for $\Xi_1[\tilde{\beta}]$. Next, if a given $\tilde{\zeta} \in \mathcal{A}_2$ describes women's strategy, then the expectation of $v_M \circ \tilde{\zeta}$ with respect to the conditional distribution over partners for a man with re-scaled type z is approximately $\int_0^\infty \varphi(z-x)(v_M \circ \tilde{\zeta})(x)dx$ in large markets. Again, this is a consequence of the asymptotic normality of men's order statistics, and of the fact that a woman with re-scaled type x is very likely to have a rank very close to $xna_{\bar{k}_n,n-1}$. In the limit, any re-scaled type z corresponds to types that converge to m_r and must therefore get the same expected utility from making his equilibrium investment and from investing 0 and remaining unmatched. This explains the formula for $\Xi_2[\tilde{\zeta}]$.

We reiterate that the above explanations are only meant to provide intuition for Theorem 2. The rigorous argument is described in Section 3.2.2, and all related proofs are given in the Appendix and the Online Appendix.

Definition 2. *Throughout the paper,* $(\tilde{\beta}_*, \tilde{\zeta}_*) \in \mathcal{A}_1 \times \mathcal{A}_2$ *denotes a pair consisting of a solution* $\tilde{\beta}_*$ *for* (22) *and of the corresponding solution for* (23)*, i.e.,* $\tilde{\zeta}_* = \Xi_1[\tilde{\beta}_*]$.

Note that Theorem 2 and Lemma 3 show in particular that a solution of (22) exists, which is a highly non-trivial result. The fixed point operators Ψ_1 and Ψ_2 are neither contractions nor monotone,[27] and topological fixed point theorems are also not easily applicable, because neither $\mathbb{R}$ nor $\mathbb{R}_+$ is compact.

While we cannot expect solutions to be unique, we show that *all* $(\tilde{\beta}_*, \tilde{\zeta}_*)$ share properties which imply that SSMBNE of sufficiently large pre-match investment games feature overinvestment. Let us define

$$\tilde{\beta}_*(\infty) := \lim_{z\to\infty} \tilde{\beta}_*(z) \quad \text{and} \quad \tilde{\zeta}_*(\infty) := \lim_{x\to\infty} \tilde{\zeta}_*(x).$$

These limits exist because $\tilde{\beta}_*$ and $\tilde{\zeta}_*$ are nondecreasing and bounded. Moreover, from (20) and (21), it is straightforward to see that

$$\tilde{\beta}_*(\infty) = m_r v_M(\tilde{\zeta}_*(\infty)) \;\text{ and }\; \tilde{\zeta}_*(\infty) = \underline{w}\left(v_W(\tilde{\beta}_*(\infty)) - \varphi * (v_W \circ \tilde{\beta}_*)(0)\right). \tag{24}$$

Thus, $(\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty)) \in \mathcal{I}_0^M(m_r)$, which reflects the obvious requirement that, in the limit, the type m_r must become indifferent between investing zero and staying unmatched and investing $\tilde{\beta}_*(\infty)$ and getting a partner with investment $\tilde{\zeta}_*(\infty)$ with certainty. Moreover, $(\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty)) \in \mathcal{I}(m_r, \underline{w})$, as $\underline{w}\varphi * (v_W \circ \tilde{\beta}_*)(0)$, which corresponds to the expected utility for the type $\underline{w}$ in the limit, is positive. In particular, it follows that

$$\tilde{\beta}_*(\infty) \leq \overline{b}_M(m_r, \underline{w}) \quad \text{and} \quad \tilde{\zeta}_*(\infty) \leq \overline{b}_W(m_r, \underline{w}). \tag{25}$$

We are now ready to state our second main result.

Theorem 3. *There is a constant* $c > 0$ *such that every solution* $\tilde{\zeta}_*$ *of* (23) *satisfies* $\tilde{\zeta}_*(\infty) \geq b_W^e(m_r, \underline{w}) + c$.

Given Theorem 2 and $(\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty)) \in \mathcal{I}_0^M(m_r)$, Theorem 3 has the following immediate implication. There are uniform constants (independent of $\tilde{\zeta}_*$) such that for any

[27] Monotonicity fails for several "independent" reasons, notably due to the strict concavity assumptions of Assumption 1, and because the term $\varphi * (v_W \circ \tilde{\beta})(0)$ is substracted in the definition of Ξ_1.

convergent subsequence of re-scaled equilibrium strategies, the investments of sufficiently large (but fixed) re-scaled types must eventually exceed the efficient investments $b^e_W(m_r, \underline{w})$ and $b^e_M(m_r, \underline{w})$ by these constants.[28]

We next explain the proof of Theorem 3, which reveals why the combination of an unbalanced limit economy and NTU matching must lead to overinvestment. The formal overinvestment result that is implied for the limits of convergent subsequences of original (not re-scaled) SSMBNE strategies is given in Theorem 4 below. Defining

$$\lambda_{\tilde{\zeta}_*} := m_r \underline{w} v'_M(\tilde{\zeta}_*(\infty)) v'_W(\tilde{\beta}_*(\infty)), \tag{26}$$

we note first that $(\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty)) \in \mathcal{I}^M_0(m_r)$, $m_r \underline{w} v'_M(b^e_W(m_r, \underline{w})) v'_W(b^e_M(m_r, \underline{w})) = 1$ and Assumption 1 together imply that

$$\tilde{\zeta}_*(\infty) > b^e_W(m_r, \underline{w}) \quad \text{if and only if} \quad \lambda_{\tilde{\zeta}_*} < 1. \tag{27}$$

Similarly, the existence of a constant $c > 0$ such that $\tilde{\zeta}_*(\infty) \geq b^e_W(m_r, \underline{w}) + c$ holds for all $\tilde{\zeta}_*$ is equivalent to the existence of a constant $\bar{\lambda}$ such that

$$\lambda_{\tilde{\zeta}_*} \leq \bar{\lambda} < 1 \quad \text{for all } \tilde{\zeta}_*. \tag{28}$$

To show (28), we rely on certain results from the theory of convolution equations on the half-line, also known as Wiener-Hopf equations, that are surveyed in Arabadzhyan and Engibaryan (1987). Given some kernel function $K \in L^1(\mathbb{R})$, $K \geq 0$, let $\mathcal{K}$ denote the linear integral operator defined by

$$(\mathcal{K}\eta)(x) = \int_0^\infty K(x - y)\eta(y)dy \quad \text{for all } x \in \mathbb{R}_+, \tag{29}$$

which maps any Lebesgue measurable function $\eta : \mathbb{R}_+ \to \mathbb{R}$ for which the integral $\int_0^\infty K(x - y)\eta(y)dy$ exists for all $x \in \mathbb{R}_+$ to a real-valued function on $\mathbb{R}_+$. A linear integral equation of the form

$$(Id - \mathcal{K})\eta = R, \tag{30}$$

where Id is the identity, $R \in L^1(\mathbb{R}_+)$, $R \neq 0$, and η is the unknown (not necessarily in $L^1(\mathbb{R}_+)$) is called *inhomogeneous Wiener-Hopf equation*. If $||K||_{L^1(\mathbb{R})} < 1$, equation (30) is called *nonsingular*. In this case, the operator $\mathcal{K}$ is a contraction in $L^1(\mathbb{R}_+)$. If $||K||_{L^1(\mathbb{R})} = 1$, equation (30) is called *conservative*. In this case, the operator $Id - \mathcal{K}$ is non-invertible as an endomorphism of $L^1(\mathbb{R}_+)$.

[28]How large these re-scaled types have to be might still depend on $\tilde{\zeta}_*$, but they correspond of course always to original types that converge to $\underline{w}$ and m_r.

We now outline how we use the theory for these linear equations to prove (28) and hence Theorem 3. The detailed proof is given in Appendix B.

If $\tilde{\zeta}_*$ solves (23), then its derivative $\tilde{\zeta}_*'$ solves the fixed point equation $\tilde{\zeta}_*' = \Psi_2[\tilde{\zeta}_*]'$. After a straightforward calculation, this yields the following identity, for all $x \in \mathbb{R}_+$:

$$\tilde{\zeta}_*'(x) = m_r \underline{w} \int_{\mathbb{R}} \varphi(x-z) v_W' \left(\Xi_2[\tilde{\zeta}_*](z)\right) \left(\varphi(z) v_M(0) + \int_0^\infty \varphi(z-y) v_M' \left(\tilde{\zeta}_*(y)\right) \tilde{\zeta}_*'(y) dy\right) dz. \tag{31}$$

Equation (31) is highly nonlinear in $\tilde{\zeta}_*'$, but another elementary calculation shows that $\tilde{\zeta}_*'$ coincides with a strictly positive[29] solution of a Wiener-Hopf equation of the form

$$\eta(x) = \min\left(1, \lambda_{\tilde{\zeta}_*}\right) \int_0^\infty \varphi_{0,2}(x-y) \eta(y) dy + R_{\tilde{\zeta}_*}(x), \tag{32}$$

where the inhomogeneous term $R_{\tilde{\zeta}_*} \in L^1(\mathbb{R}_+)$, which depends of course on $\tilde{\zeta}_*$, is strictly positive and given by (57) in Appendix B. This observation is useful for the following reasons. Equation (30) has a unique, integrable solution if $\|K\|_{L^1(\mathbb{R})} < 1$, but solutions also exist if $\|K\|_{L^1(\mathbb{R})} = 1$, for any $R \in L^1(\mathbb{R}_+)$, $R \neq 0$. Moreover, if $R \geq 0$ (and $K \geq 0$), the conservative equation has a positive solution that is minimal, with respect to the pointwise order, among all positive solutions. However, this solution is *not* integrable. As we know that $\tilde{\zeta}_*'$, which satisfies $\|\tilde{\zeta}_*'\|_{L^1(\mathbb{R}_+)} = \tilde{\zeta}_*(\infty) \leq \overline{b}_W(m_r, \underline{w})$, coincides with a positive solution of the Wiener-Hopf equation (32), and as $\|\varphi_{0,2}\|_{L^1(\mathbb{R})} = 1$ and $R_{\tilde{\zeta}_*} \geq 0$ and $R_{\tilde{\zeta}_*} \neq 0$, it follows that $\lambda_{\tilde{\zeta}_*}$ must be strictly smaller than 1.

To prove the uniform bound (28), we show that there is a function $\underline{R} \neq 0$ satisfying $0 \leq \underline{R} \leq R_{\tilde{\zeta}_*}$ for all $\tilde{\zeta}_*$. This allows us to invoke two additional results about the solutions $\eta_{\lambda,K,R} \in L^1(\mathbb{R}_+)$ of the nonsingular equations $(Id - \lambda\mathcal{K})\eta = R$, where $K \geq 0$, $\|K\|_{L^1(\mathbb{R})} = 1$ and $\lambda \in (0,1)$. First, if $R_1, R_2 \in L^1(\mathbb{R}_+)$ satisfy $0 \leq R_1 \leq R_2$, then $\eta_{\lambda,K,R_1} \leq \eta_{\lambda,K,R_2}$. Secondly, for any $R \in L^1(\mathbb{R}_+)$, $R \neq 0$, $R \geq 0$, if we let λ tend to 1 monotonically, the functions $\eta_{\lambda,K,R}$ converge monotonically to the (non-integrable) minimal positive solution of the conservative equation $(Id - \mathcal{K})\eta = R$. From these facts and $\|\tilde{\zeta}_*'\|_{L^1(\mathbb{R}_+)} \leq \overline{b}_W(m_r, \underline{w})$, it is immediate that (28) holds for

$$\bar{\lambda} = \inf\left\{\lambda \in (0,1) : \|\eta_{\lambda,\varphi_{0,2},\underline{R}}\|_{L^1(\mathbb{R}_+)} > \overline{b}_W(m_r, \underline{w})\right\} < 1, \tag{33}$$

and Theorem 3 follows.

As mentioned above, the formal overinvestment result for limits of convergent subsequences of original SSMBNE strategies is a simple corollary of Theorems 2 and 3. For $\tilde{\beta}_* \in \mathcal{L}$, let $p(\tilde{\beta}_*) := \inf\left\{p \in [0,1] : (\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty)) \in \mathcal{P}(m_{r+p(1-r)}, w_p)\right\}$, with the

[29] From (20), (21) and $v_M(0) > 0$, it is clear that $\tilde{\zeta}_*'$ is strictly positive.

convention $p(\tilde{\beta}_*) = 1$ if the set over which the infimum is taken is empty (i.e., if the investments $(\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty))$ are inefficiently high even for the pair $(\overline{m}, \overline{w})$). Theorem 3 implies $\inf_{\tilde{\beta}_* \in \mathcal{L}} p(\tilde{\beta}_*) > 0$. If a pair of nondecreasing functions $(\beta_{M,\infty}, \beta_{W,\infty})$ is the limit (pointwise at continuity points) of some convergent subsequence of SSMBNE, then $\beta^+_{M,\infty}(m_r) = \tilde{\beta}_*(\infty)$ and $\beta^+_{W,\infty}(\underline{w}) = \tilde{\zeta}_*(\infty)$ for some $(\tilde{\beta}_*, \tilde{\zeta}_*)$, and Theorem 4 pins down $\beta_{M,\infty}$ and $\beta_{W,\infty}$ for types below $m_{r+p(\tilde{\beta}_*)(1-r)}$ and $w_{p(\tilde{\beta}_*)}$.

Theorem 4. *Let $\tilde{\beta}_* \in \mathcal{L}$, and let $((\beta_{M,n_l,k_{n_l}}, \beta_{W,n_l,k_{n_l}}))$ be any subsequence of SSMBNE such that $(\tilde{\beta}_{M,n_l,k_{n_l}})$ converges to $\tilde{\beta}_*$.*[30] *Then, the sequences $(\beta_{M,n_l,k_{n_l}})$ and $(\beta_{W,n_l,k_{n_l}})$ converge pointwise on $[\underline{m}, m_{r+p(\tilde{\beta}_*)(1-r)}) \setminus \{m_r\}$ and $(\underline{w}, w_{p(\tilde{\beta}_*)})$, and the limits (on these sets) $\beta_{M,\infty} = \lim_{l\to\infty} \beta_{M,n_l,k_{n_l}}$ and $\beta_{W,\infty} = \lim_{l\to\infty} \beta_{W,n_l,k_{n_l}}$ satisfy*

$$\beta_{M,\infty}(m) = 0 \ \text{for} \ m < m_r,$$
$$(\beta_{M,\infty}(m_{r+p(1-r)}), \beta_{W,\infty}(w_p)) = (\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty)) \ \text{for} \ p \in (0, p(\tilde{\beta}_*)).$$

Thus, all types in the intervals $(m_r, m_{r+p(\tilde{\beta}_*)(1-r)})$ and $(\underline{w}, w_{p(\tilde{\beta}_*)})$ overinvest in the limit.

It is now tempting to take it for granted that if $(\beta_{M,\infty}, \beta_{W,\infty})$ is the limit of a convergent subsequence of SSMBNE, satisfying $\beta^+_{M,\infty}(m_r) = \tilde{\beta}_*(\infty)$ for $\tilde{\beta}_* \in \mathcal{L}$, pairs above $(m_{r+p(\tilde{\beta}_*)(1-r)}, w_{p(\tilde{\beta}_*)})$ must make Pareto efficient investments,[31] but this is not warranted. If we knew that $\beta_{W,\infty}$ is strictly increasing on $[w_{p(\tilde{\beta}_*)}, \overline{w}]$, the result would indeed be straightforward, by the same reasoning that shows that hedonic equilibria are efficient (see Section 2.2). However, the strategies that converge to $\beta_{W,\infty}$ become extremely flat on $(\underline{w}, w_{p(\tilde{\beta}_*)})$,[32] and it is not clear if the intense local competition in large markets combined with the facts that SSMBNE strategies are never *completely* flat and that $m_{r+p(1-r)} w_p v'_M(\tilde{\zeta}_*(\infty)) v'_W(\tilde{\beta}_*(\infty)) > 1$ for $p > p(\tilde{\beta}_*)$ suffice to ensure that $\beta_{W,\infty}$ does not remain flat beyond $w_{p(\tilde{\beta}_*)}$. It is also unclear if $\beta'_{W,\infty}(w) > 0$ for some $w > w_{p(\tilde{\beta}_*)}$ ensures that $\beta_{W,\infty}$ remains strictly increasing for higher types.

These issues prohibit a more complete characterization of the limits of SSMBNE strategies in our model, but this is arguably of secondary importance. The phenomenon that agents might underinvest just because the strictly monotone strategy for agents on the other side of the market has parts that are almost flat appears very fragile. First, this kind of problem should disappear for any model with imperfectly transferable utility (ITU).[33] Secondly, the problem disappears even if we maintain the assumption of NTU

[30] Recall that any (sub)sequence of re-scaled equilibrium strategies has a convergent subsequence.

[31] Formally, $(\beta_{M,\infty}(m_{r+p(1-r)}), \beta_{W,\infty}(w_p)) \in \mathcal{P}(m_{r+p(1-r)}, w_p)$ for all $p \geq p(\tilde{\beta}_*)$. The corresponding Pareto efficient investments for pairs $(m_{r+p(1-r)}, w_p)$ with $p \geq p(\tilde{\beta}_*)$ are of course different from those for the pairwise efficient allocation (see Section 2.2).

[32] The same is true, of course, for men's strategies on the interval $(m_r, m_{r+p(\tilde{\beta}_*)(1-r)})$.

[33] The fact that agents cannot compensate partners *at all* for a higher investment (ex post) is part of

matching, but introduce a tiny bit of signaling (see Remark 3 in Online Appendix I).

3.2 Proving the limit characterization (Theorem 2)

3.2.1 The main tools: exponential bounds and a local limit theorem

Here we present the results about concentration properties and approximate distributions of order statistics that we need to prove Theorem 6 and hence Theorem 2. Recall that we assume throughout that a c.d.f. denoted by H satisfies the assumptions of Definition 1. We first note the following well-known identities (Theorem 1.2.5 in Reiss 1989): For $1 \leq i \leq l$,

$$X_{i:l} =^d H^{-1}(U_{i:l}) \quad \text{and} \quad H(X_{i:l}) =^d U_{i:l}, \tag{34}$$

where $=^d$ denotes equality in distribution. In particular, for any $p \in [0, 1]$,

$$H_{i:l}(x_p) = P[U_{i:l} \leq p]. \tag{35}$$

Lemma 4 (Lemma 3.1.1 in Reiss 1989)**.** *For every $y \geq 0$ and $1 \leq i \leq l$,*

$$P\left[|U_{i:l} - \mu_{i,l}| \geq a_{i,l}y\right] \leq 2e^{-\frac{y^2}{3(1+y/(la_{i,l}))}}. \tag{36}$$

Lemma 4 provides *exponential bounds* for the distributions of order statistics from a uniform parent. We need Lemma 4 to prove Corollaries 1 and 2 below, which we use to show the exponential decay of the probabilities of various "rare" events, for which a given type of agent ranks "much higher" or "much lower" than expected among all realized types on his or her side of the market. We start with some basic facts about the order of magnitude of the approximate standard deviations $a_{i,l}$.

Lemma 5. *(i) (Monotonicity): For $l \in \mathbb{N}$, $a_{i,l}$ is strictly increasing in i from 1 to $\lfloor \frac{l+1}{2} \rfloor$.*

(ii) (Magnitude for extreme and intermediate OS): Let $\alpha \in [0, 1)$, and let (i_l) be an integer sequence satisfying $i_l = \Theta(l^\alpha)$. Then, $a_{i_l,l} = \Theta\left(l^{\frac{\alpha}{2}-1}\right)$.

(iii) (Magnitude for central OS): Let (i_l) be an integer sequence satisfying $i_l = \Omega(l)$ and $l - i_l = \Omega(l)$. Then, $a_{i_l,l} = \Theta\left(l^{-\frac{1}{2}}\right)$.

Corollary 1 establishes concentration inequalities for distributions, and Corollary 2 provides related bounds for densities.[34]

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the reason for the enormous multiplicity of ex post equilibria in the NTU case (see Section 2.2 and Nöldeke and Samuelson 2014). This issue does not exist if agents have some ability to make ex post utility transfers (Nöldeke and Samuelson 2015). Extending our techniques to perform a rigorous analysis of Bayesian equilibria for a finite population model with ex ante symmetric agents and with an ITU or TU matching market is an interesting direction for future work, but is beyond the scope of this paper.

[34]We actually prove stronger results, quantifying the exponential rates of decay (Online Appendix F).

Corollary 1. *(i) (Extreme and intermediate OS) For* $\alpha_1 \in [0,1)$*, let* (i_l) *be an integer sequence such that* $i_l = O(l^{\alpha_1})$*. Let* $\alpha_2 > \frac{\alpha_1}{2}$*, and let* (ε_l) *be a sequence in* $\mathbb{R}_+$ *satisfying* $\varepsilon_l = \Omega\left(l^{\alpha_2 - 1}\right)$*. Then,*

$$\max_{1 \le i \le i_l} P\left[|U_{i:l} - \mu_{i,l}| \ge \varepsilon_l\right] = \mathcal{E}(l). \tag{37}$$

(ii) (Central OS) Let (i_l) *be an integer sequence such that* $i_l = \Omega(l)$ *and* $i_l \le \frac{l}{2}$*. Let* $\alpha_2 > \frac{1}{2}$*, and let* (ε_l) *be a sequence in* $\mathbb{R}_+$ *satisfying* $\varepsilon_l = \Omega\left(l^{\alpha_2 - 1}\right)$*. Then,*

$$\max_{i_l \le i \le l - i_l} P\left[|U_{i:l} - \mu_{i,l}| \ge \varepsilon_l\right] = \mathcal{E}(l). \tag{38}$$

The fact that (37) and (38) provide asymptotic bounds that hold *uniformly* over a certain range of order statistics (see the "max" operators) will be crucial for the arguments in Section 3.2.2. A similar remark applies for the bounds in Corollary 2.

Corollary 2. *Assume that h is bounded. Then, the following bounds apply:*

(i) (Extreme and intermediate OS) For $\alpha_1 \in [0,1)$*, let* (i_l) *be an integer sequence such that* $i_l = O(l^{\alpha_1})$*. Let* $\alpha_2 > \frac{\alpha_1}{2}$*, and let* (j_l) *be an integer sequence satisfying* $j_l = \Omega(l^{\alpha_2})$*. Moreover, let* (γ_l) *be a sequence in* $\mathbb{R}_+$ *such that* $\gamma_l = O(l^{-1})$*. Then,*

$$\max_{1 \le i \le i_l} \left(\max_{j \ge i + j_l} \|h_{j:l}\|_{\infty, [\underline{x}, x_{\mu_{i,l} + \gamma_l}]} + \max_{j \le i - j_l} \|h_{j:l}\|_{\infty, [x_{\mu_{i,l} - \gamma_l}, \overline{x}]} \right) = \mathcal{E}(l). \tag{39}$$

(ii) (Central OS) Let (i_l) *be an integer sequence satisfying* $i_l = \Omega(l)$ *and* $i_l \le \frac{l}{2}$*. For* $\frac{1}{2} < \alpha_2 < 1$*, let* (j_l) *be an integer sequence such that* $j_l = \Omega(l^{\alpha_2})$*. Moreover, let* (γ_l) *be a sequence in* $\mathbb{R}_+$ *such that* $\gamma_l = O(l^{-1})$*. Then,*

$$\max_{i_l \le i \le l - i_l} \left(\max_{j \ge i + j_l} \|h_{j:l}\|_{\infty, [\underline{x}, x_{\mu_{i,l} + \gamma_l}]} + \max_{j \le i - j_l} \|h_{j:l}\|_{\infty, [x_{\mu_{i,l} - \gamma_l}, \overline{x}]} \right) = \mathcal{E}(l). \tag{40}$$

In addition to Corollaries 1 and 2, which are useful tools for eliminating terms that are negligibly small, we will also use the following *Local Limit Theorem*, Theorem 5, which is adapted from Theorem 4.7.1 in Reiss (1989).

Theorem 5 (Theorem 4.7.1 in Reiss 1989)**.** *There is a constant* $C_1 > 0$ *such that for any H whose density is strictly positive and continuously differentiable on* $[\underline{x}, \overline{x}]$ *and all* $1 \le i \le l$*, the following approximation of* $\check{h}_{i,l}$ *is valid on* $J_{i,l} := \left\{ y \in \mathbb{R} : |y| \le \frac{1}{2} \left(\frac{i(l-i)}{l} \right)^{\frac{1}{6}} \right\}$*.*[35]

$$\frac{|\check{h}_{i,l}(y) - \varphi(y)|}{\varphi(y)} \le C_1 \left(1 + |y|^3\right) \left[\left(\frac{l}{i(l-i)} \right)^{\frac{1}{2}} + a_{i,l} \frac{\|h'\|_\infty}{\left(\min_{[\underline{x}, \overline{x}]} h(x)\right)^2} \right]. \tag{41}$$

[35] Recall the definition of $\check{h}_{i,l}$ from (12).

In particular, for a fixed H, any integer sequence (i_l) satisfying $i_l = \Omega(l)$ and $i_l \leq \frac{l}{2}$, $\alpha \in \left(0, \frac{1}{6}\right)$, and any sequence (y_l) in $\mathbb{R}$ such that $y_l = \Theta\left(l^{\alpha}\right)$, it holds:

$$\max_{i_l \leq i \leq l - i_l} \max_{\{y : |y| \leq y_l\}} \frac{|\check{h}_{i,l}(y) - \varphi(y)|}{\varphi(y)} = O\left(l^{3\alpha - \frac{1}{2}}\right). \tag{42}$$

3.2.2 The main steps in the proof

Recall that $(\beta_{M,n,k_n}, \beta_{W,n,k_n})$ always denotes a SSMBNE of the game $\Gamma(n, k_n)$. By Lemma 1, β'_{M,n,k_n} and β'_{W,n,k_n} satisfy the system of equations

$$\beta'_{M,n,k_n}(m) = m \sum_{i=0}^{k_n - 1} f_{\bar{k}_n + i : n-1}(m) \left(d_{G,k_n,i}^{v_M \circ \beta_{W,n,k_n}} + v_M(0)\delta_{i0}\right) \tag{43}$$

$$\beta'_{W,n,k_n}(w) = w \sum_{i=1}^{k_n - 1} g_{i:k_n - 1}(w) d_{F,n,\bar{k}_n + i}^{v_W \circ \beta_{M,n,k_n}}. \tag{44}$$

We show first that on fixed intervals that do not contain $\overline{m}$ and $\overline{w}$, β'_{M,n,k_n} and β'_{W,n,k_n} are of order $O\left(n^{\frac{1}{2}}\right)$. The bound for β'_{M,n,k_n}, inequality (45), relies only on the concentration properties of central order statistics and on the boundedness of β_{W,n,k_n} (see Lemma 2). The bound for β'_{W,n,k_n} then follows easily as well, even though types very close to $\underline{w}$ face less uncertainty and thus "stiffer local competition" than do "central types" (the standard deviations of extreme order statistics are of order $\Theta\left(n^{-1}\right)$).[36]

Lemma 6. *For any $\tau \in \left(0, \frac{1}{2}\right)$,*

$$\left\|\beta'_{M,n,k_n}\right\|_{\infty,[\underline{m}, m_{1-\tau}]} \leq \overline{m} v_M(\overline{b}_W(\overline{m}, \overline{w})) \max_{\bar{k}_n \leq i \leq n-1} \|f_{i:n-1}\|_{\infty,[\underline{m}, m_{1-\tau}]} = O\left(n^{\frac{1}{2}}\right), \tag{45}$$

$$\max_{0 \leq i \leq (1-\tau)n} d_{F,n,i}^{v_W \circ \beta_{M,n,k_n}} = O\left(n^{-\frac{1}{2}}\right), \tag{46}$$

$$\left\|\beta'_{W,n,k_n}\right\|_{\infty,[\underline{w}, w_{1-\tau}]} = O\left(n^{\frac{1}{2}}\right) \quad \textit{and} \quad \max_{0 \leq i \leq (1-\tau)k_n} d_{G,k_n,i}^{v_M \circ \beta_{W,n,k_n}} = O\left(n^{-\frac{1}{2}}\right). \tag{47}$$

Using the identity

$$\tilde{\beta}'_{M,n,k_n}(z) = \beta'_{M,n,k_n}\left(\xi^{-1}_{F,n-1,\bar{k}_n}(z)\right) \frac{a_{\bar{k}_n,n-1}}{f(m_{\bar{k}_n/n})}, \tag{48}$$

$\frac{a_{\bar{k}_n,n-1}}{f(m_{\bar{k}_n/n})} = \Theta\left(n^{-\frac{1}{2}}\right)$, and the fact that the explicit bound in (45) is independent of the particular sequence of equilibria, we immediately obtain Corollary 3, which provides

[36] Note that Lemma 6 provides *upper* bounds for β'_{M,n,k_n} and β'_{W,n,k_n}. As $v_M(0) > 0$, it would also be easy to show directly that β'_{M,n,k_n} and β'_{W,n,k_n} must in fact be of order $\Theta\left(n^{\frac{1}{2}}\right)$ on intervals of the form $[m_{\bar{k}_n/n} - Cn^{-\frac{1}{2}}, m_{\bar{k}_n/n} + Cn^{-\frac{1}{2}}]$ and $[\underline{w}, \underline{w} + Cn^{-\frac{1}{2}}]$. However, we do not need this additional information to prove Theorem 6 (and the lower bounds on derivatives follow "ex post," from Theorems 2, 3 and 6).

the Lipschitz bound for men's re-scaled strategies that we have used to show Lemma 3.

Corollary 3. *For any sequence* (ε_n) *satisfying* $\varepsilon_n = \Theta\left(n^{\alpha-1}\right)$ *for some* $\alpha \in \left(\frac{1}{2}, 1\right)$, *there is a constant* $K < \infty$ *such that for all* n *and any SSMBNE strategy* β_{M,n,k_n} *of* $\Gamma(n, k_n)$, $\|\tilde{\beta}'_{M,n,k_n}\|_{\infty,[\xi_{F,n-1,\bar{k}_n}(m_{\bar{k}_n/n}-\varepsilon_n),\xi_{F,n-1,\bar{k}_n}(m_{\bar{k}_n/n}+\varepsilon_n)]} \leq K$.

With these preliminaries out of the way, we turn to the main quantitative result of the paper, Theorem 6. As it simplifies notation, we use intervals of re-scaled types of the following specific form to state and prove the result.

Definition 3. *For* $\alpha \in \left(\frac{1}{2}, 1\right)$, *let* $I_{n,\alpha} := \left[\underline{z}_{n,\alpha}, \bar{z}_{n,\alpha}\right]$ *and* $I^+_{n,\alpha} := [0, \bar{z}_{n,\alpha}]$, *where* $\underline{z}_{n,\alpha} := \xi_{F,n-1,\bar{k}_n}\left(m_{(\bar{k}_n-\lfloor n^\alpha\rfloor)/n}\right)$ *and* $\bar{z}_{n,\alpha} := \xi_{F,n-1,\bar{k}_n}\left(m_{(\bar{k}_n+\lfloor n^\alpha\rfloor)/n}\right)$.

From (17) and (18) (and the mean value theorem, for F^{-1}), it is clear that

$$-\underline{z}_{n,\alpha} = \Theta\left(n^{\alpha-\frac{1}{2}}\right) \quad \text{and} \quad \bar{z}_{n,\alpha} = \Theta\left(n^{\alpha-\frac{1}{2}}\right), \tag{49}$$

i.e., the boundaries of the intervals $I_{n,\alpha}$ tend to infinity like $n^{\alpha-\frac{1}{2}}$.

Theorem 6. *For* $\alpha \in \left(\frac{1}{2}, \frac{7}{12}\right)$ *and* $\varepsilon > 0$,

$$\left\|\tilde{\beta}'_{M,n,k_n} - \frac{m_{\bar{k}_n/n}}{m_r}\Psi_1[\tilde{\beta}_{M,n,k_n}]'\right\|_{\infty,I_{n,\alpha}} = O\left(n^{\frac{\alpha}{2}+\varepsilon-\frac{1}{2}}\right). \tag{50}$$

Theorem 6 provides asymptotic bounds pertaining to how close the derivatives of men's re-scaled strategies are to solving the fixed point equation $\tilde{\beta}' = \frac{m_{\bar{k}_n/n}}{m_r}\Psi_1[\tilde{\beta}]'$. It quantifies how fast the differences $\tilde{\beta}'_{M,n,k_n} - \frac{m_{\bar{k}_n/n}}{m_r}\Psi_1[\tilde{\beta}_{M,n,k_n}]'$ tend to zero on the increasingly large intervals $I_{n,\alpha}$.[37]

We now give a broad overview of the two main steps in the proof of Theorem 6, and in particular of how we use the tools of Section 3.2.1. The details are fairly involved. The complete proof, with all auxiliary results, is given in Appendix C and Online Appendix G. The reader should also consult Appendix A for additional notation, and Remark 2 (in Appendix C) for some useful comments about the proofs.

The identities (43) and (48) imply the following formula for $\tilde{\beta}'_{M,n,k_n}$:

$$\tilde{\beta}'_{M,n,k_n}(z) = \xi^{-1}_{F,n-1,\bar{k}_n}(z)\sum_{i=0}^{k_n-1}\frac{a_{\bar{k}_n,n-1}}{f(m_{\bar{k}_n/n})}f_{\bar{k}_n+i:n-1}\left(\xi^{-1}_{F,n-1,\bar{k}_n}(z)\right)\left(d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} + \nu_M(0)\delta_{i0}\right). \tag{51}$$

The first main step in the proof of Theorem 6 consists in approximating all terms $d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i}$ that are potentially relevant for the limit analysis (because men with types

[37] The order of the upper bound for the approximation error depends on α, i.e. on how fast the length of the intervals for which the uniform approximation should hold tends to infinity.

from the intervals $I_{n,\alpha}$ might receive the corresponding utility increments with probabilities that do not decay exponentially fast) by terms that involve only *men's* equilibrium strategy. Such a result is provided by Lemma 7.

Lemma 7. *For $\alpha_1 \in (0,1)$, let (i_n) be an integer sequence such that $i_n = O(n^{\alpha_1})$. Then, for any α_2 satisfying $\alpha_2 > \frac{\alpha_1}{2}$,*

$$\max_{0\le i\le i_n}\left| d_{G,k_n,i}^{v_M\circ\beta_{W,n,k_n}} - d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}} \underline{w} v_M'\left(\underline{w}\sum_{j=1}^{i} d_{F,n,\bar{k}_n+j}^{v_W\circ\beta_{M,n,k_n}}\right)\right| = O\left(n^{\alpha_2-1}\right).$$

Lemma 7 approximates each of the terms $d_{G,k_n,i}^{v_M\circ\beta_{W,n,k_n}}$ by an expression that involves only the "corresponding" increment $d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}}$, $\underline{w}$, and v_M' evaluated at a point close to $E[\beta_{W,n,k_n}(W_{i:k_n})]$. This result, which is of course based on (44), ultimately relies only on the concentration properties of lower intermediate and extreme order statistics, formalized in Corollaries 1 (i) and 2 (i), and on a lemma (Lemma 8 in Appendix C) which shows that the terms $d_{F,n,\bar{k}_n+j}^{v_W\circ\beta_{M,n,k_n}}$ cannot vary too fast with j for j close to i, because the order statistics involved in the calculation of these terms are central order statistics.

The second main step in the proof of Theorem 6 then consists in using the local limit theorem (Theorem 5) and Lemma 7 to show that the right hand side of (51) is uniformly approximated by $\frac{m_{\bar{k}_n/n}}{m_r}\Psi_1[\tilde{\beta}_{M,n,k_n}]'$ on intervals $I_{n,\alpha}$ for $\alpha < \frac{7}{12}$, with approximation errors satisfying the bounds of Theorem 6. We do this in Lemmas 9-14 and in the proof of Theorem 6 (see Online Appendix G).

Once Theorem 6 has been established, showing Theorem 2 becomes straightforward. First, as we have $\bar{z}_{n,\alpha} - \underline{z}_{n,\alpha} = \Theta\left(n^{\alpha-\frac{1}{2}}\right)$, $\tilde{\beta}_{M,n,k_n}(\underline{z}_{n,\alpha}) = \mathcal{E}(n)$ and $\Psi_1[\tilde{\beta}_{M,n,k_n}](\underline{z}_{n,\alpha}) = \mathcal{E}(n)$ (for any $\alpha > \frac{1}{2}$, see the proof of Corollary 4), Theorem 6 implies the following bounds for the differences $\tilde{\beta}_{M,n,k_n} - \frac{m_{\bar{k}_n/n}}{m_r}\Psi_1[\tilde{\beta}_{M,n,k_n}]$.

Corollary 4. *For $\alpha \in \left(\frac{1}{2}, \frac{7}{12}\right)$ and $\varepsilon > 0$,*

$$\left\|\tilde{\beta}_{M,n,k_n} - \frac{m_{\bar{k}_n/n}}{m_r}\Psi_1[\tilde{\beta}_{M,n,k_n}]\right\|_{\infty,(-\infty,\bar{z}_{n,\alpha}]} = O\left(n^{\frac{3\alpha}{2}+\varepsilon-1}\right). \tag{52}$$

Secondly, as $\frac{3\alpha}{2} < 1$ for some (in fact, all) $\alpha \in \left(\frac{1}{2}, \frac{7}{12}\right)$ and $\lim_{n\to\infty}\frac{m_{\bar{k}_n/n}}{m_r} = 1$, Corollary 4 immediately yields the following result.

Corollary 5. *The sequence $\left(\tilde{\beta}_{M,n,k_n} - \Psi_1[\tilde{\beta}_{M,n,k_n}]\right)$ converges locally uniformly to zero.*

Theorem 2 then follows easily (see Appendix C) from Corollary 5 and Lemma 3.

4 Discussion

We have approximated the continuum economy by Bayesian games with a known number of players for which the other key properties of the competitive model (a deterministic investment technology and frictionless matching) hold exactly. This approach is standard in many areas of economic theory, including the related literatures on large contests and on large double auctions, but there are of course other ways of approximating the frictionless continuum model. In particular, one could consider a model with some uncertainty about the exact numbers of men and women, add small shocks to preferences and/or investments, or introduce some search frictions. Each of these approaches corresponds to a reasonable way of making returns more noisy, and will in particular increase the range of types on the long side who face substantial uncertainty about whether they will be matched. An interesting question is therefore whether our paper's main economic insight (in unbalanced markets with NTU (and assortative) matching where all matches are acceptable and agents on the long side strictly prefer getting matched, the competition for partners always leads to overinvestment) also obtains for these alternative ways of approximating the frictionless continuum model. As each of the possible approaches requires a different set of techniques and poses significant challenges on its own, we must leave this question for future work.

Appendix

The technically standard and simple proofs for Section 2.4 and Theorem 4 are given in Online Appendices D and E. Theorem 3 is proven in Appendix B. The technical results of Section 3.2.1 are shown in Online Appendix F. Appendix C contains the proofs of Lemma 7 (the key step in the proof of Theorem 6) and of Theorem 2 (conditional on Theorem 6), whereas Online Appendix G contains the proof of Lemma 6 and the (long and mostly technical) remainder of the proof of Theorem 6 (conditional on Lemma 7). Online Appendix H contains four auxiliary lemmas used throughout the proofs.

A Additional notation for the proofs

For a vector $u \in \mathbb{R}^l$, we let $|u|_\infty = \max_{i\in\{0,\dots,l-1\}} |u_i|$ denote its maximum norm and $|u|_1 = \sum_{i=0}^{l-1} |u_i|$ its l_1-norm.[38] Next, we define for all $0 \le i \le n-1$,

$$\Delta_{i,n} := \xi_{F,n-1,\bar{k}_n}(m_{(i+1)/n}) - \xi_{F,n-1,\bar{k}_n}(m_{i/n}). \tag{53}$$

Using (17) and $a_{\bar{k}_n,n-1} = \Theta\left(n^{-\frac{1}{2}}\right)$, it follows that

$$\min_{0\le i\le n-1} \Delta_{i,n} = \Theta\left(n^{-\frac{1}{2}}\right) \quad \text{and} \quad \max_{0\le i\le n-1} \Delta_{i,n} = \Theta\left(n^{-\frac{1}{2}}\right). \tag{54}$$

[38] We use the $|\cdot|_\infty$ notation for vectors and $\|\cdot\|_\infty$ for functions to avoid confusion in some proofs.

B Proofs of Lemma 3 and Theorem 3

Proof of Lemma 3. By Lemma 2, any subsequence of re-scaled strategies $(\tilde{\beta}_{M,n_l,k_{n_l}})$ is uniformly bounded. Corollary 3 and (18) imply that the subsequence is also equicontinuous on any compact set. Applying the Arzèla-Ascoli Theorem for each compact interval $[-L, L]$, $L \in \mathbb{N}$ and using the usual (Cantor) diagonalization argument, we obtain a subsequence of $(\tilde{\beta}_{M,n_l,k_{n_l}})$ that converges uniformly on every compact set. □

Proof of Theorem 3. We start by proving (31). Given a solution $\tilde{\zeta}_*$ of (23) and recalling that $\tilde{\beta}_* = \Xi_2[\tilde{\zeta}_*]$, we differentiate $\tilde{\zeta}_* = \Psi_2[\tilde{\zeta}_*]$ and obtain, using (20), that

$$\tilde{\zeta}_*'(x) = \underline{w}\left(\varphi * (v_W \circ \tilde{\beta}_*)\right)'(x) = \underline{w}\left(\varphi * (v_W \circ \tilde{\beta}_*)'\right)(x) = \underline{w}\int_{\mathbb{R}} \varphi(x-z) v_W'\left(\tilde{\beta}_*(z)\right)\tilde{\beta}_*'(z)dz.$$

Using (21), integration by parts, and $\tilde{\zeta}_*(0) = 0$, we also have

$$\begin{aligned} \tilde{\beta}_*'(z) &= m_r \int_0^\infty \varphi'(z-y)(v_M \circ \tilde{\zeta}_*)(y)dy \\ &= m_r \left(\varphi(z) v_M(0) + \int_0^\infty \varphi(z-y) v_M'\left(\tilde{\zeta}_*(y)\right)\tilde{\zeta}_*'(y)dy\right), \end{aligned} \tag{55}$$

so that (31) follows. Next, we note that

$$\begin{aligned} &\int_{\mathbb{R}} \varphi(x-z)\int_0^\infty \varphi(z-y)\tilde{\zeta}_*'(y)dydz = \varphi * (\varphi * (\tilde{\zeta}_*'\mathbf{1}_{\mathbb{R}_+}))(x) \\ &= (\varphi * \varphi) * (\tilde{\zeta}_*'\mathbf{1}_{\mathbb{R}_+})(x) = \varphi_{0,2} * (\tilde{\zeta}_*'\mathbf{1}_{\mathbb{R}_+})(x) = \int_0^\infty \varphi_{0,2}(x-y)\tilde{\zeta}_*'(y)dy, \end{aligned} \tag{56}$$

(where we have extended the function $\tilde{\zeta}_*'$ in some arbitrary way on $(-\infty, 0)$, so that the pointwise product of $\tilde{\zeta}_*'$ and the indicator function $\mathbf{1}_{\mathbb{R}_+}$ is defined on all of $\mathbb{R}$). Here, the second equality holds as the convolution is associative, and the third equality uses the well-known identity $\varphi * \varphi = \varphi_{0,2}$. Using (31) and (56), and recalling that $\lambda_{\tilde{\zeta}_*} = m_r \underline{w} v_M'(\tilde{\zeta}_*(\infty)) v_W'(\tilde{\beta}_*(\infty))$, we obtain

$$\begin{aligned} \tilde{\zeta}_*'(x) &= m_r \underline{w} v_M(0) \int_{\mathbb{R}} \varphi(x-z)\varphi(z) v_W'\left(\tilde{\beta}_*(z)\right)dz \\ &+ m_r \underline{w}\int_{\mathbb{R}} \varphi(x-z) v_W'\left(\tilde{\beta}_*(z)\right)\int_0^\infty \varphi(z-y) v_M'\left(\tilde{\zeta}_*(y)\right)\tilde{\zeta}_*'(y)dydz \\ &= \min\left(1, \lambda_{\tilde{\zeta}_*}\right)\int_0^\infty \varphi_{0,2}(x-y)\tilde{\zeta}_*'(y)dy + R_{\tilde{\zeta}_*}(x), \end{aligned}$$

where

$$
\begin{aligned}
R_{\tilde{\zeta}_*}(x) &= m_r \underline{w} v_M(0) \int_{\mathbb{R}} \varphi(x-z)\varphi(z) v_W' \left(\tilde{\beta}_*(z)\right) dz \\
&+ m_r \underline{w} \int_{\mathbb{R}} \varphi(x-z) \left(v_W'\left(\tilde{\beta}_*(z)\right) - v_W'\left(\tilde{\beta}_*(\infty)\right)\right) \int_0^\infty \varphi(z-y) v_M'\left(\tilde{\zeta}_*(y)\right) \tilde{\zeta}_*'(y) dy dz \\
&+ m_r \underline{w} \int_{\mathbb{R}} \varphi(x-z) v_W'\left(\tilde{\beta}_*(\infty)\right) \int_0^\infty \varphi(z-y) \left(v_M'\left(\tilde{\zeta}_*(y)\right) - v_M'\left(\tilde{\zeta}_*(\infty)\right)\right) \tilde{\zeta}_*'(y) dy dz \\
&+ \max\left(\lambda_{\tilde{\zeta}_*} - 1, 0\right) \int_0^\infty \varphi_{0,2}(x-y) \tilde{\zeta}_*'(y) dy. \qquad (57)
\end{aligned}
$$

In particular, $\tilde{\zeta}_*'$ coincides with a strictly positive solution of the Wiener-Hopf equation (32). From (57) and Assumption 1, it is clear that $R_{\tilde{\zeta}_*} \geq 0$. Moreover, using $v_M(0) > 0$ and (25) we obtain a simple uniform lower bound for $R_{\tilde{\zeta}_*}$: for all $\tilde{\zeta}_*$ and all $x \in \mathbb{R}_+$,

$$
0 < m_r \underline{w} v_M(0) v_W'\left(\overline{b}_M(m_r, \underline{w})\right) \varphi_{0,2}(x) \leq R_{\tilde{\zeta}_*}(x). \qquad (58)
$$

We now give a more detailed description of the relevant results about solutions of the equation $(Id - \mathcal{K})\eta = R$ (i.e., (30)), synthesizing the results from Arabadzhyan and Engibaryan (1987, henceforth AE) and providing precise references to the corresponding sections or theorems in AE, and use these results to prove (28). We focus on the case of *symmetric* and *nonnegative* kernels, i.e., $K(x) = K(-x)$ for all $x \in \mathbb{R}_+$ and $K \geq 0$. All results have generalizations for asymmetric kernels, but only the symmetric case is relevant for us, due to the symmetry of $\varphi_{0,2}$.

If $\|K\|_{L^1(\mathbb{R})} \leq 1$, the operator $Id - \mathcal{K}$ admits a factorization

$$
Id - \mathcal{K} = (Id - \mathcal{V}_-)(Id - \mathcal{V}_+), \qquad (59)
$$

where $\mathcal{V}_+$ and $\mathcal{V}_-$ are operators of the form

$$
\begin{aligned}
(\mathcal{V}_+\eta)(x) &= \int_0^x V(x-t)\eta(t) dt \qquad (60) \\
(\mathcal{V}_-\eta)(x) &= \int_x^\infty V(t-x)\eta(t) dt, \qquad (61)
\end{aligned}
$$

and V is a nonnegative function in $L^1(\mathbb{R}_+)$ that satisfies $\|V\|_{L^1(\mathbb{R}_+)} = 1 - \sqrt{1 - \|K\|_{L^1(\mathbb{R})}}$. Equation (59) is understood as the equality of endomorphisms of $L^1(\mathbb{R}_+)$ (see sections 0 and 1 in AE, which also describe in detail how the function V can be found).

Theorem 6.1 in AE shows that for $\|K\|_{L^1(\mathbb{R})} = 1$ and any $R \in L^1(\mathbb{R}_+)$ with $R \neq 0$

and $R \geq 0$, (30) has a nonnegative solution $\eta_* \in L^1_{loc}(\mathbb{R}_+)$,[39] which can be obtained as follows (see the proof of Theorem 6.1 in AE). First, the equation $(Id - \mathcal{V}_-)\psi = R$ has a *canonical solution* $\psi \in L^1_{loc}(\mathbb{R}_+)$, i.e., a solution that is equal to the limit of the iterative process $\psi_{n+1} = R + \mathcal{V}_-\psi_n$ for $\psi_0 = 0$. This canonical solution exists by Theorem 3.1 in AE and satisfies $\psi \geq 0$ and $\psi \neq 0$ (as an element of $L^1_{loc}(\mathbb{R}_+)$). Next, the equation $(Id - \mathcal{V}_+)\eta = \psi$, has a unique solution $\eta \in L^1_{loc}(\mathbb{R}_+)$, which satisfies $\eta \geq 0$ and which, due to $\psi \neq 0$ and $\|V\|_{L^1(\mathbb{R}_+)} = 1$, cannot be in $L^1(\mathbb{R}_+)$ (see page 753 in AE for these results; the fact $\eta \notin L^1(\mathbb{R}_+)$ follows from a simple argument using the monotone convergence theorem). It is then easy to verify (see Lemma 5.1 and the sentence before Theorem 6.1 in AE) that $\eta_* = \eta$ indeed solves (30).[40]

Having constructed the solution η_* of (30) in this way, it is not difficult to see (page 765 and Theorem 6.3 in AE) that (30) has also a canonical solution η_{**}, i.e., a solution equal to the limit of the iterative process $\eta_{n+1} = R + \mathcal{K}\eta_n$ for $\eta_0 = 0$, that η_{**} is the minimal positive solution of (30), and that $\eta_{**} = \eta_*$. In particular, *positive solutions of the conservative equation are not integrable*. As $\tilde{\zeta}'_*$ is integrable ($\|\tilde{\zeta}'_*\|_{L^1(\mathbb{R}_+)} \leq \overline{b}_W(m_r, \underline{w})$) and coincides with a positive solution of the Wiener-Hopf equation with kernel $\min\left(1, \lambda_{\tilde{\zeta}_*}\right)\varphi_{0,2}$ and inhomogeneous term $R_{\tilde{\zeta}_*}$, it follows from the above results that $\lambda_{\tilde{\zeta}_*} < 1$. As described in the main text, it holds for any $K \geq 0$ with $\|K\|_{L^1(\mathbb{R})} = 1$, that $\eta_{\lambda,K,R_1} \leq \eta_{\lambda,K,R_2}$ if $0 \leq R_1 \leq R_2$, and that for any $R \in L^1(\mathbb{R}_+)$, $R \neq 0$, $R \geq 0$, the functions $\eta_{\lambda,K,R}$ converge monotonically to the minimal positive solution of the conservative equation $(Id - \mathcal{K})\eta = R$ if λ tends to 1 monotonically (see page 765 and the proof of Theorem 6.3 in AE). Using (58), (28) with $\bar{\lambda}$ given by (33) is then immediate for $\underline{R} = m_r\underline{w}v_M(0)v'_W\left(\overline{b}_M(m_r, \underline{w})\right)\varphi_{0,2}$.[41] □

C Proofs of Lemma 7 and Theorem 2

Remark 2. *(i) We often use the following basic tools without explicitly mentioning them: the triangle inequality, and its consequence that for* $y_1, y_2, y_3, y_4 \in \mathbb{R}$

$$|y_1y_2 - y_3y_4| \leq |y_1 - y_3||y_2| + |y_3||y_2 - y_4|, \tag{62}$$

the mean value theorem, and inequalities that bound an inner product of two vectors (functions) by the product of the l_1-norm (L^1-norm) of one of the vectors (functions) and the l_∞-norm (L^∞-norm) of the other vector (function).

[39] $L^1_{loc}(\mathbb{R}_+)$ is the space of *locally integrable* (integrable on every compact set) functions on $\mathbb{R}_+$.

[40] The additional verification step is necessary because (59) is an identity of endomorphisms of $L^1(\mathbb{R}_+)$, but the solution found is only in $L^1_{loc}(\mathbb{R}_+)$.

[41] One could of course try to prove tighter uniform lower bounds for $R_{\tilde{\zeta}_*}$ based on (57), but this would require delving much deeper into the theory for the nonlinear equation.

(ii) As $\lim_{n\to\infty} k_n/n = 1-r > 0$, for $\alpha \in \mathbb{R}$ a sequence of real numbers is of order $\Theta(n^\alpha)$ if and only if it is of order $\Theta(k_n^\alpha)$.

Lemma 8. *For any $\tau > 0$ such that $r \in (\tau, 1-\tau)$,*

$$\max_{\bar{k}_n < i < (1-\tau)n} \left| d_{F,n,i}^{v_W \circ \beta_{M,n,k_n}} - d_{F,n,i-1}^{v_W \circ \beta_{M,n,k_n}} \right| = O\left(n^{-1}\right). \tag{63}$$

Proof of Lemma 8. Throughout the proof, $H^{(n)}$ denotes the c.d.f. of the random variable $v_W(\beta_{M,n,k_n}(M))$, where M is a random variable with c.d.f. F. In particular, the support of $H^{(n)}$ is $[\underline{x}^{(n)}, \overline{x}^{(n)}] = [v_W(\beta_{M,n,k_n}(\underline{m})), v_W(\beta_{M,n,k_n}(\overline{m}))]$. We let $h^{(n)}$ denote the density of $H^{(n)}$. The usual change of variables formula yields

$$f(m) = h^{(n)}(v_W(\beta_{M,n,k_n}(m)))(v_W \circ \beta_{M,n,k_n})'(m) \quad \text{for all } \; m \in [\underline{m}, \overline{m}]. \tag{64}$$

The term $d_{F,n,i}^{v_W \circ \beta_{M,n,k_n}} = E[v_W(\beta_{M,n,k_n}(M_{i+1:n}) - v_W(\beta_{M,n,k_n}(M_{i:n})]$ is the *mean of the i-th spacing* for the order statistics of n i.i.d. draws from $H^{(n)}$. We first note a general formula for *differences* between the means of adjacent spacings (David and Groeneveld 1982, equation 3).[42] For all $i \in \{1, ..., n-1\}$,

$$\begin{aligned} d_{F,n,i}^{v_W \circ \beta_{M,n,k_n}} - d_{F,n,i-1}^{v_W \circ \beta_{M,n,k_n}} &= \int_{\underline{x}^{(n)}}^{\overline{x}^{(n)}} \frac{(n+1)n}{i(n+1-i)} B_{i-1,n-1}(H^{(n)}(x))(H^{(n)}(x) - \mu_{i,n}) dx \\ &= \int_{\underline{x}^{(n)}}^{\overline{x}^{(n)}} \frac{n+1}{i(n+1-i)} h_{i:n}^{(n)}(x) \frac{H^{(n)}(x) - \mu_{i,n}}{h^{(n)}(x)} dx, \end{aligned}$$

where we used (8) for the second step. Fixing an arbitrary $\tau' \in (0, \tau)$, we obtain

$$\begin{aligned} &\max_{\bar{k}_n < i < (1-\tau)n} \left| d_{F,n,i}^{v_W \circ \beta_{M,n,k_n}} - d_{F,n,i-1}^{v_W \circ \beta_{M,n,k_n}} \right| \\ &\leq \max_{\bar{k}_n < i < (1-\tau)n} \left| \int_{\underline{x}^{(n)}}^{x_{\tau'}} \frac{(n+1)n}{i(n+1-i)} B_{i-1,n-1}(H^{(n)}(x))(H^{(n)}(x) - \mu_{i,n}) dx \right| \\ &+ \max_{\bar{k}_n < i < (1-\tau)n} \left| \int_{x_{\tau'}}^{x_{1-\tau'}} \frac{n+1}{i(n+1-i)} h_{i:n}^{(n)}(x) \frac{H^{(n)}(x) - \mu_{i,n}}{h^{(n)}(x)} dx \right| \\ &+ \max_{\bar{k}_n < i < (1-\tau)n} \left| \int_{x_{1-\tau'}}^{\overline{x}^{(n)}} \frac{(n+1)n}{i(n+1-i)} B_{i-1,n-1}(H^{(n)}(x))(H^{(n)}(x) - \mu_{i,n}) dx \right|. \end{aligned} \tag{65}$$

To prove Lemma 8, we show that the second term on the right hand side of (65) is of order $O(n^{-1})$, and that the other two terms are of order $\mathcal{E}(n)$. We turn to the main term

[42] The formula is also a straightforward consequence of the first identity in (151) from Lemma 17.

first and observe that (for all $i \geq 1$)

$$\left|\int_{x_{\tau'}}^{x_{1-\tau'}} \frac{n+1}{i(n+1-i)} h^{(n)}_{i:n}(x) \frac{H^{(n)}(x)-\mu_{i,n}}{h^{(n)}(x)} dx\right| \leq \frac{n+1}{i(n+1-i)} \frac{\int_{x_{\tau'}}^{x_{1-\tau'}} |H^{(n)}(x)-\mu_{i,n}| h^{(n)}_{i:n}(x) dx}{\min_{x\in[x_{\tau'},x_{1-\tau'}]} h^{(n)}(x)}$$
$$\leq \frac{n+1}{i(n+1-i)} \frac{\int_{\underline{x}^{(n)}}^{\overline{x}^{(n)}} |H^{(n)}(x)-\mu_{i,n}| h^{(n)}_{i:n}(x) dx}{\min_{x\in[x_{\tau'},x_{1-\tau'}]} h^{(n)}(x)} = \frac{n+1}{i(n+1-i)} \frac{E[|U_{i:n}-\mu_{i,n}|]}{\min_{x\in[x_{\tau'},x_{1-\tau'}]} h^{(n)}(x)}$$
$$\leq \frac{n+1}{i(n+1-i)} \frac{Var[U_{i:n}]^{\frac{1}{2}}}{\min_{x\in[x_{\tau'},x_{1-\tau'}]} h^{(n)}(x)} = \left(\frac{1}{i(n+1-i)(n+2)}\right)^{\frac{1}{2}} \frac{1}{\min_{x\in[x_{\tau'},x_{1-\tau'}]} h^{(n)}(x)},$$

where the equality in the second line uses (34), the last inequality uses the Cauchy-Schwarz inequality, and the final equality follows from $Var[U_{i:n}] = \frac{i(n+1-i)}{(n+1)^2(n+2)}$. As $\min_{m\in[\underline{m},\overline{m}]} f(m) > 0$, $\|v'_W\|_\infty < \infty$ and $\left\|\beta'_{M,n,k_n}\right\|_{\infty,[\underline{m},m_{1-\tau'}]} = O\left(n^{\frac{1}{2}}\right)$ (by (45) from Lemma 6), (64) yields $\frac{1}{\min_{x\in[x_{\tau'},x_{1-\tau'}]} h^{(n)}(x)} = \left\|\frac{(v_W\circ\beta_{M,n,k_n})'}{f}\right\|_{\infty,[m_{\tau'},m_{1-\tau'}]} = O\left(n^{\frac{1}{2}}\right)$.

As $0 < \lim_{n\to\infty} \frac{\bar{k}_n}{n} = r < 1-\tau$ implies $\max_{\bar{k}_n<i<(1-\tau)n} \left(\frac{1}{i(n+1-i)(n+2)}\right)^{\frac{1}{2}} = \Theta\left(n^{-\frac{3}{2}}\right)$, it then follows that $\max_{\bar{k}_n<i<(1-\tau)n} \left|\int_{x_{\tau'}}^{x_{1-\tau'}} \frac{n+1}{i(n+1-i)} h^{(n)}_{i:n}(x) \frac{H^{(n)}(x)-\mu_{i,n}}{h^{(n)}(x)} dx\right| = O\left(n^{-1}\right)$.

We still have to show that the first and the third summand in (65) are of order $\mathcal{E}(n)$. This is a simple consequence of Corollary 2 (ii). Indeed, consider the first summand, which is bounded by $\left(x_{\tau'} - \underline{x}^{(n)}\right) \max_{i>\bar{k}_n} \frac{n+1}{i(n+1-i)} \|nB_{i-1,n-1}\|_{\infty,[0,\tau']}$. Note then that $x_{\tau'} - \underline{x}^{(n)} \leq v_W(\overline{b}_M(\overline{m},\overline{w}))$ and that, by (8), $nB_{i-1,n-1}$ is the density of the i-th order statistic of n draws from $U(0,1)$. As $\bar{k}_n - \lfloor\tau' n\rfloor = \Omega(1)$ (by the assumptions on τ' and τ) Corollary 2 (ii) implies $\max_{i>\lfloor\tau' n\rfloor+\bar{k}_n-\lfloor\tau' n\rfloor} \|nB_{i-1,n-1}\|_{\infty,[0,\tau']} = \mathcal{E}(n)$. The argument for showing the exponential decay of the third summand in (65) is analogous. □

Proof of Lemma 7. Given $\alpha_1 \in (0,1)$ and a sequence $i_n = O(n^{\alpha_1})$, we assume without loss of generality that $\alpha_2 \in \left(\frac{\alpha_1}{2}, \alpha_1\right)$. We start by fixing an integer sequence $j_n = \Theta(n^{\alpha_2})$ and define the events $S_{i,n} := \{w_{\mu_{i-j_n,k_n}} \leq W_{i:k_n}\} \cap \{W_{i+1:k_n} \leq w_{\mu_{i+j_n,k_n}}\}$, for $0 \leq i \leq k_n - 1$, where by convention $w_{\mu_{i-j_n,k_n}} = \underline{w}$ if $i < j_n$, and $w_{\mu_{i+j_n,k_n}} = \overline{w}$ if $i + j_n > k_n + 1$. We denote the complement of $S_{i,n}$ by $S^c_{i,n}$. Corollary 1 (i) implies

$$\max_{0\leq i\leq i_n} P[S^c_{i,n}] = \mathcal{E}(n). \tag{66}$$

Indeed, $P[S^c_{i,n}] \leq P[W_{i:k_n} < w_{\mu_{i-j_n,k_n}}] + P[W_{i+1:k_n} > w_{\mu_{i+j_n,k_n}}]$ which, by (35), is equal to $P\left[U_{i:k_n} - \mu_{i,k_n} < \max\left\{-\mu_{i,k_n}, \frac{-j_n}{k_n+1}\right\}\right] + P\left[U_{i+1:k_n} - \mu_{i+1,k_n} > \min\left\{\frac{j_n-1}{k_n+1}, 1-\mu_{i+1,k_n}\right\}\right]$. As $\frac{j_n}{k_n+1} = \Theta\left(k_n^{\alpha_2-1}\right)$ and $\alpha_2 > \frac{\alpha_1}{2}$, Corollary 1 (i) yields $\max_{0\leq i\leq i_n} P\left[U_{i:k_n} - \mu_{i,k_n} < -\frac{j_n}{k_n+1}\right] = \mathcal{E}(n)$ and $\max_{0\leq i\leq i_n} P\left[U_{i+1:k_n} - \mu_{i+1,k_n} > \frac{j_n-1}{k_n+1}\right] = \mathcal{E}(n)$, so that (using the trivial facts $P[U_{i:k_n} < 0] = P[U_{i:k_n} > 1] = 0$ as well), (66) follows. We next define the conditional expectations

$$d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i}|S_{i,n} := E\left[v_M(\beta_{W,n,k_n}(W_{i+1:k_n})) - v_M(\beta_{W,n,k_n}(W_{i:k_n}))\middle|S_{i,n}\right]. \tag{67}$$

As the random variables $v_M(\beta_{W,n,k_n}(W_{i+1:k_n}))-v_M(\beta_{W,n,k_n}(W_{i:k_n}))$ are bounded by $v_M(\overline{b}_W(\overline{m},\overline{w}))$, Lemma 16 implies that $d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i}|S_{i,n}$ is a very good approximation of $d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i}$ for all $i \leq i_n$: $\max_{0\leq i\leq i_n}\left|d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i} - d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i}|S_{i,n}\right| \leq 2v_M(\overline{b}_W(\overline{m},\overline{w})))\max_{0\leq i\leq i_n}P[S^c_{i,n}] = \mathcal{E}(n)$. With this preliminary truncation argument out of the way, we now turn to the main task, which is to show the following bound:

$$\max_{0\leq i\leq i_n}\left|d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i}|S_{i,n} - d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i}\underline{w}v'_M\left(\underline{w}\sum_{j=1}^{i}d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+j}\right)\right| = O\left(k_n^{\alpha_2-1}\right). \tag{68}$$

To this end, we invoke (44) to express $d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i}|S_{i,n}$ as

$$\begin{aligned}
d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i}|S_{i,n} &= E\left[\int_{W_{i:k_n}}^{W_{i+1:k_n}} v'_M(\beta_{W,n,k_n}(w))\beta'_{W,n,k_n}(w)dw\middle|S_{i,n}\right]\\
&= E\left[\int_{W_{i:k_n}}^{W_{i+1:k_n}} v'_M(\beta_{W,n,k_n}(w))w\sum_{j=1}^{k_n-1}g_{j:k_n-1}(w)d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+j}dw\middle|S_{i,n}\right],
\end{aligned} \tag{69}$$

and then start by showing the following approximation result, based on Lemma 8: for $\kappa^{(n)}_i := \left\|\sum_{j=1}^{k_n-1}g_{j:k_n-1}d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+j} - d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i}(k_n-1)g\right\|_{\infty,[w_{\mu_{i-j_n,k_n}},w_{\mu_{i+j_n,k_n}}]}$, it holds that

$$\max_{0\leq i\leq i_n}\kappa^{(n)}_i = O\left(k_n^{\alpha_2}\right). \tag{70}$$

To prove (70), we argue first that for $i \leq i_n$ and $w \in [w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]$, only a small number of summands with indices close to i can contribute non-negligible amounts to $\sum_{j=1}^{k_n-1}g_{j:k_n-1}(w)d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+j}$. More precisely, we show

$$\max_{0\leq i\leq i_n}\left\|\sum_{j=1}^{k_n-1}g_{j:k_n-1}d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+j} - \sum_{j=i-2j_n}^{i+2j_n}g_{j:k_n-1}d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+j}\right\|_{\infty,[w_{\mu_{i-j_n,k_n}},w_{\mu_{i+j_n,k_n}}]} = \mathcal{E}(k_n), \tag{71}$$

where by convention $g_{j:k_n-1}\equiv 0$ if $j \leq 0$. Indeed, Corollary 2 (i) applied for the sequence $i_n + j_n = O(k_n^{\alpha_1})$ yields $\max_{0\leq i\leq i_n}\max_{j\geq j_n}\|g_{i+j_n+j:k_n-1}\|_{\infty,[\underline{w},w_{\mu_{i+j_n,k_n}}]} = \mathcal{E}(k_n)$. Similarly, $\max_{0\leq i\leq i_n}\max_{j\geq j_n}\|g_{i-j_n-j:k_n-1}\|_{\infty,[w_{\mu_{i-j_n,k_n}},\overline{w}]} = \mathcal{E}(k_n)$ follows from applying Corollary 2 (i) for the sequence $(i_n - j_n)$. Thus, it follows that

$$\max_{0\le i\le i_n} \max_{j\notin\{i-2j_n,\dots,i+2j_n\}} \|g_{j:k_n-1}\|_{\infty,[w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]} = \mathcal{E}(k_n), \tag{72}$$

which, combined with $\left|d^{v_W\circ\beta_{M,n,k_n}}_{F,n}\right|_1 \le v_W(\overline{b}_M(\overline{m},\overline{w}))$, proves (71). Next, for $\tau > 0$ such that $r \in (\tau, 1-\tau)$, $i_n + 2j_n = O(k_n^{\alpha_1})$ implies $\bar{k}_n + i_n + 2j_n < (1-\tau)n$ for all but finitely many n, so that Lemma 8 yields $\max_{0\le i\le i_n} \max_{\max\{0,i-2j_n\}\le j\le i+2j_n} \left|d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+j} - d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i}\right| = O\left(\frac{2j_n}{n}\right) = O\left(k_n^{\alpha_2-1}\right)$. In particular, as $\sum_{j=1}^{k_n-1} g_{j:k_n-1}(w) = (k_n-1)g(w) < k_n\|g\|_\infty = \Theta(k_n)$, where we have used (147) for the first identity, it follows that

$$\max_{0\le i\le i_n} \left\| \sum_{j=i-2j_n}^{i+2j_n} g_{j:k_n-1} \left[d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+j} - d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i} \right] \right\|_{\infty,[w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]} = O\left(k_n^{\alpha_2}\right). \tag{73}$$

Finally, using (147), (72) and $\left|d^{v_W\circ\beta_{M,n,k_n}}_{F,n}\right|_\infty \le v_W(\overline{b}_M(\overline{m},\overline{w}))$, we also have

$$\max_{0\le i\le i_n} \left\| d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i} \left((k_n-1)g - \sum_{j=i-2j_n}^{i+2j_n} g_{j:k_n-1} \right) \right\|_{\infty,[w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]} = \mathcal{E}(k_n),$$

which, together with (71) and (73) proves (70). We then show

$$\max_{0\le i\le i_n} \left| d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i} | S_{i,n} - d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i} w_{\mu_{i,k_n}} v'_M(\beta_{W,n,k_n}(w_{\mu_{i,k_n}})) \right| = O\left(k_n^{\alpha_2-1}\right), \tag{74}$$

as the next main intermediate step in the proof of (68), and thus of Lemma 7. Recalling (69), we first consider the differences

$$\begin{aligned}
& d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i} | S_{i,n} - d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i}(k_n-1) E\left[\int_{W_{i:k_n}}^{W_{i+1:k_n}} v'_M(\beta_{W,n,k_n}(w)) w g(w) \middle| S_{i,n} \right] \\
&= E\left[\int_{W_{i:k_n}}^{W_{i+1:k_n}} v'_M(\beta_{W,n,k_n}(w)) w \left(\sum_{j=1}^{k_n-1} g_{j:k_n-1}(w) d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+j} - d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i}(k_n-1) g(w) \right) dw \middle| S_{i,n} \right].
\end{aligned}$$

From the definitions of $S_{i,n}$ and $\kappa_i^{(n)}$, we obtain

$$\begin{aligned}
&\left| d_{G,k_n,i}^{v_M \circ \beta_{W,n,k_n}} | S_{i,n} - d_{F,n,\bar{k}_n+i}^{v_W \circ \beta_{M,n,k_n}} (k_n - 1) E\left[\int_{W_{i:k_n}}^{W_{i+1:k_n}} v_M'(\beta_{W,n,k_n}(w)) w g(w) \middle| S_{i,n} \right] \right| \\
&\leq \kappa_i^{(n)} E\left[\int_{W_{i:k_n}}^{W_{i+1:k_n}} v_M'(\beta_{W,n,k_n}(w)) w dw \middle| S_{i,n} \right] \leq \kappa_i^{(n)} \|v_M'\|_\infty \overline{w}\ E[W_{i+1:k_n} - W_{i:k_n} | S_{i,n}] \\
&\leq \kappa_i^{(n)} \|v_M'\|_\infty \overline{w}\ \left(E[W_{i+1:k_n} - W_{i:k_n}] + 2(\overline{w} - \underline{w}) P[S_{i,n}^c] \right) = O\left(k_n^{\alpha_2 - 1}\right). \qquad (75)
\end{aligned}$$

Here the inequality in the last line uses (150), and the final bound is then immediate from (66), (70) and $\max_{0 \leq i \leq i_n} E[W_{i+1:k_n} - W_{i:k_n}] \leq \frac{1}{(k_n+1)\min_{w \in [\underline{w},\overline{w}]} g(w)} = \Theta(k_n^{-1})$ (by (152)). As $\max_{0 \leq i \leq i_n} d_{F,n,\bar{k}_n+i}^{v_W \circ \beta_{M,n,k_n}} = O\left(n^{-\frac{1}{2}}\right)$ (by (46)), we see from (75) that (74) follows if we can show that

$$\begin{aligned}
&\max_{0 \leq i \leq i_n} \left| (k_n - 1) E\left[\int_{W_{i:k_n}}^{W_{i+1:k_n}} v_M'(\beta_{W,n,k_n}(w)) w g(w) dw \middle| S_{i,n} \right] - w_{\mu_{i,k_n}} v_M'(\beta_{W,n,k_n}(w_{\mu_{i,k_n}})) \right| \\
&= O\left(k_n^{\alpha_2 - \frac{1}{2}}\right). \qquad (76)
\end{aligned}$$

To prove (76), we note first that

$$\max_{0 \leq i \leq i_n} \left\| Id - w_{\mu_{i,k_n}} \right\|_{\infty, [w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]} \leq \frac{j_n}{k_n + 1} \frac{1}{\min_{w \in [\underline{w},\overline{w}]} g(w)} = O\left(k_n^{\alpha_2 - 1}\right), \qquad (77)$$

where Id is the identity map on $\mathbb{R}$. Moreover, we have

$$\max_{0 \leq i \leq i_n} \left\| g - g(w_{\mu_{i,k_n}}) \right\|_{\infty, [w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]} \leq \|g'\|_\infty \max_{0 \leq i \leq i_n} \left\| Id - w_{\mu_{i,k_n}} \right\|_{\infty, [w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]} = O\left(k_n^{\alpha_2 - 1}\right), \qquad (78)$$

and (using that $\left\| \beta_{W,n,k_n}' \right\|_{\infty, [\underline{w}, w_{\mu_{i_n+j_n,k_n}}]} = O\left(n^{\frac{1}{2}}\right)$, by (47))

$$\begin{aligned}
&\max_{0 \leq i \leq i_n} \left\| v_M' \circ \beta_{W,n,k_n} - v_M'(\beta_{W,n,k_n}(w_{\mu_{i,k_n}})) \right\|_{\infty, [w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]} \\
&\leq \left\| v_M'' \right\|_{\infty, [0, \overline{b}_W(\overline{m}, \overline{w}))]} \left\| \beta_{W,n,k_n}' \right\|_{\infty, [\underline{w}, w_{\mu_{i_n+j_n,k_n}}]} \max_{0 \leq i \leq i_n} \left\| Id - w_{\mu_{i,k_n}} \right\|_{\infty, [w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]} = O\left(k_n^{\alpha_2 - \frac{1}{2}}\right). \qquad (79)
\end{aligned}$$

For $\gamma_i^{(n)} := \left\| (v_M' \circ \beta_{W,n,k_n})\ g\ Id - v_M'(\beta_{W,n,k_n}(w_{\mu_{i,k_n}})) w_{\mu_{i,k_n}} g(w_{\mu_{i,k_n}}) \right\|_{\infty, [w_{\mu_{i-j_n,k_n}}, w_{\mu_{i+j_n,k_n}}]}$, it now follows that

$$\max_{0\le i\le i_n}\gamma_i^{(n)} \le \overline{w}\|g\|_\infty \max_{0\le i\le i_n}\left\|v_M'\circ\beta_{W,n,k_n} - v_M'(\beta_{W,n,k_n}(w_{\mu_{i,k_n}}))\right\|_{\infty,[w_{\mu_{i-j_n,k_n}},w_{\mu_{i+j_n,k_n}}]}$$
$$+\left\|v_M'\right\|_\infty \|g\|_\infty \max_{0\le i\le i_n}\left\|Id - w_{\mu_{i,k_n}}\right\|_{\infty,[w_{\mu_{i-j_n,k_n}},w_{\mu_{i+j_n,k_n}}]}$$
$$+\left\|v_M'\right\|_\infty \overline{w}\max_{0\le i\le i_n}\left\|g - g(w_{\mu_{i,k_n}})\right\|_{\infty,[w_{\mu_{i-j_n,k_n}},w_{\mu_{i+j_n,k_n}}]} = O\left(k_n^{\alpha_2-\frac{1}{2}}\right), \tag{80}$$

where the last step uses (77)-(79). Using the definitions of $\gamma_i^{(n)}$ and $S_{i,n}$, it follows that

$$\max_{0\le i\le i_n}\left|E\left[\int_{W_{i:k_n}}^{W_{i+1:k_n}}\left(v_M'(\beta_{W,n,k_n}(w))wg(w) - v_M'(\beta_{W,n,k_n}(w_{\mu_{i,k_n}}))w_{\mu_{i,k_n}}g(w_{\mu_{i,k_n}})\right)dw\middle|S_{i,n}\right]\right|$$
$$\le \max_{0\le i\le i_n}\left(\gamma_i^{(n)}E\left[W_{i+1:k_n} - W_{i:k_n}\middle|S_{i,n}\right]\right) = O\left(k_n^{\alpha_2-\frac{3}{2}}\right),$$

where the last step uses (80) and $\max_{0\le i\le i_n}E\left[W_{i+1:k_n} - W_{i:k_n}\middle|S_{i,n}\right] = O\left(k_n^{-1}\right)$ (by the bound used in the proof of (75) above). This in turn yields

$$\max_{0\le i\le i_n}\left|(k_n-1)E\left[\int_{W_{i:k_n}}^{W_{i+1:k_n}}v_M'(\beta_{W,n,k_n}(w))wg(w)dw\middle|S_{i,n}\right]\right.$$
$$\left.- v_M'(\beta_{W,n,k_n}(w_{\mu_{i,k_n}}))w_{\mu_{i,k_n}}g(w_{\mu_{i,k_n}})(k_n-1)E\left[W_{i+1:k_n} - W_{i:k_n}\middle|S_{i,n}\right]\right| = O\left(k_n^{\alpha_2-\frac{1}{2}}\right).$$

In view of this bound and $\max_{0\le i\le i_n}\left|E\left[W_{i+1:k_n} - W_{i:k_n}\middle|S_{i,n}\right] - E\left[W_{i+1:k_n} - W_{i:k_n}\right]\right| = \mathcal{E}(k_n)$ (see above), the bound (76) follows from the fact that

$$\max_{0\le i\le i_n}\left|g(w_{\mu_{i,k_n}})(k_n-1)E\left[W_{i+1:k_n} - W_{i:k_n}\right] - 1\right| = O\left(k_n^{\alpha_2-1}\right). \tag{81}$$

To show (81),[43] we first invoke (151) and Corollary 2 (i) to obtain

$$\max_{0\le i\le i_n}\left|g(w_{\mu_{i,k_n}})E\left[W_{i+1:k_n} - W_{i:k_n}\right] - \int_{\mu_{i-j_n,k_n}}^{\mu_{i+j_n,k_n}}B_{i,k_n}(u)\frac{g(w_{\mu_{i,k_n}})}{g(G^{-1}(u))}du\right|$$
$$= \max_{0\le i\le i_n}\left|\int_0^{\mu_{i-j_n,k_n}}B_{i,k_n}(u)\frac{g(w_{\mu_{i,k_n}})}{g(G^{-1}(u))}du + \int_{\mu_{i+j_n,k_n}}^1 B_{i,k_n}(u)\frac{g(w_{\mu_{i,k_n}})}{g(G^{-1}(u))}du\right| = \mathcal{E}(k_n). \tag{82}$$

To see the application of Corollary 2 (i), recall the assumptions about $(i_n)_{n\in\mathbb{N}}$ and $(j_n)_{n\in\mathbb{N}}$, and note that $(k_n+1)B_{i,k_n}$ is the density of $U_{i+1:k_n+1}$ (and $\left\|g(w_{\mu_{i,k_n}})/g\right\|_\infty < \infty$). Next, using $\max_{0\le i\le i_n}\left\|g(w_{\mu_{i,k_n}})/g - 1\right\|_{\infty,[w_{\mu_{i-j_n,k_n}},w_{\mu_{i+j_n,k_n}}]} = O\left(k_n^{\alpha_2-1}\right)$ (by (78) and $\min_{w\in[\underline{w},\overline{w}]}g(w) > 0$)

[43] Showing that the expression is of order $O\left(k_n^{\alpha_2-\frac{1}{2}}\right)$ would of course also be sufficient.

and $\int_0^1 B_{i,k_n}(u)du = \frac{1}{k_n+1} = O\left(k_n^{-1}\right)$, it follows that

$$\max_{0\le i\le i_n}\left|\int_{\mu_{i-j_n,k_n}}^{\mu_{i+j_n,k_n}} B_{i,k_n}(u)\frac{g(w_{\mu_{i,k_n}})}{g(G^{-1}(u))}du - \int_{\mu_{i-j_n,k_n}}^{\mu_{i+j_n,k_n}} B_{i,k_n}(u)du\right|$$
$$\le \max_{0\le i\le i_n}\left(\left\|\frac{g(w_{\mu_{i,k_n}})}{g}-1\right\|_{\infty,[w_{\mu_{i-j_n,k_n}},w_{\mu_{i+j_n,k_n}}]}\int_{\mu_{i-j_n,k_n}}^{\mu_{i+j_n,k_n}} B_{i,k_n}(u)du\right) = O\left(k_n^{\alpha_2-2}\right). \quad (83)$$

Using Corollary 2 (i) once more, we also have $\max_{0\le i\le i_n}\left|\frac{1}{k_n+1} - \int_{\mu_{i-j_n,k_n}}^{\mu_{i+j_n,k_n}} B_{i,k_n}(u)du\right| = \mathcal{E}(k_n)$, which together with (82) and (83) shows

$$\max_{0\le i\le i_n}\left|(k_n-1)g(w_{\mu_{i,k_n}})E\left[W_{i+1:k_n}-W_{i:k_n}\right] - \frac{k_n-1}{k_n+1}\right| = O\left(k_n^{\alpha_2-1}\right),$$

and thus also (81). This concludes the proof of (76), and thus establishes (74).

To prove (68), and thus Lemma 7, it remains to be shown that

$$\max_{0\le i\le i_n} d_{F,n,\bar{k}_n+i}^{\nu_W\circ\beta_{M,n,k_n}}\left|w_{\mu_{i,k_n}}\nu_M'\left(\beta_{W,n,k_n}(w_{\mu_{i,k_n}})\right) - \underline{w}\nu_M'\left(\underline{w}\sum_{j=1}^{i} d_{F,n,\bar{k}_n+j}^{\nu_W\circ\beta_{M,n,k_n}}\right)\right| = O\left(k_n^{\alpha_2-1}\right). \quad (84)$$

As $\max_{0\le i\le i_n} d_{F,n,\bar{k}_n+i}^{\nu_W\circ\beta_{M,n,k_n}} = O\left(n^{-\frac{1}{2}}\right)$, (84) follows if we can show

$$\max_{0\le i\le i_n}\left|w_{\mu_{i,k_n}}\nu_M'(\beta_{W,n,k_n}(w_{\mu_{i,k_n}})) - \underline{w}\nu_M'\left(\underline{w}\sum_{j=1}^{i} d_{F,n,\bar{k}_n+j}^{\nu_W\circ\beta_{M,n,k_n}}\right)\right| = O\left(k_n^{\alpha_2-\frac{1}{2}}\right). \quad (85)$$

We also note that

$$\max_{0\le i\le i_n}\left|w_{\mu_{i,k_n}}\nu_M'(\beta_{W,n,k_n}(w_{\mu_{i,k_n}})) - \underline{w}\nu_M'\left(\underline{w}\sum_{j=1}^{i} d_{F,n,\bar{k}_n+j}^{\nu_W\circ\beta_{M,n,k_n}}\right)\right|$$
$$\le \max_{0\le i\le i_n}\left(|w_{\mu_{i,k_n}}-\underline{w}|\left\|\nu_M'\right\|_\infty + \underline{w}\left\|\nu_M''\right\|_{\infty,[0,\bar{b}_W(\overline{m},\overline{w})]}\left|\beta_{W,n,k_n}(w_{\mu_{i,k_n}}) - \underline{w}\sum_{j=1}^{i} d_{F,n,\bar{k}_n+j}^{\nu_W\circ\beta_{M,n,k_n}}\right|\right),$$

$\max_{0\le i\le i_n}|w_{\mu_{i,k_n}}-\underline{w}| \le \frac{\mu_{i_n,k_n}}{\min_{w\in[\underline{w},\overline{w}]}g(w)} = O\left(k_n^{\alpha_1-1}\right)$, and $\alpha_1 - 1 = \frac{\alpha_1}{2} + (\frac{\alpha_1}{2}-1) < \alpha_2 - \frac{1}{2}$, so that (85) follows from the bound

$$\max_{0\le i\le i_n}\left|\beta_{W,n,k_n}(w_{\mu_{i,k_n}}) - \underline{w}\sum_{j=1}^{i} d_{F,n,\bar{k}_n+j}^{\nu_W\circ\beta_{M,n,k_n}}\right| = O\left(k_n^{\alpha_2-\frac{1}{2}}\right), \quad (86)$$

which we show now. First, up to exponentially decaying error terms, we may bound $\beta_{W,n,k_n}(w_{\mu_{i,k_n}})$ from below by $\underline{w}\sum_{j=1}^{i-j_n} d_{F,n,\bar{k}_n+j}^{\nu_W\circ\beta_{M,n,k_n}}$ and from above by $w_{\mu_{i,k_n}}\sum_{j=1}^{i+j_n} d_{F,n,\bar{k}_n+j}^{\nu_W\circ\beta_{M,n,k_n}}$.

Indeed, starting from (44), we have

$$\beta_{W,n,k_n}(w_{\mu_{i,k_n}}) = \int_{\underline{w}}^{w_{\mu_{i,k_n}}} w \sum_{j=1}^{k_n-1} g_{j:k_n-1}(w) d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} dw$$
$$\geq \underline{w} \sum_{j=1}^{i-j_n} \left(\int_{\underline{w}}^{w_{\mu_{i,k_n}}} g_{j:k_n-1}(w) dw \right) d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} = \underline{w} \sum_{j=1}^{i-j_n} G_{j:k_n-1}(w_{\mu_{i,k_n}}) d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}}$$
$$= \underline{w} \sum_{j=1}^{i-j_n} P[U_{j:k_n-1} \leq \mu_{i,k_n}] d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}},$$

and, invoking Corollary 1 (i), it also follows that

$$\max_{0 \leq i \leq i_n} \left(\sum_{j=1}^{i-j_n} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} - \sum_{j=1}^{i-j_n} P[U_{j:k_n-1} \leq \mu_{i,k_n}] d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} \right) = \mathcal{E}(k_n).$$

Similarly, $\beta_{W,n,k_n}(w_{\mu_{i,k_n}}) \leq w_{\mu_{i,k_n}} \left(\sum_{j=1}^{i+j_n} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} + \sum_{j=i+j_n+1}^{k_n-1} P[U_{j:k_n-1} \leq \mu_{i,k_n}] d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} \right)$ and, invoking Corollary 1 (i) again, $\max_{0 \leq i \leq i_n} \sum_{j=i+j_n+1}^{k_n-1} P[U_{j:k_n-1} \leq \mu_{i,k_n}] d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} = \mathcal{E}(k_n)$. It follows that

$$\max_{0 \leq i \leq i_n} \left| \beta_{W,n,k_n}(w_{\mu_{i,k_n}}) - \underline{w} \sum_{j=1}^{i} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} \right|$$
$$\leq \max_{0 \leq i \leq i_n} \max \left\{ \underline{w} \sum_{j=i-j_n+1}^{i} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}},\ w_{\mu_{i,k_n}} \sum_{j=1}^{i+j_n} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} - \underline{w} \sum_{j=1}^{i} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} \right\} + \mathcal{E}(k_n).$$

Now, $j_n = O(k_n^{\alpha_2})$ and $\max_{0 \leq i \leq i_n} d_{F,n,\bar{k}_n+i}^{v_W \circ \beta_{M,n,k_n}} = O\left(n^{-\frac{1}{2}}\right)$ imply $\max_{0 \leq i \leq i_n} \sum_{j=i-j_n+1}^{i} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} = O\left(k_n^{\alpha_2 - \frac{1}{2}}\right)$. Moreover, $\max_{0 \leq i \leq i_n} \left(w_{\mu_{i,k_n}} \sum_{j=1}^{i+j_n} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} - \underline{w} \sum_{j=1}^{i} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}} \right)$ is bounded by $v_W(\overline{b}_M(\overline{m}, \overline{w})))\max_{0 \leq i \leq i_n}(w_{\mu_{i,k_n}} - \underline{w}) + \underline{w} \max_{0 \leq i \leq i_n} \sum_{j=i+1}^{i+j_n} d_{F,n,\bar{k}_n+j}^{v_W \circ \beta_{M,n,k_n}}$, and hence also of order $O\left(k_n^{\alpha_2 - \frac{1}{2}}\right)$ (the first summand is of order $O\left(k_n^{\alpha_1 - 1}\right)$ because $w_{\mu_{i_n,k_n}} - \underline{w}$ is; the second summand is of order $O\left(k_n^{\alpha_2 - \frac{1}{2}}\right)$ as $j_n = O(k_n^{\alpha_2})$ and $\max_{0 \leq i \leq i_n + j_n} d_{F,n,\bar{k}_n+i}^{v_W \circ \beta_{M,n,k_n}} = O\left(n^{-\frac{1}{2}}\right)$). This shows (86) and thereby concludes the proof. □

Proof of Corollary 4. In view of formula (43) and the definitions of $\tilde{\beta}_{M,n,k_n}$ and $\underline{z}_{n,\alpha}$, $\alpha > \frac{1}{2}$ and Corollary 2 (ii) immediately imply $\tilde{\beta}_{M,n,k_n}(\underline{z}_{n,\alpha}) = \mathcal{E}(n)$. Moreover, from the definition of Ξ_2 and the fact that $-\underline{z}_{n,\alpha} = \Theta\left(n^{\alpha - \frac{1}{2}}\right)$, $\Psi_1[\tilde{\beta}_{M,n,k_n}](\underline{z}_{n,\alpha}) = \mathcal{E}(n)$ is immediate from basic properties of the tail of a normal distribution. Consequently, (52) follows from (50), because the length of the interval $I_{n,\alpha}$ is of order $\Theta\left(n^{\alpha - \frac{1}{2}}\right)$. □

Proof of Theorem 2. Let $\tilde{\beta} \in \mathcal{L}$, and let $(\tilde{\beta}_{M,n_l,k_{n_l}})$ be a subsequence of re-scaled equilibrium strategies that converges locally uniformly to $\tilde{\beta}$. From the definitions of Ξ_1 and Ξ_2 and $\|\tilde{\beta}_{M,n_l,k_{n_l}}\|_\infty \leq \overline{b}_M(\overline{m},\overline{w})$, it is completely straightforward to show that the sequence $(\Psi_1[\tilde{\beta}_{M,n_l,k_{n_l}}])$ then converges locally uniformly to $\Psi_1[\tilde{\beta}]$ (we omit the formal proof). Thus, for any compact interval $I \subset \mathbb{R}$, $\lim_{l\to\infty} \|\tilde{\beta} - \tilde{\beta}_{M,n_l,k_{n_l}}\|_{\infty,I} = 0$ and $\lim_{l\to\infty} \|\Psi_1[\tilde{\beta}_{M,n_l,k_{n_l}}] - \Psi_1[\tilde{\beta}]\|_{\infty,I} = 0$. Invoking the key ingredient, Corollary 5, which implies that $\lim_{l\to\infty} \|\tilde{\beta}_{M,n_l,k_{n_l}} - \Psi_1[\tilde{\beta}_{M,n_l,k_{n_l}}]\|_{\infty,I} = 0$, we obtain from

$$\|\tilde{\beta} - \Psi_1[\tilde{\beta}]\|_{\infty,I} \leq \|\tilde{\beta} - \tilde{\beta}_{M,n_l,k_{n_l}}\|_{\infty,I} + \|\tilde{\beta}_{M,n_l,k_{n_l}} - \Psi_1[\tilde{\beta}_{M,n_l,k_{n_l}}]\|_{\infty,I} + \|\Psi_1[\tilde{\beta}_{M,n_l,k_{n_l}}] - \Psi_1[\tilde{\beta}]\|_{\infty,I}$$

that $\tilde{\beta} = \Psi_1[\tilde{\beta}]$ holds on I. As I was arbitrary, this concludes the proof. □

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Online Appendix

Parts D-H of this Online Appendix contain all proofs not given in the Appendix. Part I contains Remarks 3 and 4 (referred to in the main text).

D Proofs for Section 2.4

Proof of Lemma 1. We derive (13) using the standard first-order approach.[44] First, in a SSMBNE, a man with type $\underline{m}$ makes the lowest-ranking investment with probability 1. This implies $\beta_M(\underline{m}) = 0$.[45] Similarly, $\beta_W(\underline{w}) = 0$. Secondly, for a man with type $m > \underline{m}$ who assumes that all other agents use strictly increasing, differentiable strategies β_M and β_W, the problem of maximizing his expected utility is to choose $s \in [\underline{m}, \overline{m}]$ in order to maximize $m \sum_{i=0}^{k-1} F_{n-k+i:n-1}(s) \left(d_{G,k,i}^{v_M \circ \beta_W} + v_M(0)\delta_{i0}\right) - \beta_M(s)$. This implies the equilibrium first order condition $\beta_M'(m) = m \sum_{i=0}^{k-1} f_{n-k+i:n-1}(m) \left(d_{G,k,i}^{v_M \circ \beta_W} + v_M(0)\delta_{i0}\right)$. Integrating, we obtain (13). The proof of (14) is analogous. □

Proof of Lemma 2. The equilibrium interim expected utilities of the types $\overline{m}$ and $\overline{w}$, $u_M(\overline{m})$ and $u_W(\overline{w})$, satisfy[46] $0 \le u_M(\overline{m}) = \overline{m} E[v_M(\beta_W(W_{k:k}))] - \beta_M(\overline{m}) < \overline{m} v_M(\beta_W(\overline{w})) - \beta_M(\overline{m})$ and $0 \le u_W(\overline{w}) = \overline{w} E[v_W(\beta_M(M_{n:n}))] - \beta_W(\overline{w}) < \overline{w} v_W(\beta_M(\overline{m})) - \beta_W(\overline{w})$. In particular, $(\beta_M(\overline{m}), \beta_W(\overline{w})) \in \mathcal{I}(\overline{m}, \overline{w})$, which implies (15). □

Proof of Theorem 1. The first part of the proof is analogous to the one of Lemma 1 in Dizdar, Moldovanu and Szech (2019). We define $T : \mathbb{R}_+^k \to \mathbb{R}_+^k$ and $S : \mathbb{R}_+^k \to \mathbb{R}_+^k$ as follows: for any $y \in \mathbb{R}_+^k$ and $i \in \{0, ..., k-1\}$,

$$
\begin{aligned}
T_i(y) &:= E\left[v_W\left(\int_{\underline{m}}^{M_{n-k+i+1:n}} s \sum_{j=0}^{k-1} f_{n-k+j:n-1}(s)\left(y_j + v_M(0)\delta_{j0}\right) ds\right)\right] \\
&\quad - E\left[v_W\left(\int_{\underline{m}}^{M_{n-k+i:n}} s \sum_{j=0}^{k-1} f_{n-k+j:n-1}(s)\left(y_j + v_M(0)\delta_{j0}\right) ds\right)\right], \\
S_i(y) &:= E\left[v_M\left(\int_{\underline{w}}^{W_{i+1:k}} s \sum_{j=1}^{k-1} g_{j:k-1}(s) y_j\, ds\right) - v_M\left(\int_{\underline{w}}^{W_{i:k}} s \sum_{j=1}^{k-1} g_{j:k-1}(s) y_j\, ds\right)\right].
\end{aligned}
$$

The mapping $\iota : \beta_W \mapsto d_{G,k}^{v_M \circ \beta_W}$ is a bijection between the set of SSMBNE and the set of fixed points of $S \circ T$ (given β_W, β_M is determined by (13)). First, if β_W is an equilibrium strategy, it is immediate from (13), (14) and the definitions of T and S that $d_{G,k}^{v_M \circ \beta_W}$ is a fixed point of $S \circ T$. Next, the mapping is one-to-one (if $\iota(\beta_W^1) = \iota(\beta_W^2)$ for two equilibrium strategies β_W^1 and β_W^2, then (13) and (14) imply $\beta_W^1 = \beta_W^2$) and onto: if y^*

[44] See Appendices A and C of Moldovanu and Sela (2001) for details on sufficiency.

[45] Otherwise, the type $\underline{m}$ could decrease his investment without changing his expected match.

[46] Note that each agent can ensure a nonnegative expected utility by making a zero investment.

is a fixed point of $S \circ T$, then $\beta_M(m) := \int_{\underline{m}}^{m} s \sum_{j=0}^{k-1} f_{n-k+j:n-1}(s)\left(y_j^* + v_M(0)\delta_{j0}\right) ds$ and $\beta_W(w) := \int_{\underline{w}}^{w} s \sum_{j=1}^{k-1} g_{j:k-1}(s) d_{F,n,n-k+j}^{v_W \circ \beta_M} ds$ are equilibrium strategies and $d_{G,k}^{v_M \circ \beta_W} = y^*$.

The existence of a fixed point now follows from Brouwer's Theorem (as in Peters 2007, 2011). Indeed, $S \circ T$ is continuous and, as shown by the following argument, it maps the compact and convex set $\{y \in \mathbb{R}_+^k \mid |y|_1 \leq v_M\left(\overline{b}_W(\overline{m}, \overline{w})\right) - v_M(0)\}$ into itself. First, we have for all $y \in \mathbb{R}_+^k$:

$$
\begin{aligned}
|S(T(y))|_1 &= \sum_{i=0}^{k-1} S_i(T(y)) = E\left[v_M\left(\int_{\underline{w}}^{W_{k:k}} s \sum_{j=1}^{k-1} g_{j:k-1}(s) T_j(y) ds\right)\right] - v_M(0) \\
&\leq v_M\left(\overline{w}|T(y)|_1\right) - v_M(0) \leq v_M\left(\overline{w} v_W\left(\overline{m}(|y|_1 + v_M(0))\right)\right) - v_M(0).
\end{aligned}
$$

Consequently, for any $y \in \mathbb{R}_+^k$ that satisfies $|y|_1 \leq v_M(\overline{b}_W(\overline{m}, \overline{w})) - v_M(0)$,

$$
\begin{aligned}
|S(T(y))|_1 &\leq v_M\left(\overline{w} v_W\left(\overline{m} v_M\left(\overline{b}_W(\overline{m}, \overline{w})\right)\right)\right) - v_M(0) \\
&= v_M\left(\overline{w} v_W\left(\overline{b}_M(\overline{m}, \overline{w})\right)\right) - v_M(0) = v_M\left(\overline{b}_W(\overline{m}, \overline{w})\right) - v_M(0),
\end{aligned}
$$

where the two last identities use (4). This concludes the proof. □

E Proof of Theorem 4

Proof of Theorem 4. Let (γ_M, γ_W) be a pair of nondecreasing functions for which there is a subsequence $((\beta_{M,n_{l_j},k_{n_{l_j}}}, \beta_{W,n_{l_j},k_{n_{l_j}}}))$ of $((\beta_{M,n_l,k_{n_l}}, \beta_{W,n_l,k_{n_l}}))$ such that $\lim_{j\to\infty} \beta_{M,n_{l_j},k_{n_{l_j}}}(m) = \gamma_M(m)$ at all continuity points of γ_M and $\lim_{j\to\infty} \beta_{W,n_{l_j},k_{n_{l_j}}}(w) = \gamma_W(w)$ at all continuity points of γ_W. We will show $\gamma_M(m) = 0$ for $m < m_r$ and $(\gamma_M(m_{r+p(1-r)}), \gamma_W(w_p)) = (\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty))$ for $p \in (0, p(\tilde{\beta}_*))$. In view of Helly's selection theorem, this proves Theorem 4. As $\tilde{\beta}_{M,n_{l_j},k_{n_{l_j}}}$ converges locally uniformly to $\tilde{\beta}_* \in \mathcal{L}$, it is immediate from Theorem 2 and $\lim_{n\to\infty} \bar{k}_n/n = r$ that $\gamma_M(m) = 0$ for $m < m_r$, and that $\gamma_M(m_{r+p(1-r)}) \geq \tilde{\beta}_*(\infty)$ and $\gamma_W(w_p) \geq \tilde{\zeta}_*(\infty)$ for all $p > 0$. As γ_M and γ_W are nondecreasing, they have at most countably many discontinuities. In particular, the set of values of p such that γ_M is continuous at $m_{r+p(1-r)}$ and γ_W is continuous at w_p is dense in $(0, p(\tilde{\beta}_*))$. Consider any such joint continuity point $p \in (0, p(\tilde{\beta}_*))$. It follows that as $j \to \infty$, the equilibrium utility of $m_{r+p(1-r)}$ converges to $m_{r+p(1-r)} v_M(\gamma_W(w_p)) - \gamma_M(m_{r+p(1-r)})$, and that the equilibrium utility of w_p converges to $w_p v_W(\gamma_M(m_{r+p(1-r)})) - \gamma_W(w_p)$ (due to the continuity of γ_M and γ_W, and because agents' uncertainty about the (non-rescaled) *type* of their equilibrium partner vanishes in the limit). As $(\tilde{\beta}_{M,n_{l_j},k_{n_{l_j}}})$ converges to $\tilde{\beta}_*$, we also know that the expected return for the investment $\tilde{\beta}_*(\infty)$ converges to the return from a deterministic match with a partner with investment $\tilde{\zeta}_*(\infty)$, and vice versa. Using that equilibrium investments maximize expected utilities and letting $j \to \infty$, we obtain

$$m_{r+p(1-r)} v_M(\gamma_W(w_p)) - \gamma_M(m_{r+p(1-r)}) \geq m_{r+p(1-r)} v_M(\tilde{\zeta}_*(\infty)) - \tilde{\beta}_*(\infty), \tag{87}$$

$$w_p v_W(\gamma_M(m_{r+p(1-r)})) - \gamma_W(w_p) \geq w_p v_W(\tilde{\beta}_*(\infty)) - \tilde{\zeta}_*(\infty). \tag{88}$$

As $p \in (0, p(\tilde{\beta}_*))$, the investments $(\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty))$ correspond to overinvestment for the types $m_{r+p(1-r)}$ and w_p (the pair of investments lies "above" the set $\mathcal{P}(m_{r+p(1-r)}, w_p)$), so that (87) and (88) imply $(\gamma_M(m_{r+p(1-r)}), \gamma_W(w_p)) \leq (\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty))$ (there is no $(b_M, b_W) \neq (\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty))$ with $(b_M, b_W) \geq (\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty))$ that yields weakly higher utilities for both agents). Thus, $(\gamma_M(m_{r+p(1-r)}), \gamma_W(w_p)) = (\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty))$ at any joint continuity point and hence (using monotonicity and that the set of continuity points is dense), $(\gamma_M(m_{r+p(1-r)}), \gamma_W(w_p)) = (\tilde{\beta}_*(\infty), \tilde{\zeta}_*(\infty))$ for all $p \in (0, p(\tilde{\beta}_*))$. □

F Proofs for Section 3.2.1

Proof of Lemma 4. See Lemma 3.1.1 in Reiss (1989). □

Proof of Lemma 5. (i) This follows from $a_{i,l} = \left(\frac{B_{1,2}(\mu_{i,l})}{2l}\right)^{\frac{1}{2}}$, $B_{1,2}(u) = 2u(1-u)$ and $\mu_{i,l} = \frac{i}{l+1}$.

(ii) As $\alpha < 1$ and $i_l = \Theta(l^{\alpha})$ imply $\frac{l+1-i_l}{l+1} = \Theta(1)$, we have $a_{i_l,l} = \left(\frac{i_l(l+1-i_l)}{(l+1)^2 l}\right)^{\frac{1}{2}} = \Theta\left(\left(\frac{i_l}{(l+1)l}\right)^{\frac{1}{2}}\right) = \Theta\left(l^{\frac{\alpha}{2}-1}\right)$.

(iii) As $B_{1,2}(\mu_{i_l,l}) = \Theta(1)$, we obtain $a_{i_l,l} = \Theta\left(l^{-\frac{1}{2}}\right)$. □

Proof of Corollary 1. (i) We show a stronger result that quantifies the exponential rate of decay: for any $\alpha_2 \in (\frac{\alpha_1}{2}, 1]$ and any sequence (ε_l) such that $\varepsilon_l = \Theta\left(l^{\alpha_2 - 1}\right)$,

$$\min_{1 \leq i \leq i_l} \left(-\ln P\left[|U_{i:l} - \mu_{i,l}| \geq \varepsilon_l\right]\right) = \Omega\left(l^{\min\{2\alpha_2 - \alpha_1, \alpha_2\}}\right). \tag{89}$$

Clearly, (89) implies (37). To prove (89), we first set $y_{i,l} := \frac{\varepsilon_l}{a_{i,l}}$ and apply the exponential bound (36). This yields $P\left[|U_{i:l} - \mu_{i,l}| \geq \varepsilon_l\right] = P[|U_{i:l} - \mu_{i,l}| \geq y_{i,l} a_{i,l}] \leq 2e^{-\frac{y_{i,l}^2}{3(1+y_{i,l}/(la_{i,l}))}}$ for any $i \in \{1, ..., l\}$. Thus,

$$-\ln P\left[|U_{i:l} - \mu_{i,l}| \geq \varepsilon_l\right] \geq -\ln 2 + \frac{y_{i,l}^2}{3(1 + y_{i,l}/(la_{i,l}))} = -\ln 2 + \frac{1}{3}\left(\frac{a_{i,l}^2}{\varepsilon_l^2} + \frac{1}{l\varepsilon_l}\right)^{-1}. \tag{90}$$

We have $\frac{1}{l\varepsilon_l} = \Theta\left(l^{-\alpha_2}\right)$ and, using $\alpha_1 < 1$, $i_l = O\left(l^{\alpha_1}\right)$ and parts (i) and (ii) of Lemma 5, also $\max_{1 \leq i \leq i_l} \frac{a_{i,l}^2}{\varepsilon_l^2} = O\left(l^{\alpha_1 - 2\alpha_2}\right)$. Thus, $\max_{1 \leq i \leq i_l}\left(\frac{a_{i,l}^2}{\varepsilon_l^2} + \frac{1}{l\varepsilon_l}\right) = O\left(l^{\max\{\alpha_1 - 2\alpha_2, -\alpha_2\}}\right)$, i.e., $\min_{1 \leq i \leq i_l}\left(\frac{a_{i,l}^2}{\varepsilon_l^2} + \frac{1}{l\varepsilon_l}\right)^{-1} = \Omega\left(l^{\min\{2\alpha_2 - \alpha_1, \alpha_2\}}\right)$. In view of (90), this proves (89).

(ii) We show a stronger result: for $\alpha_2 \in (\frac{1}{2}, 1]$ and $\varepsilon_l = \Theta\left(l^{\alpha_2-1}\right)$,

$$\min_{i_l \le i \le l-i_l} \left(-\ln P\left[|U_{i:l} - \mu_{i,l}| \ge \varepsilon_l\right]\right) = \Omega\left(l^{2\alpha_2-1}\right). \tag{91}$$

The estimate (90) still applies, and we have $\frac{1}{l\varepsilon_l} = \Theta\left(l^{-\alpha_2}\right)$. Lemma 5 (iii) implies $\max_{i_l \le i \le l-i_l} \frac{a_{i,l}^2}{\varepsilon_l^2} = \Theta\left(l^{1-2\alpha_2}\right)$. As $\alpha_2 \le 1$, this yields $\max_{i_l \le i \le l-i_l}\left(\frac{a_{i,l}^2}{\varepsilon_l^2} + \frac{1}{l\varepsilon_l}\right) = \Theta\left(l^{1-2\alpha_2}\right)$, i.e., $\min_{i_l \le i \le l-i_l}\left(\frac{a_{i,l}^2}{\varepsilon_l^2} + \frac{1}{l\varepsilon_l}\right)^{-1} = \Theta\left(l^{2\alpha_2-1}\right)$, so that (91) follows. □

Proof of Corollary 2. (i) We show a stronger result: for any $\alpha_2 \in (\frac{\alpha_1}{2}, \alpha_1]$ and any sequence (j_l) satisfying $j_l = \Theta\left(l^{\alpha_2}\right)$,

$$\min_{1 \le i \le i_l}\left(-\ln\left(\max_{j \ge i+j_l} \|h_{j:l}\|_{\infty,[\underline{x}, x_{\mu_{i,l}+\gamma_l}]} + \max_{j \le i-j_l} \|h_{j:l}\|_{\infty,[x_{\mu_{i,l}-\gamma_l}, \overline{x}]}\right)\right) = \Omega\left(l^{2\alpha_2-\alpha_1}\right). \tag{92}$$

This result, which we will derive using the bound (89), clearly implies Corollary 2 (i). To prove (92), we note first that for any i,

$$\begin{aligned}
&\max_{j \ge i+j_l} \|h_{j:l}\|_{\infty,[\underline{x}, x_{\mu_{i,l}+\gamma_l}]} = \sup_{[\underline{x}, x_{\mu_{i,l}+\gamma_l}]} \max_{j \ge i+j_l} h_{j:l}(x) \le \sup_{[\underline{x}, x_{\mu_{i,l}+\gamma_l}]} \sum_{j=i+j_l}^{l} h_{j:l}(x)\\
&\le l\|h\|_\infty H_{i+j_l-1:l-1}(x_{\mu_{i,l}+\gamma_l}) = l\|h\|_\infty P[U_{i+j_l-1:l-1} \le \mu_{i,l} + \gamma_l]\\
&= l\|h\|_\infty P[U_{i+j_l-1:l-1} - \mu_{i+j_l-1,l-1} \le -(\mu_{i+j_l-1,l-1} - (\mu_{i,l} + \gamma_l))].
\end{aligned} \tag{93}$$

Here, the inequality in the second line uses (147) and the monotonicity of $H_{i+j_l-1:l-1}$, and the subsequent equality uses (35). Using (148), we similarly bound the expression $\max_{j \le i-j_l} \|h_{j:l}\|_{\infty,[x_{\mu_{i,l}-\gamma_l}, \overline{x}]}$ (if $i \le j_l$, the term is of course trivially equal to zero):

$$\begin{aligned}
&\max_{j \le i-j_l} \|h_{j:l}\|_{\infty,[x_{\mu_{i,l}-\gamma_l}, \overline{x}]} = \sup_{[x_{\mu_{i,l}-\gamma_l}, \overline{x}]} \max_{j \le i-j_l} h_{j:l}(x) \le \sup_{[x_{\mu_{i,l}-\gamma_l}, \overline{x}]} \sum_{j=1}^{i-j_l} h_{j:l}(x)\\
&\le l\|h\|_\infty (1 - H_{i-j_l:l-1}(x_{\mu_{i,l}-\gamma_l})) = l\|h\|_\infty P[U_{i-j_l:l-1} > \mu_{i,l} - \gamma_l]\\
&= l\|h\|_\infty P\left[U_{i-j_l:l-1} - \mu_{i-j_l,l-1} > \mu_{i,l} - \gamma_l - \mu_{i-j_l,l-1}\right].
\end{aligned} \tag{94}$$

As $\frac{\alpha_1}{2} < \alpha_2 \le \alpha_1$ and $j_l = \Theta(l^{\alpha_2})$, we have $i_l + j_l - 1 = O\left(l^{\alpha_1}\right)$ (where $\alpha_1 < 1$). We then apply the bound (89) from the proof of Corollary 1 (i) with respect to the sequences $(i_l + j_l - 1)_{l \in \mathbb{N}}$ and $\varepsilon_l = \min_{1 \le i \le i_l}(\mu_{i+j_l-1,l-1} - (\mu_{i,l} + \gamma_l)) = \Theta\left(l^{\alpha_2-1}\right)$. This implies

$$\min_{1 \le i \le i_l}\left(-\ln P\left[U_{i+j_l-1:l-1} - \mu_{i+j_l-1,l-1} \le -(\mu_{i+j_l-1,l-1} - (\mu_{i,l} + \gamma_l))\right]\right) = \Omega\left(l^{2\alpha_2-\alpha_1}\right).$$

Similarly, applying the bound (89) with respect to the sequences $i_l - j_l = O\left(l^{\alpha_1}\right)$ and

$\varepsilon_l = \min_{j_l<i\le i_l}(\mu_{i,l} - \gamma_l - \mu_{i-j_l,l-1}) = \Theta\left(l^{\alpha_2-1}\right)$ yields

$$\min_{j_l<i\le i_l}\left(-\ln P\left[U_{i-j_l:l-1} - \mu_{i-j_l,l-1} > \mu_{i,l} - \gamma_l - \mu_{i-j_l,l-1}\right]\right) = \Omega\left(l^{2\alpha_2-\alpha_1}\right).$$

In view of (93) and (94), this proves (92).

(ii) We show: if $j_l = \Theta(l^{\alpha_2})$, then

$$\min_{i_l\le i\le l-i_l}\left(-\ln\left(\max_{j\ge i+j_l}\|h_{j:l}\|_{\infty,[\underline{x},x_{\mu_{i,l}+\gamma_l}]} + \max_{j\le i-j_l}\|h_{j:l}\|_{\infty,[x_{\mu_{i,l}-\gamma_l},\overline{x}]}\right)\right) = \Omega\left(l^{2\alpha_2-1}\right). \tag{95}$$

Clearly, (95) implies (40). The bounds (93) and (94) still apply. As $i_l = \Omega(l)$, $i_l \le \frac{l}{2}$ and $\alpha_2 < 1$, there is a sequence (i'_l) with $i'_l = \Omega(l)$ and $i'_l \le \frac{l}{2}$, such that $i'_l < i_l - j_l$ and $i_l+j_l-1 < l-i'_l$ for all but finitely many l. In view of (93) and (94), (95) then follows from applying the bound (91) from the proof of Corollary 1 (ii), with respect to the sequences $(i'_l)_{l\in\mathbb{N}}$ and $\varepsilon_l := \min_{i_l\le i\le l-i_l}\min\{\mu_{i,l} - \gamma_l - \mu_{i-j_l,l-1}, \mu_{i+j_l-1,l-1} - (\mu_{i,l}+\gamma_l))\} = \Theta\left(l^{\alpha_2-1}\right)$. □

Proof of Theorem 5. For any given H and $1 \le i \le l$, we let $Z_{i,l}$ denote the function

$$Z_{i,l}(y) = \frac{1}{a_{i,l}}\left[H\left(x_{\mu_{i,l}} + y\frac{a_{i,l}}{h(x_{\mu_{i,l}})}\right) - \mu_{i,l}\right].$$

By the case $m = 1$ of Theorem 4.7.1 in Reiss (1989) there is a constant $C_1 > 0$ such that for all H satisfying the assumptions of the Theorem and all $1 \le i \le l$, we have $|\check{h}_{i,l}(y) - \varphi(y)| \le C_1\varphi(y)\left(1+|y|^3\right)\left[\left(\frac{l}{i(l-i)}\right)^{\frac{1}{2}} + \|Z''_{i,l}\|_{\infty,J_{i,l}}\right]$ for all $y \in J_{i,l}$.[47] As $Z''_{i,l}(y) = h'\left(x_{\mu_{i,l}} + y\frac{a_{i,l}}{h(x_{\mu_{i,l}})}\right)\frac{a_{i,l}}{h^2(x_{\mu_{i,l}})}$, it follows that for all $y \in J_{i,l}$,

$$|\check{h}_{i,l}(y) - \varphi(y)| \le C_1\varphi(y)\left(1+|y|^3\right)\left[\left(\frac{l}{i(l-i)}\right)^{\frac{1}{2}} + a_{i,l}\frac{\|h'\|_\infty}{\left(\min_{[\underline{x},\overline{x}]} h(x)\right)^2}\right].$$

This proves (41). The estimate (42) is a simple corollary: given the assumptions about i_l, we have $\max_{i_l\le i\le l-i_l}\left(\frac{l}{i(l-i)}\right)^{\frac{1}{2}} = \Theta\left(l^{-\frac{1}{2}}\right)$ and $\max_{i_l\le i\le l-i_l} a_{i,l} = \Theta\left(l^{-\frac{1}{2}}\right)$ (by parts (i) and (iii) of Lemma 5). Thus, for any given H,

$$\max_{i_l\le i\le l-i_l}\left[\left(\frac{l}{i(l-i)}\right)^{\frac{1}{2}} + a_{i,l}\frac{\|h'\|_\infty}{\left(\inf_{[\underline{x},\overline{x}]} h(x)\right)^2}\right] = \Theta\left(l^{-\frac{1}{2}}\right). \tag{96}$$

[47] Theorem 4.7.1 in Reiss (1989) requires only that $h(x_{\mu_{i,l}}) > 0$ and that $Z_{i,l}$ is twice differentiable on a sufficiently large interval around 0. If this interval is smaller than $J_{i,l}$, the approximation is only ensured on the smaller interval. Under our assumption that h is non-zero and differentiable on the entire support, this issue does not occur.

Moreover, $\min_{i_l \le i \le l - i_l} \left(\frac{i(l-i)}{l}\right)^{\frac{1}{6}} = \Theta\left(l^{\frac{1}{6}}\right)$, so that $y_l = \Theta(l^\alpha)$ for $\alpha < \frac{1}{6}$ implies that $[-y_l, y_l] \subset \cap_{i_l \le i \le l-i_l} J_{i,l}$ for all but finitely many l. Applying (41) and using (96) and $\max_{y \in [-y_l, y_l]} 1 + |y|^3 = \Theta\left(l^{3\alpha}\right)$, we obtain (42). □

G Proofs of Lemma 6 and Theorem 6

Proof of Lemma 6. The inequality in (45) is a straightforward implication of (43) and

$$\left| d_{G,k_n}^{v_M \circ \beta_{W,n,k_n}} \right|_1 = E[v_M(\beta_{W,n,k_n}(W_{k_n:k_n})) - v_M(\beta_{W,n,k_n}(W_{0:k_n}))] \le v_M(\bar{b}_W(\overline{m}, \overline{w})) - v_M(0),$$

where the last step uses Lemma 2. Thus, (45) is implied by the bound

$$\max_{\bar{k}_n \le i \le n-1} \|f_{i:n-1}\|_{\infty, [\underline{m}, m_{1-\tau}]} = O\left(n^{\frac{1}{2}}\right), \tag{97}$$

which is just a general fact about densities of order statistics. To prove (97), we fix two integer sequences (i_n) and (j_n), such that $\mu_{n-i_n,n} - (1-\tau) = O(n^{-1})$ and $j_n = \Omega(n^\alpha)$ for some $\alpha \in \left(\frac{1}{2}, 1\right)$. Corollary 2 (ii) implies that $\max_{j > n-i_n+j_n} \|f_{j:n-1}\|_{\infty, [\underline{m}, m_{1-\tau}]} = \mathcal{E}(n)$, so that (97) follows from

$$\max_{\bar{k}_n \le i \le n-i_n+j_n} \|f_{i:n-1}\|_{\infty, [\underline{m}, m_{1-\tau}]} \le \max_{\bar{k}_n \le i \le n-i_n+j_n} \|f_{i:n-1}\|_\infty = O\left(n^{\frac{1}{2}}\right), \tag{98}$$

which holds as $\bar{k}_n = \Omega(n)$, $i_n - j_n = \Omega(n)$, and the supremum norms of the densities of central order statistics from a distribution with bounded density are of order $O\left(n^{\frac{1}{2}}\right)$.[48] More precisely, Lemma 18 in Online Appendix H shows that there is a universal constant C such that $\max_{\bar{k}_n \le i \le n-i_n+j_n} \|f_{i:n-1}\|_\infty \le C\|f\|_\infty (n-1)^{\frac{1}{2}} (\min_{\bar{k}_n \le i \le n-i_n+j_n} B_{1,2}(\mu_{i-1,n-3}))^{-\frac{1}{2}}$, so that (98) follows because $\min_{\bar{k}_n \le i \le n-i_n+j_n} B_{1,2}(\mu_{i-1,n-3}) = \Omega(1)$. This concludes the proof of (45). To prove (46), we first fix an arbitrary $\tau' \in (0, \tau)$. Then,

$$\begin{aligned} &\max_{0 \le i \le (1-\tau)n} P[M_{i+1:n} > m_{1-\tau'}] \le P[M_{\lceil (1-\tau)n \rceil + 1:n} > m_{1-\tau'}] = P[U_{\lceil (1-\tau)n \rceil + 1:n} > 1 - \tau'] \\ &= P[U_{\lceil (1-\tau)n \rceil + 1:n} - \mu_{\lceil (1-\tau)n \rceil + 1,n} > 1 - \tau' - \mu_{\lceil (1-\tau)n \rceil + 1,n}] = \mathcal{E}(n). \end{aligned} \tag{99}$$

Here, the first inequality uses the fact that higher order statistics first order stochastically dominate lower order statistics, and the final step uses Corollary 1 (ii). As the random variables $v_W(\beta_{M,n,k_n}(M_{i+1:n})) - v_W(\beta_{M,n,k_n}(M_{i:n}))$ are all bounded by $v_W(\bar{b}_M(\overline{m}, \overline{w}))$, (99)

[48]If $1 - \tau < r$, the set $\{\bar{k}_n, ..., n - i_n + j_n\}$ is empty for all but finitely many n, so that $\|\beta'_{M,n,k_n}\|_{\infty, [\underline{m}, m_{1-\tau}]} = \mathcal{E}(n)$ in this case. The relevant case for us is of course $r < 1 - \tau$, as we need to bound β'_{M,n,k_n} also for types above the marginal type m_r.

and the elementary estimate (150) from Lemma 16 in Online Appendix H imply

$$\max_{0\le i\le(1-\tau)n}\left|d_{F,n,i}^{v_W\circ\beta_{M,n,k_n}}-E\left[v_W(\beta_{M,n,k_n}(M_{i+1:n}))-v_W(\beta_{M,n,k_n}(M_{i:n}))\middle|M_{i+1:n}\le m_{1-\tau'}\right]\right|$$
$$\le 2v_W(\overline{b}_M(\overline{m},\overline{w}))\max_{0\le i\le(1-\tau)n}P[M_{i+1:n}>m_{1-\tau'}]=\mathcal{E}(n).$$

Thus (46) follows from

$$\max_{0\le i\le(1-\tau)n}E\left[v_W(\beta_{M,n,k_n}(M_{i+1:n}))-v_W(\beta_{M,n,k_n}(M_{i:n}))\middle|M_{i+1:n}\le m_{1-\tau'}\right]=O\left(n^{-\frac{1}{2}}\right),\quad(100)$$

which is a simple implication of (45) and the mean value theorem. Indeed,

$$\begin{aligned}
&\max_{0\le i\le(1-\tau)n}E\left[v_W(\beta_{M,n,k_n}(M_{i+1:n}))-v_W(\beta_{M,n,k_n}(M_{i:n}))\middle|M_{i+1:n}\le m_{1-\tau'}\right]\\
&=\max_{0\le i\le(1-\tau)n}E\left[\int_{M_{i:n}}^{M_{i+1:n}}v'_W(\beta_{M,n,k_n}(m))\beta'_{M,n,k_n}(m)dm\middle|M_{i+1:n}\le m_{1-\tau'}\right]\\
&\le\|v'_W\|_\infty\left\|\beta'_{M,n,k_n}\right\|_{\infty,[\underline{m},m_{1-\tau'}]}\max_{0\le i\le(1-\tau)n}E\left[M_{i+1:n}-M_{i:n}\middle|M_{i+1:n}\le m_{1-\tau'}\right]\\
&=\|v'_W\|_\infty\left\|\beta'_{M,n,k_n}\right\|_{\infty,[\underline{m},m_{1-\tau'}]}\left(\max_{0\le i\le(1-\tau)n}E[M_{i+1:n}-M_{i:n}]+\mathcal{E}(n)\right),
\end{aligned}$$

where we have used (99) and (150) again in the final step. Using (45) and the bound $\max_{0\le i\le(1-\tau)n}E[M_{i+1:n}-M_{i:n}]=O\left(n^{-1}\right)$, which holds by (152) from Lemma 17 in Online Appendix H, (100) and thus (46) follow.

To show the first bound in (47), we again fix some $\tau'\in(0,\tau)$ and let $i_n=\lfloor(1-\tau')k_n\rfloor$. Starting from (44), we bound $\left\|\beta'_{W,n,k_n}\right\|_{\infty,[\underline{w},w_{1-\tau}]}$ as follows:

$$\begin{aligned}
\left\|\beta'_{W,n,k_n}\right\|_{\infty,[\underline{w},w_{1-\tau}]}&\le\overline{w}\left\|\sum_{i=1}^{i_n}g_{i:k_n-1}d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}}\right\|_\infty+\overline{w}\left\|\sum_{i=i_n+1}^{k_n-1}g_{i:k_n-1}d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}}\right\|_{\infty,[\underline{w},w_{1-\tau}]}\\
&\le\overline{w}(k_n-1)\|g\|_\infty\max_{1\le i\le i_n}d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}}+\overline{w}\max_{i>i_n}\|g_{i:k_n-1}\|_{\infty,[\underline{w},w_{1-\tau}]}\left|d_{F,n}^{v_W\circ\beta_{M,n,k_n}}\right|_1,
\end{aligned}\quad(101)$$

where the second inequality uses (148) from Lemma 15 (to bound the first summand in the first line by the first summand in the second line). As $n-(\bar{k}_n+i_n)=k_n-i_n=\Theta(k_n)=\Theta(n)$, (46) implies $\max_{1\le i\le i_n}d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}}=O\left(n^{-\frac{1}{2}}\right)$. It follows that the first summand in (101) is of order $O\left(n^{\frac{1}{2}}\right)$. The second summand is of order $\mathcal{E}(n)$, because a simple application of Corollary 2 (ii) yields $\max_{i>i_n}\|g_{i:k_n-1}\|_{\infty,[\underline{w},w_{1-\tau}]}=\mathcal{E}(n)$. This proves the first bound in (47).[49] The second bound in (47) then follows by an argument analogous

[49]Note that we had to prove (46) first. Establishing (47) by an argument analogous to the one we used to prove (45) is not feasible because, as is easy to see, the supremum norm of the density of extreme

to the one used above to prove (46) from (45). $\square$

We prepare the proof of Theorem 6 in a series of Lemmas (Lemmas 9-14). The key approximation result of Lemma 7 enters the argument in Lemma 14.

Lemma 9. *For any $\tilde{\beta} \in \mathcal{A}_1$ and $z \in \mathbb{R}$,*

$$\Psi_1[\tilde{\beta}]'(z) = m_r \left(v_M(0)\varphi(z) + \int_0^\infty \varphi(z-x) \left(v_M \circ \Xi_1[\tilde{\beta}] \right)'(x)\, dx \right)$$
$$= m_r v_M(0)\varphi(z) + m_r \underline{w} \int_0^\infty \varphi(z-x) v_M' \left(\Xi_1[\tilde{\beta}](x) \right) \left(\varphi * (v_W \circ \tilde{\beta})'(x) \right) dx. \tag{102}$$

Proof of Lemma 9. This is immediate from the calculation in (55) and the calculation preceding it in the proof of Theorem 3. $\square$

Lemma 10. *Let $\xi_1(y) = (y - y_1)c_1$ and $\xi_2(y) = (y - y_2)c_2$ be two affine functions on $\mathbb{R}$ with $c_2 \neq 0$. Then, for all $y \in \mathbb{R}$, $\xi_1 \circ \xi_2^{-1}(y) = \frac{c_1}{c_2}(y - \xi_2(y_1))$.*

Proof of Lemma 10. This is completely straightforward:

$$\xi_1\left(\xi_2^{-1}(y)\right) = \xi_1\left(y_2 + \frac{y}{c_2}\right) = \frac{c_1}{c_2}y - c_1(y_1 - y_2) = \frac{c_1}{c_2}(y - \xi_2(y_1)).$$

$\square$

Definition 4. *For $n \geq 2$, $l \in \mathbb{N}$ and $1 \leq j \leq l$, let*

$$C_{j,l,n} := \frac{f(m_{\mu_{\bar{k}_n,n-1}})a_{j,l}}{f(m_{\mu_{j,l}})a_{\bar{k}_n,n-1}}, \tag{103}$$

i.e., $C_{j,l,n}$ is the ratio of the slopes of $\xi_{F,n-1,\bar{k}_n}$ and $\xi_{F,l,j}$.[50] *For any $\alpha \in [0,1)$, let*

$$\overline{K}_{\alpha,n} := \max_{0\leq i\leq \lfloor n^\alpha \rfloor} C_{\bar{k}_n+i,n-1,n} \quad \text{and} \quad \underline{K}_{\alpha,n} := \min_{0\leq i\leq \lfloor n^\alpha \rfloor} C_{\bar{k}_n+i,n-1,n}. \tag{104}$$

Lemma 11. *For $\alpha \in [0,1)$, let (i_n) be an integer sequence satisfying $i_n = O(n^\alpha)$. Moreover, for any n, let $l, l' \in \{n-1, n, n+1\}$. Then,*

order statistics (of women's types) is of order $\Theta(k_n)$.

[50]We suppress the dependence on k_n in the notation.

$$\max_{(m,z)\in[m_{\mu_{\bar{k}_n-i_n,l}},m_{\mu_{\bar{k}_n+i_n,l}}]\times[m_{\mu_{\bar{k}_n-i_n,l'}},m_{\mu_{\bar{k}_n+i_n,l'}}]}\left|\frac{f(m)}{f(z)}-1\right| = O\left(n^{\alpha-1}\right), \tag{105}$$

$$\max_{0\le i\le i_n}\left|\frac{a_{\bar{k}_n+i,l}}{a_{\bar{k}_n,n-1}}-1\right| = O\left(n^{\alpha-1}\right), \tag{106}$$

$$\max_{0\le i\le i_n}\left|\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,l}})-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}})-\frac{1}{l'a_{\bar{k}_n,n-1}}\right| = O\left(n^{\alpha-\frac{3}{2}}\right), \tag{107}$$

$$\max_{0\le i\le i_n}|C_{\bar{k}_n+i,l,n}-1| = O\left(n^{\alpha-1}\right), \tag{108}$$

$$|\underline{K}_{\alpha,n}-1| = O\left(n^{\alpha-1}\right),\quad |\overline{K}_{\alpha,n}-1| = O\left(n^{\alpha-1}\right)\quad \textit{and}\quad \left|\frac{\overline{K}_{\alpha,n}}{\underline{K}_{\alpha,n}}-1\right| = O\left(n^{\alpha-1}\right). \tag{109}$$

Proof of Lemma 11. From $m_{\mu_{j,l}} = F^{-1}\left(\frac{j}{l+1}\right)$ and $\|\left(F^{-1}\right)'\|_\infty < \infty$, it follows that

$$\max_{(m,z)\in[m_{\mu_{\bar{k}_n-i_n,l}},m_{\mu_{\bar{k}_n+i_n,l}}]\times[m_{\mu_{\bar{k}_n-i_n,l'}},m_{\mu_{\bar{k}_n+i_n,l'}}]}|z-m| = O\left(n^{\alpha-1}\right).$$

(105) then follows from $f(m)/f(z)-1 = (f(m)-f(z))/f(z)$, $\min_{[\underline{m},\overline{m}]} f(z) > 0$ and $\|f'\|_\infty < \infty$. Next, we note that

$$\left(\frac{l}{n-1}\right)^{\frac{1}{2}}\frac{a_{\bar{k}_n+i,l}}{a_{\bar{k}_n,n-1}} = \left(\frac{B_{1,2}\left(\frac{\bar{k}_n+i}{l+1}\right)}{B_{1,2}\left(\frac{\bar{k}_n}{n}\right)}\right)^{\frac{1}{2}} = 1+\frac{\sqrt{B_{1,2}}\left(\frac{\bar{k}_n+i}{l+1}\right)-\sqrt{B_{1,2}}\left(\frac{\bar{k}_n}{n}\right)}{\sqrt{B_{1,2}}\left(\frac{\bar{k}_n}{n}\right)}.$$

On any compact interval that does not contain 0 or 1, $\sqrt{B_{1,2}}$ is bounded away from zero and has a bounded derivative. As $\lim_{n\to\infty}\frac{\bar{k}_n}{n} = r \in (0,1)$ and $i_n = O(n^\alpha)$ for $\alpha < 1$, it follows that $\max_{0\le i\le i_n}\frac{\sqrt{B_{1,2}}\left(\frac{\bar{k}_n+i}{l+1}\right)-\sqrt{B_{1,2}}\left(\frac{\bar{k}_n}{n}\right)}{\sqrt{B_{1,2}}\left(\frac{\bar{k}_n}{n}\right)} = O\left(n^{\alpha-1}\right)$, which (together with $|(l/n-1)^{1/2}-1| = \Theta\left(n^{-1}\right)$) implies (106). Next,

$$\begin{aligned}
&\max_{0\le i\le i_n}\left|\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,l}})-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}})-\frac{1}{l'a_{\bar{k}_n,n-1}}\right| \\
&= \max_{0\le i\le i_n}\left|\frac{1}{a_{\bar{k}_n,n-1}}\int_{\mu_{\bar{k}_n+i,l}}^{\mu_{\bar{k}_n+i+1,l}}\left(\frac{f(m_{\mu_{\bar{k}_n,n-1}})}{f(F^{-1}(u))}-\frac{l+1}{l'}\right)du\right| \\
&\le \frac{1}{(l+1)a_{\bar{k}_n,n-1}}\max_{0\le i\le i_n}\left\|\frac{f(m_{\mu_{\bar{k}_n,n-1}})}{f}-\frac{l+1}{l'}\right\|_{\infty,[m_{\mu_{\bar{k}_n+i,l}},m_{\mu_{\bar{k}_n+i+1,l}}]} = O\left(n^{\alpha-\frac{3}{2}}\right),
\end{aligned}$$

where the final step uses (105), $|(l+1)/l'-1| = \Theta\left(n^{-1}\right)$, $a_{\bar{k}_n,n-1} = \Theta\left(n^{-\frac{1}{2}}\right)$ and $1/(l+1) = \Theta\left(n^{-1}\right)$. This shows (107). The bound (108) is straightforward from of (103), (105) and (106), and the bounds in (109) are immediate from (108). □

Lemma 12 (Bounds on the L^1-distance between normal distributions)**.** *For $t_1, t_2 \in \mathbb{R}$, $\|\varphi_{t_1,1} - \varphi_{t_2,1}\|_{L^1(\mathbb{R})} \leq \sqrt{2}|t_1 - t_2|$. Moreover, there is a constant $C > 0$ such that for all $t \in \mathbb{R}$ and all $\sigma_1, \sigma_2 > 0$, $\|\varphi_{t,\sigma_1^2} - \varphi_{t,\sigma_2^2}\|_{L^1(\mathbb{R})} \leq C|\sigma_1^2/\sigma_2^2 - 1|$.*

Proof of Lemma 12. The first bound is the case $p = 1$ of Example 2.3 in DasGupta (2008). The second bound is the case $n = 1$ of Lemma 4.8 in Klartag (2007). $\square$

Lemma 13. *(i) Let $n \geq 2$, $l \in \mathbb{N}$ and $1 \leq j \leq l$. Then, for all $z \in \mathbb{R}$,*

$$f_{j:l}\left(\xi_{F,n-1,\bar{k}_n}^{-1}(z)\right) = \frac{f(m_{\mu_{\bar{k}_n,n-1}})}{a_{\bar{k}_n,n-1}} C_{j,l,n}^{-1} \check{f}_{j:l}\left(\frac{z - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{j,l}})}{C_{j,l,n}}\right). \tag{110}$$

(ii) Let $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$ and $\alpha' \in \left(\frac{1}{2}, \frac{2}{3}\right)$, and let (i_n) be an integer sequence satisfying $i_n = \Theta(n^\alpha)$. Moreover, for any n, let $l \in \{n-1, n, n+1\}$. Then,

$$\max_{0\leq i\leq i_n} \left\| \frac{C_{\bar{k}_n+i,l,n}^{-1} \check{f}_{\bar{k}_n+i:l}\left(\frac{Id - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}})}{C_{\bar{k}_n+i,l,n}}\right) - \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}}),C_{\bar{k}_n+i,l,n}^2}}{\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}}),C_{\bar{k}_n+i,l,n}^2}} \right\|_{\infty,I_{n,\alpha'}} = O\left(n^{3\max\{\alpha,\alpha'\}-2}\right), \tag{111}$$

$$\max_{0\leq i\leq i_n} \left\| C_{\bar{k}_n+i+1,l,n}^{-1} \check{f}_{\bar{k}_n+i+1:l}\left(\frac{Id - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,l}})}{C_{\bar{k}_n+i+1,l,n}}\right) - \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),1} \right\|_{L^1(I_{n,\alpha'})} = O\left(n^{3\max\{\alpha,\alpha'\}-2}\right). \tag{112}$$

(iii) For any $\alpha \in [0, 1)$, $0 \leq i \leq \lfloor n^\alpha \rfloor$ and $y \in \mathbb{R}$,

$$\overline{K}_{\alpha,n}^{-1} \varphi\left(\frac{y}{\underline{K}_{\alpha,n}}\right) \leq C_{\bar{k}_n+i,n-1,n}^{-1} \varphi\left(\frac{y}{C_{\bar{k}_n+i,n-1,n}}\right) \leq \underline{K}_{\alpha,n}^{-1} \varphi\left(\frac{y}{\overline{K}_{\alpha,n}}\right).$$

Proof of Lemma 13. (i) Recall from (12) that $\check{f}_{j:l}$ is the density of the distribution of $\xi_{F,l,j}(M_{j:l})$, i.e., $f_{j:l}(m) = \check{f}_{j:l}\left(\xi_{F,l,j}(m)\right) f(m_{\mu_{j,l}})/a_{j,l}$. Lemma 10 and (103) yield

$$\begin{aligned} f_{j:l}\left(\xi_{F,n-1,\bar{k}_n}^{-1}(z)\right) &= \frac{f(m_{\mu_{j,l}})}{a_{j,l}} \check{f}_{j:l}\left(\xi_{F,l,j}\left(\xi_{F,n-1,\bar{k}_n}^{-1}(z)\right)\right) \\ &= \frac{f(m_{\mu_{j,l}})}{a_{j,l}} \check{f}_{j:l}\left(\frac{f(m_{\mu_{j,l}}) a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}}) a_{j,l}}\left(z - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{j,l}})\right)\right) = \frac{f(m_{\mu_{\bar{k}_n,n-1}})}{a_{\bar{k}_n,n-1}} C_{j,l,n}^{-1} \check{f}_{j:l}\left(\frac{z - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{j,l}})}{C_{j,l,n}}\right). \end{aligned}$$

(ii) Note that (49) implies $\max_{0\leq i\leq i_n} \max_{z\in I_{n,\alpha'}} \left|z - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}})\right| = \Theta\left(n^{\max\{\alpha,\alpha'\}-\frac{1}{2}}\right)$. Moreover, by (108), $\max_{0\leq i\leq i_n} |C_{\bar{k}_n+i,l,n} - 1| = O\left(n^{\alpha-1}\right)$. Hence, the maximal modulus of the values for which one of the densities $\check{f}_{\bar{k}_n+i:l}$ is evaluated in (111) satisfies

$$\max_{0\le i\le i_n}\max_{z\in I_{n,\alpha'}}\left|\frac{z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}})}{C_{\bar{k}_n+i,l,n}}\right| = \Theta\left(n^{\max\{\alpha,\alpha'\}-\frac{1}{2}}\right).$$

As $0<\max\{\alpha,\alpha'\}-\frac{1}{2}<\frac{1}{6}$, $\bar{k}_n=\Omega(n)$ and $n-(\bar{k}_n+i_n)=\Omega(n)$, the bound (42) from Theorem 5 then implies

$$\max_{0\le i\le i_n}\max_{z\in I_{n,\alpha'}}\frac{\left|\check{f}_{\bar{k}_n+i:l}\left(\frac{z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}})}{C_{\bar{k}_n+i,l,n}}\right)-\varphi\left(\frac{z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}})}{C_{\bar{k}_n+i,l,n}}\right)\right|}{\varphi\left(\frac{z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,l}})}{C_{\bar{k}_n+i,l,n}}\right)} = O\left(n^{3\max\{\alpha,\alpha'\}-2}\right).$$

Expanding the fractions by $C^{-1}_{\bar{k}_n+i,l,n}$, we obtain (111). For (112), we note first that (111) and $\|\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,l}}),C^2_{\bar{k}_n+i+1,l,n}}\|_{L^1(I_{n,\alpha'})}\le 1$ imply (via the obvious L^1-L^∞ estimate) that

$$\begin{aligned}\max_{0\le i\le i_n}\Bigg\| & C^{-1}_{\bar{k}_n+i+1,l,n}\check{f}_{\bar{k}_n+i+1:l}\left(\frac{Id-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,l}})}{C_{\bar{k}_n+i+1,l,n}}\right)\\ & -\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,n-1}}),C^2_{\bar{k}_n+i+1,l,n}}\Bigg\|_{L^1(I_{n,\alpha'})} = O\left(n^{3\max\{\alpha,\alpha'\}-2}\right).\end{aligned} \tag{113}$$

Next, (108) implies $\max_{0\le i\le i_n}|C^2_{\bar{k}_n+i+1,l,n}-1| = O\left(n^{\alpha-1}\right)$, so that, by Lemma 12,

$$\max_{0\le i\le i_n}\|\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,l}}),C^2_{\bar{k}_n+i+1,l,n}}-\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,l}}),1}\|_{L^1(\mathbb{R})} = O\left(n^{\alpha-1}\right). \tag{114}$$

Moreover, $\max_{0\le i\le i_n}|\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,l}})-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})| = O\left(n^{-\frac{1}{2}}\right)$, so that we also obtain, from the other estimate in Lemma 12, that

$$\max_{0\le i\le i_n}\|\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,l}}),1}-\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),1}\|_{L^1(\mathbb{R})} = O\left(n^{-\frac{1}{2}}\right). \tag{115}$$

Together, (113), (114) and (115) imply (112), because $3\max\{\alpha,\alpha'\}-2>\alpha-1>-\frac{1}{2}$.

(iii) The bounds are immediate from (104), which implies $\underline{K}_{\alpha,n}\le C_{\bar{k}_n+i,n-1,n}\le\overline{K}_{\alpha,n}$, combined with the fact that $\varphi(y)$ is decreasing in $|y|$. □

Lemma 14. *Let $\alpha\in\left(\frac{1}{2},\frac{7}{12}\right)$. Then, for any $\varepsilon>0$,*

$$\max_{0\le i\le\lfloor n^\alpha\rfloor}|d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i}-\Delta_{\bar{k}_n+i,n}\left(\nu_M\circ\Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}))| = O\left(n^{\frac{\alpha}{2}+\varepsilon-1}\right). \tag{116}$$

Proof of Lemma 14. From the definition of Ξ_1 (and using the symmetry of φ and the basic formula for differentiating a convolution), we have for all $x \geq 0$,

$$\left(v_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(x) = v_M'\left(\Xi_1[\tilde{\beta}_{M,n,k_n}](x)\right)\Xi_1[\tilde{\beta}_{M,n,k_n}]'(x)$$
$$= v_M'\left(\underline{w}\int_{\mathbb{R}}(v_W \circ \tilde{\beta}_{M,n,k_n})(z)(\varphi(z-x)-\varphi(z))dz\right)\underline{w}\int_{\mathbb{R}}(v_W \circ \tilde{\beta}_{M,n,k_n})'(z)\varphi(z-x)dz. \tag{117}$$

In particular, this formula applies for $x = \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}) \geq 0$ (for all $i \geq 0$). According to Lemma 7, we have the following bound, for any $\varepsilon > 0$:

$$\max_{0\leq i\leq \lfloor n^\alpha \rfloor}\left|d_{G,k_n,i}^{v_M\circ\beta_{W,n,k_n}} - d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}}\underline{w}v_M'\left(\underline{w}E[v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+i+1:n})) - v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+1:n}))]\right)\right|$$
$$= O\left(n^{\frac{\alpha}{2}+\varepsilon-1}\right). \tag{118}$$

Fixing an arbitrary $\alpha' \in \left(\alpha, \frac{7}{12}\right)$, we prove the following approximation results below:

$$\max_{0\leq i\leq \lfloor n^\alpha \rfloor}\left|v_M'\left(\underline{w}E[v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+i+1:n})) - v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+1:n}))]\right)\right.$$
$$\left. - v_M'\left(\underline{w}\int_{I_{n,\alpha'}}(v_W \circ \tilde{\beta}_{M,n,k_n})(z)\left(\varphi\left(z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})\right)-\varphi(z)\right)dz\right)\right| = O\left(n^{3\alpha'-2}\right), \tag{119}$$

$$\max_{0\leq i\leq \lfloor n^\alpha \rfloor}\left|d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}} - \Delta_{\bar{k}_n+i,n}\int_{I_{n,\alpha'}}(v_W \circ \tilde{\beta}_{M,n,k_n})'(z)\varphi\left(z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})\right)dz\right| = O\left(n^{3\alpha'-\frac{5}{2}}\right). \tag{120}$$

As $\max_{0\leq i\leq\lfloor n^\alpha\rfloor}\int_{\mathbb{R}\setminus I_{n,\alpha'}}\varphi\left(z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})\right) = \mathcal{E}(n)$ (and $\|v_M''\|_\infty < \infty$, $\|v_W'\|_\infty < \infty$ and $v_W\circ\tilde{\beta}_{M,n,k_n} \leq v_W(\overline{b}_M(\overline{m},\overline{w}))$), it is then immediate that the additional errors from replacing the integrals over $I_{n,\alpha'}$ in (119) and (120) by integrals over $\mathbb{R}$ is of order $\mathcal{E}(n)$. In view of this, the approximations (119) and (120) and formula (117), we obtain

$$\max_{0\leq i\leq \lfloor n^\alpha \rfloor}\left|d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}}\underline{w}v_M'\left(\underline{w}E[v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+i+1:n})) - v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+1:n}))]\right)\right.$$
$$\left. - \Delta_{\bar{k}_n+i,n}\left(v_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}))\right| = O\left(n^{3\alpha'-\frac{5}{2}}\right). \tag{121}$$

Indeed, if we approximate the product of $\underline{w}v_M'\left(\underline{w}E[v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+i+1:n})) - v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+1:n}))]\right)$ and $d_{F,n,\bar{k}_n+i}^{v_W\circ\beta_{M,n,k_n}}$ by the product of the respective approximating terms (from (119) and (120), with the integrals replaced by integrals over $\mathbb{R}$), applying a bound as in (62) for the error, both summands in the error bound are of order $O\left(n^{3\alpha'-\frac{5}{2}}\right)$, because of

$\|v'_M\|_\infty < \infty$ and $\max_{0\le i\le\lfloor n^\alpha\rfloor} d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i} = O\left(n^{-\frac{1}{2}}\right)$ (by (46)).

As $\alpha' < \frac{7}{12}$ is equivalent to $3\alpha' - \frac{5}{2} < -\frac{3}{4}$ and, moreover, $\frac{\alpha}{2} + \varepsilon - 1 > -\frac{3}{4}$ for any $\varepsilon > 0$, the bounds (118) and (121) together imply (116).

Of course, we still have to prove (119) and (120). Starting from

$$E[v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+i+1:n}))] = \int_{\underline{m}}^{\overline{m}} \left(v_W \circ \beta_{M,n,k_n}\right)(m) f_{\bar{k}_n+i+1:n}(m)dm, \tag{122}$$

we first truncate the above integral at $m_{\mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1}}$ and $m_{\mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1}}$ and use (35) and Corollary 1 (ii) to bound the resulting error:

$$\begin{aligned}
&\max_{0\le i\le\lfloor n^\alpha\rfloor}\left|E[v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+i+1:n}))] - \int_{m_{\mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1}}}^{m_{\mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1}}} \left(v_W \circ \beta_{M,n,k_n}\right)(m) f_{\bar{k}_n+i+1:n}(m)dm\right| \\
&\le v_W(\overline{b}_M(\overline{m},\overline{w})) \max_{0\le i\le\lfloor n^\alpha\rfloor} P\left[U_{\bar{k}_n+i+1:n} - \mu_{\bar{k}_n+i+1,n} \ge \mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1} - \mu_{\bar{k}_n+i+1,n}\right] \\
&+ v_W(\overline{b}_M(\overline{m},\overline{w})) \max_{0\le i\le\lfloor n^\alpha\rfloor} P\left[U_{\bar{k}_n+i+1:n} - \mu_{\bar{k}_n+i+1,n} \le \mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1} - \mu_{\bar{k}_n+i+1,n}\right] = \mathcal{E}(n),
\end{aligned} \tag{123}$$

where the last step uses Corollary 1 (ii) (note that $\bar{k}_n = \Omega(n)$, $n - (\bar{k}_n + \lfloor n^\alpha\rfloor) = \Omega(n)$, $\mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1} - \mu_{\bar{k}_n+\lfloor n^\alpha\rfloor+1,n} = \Theta\left(n^{\alpha'-1}\right)$[51] and $\mu_{\bar{k}_n+1,n} - \mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1} = \Theta\left(n^{\alpha'-1}\right)$).

We change variables via $z = \xi_{F,n-1,\bar{k}_n}(m)$ (in particular, "$dm = \frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})}dz$") and use (110) to obtain

$$\begin{aligned}
&\int_{m_{\mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1}}}^{m_{\mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1}}} \left(v_W \circ \beta_{M,n,k_n}\right)(m) f_{\bar{k}_n+i+1:n}(m)dm \\
&= \int_{I_{n,\alpha'}} \left(v_W \circ \tilde{\beta}_{M,n,k_n}\right)(z)\, C^{-1}_{\bar{k}_n+i+1,n,n} \check{f}_{\bar{k}_n+i+1:n}\left(\frac{z - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,n}})}{C^{-1}_{\bar{k}_n+i+1,n,n}}\right)dz.
\end{aligned}$$

Using $\|v_W \circ \tilde{\beta}_{M,n,k_n}\|_{\infty,I_{n,\alpha'}} \le v_W(\overline{b}_M(\overline{m},\overline{w}))$ and the bound (112), it follows that

$$\begin{aligned}
&\max_{0\le i\le\lfloor n^\alpha\rfloor}\left|\int_{m_{\mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1}}}^{m_{\mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1}}} \left(v_W \circ \beta_{M,n,k_n}\right)(m) f_{\bar{k}_n+i+1:n}(m)dm\right. \\
&\left.- \int_{I_{n,\alpha'}} \left(v_W \circ \tilde{\beta}_{M,n,k_n}\right)(z)\varphi\left(z - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})\right)dz\right| = O\left(n^{3\alpha'-2}\right).
\end{aligned} \tag{124}$$

[51] This is true because $\alpha' > \alpha$.

Taken together, (123) and (124) establish

$$\max_{0\le i\le\lfloor n^\alpha\rfloor}\Big|E[v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+i+1:n}))] - \int_{I_{n,\alpha'}}\left(v_W\circ\tilde{\beta}_{M,n,k_n}\right)(z)\varphi\left(z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})\right)dz\Big| = O\left(n^{3\alpha'-2}\right). \qquad (125)$$

In particular, as $\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n,n-1}}) = 0$, it follows from (125) (and applying the triangle inequality) that

$$\max_{0\le i\le\lfloor n^\alpha\rfloor}\Big|\underline{w}E[v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+i+1:n})) - v_W(\beta_{M,n,k_n}(M_{\bar{k}_n+1:n}))] - \underline{w}\int_{I_{n,\alpha'}}\left(v_W\circ\tilde{\beta}_{M,n,k_n}\right)(z)\left(\varphi\left(z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})\right)-\varphi(z)\right)dz\Big| = O\left(n^{3\alpha'-2}\right).$$

As $\|v_M''\|_{\infty,[0,\bar{b}_W(\overline{m},\overline{w})]}<\infty$, this implies (119). To prove (120), we note first that

$$\begin{aligned} d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i} &= \int_{\underline{m}}^{\overline{m}}\left(v_W\circ\beta_{M,n,k_n}\right)(m)(f_{\bar{k}_n+i+1:n}(m)-f_{\bar{k}_n+i:n}(m))dm\\ &= \int_{\underline{m}}^{\overline{m}}\left(v_W\circ\beta_{M,n,k_n}\right)'(m)(F_{\bar{k}_n+i:n}(m)-F_{\bar{k}_n+i+1:n}(m))dm\\ &= \int_{\underline{m}}^{\overline{m}}\left(v_W\circ\beta_{M,n,k_n}\right)'(m)\,\frac{f_{\bar{k}_n+i+1:n+1}(m)}{(n+1)f(m)}dm, \end{aligned} \qquad (126)$$

where the first equation uses only the definition of $d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i}$, the second equation follows from integrating by parts, and the final equation uses (7) and (8). Therefore (compare (126) and (122)), the proof of (120) is similar to, albeit a bit more involved than, the proof of (125): first, from (126) and Corollary 2 (ii), it follows that

$$\begin{aligned} &\max_{0\le i\le\lfloor n^\alpha\rfloor}\left|d^{v_W\circ\beta_{M,n,k_n}}_{F,n,\bar{k}_n+i} - \int_{m_{\mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1}}}^{m_{\mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1}}}\left(v_W\circ\beta_{M,n,k_n}\right)'(m)\frac{f_{\bar{k}_n+i+1:n+1}(m)}{(n+1)f(m)}dm\right|\\ &\le \frac{v_W(\overline{b}_M(\overline{m},\overline{w}))-v_W(0)}{(n+1)\min_{m\in[\underline{m},\overline{m}]}f(m)}\max_{0\le i\le\lfloor n^\alpha\rfloor}\left\|f_{\bar{k}_n+i+1:n+1}\right\|_{\infty,[\underline{m},m_{\mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1}}]\cup[m_{\mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1}},\overline{m}]} = \mathcal{E}(n). \end{aligned} \qquad (127)$$

Using $\left(v_W\circ\tilde{\beta}_{M,n,k_n}\right)'(z)\ f(m_{\mu_{\bar{k}_n,n-1}})/a_{\bar{k}_n,n-1} = \left(v_W\circ\beta_{M,n,k_n}\right)'(\xi^{-1}_{F,n-1,\bar{k}_n}(z))$, which holds by the definition of $\tilde{\beta}_{M,n,k_n}$ (see also (48)), and (110), it follows that

$$\int_{m_{\mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1}}}^{m_{\mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1}}} (v_W\circ\beta_{M,n,k_n})'(m)\frac{f_{\bar{k}_n+i+1:n+1}(m)}{(n+1)f(m)}dm$$
$$=\frac{f(m_{\mu_{\bar{k}_n,n-1}})}{(n+1)a_{\bar{k}_n,n-1}}\int_{I_{n,\alpha'}}\left(v_W\circ\tilde{\beta}_{M,n,k_n}\right)'(z)\frac{C^{-1}_{\bar{k}_n+i+1,n+1,n}\check{f}_{\bar{k}_n+i+1:n+1}\left(\frac{z-\xi_{F,n-1,k_n^c}(m_{\mu_{\bar{k}_n+i+1,n+1}})}{C_{\bar{k}_n+i+1,n+1,n}}\right)}{f\left(\xi^{-1}_{F,n-1,\bar{k}_n}(z)\right)}dz.$$

Using $(n+1)a_{\bar{k}_n,n-1}=\Theta\left(n^{\frac{1}{2}}\right)$, $\|(v_W\circ\tilde{\beta}_{M,n,k_n})'\|_{\infty,I_{n,\alpha'}}=O(1)$ (by Corollary 3) and $\|f(m_{\mu_{\bar{k}_n,n-1}})/(f\circ\xi^{-1}_{F,n-1,\bar{k}_n})\|_{\infty,I_{n,\alpha'}}=O(1)$, we have $\left\|\frac{f(m_{\mu_{\bar{k}_n,n-1}})(v_W\circ\tilde{\beta}_{M,n,k_n})'}{(n+1)a_{\bar{k}_n,n-1}\left(f\circ\xi^{-1}_{F,n-1,\bar{k}_n}\right)}\right\|_{\infty,I_{n,\alpha'}}=O\left(n^{-\frac{1}{2}}\right)$. Combining this with the L^1-bound (112) from Lemma 13 ii), we obtain

$$\max_{0\le i\le\lfloor n^{\alpha}\rfloor}\Bigg|\int_{m_{\mu_{\bar{k}_n-\lfloor n^{\alpha'}\rfloor,n-1}}}^{m_{\mu_{\bar{k}_n+\lfloor n^{\alpha'}\rfloor,n-1}}} (v_W\circ\beta_{M,n,k_n})'(m)\frac{f_{\bar{k}_n+i+1:n+1}(m)}{(n+1)f(m)}dm$$
$$-\frac{f(m_{\mu_{\bar{k}_n,n-1}})}{(n+1)a_{\bar{k}_n,n-1}}\int_{I_{n,\alpha'}}\left(v_W\circ\tilde{\beta}_{M,n,k_n}\right)'(z)\frac{\varphi\left(z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})\right)}{f\left(\xi^{-1}_{F,n-1,\bar{k}_n}(z)\right)}dz\Bigg|=O\left(n^{3\alpha'-\frac{5}{2}}\right).\tag{128}$$

Next, $\|f(m_{\mu_{\bar{k}_n,n-1}})/(f\circ\xi^{-1}_{F,n-1,\bar{k}_n})-1\|_{\infty,I_{n,\alpha'}}=O\left(n^{\alpha'-1}\right)$ (by (105)), $(n+1)a_{\bar{k}_n,n-1}=\Theta\left(n^{\frac{1}{2}}\right)$ and $\max_{0\le i\le\lfloor n^{\alpha}\rfloor}|\Delta_{\bar{k}_n+i,n}-(n+1)^{-1}a^{-1}_{\bar{k}_n,n-1}|=O\left(n^{\alpha-\frac{3}{2}}\right)$ (by (107)) imply

$$\max_{0\le i\le\lfloor n^{\alpha}\rfloor}\left\|\frac{1}{(n+1)a_{\bar{k}_n,n-1}}\frac{f(m_{\mu_{\bar{k}_n,n-1}})}{f\circ\xi^{-1}_{F,n-1,\bar{k}_n}}-\Delta_{\bar{k}_n+i,n}\right\|_{\infty,I_{n,\alpha'}}=O\left(n^{\alpha'-\frac{3}{2}}\right).$$

Combined with $\max_{0\le i\le\lfloor n^{\alpha}\rfloor}\|\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),1}(v_W\circ\tilde{\beta}_{M,n,k_n})'\|_{L^1(I_{n,\alpha'})}=O(1)$ (as $\|(v_W\circ\tilde{\beta}_{M,n,k_n})'\|_{\infty,I_{n,\alpha'}}=O(1)$, by Corollary 3), this yields

$$\max_{0\le i\le\lfloor n^{\alpha}\rfloor}\Bigg|\frac{f(m_{\mu_{\bar{k}_n,n-1}})}{(n+1)a_{\bar{k}_n,n-1}}\int_{I_{n,\alpha'}}\left(v_W\circ\tilde{\beta}_{M,n,k_n}\right)'(z)\frac{\varphi\left(z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})\right)}{f\left(\xi^{-1}_{F,n-1,\bar{k}_n}(z)\right)}dz$$
$$-\Delta_{\bar{k}_n+i,n}\int_{I_{n,\alpha'}}\left(v_W\circ\tilde{\beta}_{M,n,k_n}\right)'(z)\varphi\left(z-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})\right)dz\Bigg|=O\left(n^{\alpha'-\frac{3}{2}}\right).\tag{129}$$

Taken together, (127), (128) and (129) prove (120) (note that $\alpha'-\frac{3}{2}<3\alpha'-\frac{5}{2}$), and thus Lemma 14. □

Proof of Theorem 6. Given $\varepsilon>0$, we fix some $\alpha'\in\left(\alpha,\frac{7}{12}\right)$ satisfying $\frac{\alpha'}{2}<\frac{\alpha}{2}+\varepsilon$ and note first that (51) and $\max_{j\ge\bar{k}_n+\lfloor n^{\alpha'}\rfloor}\|f_{j:n-1}\|_{\infty,[\underline{m},m_{\mu_{\bar{k}_n+\lfloor n^{\alpha}\rfloor,n-1}}]}=\mathcal{E}(n)$ (which holds by

Corollary 2 (ii), as $\lfloor n^{\alpha'} \rfloor - \lfloor n^{\alpha} \rfloor = \Theta\left(n^{\alpha'}\right)$ and $\alpha' > \frac{1}{2}$), we have

$$\left\| \tilde{\beta}'_{M,n,k_n} - \xi^{-1}_{F,n-1,\bar{k}_n} \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor - 1} \left(d^{\nu_M \circ \beta_{W,n,k_n}}_{G,k_n,i} + \nu_M(0)\delta_{i0} \right) \frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})} f_{\bar{k}_n+i:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n} \right\|_{\infty,I_{n,\alpha}}$$
$$= \mathcal{E}(n). \tag{130}$$

Next, as $\|\xi^{-1}_{F,n-1,\bar{k}_n}\|_{\infty,I_{n,\alpha}} = \Theta(1)$ and $\|\tilde{\beta}'_{M,n,k_n}\|_{\infty,I_{n,\alpha}} = O(1)$ (by Corollary 3), (130) implies in particular that

$$\left\| \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor - 1} \left(d^{\nu_M \circ \beta_{W,n,k_n}}_{G,k_n,i} + \nu_M(0)\delta_{i0} \right) \frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})} f_{\bar{k}_n+i:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n} \right\|_{\infty,I_{n,\alpha}} = O(1).$$

Using that $\|\xi^{-1}_{F,n-1,\bar{k}_n} - m_{\mu_{\bar{k}_n,n-1}}\|_{\infty,I_{n,\alpha}} = \Theta\left(n^{\alpha-1}\right)$, it thus follows that

$$\left\| \left(\xi^{-1}_{F,n-1,\bar{k}_n} - m_{\mu_{\bar{k}_n,n-1}}\right) \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor - 1} \left(d^{\nu_M \circ \beta_{W,n,k_n}}_{G,k_n,i} + \nu_M(0)\delta_{i0} \right) \frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})} f_{\bar{k}_n+i:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n} \right\|_{\infty,I_{n,\alpha}}$$
$$= O\left(n^{\alpha-1}\right). \tag{131}$$

Fixing some $\varepsilon' > 0$ such that $\frac{\alpha'}{2} + \varepsilon' < \frac{\alpha}{2} + \varepsilon$, we show below that

$$\left\| \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor - 1} \left(d^{\nu_M \circ \beta_{W,n,k_n}}_{G,k_n,i} + \nu_M(0)\delta_{i0} \right) \frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})} f_{\bar{k}_n+i:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n} - \frac{\Psi_1[\tilde{\beta}_{M,n,k_n}]'}{m_r} \right\|_{\infty,I_{n,\alpha}}$$
$$= O\left(n^{\frac{\alpha'}{2}+\varepsilon'-\frac{1}{2}}\right). \tag{132}$$

The bounds (130), (131) and (132) imply (50), and hence the claim of Theorem 6, because $\alpha - 1 < \frac{\alpha'}{2} + \varepsilon' - \frac{1}{2}$ (as $\alpha - 1 < \frac{\alpha-1}{2} < \frac{\alpha'-1}{2}$) and $\frac{\alpha'}{2} + \varepsilon' < \frac{\alpha}{2} + \varepsilon$.

To prove (132), we note first that by (110)

$$\frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})} f_{\bar{k}_n+i:n-1}\left(\xi^{-1}_{F,n-1,\bar{k}_n}(z)\right) = C^{-1}_{\bar{k}_n+i,n-1,n} \check{f}_{\bar{k}_n+i:n-1}\left(\frac{z - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})}{C_{\bar{k}_n+i,n-1,n}}\right).$$

for all $z \in \mathbb{R}$. Thus, the bound (111) yields

$$\max_{0 \le i \le \lfloor n^{\alpha'} \rfloor} \left\| \frac{\frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})} f_{\bar{k}_n+i:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n}}{\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),C^2_{\bar{k}_n+i,n-1,n}}} - 1 \right\|_{\infty,I_{n,\alpha}} = O\left(n^{3\alpha'-2}\right). \tag{133}$$

In particular, $\|(f_{\bar{k}_n:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n}) a_{\bar{k}_n,n-1}/(\varphi f(m_{\mu_{\bar{k}_n,n-1}})) - 1\|_{\infty,I_{n,\alpha}} = O\left(n^{3\alpha'-2}\right)$ (as $\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n,n-1}}) =$

0 and $C_{\bar{k}_n,n-1,n} = 1$), which (as φ is bounded) implies

$$\left\| v_M(0) \frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})} f_{\bar{k}_n:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n} - v_M(0)\varphi \right\|_{\infty,I_{n,\alpha}} = O\left(n^{3\alpha'-2}\right).$$

In view of equation (102) for $\Psi_1[\tilde{\beta}_{M,n,k_n}]'$ and $3\alpha' - 2 < \frac{\alpha'}{2} - \frac{1}{2}$, it follows that to prove (132), we still have to show that

$$\Bigg\| \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor - 1} d^{v_M \circ \beta_{W,n,k_n}}_{G,k_n,i} \frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})} f_{\bar{k}_n+i:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n}$$
$$- \int_0^\infty \varphi(Id - x)\left(v_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(x)dx \Bigg\|_{\infty,I_{n,\alpha}} = O\left(n^{\frac{\alpha'}{2}+\varepsilon'-\frac{1}{2}}\right). \tag{134}$$

The proof of (134) relies on Lemma 14 and on the bound (133). To deal with the minor nuisance that the approximation of the terms $(f_{\bar{k}_n+i:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n})a_{\bar{k}_n,n-1}/f(m_{\mu_{\bar{k}_n,n-1}})$ provided by (133) involves normal distributions with different variances, we will use Lemma 13 (iii), which ensures that for all $0 \le i \le \lfloor n^{\alpha'} \rfloor$ and all $z \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$\frac{\underline{K}_{\alpha',n}}{\overline{K}_{\alpha',n}} \varphi_{t,\underline{K}^2_{\alpha',n}}(z) \le \varphi_{t,C^2_{\bar{k}_n+i,n-1,n}}(z) \le \frac{\overline{K}_{\alpha',n}}{\underline{K}_{\alpha',n}} \varphi_{t,\overline{K}^2_{\alpha',n}}(z). \tag{135}$$

As a first step towards proving (134), we show that

$$\left\| \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor - 1} d^{v_M \circ \beta_{W,n,k_n}}_{G,k_n,i} \left(\frac{a_{\bar{k}_n,n-1}}{f(m_{\mu_{\bar{k}_n,n-1}})} f_{\bar{k}_n+i:n-1} \circ \xi^{-1}_{F,n-1,\bar{k}_n} - \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),C^2_{\bar{k}_n+i,n-1,n}} \right) \right\|_{\infty,I_{n,\alpha}}$$
$$= O\left(n^{3\alpha'-2}\right). \tag{136}$$

To this end, we note first that

$$\left\| \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor - 1} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\overline{K}^2_{\alpha',n}} \right\|_{\infty,I_{n,\alpha}} = O\left(n^{\frac{1}{2}}\right). \tag{137}$$

Indeed, $0 = \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n,n-1}}) < \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+1,n-1}}) < ... < \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+\lfloor n^{\alpha'} \rfloor,n-1}})$ defines a partition of the interval $I^+_{n,\alpha'}$, and (54) shows that the maximum and the minimum length of the corresponding subintervals are of order $\Theta\left(n^{-\frac{1}{2}}\right)$. In particular,

$$\left\| \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor - 1} \Delta_{\bar{k}_n+i,n} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\overline{K}^2_{\alpha',n}} \right\|_{\infty,I_{n,\alpha}} = O(1), \tag{138}$$

because for every n and $z \in I_{n,\alpha}$, $\sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} \Delta_{\bar{k}_n+i,n} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\overline{K}^2_{\alpha',n}}(z)$ is a (left) Riemann sum for the integral $\int_{I^+_{n,\alpha'}} \varphi_{0,\overline{K}^2_{\alpha',n}}(z-x)dx$. (138) and (54) show (137).

From (137), the second inequality in (135), $\max_{0\le i\le \lfloor n^{\alpha'} \rfloor} d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} = O\left(n^{-\frac{1}{2}}\right)$ (by (47)) and $\frac{\overline{K}_{\alpha',n}}{\underline{K}_{\alpha',n}} = O(1)$ (by (109)), it then follows that

$$\left\| \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),C^2_{\bar{k}_n+i,n-1,n}} \right\|_{\infty,I_{n,\alpha}}$$
$$\le \left(\max_{0\le i\le \lfloor n^{\alpha'} \rfloor -1} d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} \right) \frac{\overline{K}_{\alpha',n}}{\underline{K}_{\alpha',n}} \left\| \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\overline{K}^2_{\alpha',n}} \right\|_{\infty,I_{n,\alpha}} = O(1).$$

The latter result and (133) imply (136) (via the obvious $l^1 - l^\infty$ bound for the sum in $\mathbb{R}^{\lfloor n^{\alpha'} \rfloor}$, using that all terms $d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),C^2_{\bar{k}_n+i,n-1,n}}(z)$ are nonnegative). Next, as all $d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i}$ are nonnegative, (135) implies for all $z \in \mathbb{R}$,

$$\frac{\underline{K}_{\alpha',n}}{\overline{K}_{\alpha',n}} \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\underline{K}^2_{\alpha',n}}(z) \le \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),C^2_{\bar{k}_n+i,n-1,n}}(z)$$
$$\le \frac{\overline{K}_{\alpha',n}}{\underline{K}_{\alpha',n}} \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\overline{K}^2_{\alpha',n}}(z). \tag{139}$$

Given (136), (139), and $3\alpha' - 2 < \frac{\alpha'}{2} - \frac{1}{2}$, we see that (134) follows if we can show both

$$\left\| \frac{\underline{K}_{\alpha',n}}{\overline{K}_{\alpha',n}} \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\underline{K}^2_{\alpha',n}} - \int_0^\infty \varphi(Id - x) \left(\nu_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(x)dx \right\|_{\infty,I_{n,\alpha}} = O\left(n^{\frac{\alpha'}{2}+\varepsilon'-\frac{1}{2}}\right) \tag{140}$$

and

$$\left\| \frac{\overline{K}_{\alpha',n}}{\underline{K}_{\alpha',n}} \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} d^{\nu_M\circ\beta_{W,n,k_n}}_{G,k_n,i} \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\overline{K}^2_{\alpha',n}} - \int_0^\infty \varphi(Id - x) \left(\nu_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(x)dx \right\|_{\infty,I_{n,\alpha}} = O\left(n^{\frac{\alpha'}{2}+\varepsilon'-\frac{1}{2}}\right). \tag{141}$$

We now prove (141). The proof of (140) is completely analogous. First, using Lemma

14 (for α' and ε') and (137), we obtain

$$\left\| \sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} \left(d_{G,k_n,i}^{v_M \circ \beta_{W,n,k_n}} - \Delta_{\bar{k}_n+i,n} \left(v_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}] \right)' (\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})) \right) \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\overline{K}^2_{\alpha',n}} \right\|_{\infty, I_{n,\alpha}}$$
$$= O\left(n^{\frac{\alpha'}{2}+\varepsilon'-\frac{1}{2}} \right). \tag{142}$$

Next, we note that (for any n and any $z \in I_{n,\alpha}$),

$$\sum_{i=0}^{\lfloor n^{\alpha'} \rfloor -1} \Delta_{\bar{k}_n+i,n} \left(v_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}] \right)' (\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})) \varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\overline{K}^2_{\alpha',n}}(z)$$

is a (left) Riemann sum for the integral

$$\int_{I^+_{n,\alpha'}} \varphi_{0,\overline{K}^2_{\alpha',n}}(z-x) \left(v_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}] \right)' (x) dx. \tag{143}$$

By (54), the mesh size of the corresponding partition, i.e., $\max_{0 \le i \le \lfloor n^{\alpha'} \rfloor -1} \Delta_{\bar{k}_n+i,n}$ is of order $\Theta\left(n^{-\frac{1}{2}}\right)$, and it is easy to see that the derivative of the integrand in (143) is of order $O(1)$: More precisely,

$$\|\Xi_1[\tilde{\beta}_{M,n,k_n}]'\|_\infty = \underline{w} \|\varphi' * (v_W \circ \tilde{\beta}_{M,n,k_n})\|_\infty \le \underline{w} v_W(\overline{b}_M(\overline{m}, \overline{w})) \|\varphi'\|_{L^1(\mathbb{R})} < \infty,$$
$$\|\Xi_1[\tilde{\beta}_{M,n,k_n}]''\|_\infty = \underline{w} \|\varphi'' * (v_W \circ \tilde{\beta}_{M,n,k_n})\|_\infty \le \underline{w} v_W(\overline{b}_M(\overline{m}, \overline{w})) \|\varphi''\|_{L^1(\mathbb{R})} < \infty,$$

$\overline{K}_{\alpha',n} = \Theta(1)$, $\|\varphi\|_\infty < \infty$, $\|\varphi'\|_\infty < \infty$, $\|v_M'\|_\infty < \infty$ and $\|v_M''\|_{\infty,[0,\bar{b}_W(\overline{m},\overline{w})]} < \infty$ imply

$$\max_{z \in I_{n,\alpha}} \max_{x \in I^+_{n,\alpha'}} \frac{d}{dx} \left(\varphi_{0,\overline{K}^2_{\alpha',n}}(z-x) \left(v_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}] \right)' \right) = O(1).$$

Thus, (54) and the mean value theorem yield

$$\max_{0 \le i \le \lfloor n^{\alpha'} \rfloor -1} \Bigg\| \int_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})}^{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i+1,n-1}})} \varphi_{0,\overline{K}^2_{\alpha',n}}(Id - x) \left(v_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}] \right)' (x) dx$$
$$- \Delta_{\bar{k}_n+i,n} \varphi_{0,\overline{K}^2_{\alpha',n}}(Id - \xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})) \left(v_M \circ \Xi_1[\tilde{\beta}_{M,n,k_n}] \right)' (\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}})) \Bigg\|_{\infty, I_{n,\alpha}} = O\left(n^{-1}\right),$$

and hence also

$$\Big\| \int_{I^+_{n,\alpha'}} \varphi_{0,\overline{K}^2_{\alpha',n}}(Id-x)\left(v_M\circ\Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(x)dx$$
$$-\sum_{i=0}^{\lfloor n^{\alpha'}\rfloor-1}\Delta_{\bar{k}_n+i,n}\varphi_{0,\overline{K}^2_{\alpha',n}}(Id-\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}))\left(v_M\circ\Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}))\Big\|_{\infty,I_{n,\alpha}} = O\left(n^{\alpha'-1}\right),$$

which, combined with (142) and using $\alpha'-1<\frac{\alpha'}{2}-\frac{1}{2}$ yields

$$\Big\|\sum_{i=0}^{\lfloor n^{\alpha'}\rfloor-1} d^{v_M\circ\beta_{W,n,k_n}}_{G,k_n,i}\varphi_{\xi_{F,n-1,\bar{k}_n}(m_{\mu_{\bar{k}_n+i,n-1}}),\overline{K}^2_{\alpha',n}}-\int_{I^+_{n,\alpha'}}\varphi_{0,\overline{K}^2_{\alpha',n}}(Id-x)\left(v_M\circ\Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(x)dx\Big\|_{\infty,I_{n,\alpha}}$$
$$= O\left(n^{\frac{\alpha'}{2}+\varepsilon'-\frac{1}{2}}\right). \tag{144}$$

Finally, we also note that, in view of $|\overline{K}^2_{\alpha',n}-1| = O\left(n^{\alpha'-1}\right)$ (by (109)), applying the second bound in Lemma 12 (for each fixed z, using the symmetry of φ and $\|(v_M\circ\Xi_1[\tilde{\beta}_{M,n,k_n}])'\|_{\infty,I^+_{n,\alpha'}} = O(1)$ as well) yields

$$\Big\|\int_{I^+_{n,\alpha'}}\left(\varphi_{0,\overline{K}^2_{\alpha',n}}(Id-x)-\varphi(Id-x)\right)\left(v_M\circ\Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(x)dx\Big\|_{\infty,I_{n,\alpha}} = O\left(n^{\alpha'-1}\right), \tag{145}$$

while $|\overline{K}_{\alpha',n}/\underline{K}_{\alpha',n}-1| = O\left(n^{\alpha'-1}\right)$ (by (109)) and $\|(v_M\circ\Xi_1[\tilde{\beta}_{M,n,k_n}])'\|_{\infty,I^+_{n,\alpha'}} = O(1)$ also imply

$$\left\|\left(\frac{\underline{K}_{\alpha',n}}{\overline{K}_{\alpha',n}}-1\right)\int_{I^+_{n,\alpha'}}\varphi(Id-x)\left(v_M\circ\Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(x)dx\right\|_{\infty,I_{n,\alpha}} = O\left(n^{\alpha'-1}\right), \tag{146}$$

Recalling $\alpha'-1<\frac{\alpha'}{2}-\frac{1}{2}$ once again, (144), (145), (146) and

$$\left\|\int_{\mathbb{R}_+\setminus I^+_{n,\alpha'}}\varphi(Id-x)\left(v_M\circ\Xi_1[\tilde{\beta}_{M,n,k_n}]\right)'(x)dx\right\|_{\infty,I_{n,\alpha}} = \mathcal{E}(n)$$

(which is obvious from the decay properties of φ) imply (141). As mentioned above, the proof of (140) is analogous. This concludes the proof of (134), and hence of (50). □

H Auxiliary mathematical results

Lemma 15. *For* $1 \le i \le l$ *and* $x \in [\underline{x}, \overline{x}]$*:*

$$\sum_{j=i}^{l} h_{j:l}(x) = lh(x)H_{i-1:l-1}(x) \tag{147}$$

$$\sum_{j=1}^{i-1} h_{j:l}(x) = lh(x)(1 - H_{i-1:l-1}(x)). \tag{148}$$

Proof of Lemma 15. We have $\sum_{j=i}^{l} h_{j:l}(x) = lh(x) \sum_{j=i-1}^{l-1} B_{j,l-1}(H(x)) = lh(x)H_{i-1:l-1}(x)$. Here, the first equality uses (8). For $i \ge 2$, the second equality is immediate from (7). For $i = 1$, it follows from $H_{0:l-1} \equiv 1$ on $[\underline{x}, \overline{x}]$ and the fact that the Bernstein polynomials form a partition of unity. This proves (147). Substracting (147) for general i from (147) for $i = 1$, we also obtain (148). □

Lemma 16. *Let* X *be a real-valued random variable on some probability space, with* $E[|X|] < \infty$*. Let* S *be an event, with complement* S^c*, and assume* $0 < P[S] < 1$*. Then, it holds that*

$$|E[X] - E[X|S]| = |E[X|S^c] - E[X|S]|P[S^c], \tag{149}$$

In particular, if there is some $K > 0$ *such that* $|X| \le K$*, then*

$$|E[X] - E[X|S]| \le 2K(1 - P[S]). \tag{150}$$

Proof of Lemma 16. From the basic identity $E[X] = E[X|S]P[S] + E[X|S^c]P[S^c]$ we obtain $E[X] - E[X|S] = P[S^c](E[X|S^c] - E[X|S])$, and thus also (149) and (150). □

Lemma 17. *If* h *is continuous on* $[\underline{x}, \overline{x}]$ *and* $\min_{[\underline{x},\overline{x}]} h(x) > 0$*, then the* expected spacings $E[X_{i+1:l} - X_{i:l}]$, $i \in \{0, ..., l-1\}$ *(recall that* $X_{0:l} \equiv \underline{x}$*) satisfy*

$$E[X_{i+1:l} - X_{i:l}] = \int_{\underline{x}}^{\overline{x}} B_{i,l}(H(x))dx = \int_0^1 \frac{B_{i,l}(u)}{h(H^{-1}(u))}du. \tag{151}$$

In particular, for all $i \in \{0, ..., l-1\}$*,*

$$\frac{1}{(l+1)||h||_\infty} \le E[X_{i+1:l} - X_{i:l}] \le \frac{1}{(l+1)\min_{[\underline{x},\overline{x}]} h(x)}. \tag{152}$$

Proof of Lemma 17. The first identity in (151) is a classical representation (e.g., equation (2) in David and Groeneveld 1982).[52] The second identity in (151) is immediate

[52] David and Groeneveld (1982) state the result only for $1 \le i \le l-1$. The case $i = 0$ follows from $X_{0:l} = \underline{x}$ and $E[X_{1:l}] = \underline{x} + \int_{\underline{x}}^{\overline{x}} 1 - H_{1:l}(x)dx = \underline{x} + \int_{\underline{x}}^{\overline{x}} B_{0,l}(H(x))dx$.

from the change of variables $u = H(x)$, and (152) then follows from

$$\frac{1}{||h||_\infty}\int_0^1 B_{i,l}(u)du \leq \int_0^1 \frac{B_{i,l}(u)}{h(H^{-1}(u))}du \leq \frac{1}{\min_{[\underline{x},\bar{x}]} h(x)}\int_0^1 B_{i,l}(u)du$$

and $\int_0^1 B_{i,l}(u)du = E[U_{i+1:l} - U_{i:l}] = \frac{1}{l+1}$. □

Lemma 18. *There is a constant $C > 0$ such that for any H and any $1 < i < l$:*

$$h_{i:l}(x) \leq Ch(x)\left(\frac{l}{B_{1,2}(\mu_{i-1,l-2})}\right)^{\frac{1}{2}}.$$

Proof of Lemma 18. For any $l \in \mathbb{N}$ and $0 \leq i \leq l$, $B_{i,l}$ is maximized at $\frac{i}{l} = \mu_{i,l-1}$. Next, by Robbins' (1955) error bound for Stirling's approximation, it holds for all $N \in \mathbb{N}$:

$$\sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N}e^{\frac{1}{12N+1}} < N! < \sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N}e^{\frac{1}{12N}}.$$

This implies that for $1 \leq i < l$, we have $\binom{l}{i} < \frac{l^{l+\frac{1}{2}}e^{\frac{1}{12l}}}{\sqrt{2\pi}i^{i+\frac{1}{2}}e^{\frac{1}{12i+1}}(l-i)^{l-i+\frac{1}{2}}e^{\frac{1}{12(l-i)+1}}}$, which is equivalent to

$$B_{i,l}(\mu_{i,l-1}) = \binom{l}{i}\left(\frac{i}{l}\right)^i\left(1-\frac{i}{l}\right)^{l-i} < \frac{1}{\sqrt{2\pi}}\left(\frac{l}{i(l-i)}\right)^{\frac{1}{2}}e^{\frac{1}{12l}-\frac{1}{12i+1}-\frac{1}{12(l-i)+1}}.$$

In particular, as $\frac{1}{12l} - \frac{1}{12i+1} - \frac{1}{12(l-i)+1} < 0$, it follows for any $1 \leq i < l$:

$$B_{i,l}(\mu_{i,l-1}) < \frac{1}{\sqrt{2\pi}}\left(\frac{l}{i(l-i)}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}\left(\frac{2}{lB_{1,2}(\mu_{i,l-1})}\right)^{\frac{1}{2}} = \left(\pi l B_{1,2}(\mu_{i,l-1})\right)^{-\frac{1}{2}}.$$

Using equation (8), we therefore obtain for any $1 < i < l$ and all x:

$$h_{i:l}(x) = lh(x)B_{i-1,l-1}(H(x)) \leq lh(x)B_{i-1,l-1}(\mu_{i-1,l-2}) \leq \frac{lh(x)}{((l-1)\pi B_{1,2}(\mu_{i-1,l-2}))^{\frac{1}{2}}}.$$

This proves the claim of the lemma for $C = \sup_{l\geq 3}\left(\frac{l}{(l-1)\pi}\right)^{\frac{1}{2}} = \left(\frac{3}{2\pi}\right)^{\frac{1}{2}}$. □

I Two Remarks

Remark 3. *Assume that investments have both a signaling and a productive function, as in Dizdar, Moldovanu and Szech (2019). More precisely, let us assume that $U_M(b_M, b_W, m, w) = m(\varepsilon w + v_M(b_W)) - b_M$ and $U_W(b_W, b_M, w, m) = w(\varepsilon m + v_W(b_M)) - b_W$ for some $\varepsilon > 0$.[53] Arguments analogous to those for Theorem 2 show that the limits of re-scaled SSMBNE strategies for a pre-match investment game as in Dizdar, Moldovanu*

[53] All other assumptions, about preferences and type distributions, remain unchanged.

and Szech (2019) (with positive assortative matching based on investments) must satisfy (22)*, with the only difference that* $(v_M \circ \tilde{\zeta})(x)$ *is replaced by* $(v_M \circ \tilde{\zeta})(x) + \varepsilon \underline{w}$ *in* (21)*. The limits of the original strategies must be strictly increasing on* $(m_r, \overline{m}]$ *and* $[\underline{w}, \overline{w}]$ *in this case, and it is then easy to see that as* $\varepsilon \to 0$*, these functions are approximately of the form that pairs below* $(m_{r+p(\tilde{\beta}_*)(1-r)}, w_{p(\tilde{\beta}_*)})$ *overinvest while higher types make investments that are approximately efficient.*

Remark 4. *Extending our results to models with type-dependent autarchy investments and preferences that are additively separable in* b_M *and* b_W *(as in Peters and Siow 2002, Peters 2011, and the example in Nöldeke and Samuelson 2014) is conceptually straightforward. A formulation that nests the existing models could posit an autarchy utility function* $U_M^{\emptyset}$*, concave in* $b_M \geq 0$*, for which autarchy investments* $b_M^{\emptyset}(m) > 0$ *are unique, nondecreasing and differentiable in m, and separable preferences of the form* $U_M(b_M, b_W, m, w) = \hat{U}_M(b_W, b_M, m)$*, where either* $\hat{U}_M(x, b_M, m) = x + U_M^{\emptyset}(b_M, m)$ *or* $\hat{U}_M(x, b_M, m) = mx + U_M^{\emptyset}(b_M, m)$*. Similar assumptions should be made for women, and preferences should satisfy strict outer single crossing. Our observations regarding agents' uncertainty about their equilibrium partners apply without changes and, under mild technical conditions, the types whose behavior determines the limit equation for re-scaled SSMBNE strategies have very similar preferences.*[54] *Assuming that there is some constant* $\delta > 0$ *such that* $\left(\frac{\partial U_M^{\emptyset}}{\partial b_M}\right)^+ (b_M^{\emptyset}(m), m) \leq -\delta$ *and* $\left(\frac{\partial U_W^{\emptyset}}{\partial b_W}\right)^+ (b_W^{\emptyset}(w), w) \leq -\delta$ *for all types,*[55] *one can show an exact analogue of Theorem 2 based on the simple idea of approximating the investment costs of the relevant types by those of the types* $m_{\bar{k}_n/n}$ *and* $\underline{w}$ *in the equilibrium first order conditions. As the additional technicalities are extremely tedious and do not provide further insights, we have chosen the standard "multiplicative in types, quasi-linear in own investment" form of utilities for our analysis.*

[54]For any $\alpha > 1/2$, all relevant types are contained in intervals with length of order $O(n^{\alpha-1})$.

[55]This assumption, which essentially says that autarchy utility functions have a kink at the autarchy investment levels (so that marginal costs are bounded away from 0), is needed to avoid problems with SSMBNE existence due to a failure of the standard (Lipschitz) continuity conditions that ensure the existence (and uniqueness) of solutions for ordinary differential equations.