# The Mobility Curve:
# Measuring the Impact of Income Changes on Welfare†

James E. Foster* and Jonathan Rothbaum†

December 4, 2014

*Work in Progress - Please do not cite.*

**Abstract**

This paper examines how income mobility affects social welfare. We derive conditions under which the welfare impact of either upward or downward mobility is unambiguously greater in one society than another for broad classes of utility functions. From this analysis, we construct mobility dominance orderings which are analogous to stochastic dominance. The mobility dominance orderings motivate a new framework for mobility measurement, which we call the mobility curve. This framework allows unambiguous comparisons of mobility across societies and over time with a clear graphical representation. Our approach builds on distance-based measures of mobility, which evaluate how much change has occurred, by also incorporating where in the distribution mobility is occurring, as transition matrices do, but without the censorship of income movements inherent in transition matrices. We also relate the mobility curve approach to changes in the FGT class of poverty measures. Finally, we use mobility curves to analyze intragenerational income mobility in the United States over various six-year periods from 1974 to 2004.

* Department of Economics and Institute for International Economic Policy, George Washington University, Washington DC 20052. Email: fosterje@gwu.edu

† Social and Economic Housing Statistics Division, U.S. Census Bureau. Email: jonathan.l.rothbaum@census.gov

# 1 Introduction

Are children better or worse off than their parents? How much better or worse off? These are central questions in the measurement of intergenerational mobility, which has received considerable attention in the press[1] and the academic literature.[2]

To answer these questions, we must define what is meant by "better." The first step in this process is to decide: better in what? We must select the domain or general space for mobility comparisons, such as income, education, occupation, or some other variable of socioeconomic status or achievement. We can then construct a specific variable within that space to make the comparison. For example, for income we could use absolute income, rank, income relative to the mean, or a coarse categorical variable such as quintile.

With the domain and variable chosen, we can determine whether children have more than, less than, or the same as their parents. This step, which we call identification, yields a simple first measure, the mobility headcount. For example, the number of upwardly mobile children tells us how many children are better off than their parents. However, a headcount measure of mobility is very crude, as it would be the same whether each child was slightly better off than their parents or much better off. Once we have identified the upwardly mobile ($H_U$), the downwardly mobile ($H_D$), and the immobile children ($H_I$), we would like to measure how much better or worse off these children are than their parents.

For the moment, let us focus on upward mobility. How much did upwardly mobile children advance relative to their parents? One straightforward approach is to compare the mean of child

[1] Select from: http://www.economist.com/node/15908469
http://www.pbs.org/newshour/rundown/the-great-gatsby-curve-inequality-and-the-end-of-upward-mobility-in-america-1/
http://www.washingtonpost.com/opinions/the-downward-path-of-upward-mobility/2011/11/09/gIQAegpS6M_story.html
www.bbc.com/news/business-20154358
http://www.newrepublic.com/article/politics/magazine/100516/inequality-mobility-economy-america-recession-divergence
http://www.theatlantic.com/business/archive/2012/10/how-to-make-sure-the-next-generation-is-better-off-than-we-are/263579/
http://www.huffingtonpost.ca/2014/05/17/millennials-better-off-than-parents-not_n_5340280.html
http://economix.blogs.nytimes.com/2012/07/11/only-half-of-americans-exceed-parents-wealth/?_php=true&_type=blogs&_r=0
www.forbes.com/sites/learnvest/2013/12/04/american-dream-2-0-are-we-better-off-than-our-parents/
http://www.nytimes.com/2013/07/22/business/in-climbing-income-ladder-location-matters.html?pagewanted=all
http://www.project-syndicate.org/commentary/richard-n--haass-cautions-against-responses-to-growing-inequality-that-merely-shift-wealth--rather-than-creating-it

[2] Citations – Solon, Reardon, Corak, Chetty et al. etc.

achievements and the mean of parent achievements for the upwardly mobile. This measure is equivalent to comparing these children's average gains relative to their parents. In fact, this measure is a decomposition of the distance-based approach proposed by Fields and Ok (1996). This reasonable first order approach gives a sense of the extent of change.

We could also consider using a function of distance itself, without paying particular attention to the parent or child income levels per se, as proposed by Mitra and Ok (1998). However by focusing only on distance, a unit of upward mobility for the poorest and richest children are treated identically. In other words, mobility is the same whether the parent earned and 1 and the child 11 or if the parent earned 1,000,000 and the child 1,000,010.

Instead, it may make sense to transform the underlying variable such as income to rescale the distance more in accordance with intuition about how much better off each change would make the child than the parent. Fields and Ok (1999) do this using the log function to emphasize changes in income at the lower end of the distribution over changes at the upper end. They note in particular the relationship between log income and utility, as captured by Daniel Bernoulli in his original treatise on expected utility. This of course leads to a distinct measure based on the difference between the log of child income and the log of parent income, and this clearly evaluates mobility differently than the aforementioned distance measures. For example, mobility is greater for the 1→11 parent-child pair than for the 1,000,000→1,000,010 pair.

We can imagine a broad array of alternative measures, each having a different way of valuing achievement levels at different points in the distribution, which still agree that more is better. For example, in addition to $\log(x)$, we could use $x^2$ or $\sqrt{x}$ depending on whether emphasis should be given to higher or lower incomes. Each monotonic transformation (or $V$-function) would lead to a different plausible measure of upward and downward mobility.

Given the wide array of possible measures, it is natural to explore the cases where all such measures would agree. We consider the cases where there is unanimous agreement that upward mobility is higher or lower according to *all* measures of this form. These measures are defined by their focus on the extent to which children progress beyond their parents. We identify the resulting upward mobility dominance ordering and show that it can be represented with the help of an upward mobility curve. The curve also links to and graphically represents a specific measure, the Fields and Ok (1996) distance-based mobility measure described above. The area below the upward mobility curve is equal to the distance-based upward mobility decomposition. The entire discussion is completely analogous for downward mobility.

The upward and downward mobility measures have an array of properties that help describe what they are measuring. We propose a set of axioms that characterize our dominance ordering and define which changes lead to increases in mobility and which do not affect mobility in our approach. We also show that our mobility ordering is equivalent to a result that we call flip vector dominance. For two societies with the same number of parent-child pairs, we can construct two "flipped" vectors with the parent achievements from one society and the child achievements from the other. We show that our mobility dominance ordering is equivalent to vector (or stochastic) dominance between the two flipped vectors.

The first order mobility dominance criteria can also be applied to variables that are not cardinally meaningful, such as education or health status. In these cases, any arbitrary rescaling of the underlying ordinal variable may be as valid as the original cardinalization. Unanimity among all possible $V$-functions for ordinal variables can be interpreted as which comparisons are meaningful for all possible cardinalizations of the variable. That means we can make mobility comparisons of ordinal variables without having to assume and defend an arbitrary cardinal representation of that variable.

If the variable is cardinal however, we can make further reasonable assumptions to generate a more complete mobility ordering. The set of allowable $V$-functions could be restricted to the subset of monotonic functions that are concave. These functions give more value to achievements at the lower end of the distribution than the upper end, as for Fields and Ok's (1999) use of log income. However, as we are not sure that log is the only reasonable function in this class, we derive second order mobility dominance to compares mobility under all possible monotonic concave transformations of the cardinal achievements. This ordering also has a corresponding second order mobility curve and flip vector dominance result.

Many evaluations of mobility are conducted using transformations of the basic domain variables that fundamentally alter the levels of child achievements that are seen as equivalent to levels of parent achievements. Examples of such transformation include rank or income group in a transition matrix. In these cases, children with more income than their parents could be upwardly mobile, downwardly mobile, or immobile depending on other incomes in both the parent and child distributions. We discuss some issues and concerns that must be considered when analyzing mobility of these transformed variables.

Some authors combine upward and downward mobility to obtain a single aggregate measure. We do not focus on such aggregated measures in this paper. However, the combined area below the upward and downward mobility curves sum to the Fields and Ok (1996) distance-based measure, one such aggregate. One could also argue that downward mobility, being a net loss, should be subtracted from upward mobility. If we subtract the downward from the upward mobility curve, we obtain a curve that can be used to test for stochastic dominance between the parent and child income distributions as in Bawa (1975) or for poverty dominance as in Foster and Shorrocks (1988). Another option is that one might be interested in upward or downward mobility but not the other. All are possible with our general approach. We explore each but leave open all of the possibilities.

INTRO EMPIRICAL DISCUSSION…

In section 2, we formalize our discussion of how we define "better" off and derive the mobility dominance ordering. In section 3, we discuss the axioms that this approach satisfies and introduce the graphical representation of our mobility ordering, mobility curves. In section 4, we discuss the application of our approach to cardinal and ordinal variables and propose the more complete second order dominance for mobility of cardinal variables. In section 5, we discuss extensions to our approach and other considerations. In section 6, we apply our approach to the measurement of intergenerational mobility in the United States. Section 7 concludes.

## 2 Mobility as Utility Gains and Losses Relative to Parents

A mobility pair is an ordered pair $m = (x, y)$ of distributions with the same population $n = n(x) = n(y)$, where $x_i$ is the income of parent $i$ and $y_i$ is the income of child $i$ for $i = 1, \dots, n$. At this stage, we assume $x_i$ and $y_i$ are cardinally meaningful.[3] We discuss mobility of ordinal variables in Section 4.

We have already discussed a first, basic measure of mobility, the mobility headcount ratio. Let $I(\alpha)$ be an indicator function, where $I(\alpha) = 1$ if condition $\alpha$ is true and $I(\alpha) = 0$ if condition $\alpha$ is false. The mobility headcount ratios are $H_U = \frac{1}{n}\sum_{i=1}^{n} I(y_i > x_i)$, $H_D =$

[3] In the exposition of this paper, we use the terms income and achievement interchangeably for convenience.

$\frac{1}{n}\sum_{i=1}^{n} I(y_i < x_i)$, and $H_I = \frac{1}{n}\sum_{i=1}^{n} I(y_i = x_i)$. While this gives a count of how many children are better or worse off than their parents, it does not tell us anything about how much better or worse off they are.

To measure that, we start by focusing on upward mobility. Let $U$ be a measure of how much better off upwardly mobile children are than their parents so that for a single parent achievement $x_i$ and child achievement $y_i$, the value or utility gained by child $i$ from upward mobility is $U(x_i, y_i)$. In the introduction, we considered potential functions, including $U(x_i, y_i) = V(y_i - x_i)$ used in the distance-based mobility literature by Fields and Ok (1996) and Mitra and Ok (1998).[4] However, under this approach the value gained by a child due to upward mobility is a function only of the distance between the child and parent incomes as $V(y_i - x_i) = V\big(y_i + \alpha - (x_i + \alpha)\big)$ for any $\alpha \in \mathbb{R}$. Therefore, $V(11 - 1) = V(1{,}000{,}010 - 1{,}000{,}000) = V(10)$ under any function in this class. This is an unappealing property as it means that this approach is indifferent to the parent income levels of mobile children. Mobility is equal whether an additional dollar of upward mobility is given to the poor child ($y_i = 10$) or the rich child ($y_i = 1{,}000{,}010$).

Instead, we propose a transformation of the original variable to rescale the distance, so that $U(x_i, y_i) = V(y_i) - V(x_i)$. This formulation has a simple and intuitive interpretation: how much additional utility or value has the child gained relative to their parents due to upward mobility (or lost due to downward mobility)? This gives a concrete and specific meaning to our initial question of how much better or worse off are children due to mobility. We are measuring better and worse off in the space of utility or value. This general approach is used by Fields and Ok (1999) using the log function as $V$ where $U(x_i, y_i) = \log(y_i) - \log(x_i)$. Their approach has the advantage of explicitly valuing changes in income at the lower end of the distribution over changes at the upper end, so that the measure is not indifferent between a change from $1 \rightarrow 11$ and another from $1{,}000{,}000 \rightarrow 1{,}000{,}010$.

However, log is only one of a broad array of potential transformations of the achievement variable into value or utility which are monotonically increasing. As we discussed in the introduction, in addition to $\log(x)$, we could use $x^2$ or $\sqrt{x}$ depending on whether emphasis

[4] Mitra and Ok (1998) define distance as a function of the individual distances where $d(x, y) = (\sum_{i=1}^{n} |y_i - x_i|^{\alpha})^{\frac{1}{\alpha}}$. For simplicity, we focus on each individual's distance term $|y_i - x_i|^{\alpha}$ as, for a given $\alpha$, the outer $1/\alpha$ exponent does not affect the distance ordering between any two societies of the same population size.

should be given to higher or lower incomes. Given the wide variety of potential $V$-functions, we explore the cases where all such measures would agree. In other words, we characterize the conditions under which mobility is higher or lower according to *all* the upward and downward mobility measures of this form.

## 2.1 First Order Mobility Dominance

To define our first order dominance, we turn to the more general case of potentially continuous income distributions. Random variables $X$ and $Y$ have the cumulative distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$ respectively, where $X$ represents parent incomes and $Y$ represents child incomes in a given society. The distributions may be discrete (as with the aforementioned mobility pair $m$), continuous, or mixed under a closed interval $[0, b]$, $0 < b$.[5] They are non-decreasing continuous on the right with $F(0) = 0$ and $F(b) = 1$. Let $F_{XY}(X, Y)$ be the joint distribution of the parent and child incomes, where $F_{XY}(X, Y) = \int_0^b \int_0^b f_{XY}(X, Y) dF(Y) dF(X)$.[6]

Because our measure of value $V(x_i) - V(y_i)$ is an additively separable function of parent and child incomes, we can separate the joint distribution into the parent and child marginal distributions. As we discussed in the introduction, we are interested in measuring the utility gained from upward mobility and the utility lost from downward mobility separately. Therefore for the measure of upward mobility, because none of the downwardly mobile are better off than their parents, we treat them as immobile. To do that, we censor the incomes of downwardly mobile children to be equal to their parents. We define the upward mobility marginal distributions as

$$F_X^U(X) = F_X(X) = \int_0^b f_{XY}(X, Y) dF_Y(Y) \tag{2.1.1}$$

and

$$F_Y^U(Y) = \int_0^b f_{XY}(X, Y) I(x \geq y) dF_X(X) + \int_0^b f_{XY}(X, Y) I(x < y) dF_Y(Y). \tag{2.1.2}$$

We use the notation $m_U = \left(F_X(X), F_Y^U(Y)\right)$ as the pair of parent and upward mobility censored child marginal distributions. Given $m_U$, the average gain in utility from upward mobility is

[5] The lower bound could be any value $a < b$, but we will set the minimum income to 0 for simplicity.

[6] All integrals are Lebesgue-Stieltjes integrals and are assumed to be bounded.

$$U(m_U) = \int_0^b V(y) dF_Y^U(Y) - \int_0^b V(x) dF_X(X). \tag{2.1.3}$$

After integration by parts, this becomes

$$\begin{aligned} U(m_U) &= \int_0^b V'(c) F_X(c) dc - \int_0^b V'(c) F_Y^U(c) dc \\ &= \int_0^b V'(c)[F_X(c) - F_Y^U(c)] dc = \int_0^b V'(c) M_U(c) dc. \end{aligned} \tag{2.1.4}$$

$F_X(c) - F_Y^U(c)$, which we denote as $M_U(c)$, is the share of the population with parent income below $c$ and child income above $c$. We call this the share of the population that is upwardly mobile across income cutoff $c$. One way to think about this is to consider $c$ a poverty line. In this context, for a given poverty line $c$, upwardly mobile children are those that are not in poverty themselves but came from poor parent households.

With this simple framework, we can compare two societies to see if upward mobility has increased average utility or value more in one than in another. Stated in terms of our original question, are children in one society better off on average than children in another due to upward mobility? We propose an upward mobility dominance ordering that is analogous to the well-known stochastic dominance ordering. For societies $A$ and $B$, $B$ upward mobility dominates $A$ if and only if for all possible value functions $V$ in a given set, the average value gained from upward mobility is greater in $B$ than $A$, or

$$B \succ_M^U A \text{ if and only if } U(m_U^B) > U(m_U^A)\ \forall V \in \mathcal{V}. \tag{2.1.5}$$

We first restrict our attention to the set of monotonically increasing functions $\mathcal{V}_1$ where if $V \in \mathcal{V}_1$, then $V' > 0$. For societies $A$ and $B$, from (2.1.4) the difference in utility gained from upward mobility is

$$U(m_U^B) - U(m_U^A) = \int_0^b V'(c)[M_U^B(c) - M_U^A(c)] dc. \tag{2.1.6}$$

First order upward mobility dominance ($\succ_{M1}^U$) holds for $A$ and $B$ if for all possible monotonically increasing $V$-functions, there is a greater average increase in utility in $B$ than $A$ due to upward mobility.

**Theorem 1.** The following statements are equivalent:

1. $B \succ_{M1}^U A$
2. $U(m_U^B) > U(m_U^A) \forall V \in \mathcal{V}_1$, where $V' > 0$

3. $M_U^B(c) \geq M_U^A(c) \forall c \in [0, b]$, and for some $c$, $M_U^B(c) > M_U^A(c)$

**Proof:** The proof of Theorem 1 is virtually identical to the first order stochastic dominance proof in Bawa (1975). We show that for some arbitrary $c_0$ with $0 \leq c_0 < c_0 + \delta \leq b$, if $M_U^B(c_0) < M_U^A(c_0)$, there exists a utility function $V \in \mathcal{V}_1$ where $U(m_U^B) < U(m_U^A)$. Let $\epsilon = \gamma\delta$ and

$$\phi(c) = \begin{cases} \epsilon & 0 \leq c \leq c_0 \\ \epsilon - \gamma(c - c_0) & c_0 \leq c \leq c_0 + \delta \\ 0 & c_0 + \delta \leq b. \end{cases} \tag{2.1.7}$$

We define a utility function where $V_1(y) = ky - \phi(y)$ and $V_1{}'(y) = k - \phi'(y)$, where $k > 0$. Under this utility function, $V_1{}'(y) > 0 \; \forall y$ and therefore $V_1 \in \mathcal{V}_1$.[7] With this function:

$$\begin{aligned} &U(m_U^B) - U(m_U^A) \\ &\qquad = k \int_0^b \left(M_U^B(c) - M_U^A(c)\right) dc + \gamma \int_{c_0}^{c_0+\delta} \left(M_U^B(c) - M_U^A(c)\right) dc \end{aligned} \tag{2.1.8}$$

Given that at $c_0$ $M_B^U(c) < M_A^U(c)$, by choosing a sufficiently large $\gamma$, one can make $U(m_U^B) < U(m_U^A)$. This completes the proof of Theorem 1.

$B$ first order upward mobility dominates $A$ if and only if an equal or greater share of the population is upwardly mobile across ***all possible cutoffs*** in $B$ than $A$ and the inequality is strict for some cutoffs.

This entire discussion is also valid for downward mobility. In that case, we censor the incomes of upwardly mobile children and treat them as immobile so that

$$F_Y^D(Y) = \int_0^b f_{XY}(X,Y) I(x < y) dF_X(X) + \int_0^b f_{XY}(X,Y) I(x \geq y) dF_Y(Y). \tag{2.1.9}$$

We use the notation $m_D = \left(F_X(X), F_Y^D(Y)\right)$ as the pair of parent and downward mobility censored child marginal distributions. The loss in utility from downward mobility is

$$U(m_D) = \int_0^b V(y) dF_Y^D(Y) - \int_0^b V(x) dF_X(X) \tag{2.1.10}$$

which after integration by parts, becomes

$$U(m_D) = \int_0^b V'(c)[F_X(c) - F_Y^D(c)] dc = \int_0^b V'(c) M_D(c) dc. \tag{2.1.11}$$

[7] As noted in Bawa (1975), the differentiability requirements for the utility function are satisfied by rounding the edges at the points of discontinuity which does not affect the analysis in these proofs.

Analogous to upward mobility, $M_D(c)$ is the share of the population with parent income above $c$ and child income below $c$. In other words, it is the share of the population downwardly mobile across income cutoff $c$.

We can derive first order downward mobility dominance ($\succ_{M1}^{D}$) as well for $V \in \mathcal{V}_1$. In this case, we are comparing the utility lost due to downward mobility in two societies. For societies $A$ and $B$, the difference in utility lost from downward mobility is

$$U(m_D^B) - U(m_D^A) = \int_0^b V'(c)[M_D^B(c) - M_D^A(c)]dc. \quad (2.1.12)$$

We define downward mobility dominance for utility losses, so that $B$ downward mobility dominates $A$ if a greater amount of utility is lost in $B$ due to downward mobility. For first order downward mobility dominance, we establish conditions under which $\forall V \in \mathcal{V}_1$, $U(m_D^B) < U(m_D^A)$, or since in both cases utility is lost, $|U(m_D^B)| > |U(m_D^A)|$.

**Theorem 1a.** The following statements are equivalent:

1. $B \succ_{M1}^{D} A$
2. $U(m_D^B) < U(m_D^A) \forall V \in \mathcal{V}_1$, where $V' > 0$
3. $M_D^B(c) \geq M_D^A(c) \forall c \in [0, b]$, and for some $c$, $M_D^B(c) > M_D^A(c)$

**Proof.** Identical to proof of Theorem 1 with $M_D$ in place of $M_U$ with the signs on inequalities of the $U(m_D^B)$ and $U(m_D^A)$ comparisons flipped.

Again, $B$ first order downward mobility dominates $A$ if and only if an equal or greater share of the population is downwardly mobile across ***all possible cutoffs*** in $B$ than $A$ and the inequality is strict for some cutoffs.

## 2.2 Mobility Curves

First order mobility dominance lends itself to a simple graphical representation. Recall from (2.1.4) that $M_U(c)$ is equal to the share of the population that is upwardly mobile across an income cutoff $c$. From Theorem 1, upward mobility dominance holds if and only if for societies $A$ and $B$, $M_U^B(c) \geq M_U^A(c)$ for all possible cutoffs and for some $c$ the inequality is strict. By plotting $M_U(c)$ at all cutoffs, we can easily compare mobility in two societies for first order dominance. We define the upward mobility curve as $M_U(c)$ plotted at all possible income cutoffs $c$, which given the discrete form of income data, we express as

$$M_U(c) = \frac{1}{n}\sum_{i=1}^{n} I(x_i \leq c)I(y_i > c). \tag{2.2.1}$$

A few simple examples help illustrate how mobility curves are constructed and how they facilitate mobility dominance comparisons. Let $A$ and $B$ be societies with two parent-child pairs. In both societies, the incomes for the parent generation are $x^A = x^B = (1,5)$. In society A, the child from the poor household earns 1 more than their parents so that the incomes for the children are $y^A = (2,5)$. The first child is upwardly mobile from 1 to 2 and the second child is immobile (5 to 5). In society B, the poor child earns 2 more than their parents and $y^B = (3,5)$. For any monotonically increasing utility function, the utility gain from upward mobility is clearly greater in society $B$ than $A$, as the poor child in $B$ is upwardly mobile by 2 as compared to 1 in $A$ from the same parent income level.

Figure 1 Panel A shows the upward curve generated by plotting $M_U(c)$ for societies $A$ and $B$ across all possible cutoffs. The mobility curve is a step function where each parent-child pair contributes $\frac{1}{n}$ to the height of the curve at all cutoffs between the parent and child incomes. In society $A$, the curve is equal to 0.5 for all cutoffs between the parent income of 1 and the child income of 2 for the upwardly mobile child and zero everywhere else. In society $B$, the curve is 0.5 for all cutoffs between the parent income of 1 and the child income of 3 for the upwardly mobile child and zero everywhere else.

The upward mobility curves for $A$ and $B$ show that the share of the population that is upwardly mobile at all cutoffs in $B$ is greater than or equal to the share in $A$. Therefore, by Theorem 1, $B \succ_{M1}^{U} A$ and $U(m_U^B) > U(m_U^A)$ for all monotonically increasing utility functions.

Figure 1 Panel B shows the upward mobility curve for an example where there is no first order dominance. In this case, the child incomes in $B$ are $(1,6)$. In each society, one child earns 1 more than their parents and the other is immobile. In society $A$ it is the poor child that is upwardly mobile, and in society $B$ it is the rich child. As Panel B shows, there is no first order upward mobility dominance. Neither mobility curve is greater than or equal to the other at all cutoffs. Since we only assume monotonic utility, we cannot determine unambiguously that children in one society are better off relative to their parents than children in the other due to upward mobility. In our approach, this lack of first order dominance is true for any nonoverlapping income gains, no matter how large or small. This stands in contrast to distance-based approaches, which would evaluate the mobility in Panel B as equal in the two societies.

The curve also links to and graphically represents a specific measure, the distance-based upward mobility decomposition of Fields and Ok (1996). The area below the mobility curve is equal to the decomposed Fields and Ok measure as

$$\int_0^b \frac{1}{n}\sum_{i=1}^{n} I(x_i \le c)I(y_i > c)\, dc = \frac{1}{n}\sum_{i=1}^{n}\left(y_{U,i} - x_i\right). \tag{2.2.2}$$

The downward mobility curve is entirely analogous. For downward mobility, the mobility curve at each cutoff given discrete incomes is

$$M_D(c) = \frac{1}{n}\sum_{i=1}^{n} I(x_i > c)I(y_i \le c). \tag{2.2.3}$$

Figure 2 shows two examples of downward mobility comparisons, where downward mobility is plotted below the $x$-axis. In Panel A, $x^A = x^B = (1,5)$ as in the previous example, but $y^A = (1,4)$ and $y^B = (1,3)$. $B$ first order downward mobility dominates $A$ as in both cases the child of the rich parents is downwardly mobile, but by 2 in $B$ and by 1 in $A$, as shown by the downward mobility curve. Panel B shows a case with no first order downward mobility dominance with $y^B = (0,5)$. In this example, in both societies a child is downwardly mobile by 1, in $A$ the child from the rich parents and in $B$ the child from the poor parents.

## 3 Axioms

In order to clarify what we are measuring, we propose a number of axioms that describe our approach. These properties fit into three general categories: invariance, dominance, and subgroup. Invariance properties define what changes a measure ignores. In this case, for each invariance axiom where $m_U$ is the initial mobility pair and $m_U'$ is the pair after the change, $U(m_U) = U(m_U')$. Dominance conditions focus on changes that unambiguously lead to increases or decreases in the measures so that $U(m_U) > U(m_U')$ or $U(m_U) < U(m_U')$. Subgroup properties define how the measures computed from subgroups relate to the measures computed from the combined population. These axioms are defined for discrete income distributions, the form in which income data is available.

## 3.1 Upward Mobility

Again, we begin by focusing our attention on upward mobility. Given a mobility pair $m = (x, y)$, we censor all movement downward by replacing each $y_i$ with $y_{U,i} = \max(x_i, y_i)$, $y_U = (y_{U,1}, \ldots, y_{U,n})$, so that $m_U = (x, y_U)$. From (2.1.4) with discrete incomes, $U(m_U) = \frac{1}{n}\sum_{i=1}^{n}\left[V(y_{U,i}) - V(x_i)\right]$.

**Axiom 1: Symmetry.** If $m'_U = (x', y'_U)$ is obtained from $m_U = (x, y_U)$ by a permutation of identities of both parents and children simultaneously, then $U(m_U) = U(m'_U)$.

**Axiom 2: Replication Invariance.** If $m'_U = (x', y'_U)$ is obtained from $m_U = (x, y_U)$ by a replication, then $U(m_U) = U(m'_U)$.

**Axiom 3: Normalization.** If $x_i = y_{Ui} \forall\ i = 1, \ldots, n$, then $U(m_U) = 0$.

These invariance conditions are straightforward: mobility should not depend on the size of the population or the ordering of the parent-child pairs. Also, if there are no differences in achievement between any censored parent-child pair, then there is no upward mobility and no gain in utility.

**Axiom 4: Decomposability.** If $m_U$ with population $n$ is decomposed into subgroups $m_U^A$ and $m_U^B$ with populations $n^A$ and $n^B$ respectively then $U(m_U) = \frac{n^A}{n}U(m_U^A) + \frac{n^B}{n}U(m_U^B)$.

This subgroup condition is a strong one as it rules out all mobility measures where the mobility or achievements of one parent-child pair affects the mobility or achievement of another.[8]

**Axiom 5: Simple Child Increment.** If $m'_U$ is obtained from $m_U$ be a simple increment to the income of immobile or upwardly mobile child $i$ so that $y'_{Ui} = y_{Ui} + \alpha$, $\alpha > 0$, then $U(m'_U) > U(m_U)$.

**Axiom 6: Simple Parent Decrement.** If $m'_U$ is obtained from $m_U$ be a simple decrement to the parent income of immobile or upwardly mobile child $i$ where $y_i \geq x_i$ so that $x'_i = x_i - \alpha$, $\alpha > 0$, then $U(m'_U) > U(m_U)$.

[8] For example, there is a class of mobility measures based on how inequality of multi-period income (including parent and child income aggregated) is related to inequality of income in a given period (Shorrocks 1978; Maasoumi and Zandvakili 1986; Yitzhaki and Wodon 2004; Fields 2010). Since each parent-child pair's contribution to these measures is a function of the single and multi-period inequalities, they are not decomposable or subgroup consistent. Under our approach, by Decomposability and Normalization, adding an immobile parent-child pair decreases or does not affect mobility. However, under an inequality-based approach, adding an immobile pair could decrease mobility, increase mobility, or leave mobility unchanged.

These dominance axioms are straightforward. If a child gains more utility or value relative to their parents, then upward mobility should increase. A larger difference between parent and child utility can be created by either increasing the child's achievement or decreasing the parent's achievement. With these axioms, we are explicitly defining our class of mobility measures to value increases in achievements regardless of the parent income level of the child that achieves them.

**Axiom 7: Upward Switch Independence.** If $m_U'$ is obtained from $m_U$ by an upward mobility switch of incomes for children $i$ and $j$ where $y_i$ and $y_j$ are both greater than or equal to $x_i$ and $x_j$, then $U(m_U) = U(m_U')$.

This axiom is based on an assumption of time (or generational) separability in utility. Our research question is to measure how much better off children are than their parents. Under this axiom, we assume that for a given achievement level $y_i$, each child receives the same utility and that this utility is independent of their parent achievement level. As a result, to measure how much better off each child is than their parent, we need only subtract parent utility from child utility. For an upwardly mobile subgroup of two individuals $i$ and $j$ the average utility gain from upward mobility is $U\left(m_U^{ij}\right) = \frac{1}{2}\left[V\left(y_{U,i}\right) + V\left(y_{U,j}\right) - V(x_i) - V\left(x_j\right)\right]$. This gain is not affected by a switch of incomes between children $i$ and $j$ as long as both would remain upwardly mobile or immobile after the switch.

**Theorem 2.** An upward mobility measure that satisfies Symmetry, Replication Invariance, Normalization, Decomposability, Simple Child Increment, Simple Parent Decrement, and Upward Switch Independence is consistent with first order upward mobility dominance.

**Proof.** Given mobility pairs $m_U^A$ and $m_U^B$, where $n_A = n(m^A)$, $n_B = n(m^B)$, and $n_A = n_B$, then $B \succ_{M1}^U A$ if $m_U^A$ is obtained from $m_U^B$ by a series of steps which satisfy these axioms. This can be shown by the following steps:

1. For each society, add $n$ immobile parent-child pairs whose incomes are equal to the parent incomes in the other society. For $A$, we add the mobility pair $(x^B, x^B)$ to create $A'$ where $m_U^{A\prime} = (\{x^A, x^B\}, \{y_U^A, x^B\})$. Do the same for society $B$ with the parent incomes from society $A$ to create $B'$ where $m_U^{B\prime} = (\{x^B, x^A\}, \{y_U^B, x^A\})$. This creates societies $A'$ and $B'$ that have the same parent income distribution $x' = (x^A, x^B)$ (in $B$ after permuting

the parent-child pairs by Symmetry). By Decomposability and Normalization, if $B \succ_{M1}^{U} A$, then $B' \succ_{M1}^{U} A'$.

2. Permute parent and child incomes in each society so that the parent incomes are ordered from lowest to highest. By Symmetry, if $B \succ_{M1}^{U} A$, then $B' \succ_{M1}^{U} A'$.
3. Through a series of upward mobility switches of child incomes, order child incomes from lowest to highest in each society. Note that by definition for each $i$, $x_i \leq y_{U,i}$ and that we have already arranged parent incomes so that if $i < j$, then $x_i \leq x_j$. For each child $i$ and $j$ where $i < j$, there are three possible cases.

   **Case 1:** $\boldsymbol{x_i \leq y_{U,i} \leq x_j \leq y_{U,j}}$ No switch necessary as $y_i \leq y_j$ already.

   **Case 2:** $\boldsymbol{x_i \leq x_j \leq y_{U,i} \leq y_{U,j}}$ No switch necessary as $y_i \leq y_j$ already.

   **Case 3:** $\boldsymbol{x_i \leq x_j \leq y_{U,j} \leq y_{U,i}}$ Upward switch where $U(m_U) = U(m_U')$ by Upward Switch Independence.

   As a result, in the only possible case where $y_{U,i} > y_{U,j}$, we can use the Upward Switch Independence axiom to order the two child incomes. Through a series of these switches, we can order $y_U$ from smallest to largest child income. By Upward Switch Independence, with each switch $U(m_U) = U(m_U')$ and comparing the pre- and post-switch societies, if $B \succ_{M1}^{U} A$, then $B^{'} \succ_{M1}^{U} A'$.
4. For each $y_{U,i}^{A} < y_{U,i}^{B}$, increment $y_{U,i}^{A}$ until they are equal. With each increment, $U(m_U^{A\prime}) > U(m_U^{A})$ and $A' \succ_{M1}^{U} A$ so that after all necessary increments to equalize $y_U^A$ and $y_U^B$, we have shown that $B \succ_{M1}^{U} A$. By adding immobile pairs so that $x^{A\prime} = x^{B\prime} = (x^A, x^B)$ and ordering each parent and child income from lowest to highest, we have ensured that if there exists an individual $i$ in this step where $y_{U,i}^{B} < y_{U,i}^{A}$, then there is no dominance by Theorem 1.

This completes the proof. [9]

It is trivial to generalize Theorem 2 to two societies of arbitrary population sizes using the Replication Invariance axiom as both societies can be replicated to equalize their populations.

[9] Theorem 2 could also have been proven by supplementing each society with the incomes of the children from the opposite society (instead of the parents) as immobile parent-child pairs in Step 1 so that $m_U^{A\prime} = (\{x^A, y_U^B\}, \{y_U^A, y_U^B\})$ and $m_U^{B\prime} = (\{x^B, y_U^A\}, \{y_U^B, y_U^A\})$. In this case, because the two child distributions are now equal, Step 4 would use the Simple Parent Decrement instead of Simple Child Increment axiom to decrement each parent income in $A$ until the two societies were equal.

## 3.2 Flip Vector Dominance

Another way of understanding this result is that for two societies of the same population size, you can compare the "flip" vectors $z_U^A = (y_U^A, x_B)$ and $z_U^B = (y_U^B, x_A)$ for mobility dominance. If flip vector $z_U^B$ vector dominates $z_U^A$, then $B \succ_{M1}^U A$.[10]

This may seem counterintuitive. However, by setting $x^{A\prime} = x^{B\prime} = (x^A, x^B)$ in Step 1, we have set the initial income distributions to be the same in the two transformed societies without affecting the mobility ordering. For two societies with the same parent income distributions, the utility gained by upward mobility can be assessed by comparing only the censored child income distributions because if $F_X^A(X) = F_X^B(X)$ at all possible $X$, (2.1.4) reduces to

$$U(m_U^B) - U(m_U^A) = \int_0^b V'(c)[F_{UY}^A(c) - F_{UY}^B(c)]dc. \tag{3.2.1}$$

Comparing mobility in (3.2.1) is the same as comparing the two child income distributions for first order stochastic dominance. The child income distributions after Step 1 are the flip vectors. Therefore, in evaluating mobility in the discrete case with equal population sizes, we only have to evaluate the flip vectors for vector dominance, a result we call flip vector dominance.

The intuition behind flip vector dominance is that upward mobility depends on both how rich the children are and how poor the parents are. By creating the flip vectors, we are simultaneously comparing two income distributions where, all else equal, higher incomes are always preferable in the mobility comparison. For the child incomes, higher incomes imply more mobility. For parent incomes, higher incomes for the other society's parents imply a mobility advantage for this society.

As an example of flip vector dominance, we return to societies $A$ and $B$ in the two panels of Figure 1. In Panel A, with $x^A = x^B = (1,5)$, $y^A = (2,5)$, and $y^B = (3,5)$, the ordered flip vector for $A$ is $z^A = (1,2,5,5)$ and for $B$ is $z^B = (1,3,5,5)$. Since $z^B$ vector dominates $z^A$, $B$ first order upward mobility dominates $A$. In the Panel B example, with $y^B = (1,6)$ and the rest

[10] $z_U^B$ vector dominates $z_U^A$ if after ordering both from lowest to highest incomes, $z_{U,i}^B \geq z_{U,i}^A \ \forall i = 1, \ldots, 2n$ and $\exists i$ where $z_{U,i}^B > z_{U,i}^A$. Using the alternative proof in Footnote 9, $z_U^A$ becomes the parent income distribution for society $B$ and $z_U^B$ becomes the parent income distribution for society $A$. If $z_U^B$ vector dominates $z_U^A$, the intuition is that society A had higher income parents than society $B$ for the same child incomes, so $B \succ_{M1}^U A$.

unchanged, the ordered flip vector for $A$ is the same and for $B$ is now $z^B = (1,1,5,6)$. There is no vector dominance in this case, and therefore there is no first order mobility dominance.[11]

The flip vector dominance result can also be generalized to societies of different population sizes. In this case, the difference in average utility gained due to upward mobility is

$$U(m_U^B) - U(m_U^A) = \frac{1}{n_B}\sum_{i=1}^{n_B}\left[V\left(y_{U,i}^B\right) - V(x_i^B)\right] - \frac{1}{n_A}\sum_{i=1}^{n_A}\left[V\left(y_{U,i}^A\right) - V\left(x_i^A\right)\right]. \qquad (3.2.2)$$

This can be rewritten into a general weighted flip vector dominance result

$$U(m_U^B) - U(m_U^A) = \frac{1}{n_B}\sum_{i=1}^{n_B} V\left(y_{U,i}^B\right) + \frac{1}{n_A}\sum_{i=1}^{n_A} V\left(x_i^A\right) - \left\{\frac{1}{n_A}\sum_{i=1}^{n_A} V\left(y_{U,i}^A\right) + \frac{1}{n_B}\sum_{i=1}^{n_B} V\left(x_i^B\right)\right\} \qquad (3.2.3)$$

so that comparing weighted flip vectors for first order stochastic dominance is equivalent to comparing mobility pairs for first order mobility dominance.

## 3.3 Downward Mobility

For downward mobility, we censor all upward movements by replacing each $y_i$ with $y_{D,i} = \min(x_i, y_i)$, $y_D = \left(y_{D,1}, \ldots, y_{D,n}\right)$, and $m_D = (x, y_D)$. We replace dominance axioms 5, 6, and 7 with their downward mobility counterparts. All of the others are maintained with the only change being replacing $m_U$ and $y_U$ with their downward mobility counterparts $m_D$ and $y_D$.

**Axiom 5a: Simple Parent Increment.** If $m'_D$ is obtained from $m_D$ be a simple increment to the income of the parent of an immobile or downwardly mobile child $i$ so that $x_i = x_i + \alpha$, $\alpha > 0$, then $U(m'_D) < U(m_D)$.

**Axiom 6a: Simple Child Decrement.** If $m'_D$ is obtained from $m_D$ by a simple decrement of the income of an immobile or downwardly mobile child $i$ so that $y'_{Di} = y_{Di} - \alpha$, $\alpha > 0$, then $U(m'_U) < U(m_U)$.

Again, these dominance axioms are straightforward. If a child loses more utility or value relative to their parents, then downward mobility should increase. A larger difference between

[11] Since in both examples, the two societies have the same parent distributions, using the flip vectors is actually unnecessary. Comparing the upward mobility censored child income vectors for dominance is sufficient. However, for consistency and clarity we use the flip vectors in the exposition.

parent and child utility can be created by either decreasing the child's achievement or increasing the parents' achievement.

**Axiom 7a: Downward Switch Independence.** If $m_D'$ is obtained from $m_D$ by a downward mobility switch of incomes for children $i$ and $j$ where $y_i$ and $y_j$ are both less than or equal to $x_i$and $x_j$, then $U(m_D) = U(m_D')$.

Theorem 2 holds for downward mobility dominance using Axioms 5a, 6a, and 7a in place of 5, 6, and 7. With $z_D^A = (y_D^A, x_B)$ and $z_D^B = (y_D^B, x_A)$, the flip vector dominance result is reversed for downward mobility. If $z_D^A$ vector dominates $z_D^B$, then $B \succ_{M1}^D A$.

# 4 Ordinality, Cardinality, and Second Order Mobility Dominance

## 4.1 Ordinal Variables

Until now, we have treated the achievement variable as if it were cardinally meaningful, as income is generally assumed to be. However, some variables are not cardinally meaningful and any arbitrary rescaling could be equally valid as the original variable. For example, education and socioeconomic status are not customarily considered to have cardinal meaning. In this context, our entire approach is valid in that our mobility ordering holds for any possible rescaling of the ordinal variable. We can make mobility comparisons for ordinal variables without having to assume and defend an arbitrary cardinal representation. For example, in studying intergenerational mobility of education, if a particular cardinal representation is assumed, the results may depend on that representation.

However, the possibility of such comparisons must be tempered by the observation that if the parent distributions are too different, then it might be difficult to make comparisons with this mobility ordering. For example, if we were interested in studying income or educational mobility between the US and Mexico, the parent income and education levels in Mexico may be far to the left of the US distributions. The mobility curve or flip vector dominance comparisons would be very incomplete in this example. In this and similar cases, it may be necessary to rescale cardinal variables by a relevant and reasonable standard to facilitate comparisons, such as comparing income relative to the mean or median rather than absolute income. With ordinal variables, such a rescaling may not be possible.

## 4.2 Cardinal Variables and Second Order Mobility Dominance

If the achievement variable is cardinal, there may be additional analysis that can be brought to bear to generate a more complete mobility ordering. In particular, the set of allowable $V$-functions might reasonably be restricted to a subset of all monotonic functions. A particularly interesting subset is that of concave functions, which give more value to achievements at the lower end of the distribution. For example, suppose we agree with Fields and Ok (1999) that a monotonic and concave transformation of the achievement such as $\log(x)$ makes sense. However, we are not sure log is the only reasonable function in this class. To examine the robustness of the mobility ranking, we derive a second order dominance that compares mobility under all possible monotonic concave transformations of the cardinal achievements. This ordering has a corresponding second order mobility curve that can be used to visually represent and make mobility dominance comparisons. It also has an equivalent second order flip vector dominance result.

We restrict our attention to the set of monotonically increasing concave functions $\mathcal{V}_2$ where if $V \in \mathcal{V}_2$, then $V' > 0$ and $V'' < 0$. We can integrate (2.1.6) by parts to get

$$
\begin{aligned}
&U(m_U^B) - U(m_U^A) \\
&\qquad = V'(b)\left(\int_0^c M_U^B(\hat{c})d\hat{c} - \int_0^c M_U^A(\hat{c})d\hat{c}\right) \\
&\qquad - \int_0^b \left(\int_0^c M_U^B(\hat{c})d\hat{c} - \int_0^c M_U^A(\hat{c})d\hat{c}\right) V''(c)dc \\
&\qquad = V'(b)\left(\widehat{M}_U^B(b) - \widehat{M}_U^A(b)\right) - \int_0^b \left(\widehat{M}_U^B(c) - \widehat{M}_U^A(c)\right) dc\,.
\end{aligned}
\tag{4.2.1}
$$

We have defined $\widehat{M}_U(c)$ as the integral of the mobility curve up to cutoff $c$. We now define our second order upward mobility dominance ($\succ_{M2}^U$).

**Theorem 3.** The following statements are equivalent:

1. $B \succ_{M2}^U A$
2. $U(m_U^B) > U(m_U^A) \forall V \in \mathcal{V}_2$, where $V' > 0$ and $V'' < 0$
3. $\widehat{M}_U^B(c) \geq \widehat{M}_U^A(c) \forall c \in [0, b]$, and for some $c$, $\widehat{M}_U^B(c) > \widehat{M}_U^A$

**Proof:** The proof of Theorem 3 is virtually identical to the second order stochastic dominance proof in Bawa (1975). In this case, let $c_0$ be an arbitrary cutoff where $\widehat{M}_U^B(c) < \widehat{M}_U^A$.

We define a utility function $V_2$ where $V_2''(y) = -k + \phi'(y)$, so that by construction, $V_2'' < 0$ with $\phi(c)$ defined in (2.1.7). We also define $V_2'(y) = k_1 - ky + \phi(y)$ so that with a sufficiently large $k_1$, $V_2'(y) > 0\ \forall y$ and $V_2 \in \mathcal{V}_2$. We also normalize $V_2(0) = 0$. With this utility function, (4.2.1) becomes:

$$\begin{aligned} U(m_U^B) - U(m_U^A) & \\ &= \left(\widehat{M}_U^B(b) - \widehat{M}_U^A(b)\right)(k_1 - kb) \\ &+ k\int_0^b \left(\widehat{M}_U^B(c) - \widehat{M}_U^A(c)\right)dc \\ &+ \gamma\int_{c_0}^{c_0+\delta} \left(\widehat{M}_U^B(c) - \widehat{M}_U^A(c)\right)dc \end{aligned} \tag{4.2.2}$$

Given that at $c_0$ $\widehat{M}_U^B < \widehat{M}_U^A$, by choosing a sufficiently large $\gamma$, one can make $U(m_U^B) < U(m_U^A)$. This completes the proof of Theorem 3.

Because $\widehat{M}_U(c)$ is the integral of the mobility curve up to a given income level $c$, first order upward mobility dominance implies second order upward mobility dominance.

The same results hold for $\widehat{M}_D(c)$ and second order downward mobility dominance ($\succ_{M2}^D$). $U(m_D^B) < U(m_D^A) \forall V \in \mathcal{V}_2$ if and only if $\widehat{M}_D^B(c) \geq \widehat{M}_D^A(c) \forall c \in [0, b]$, and for some $c$, $\widehat{M}_D^B(c) > \widehat{M}_D^A$.

This suggests the second order mobility curve, which in the discrete case is

$$\widehat{M}_U(c) = \frac{1}{n}\int_0^c \sum_{i=1}^n I(x_i \leq c)I(y_i > c)dc \tag{4.2.3}$$

for upward mobility[12] and for downward mobility is

$$\widehat{M}_D(c) = \frac{1}{n}\int_0^c \sum_{i=1}^n I(x_i > c)I(y_i \geq c)dc. \tag{4.2.4}$$

By plotting $\widehat{M}_U(c)$ or $\widehat{M}_D(c)$ for all possible cutoffs, we can easily compare mobility in two societies for second order upward and downward mobility dominance.

[12] The upward mobility curve with discrete incomes could also be written without integrals as

$$\widehat{M}_U(c) = \sum_i^n \begin{matrix} y_i - x_i & if\ y_i > x_i \text{ and } c \geq y_i \\ c - x_i & if\ y_i > x_i \text{ and } c < y_i \\ 0 & \text{otherwise} \end{matrix}$$

with a corresponding representation for downward mobility.

## 4.3 Second Order Mobility Dominance Axioms and Second Order Flip Vector Dominance

Assuming concave utility also implies additional dominance axioms for upward and downward mobility comparisons.

**Axiom 8: Upward Child Equalizing Transfer.** If $m_U'$ is obtained from $m_U$ by an equalizing transfer of $\delta$ between children $i$ and $j$ after which both children are upwardly mobile or immobile where $y_i < y_j$, $y_i' = y_i + \delta$, $y_j' = y_j - \alpha$, $x_j \leq y_j'$ and $y_i' \leq y_j'$, then $U(m_U) < U(m_U')$.

**Axiom 9: Upward Parent Disequalizing Transfer.** If $m_U'$ is obtained from $m_U$ by a disequalizing transfer of $\delta$ between parents $i$ and $j$ after which both children are upwardly mobile or immobile where $x_i \leq x_j$, $x_i' = x_i - \delta$, $x_j' = x_j + \delta$, and $x_j' \leq y_j$, then $U(m_U) < U(m_U')$.

In both Axioms 8 and 9, the utility gain to upward mobility is greater if for a given average income gain, the gains occur at lower income levels. This can be achieved either by equalizing the incomes of upwardly mobile children (Axiom 8), or disequalizing the parent incomes of upwardly mobile children (Axiom 9). In each case, more of the mobility occurred at lower child income levels, and therefore, greater utility was gained due to that mobility given a concave $V$-function.

An example with a specific $V$-function may be helpful to understand these axioms. Suppose we have a mobility pair $m$ with parent incomes $x = (2,2)$ and child incomes $y = (3,5)$ and we evaluate utility with the log function so that $U(m_U) = \sum_{i=1}^{n}\left(\log\left(y_{U,i}\right) - \log(x_i)\right)$. Let $m_U'$ be the society after an Upward Child Equalizing Transfer of 1, so that $y' = (4,4)$. The utility gains in $m_U$ and $m_U'$ are $U(m_U) = \frac{1}{2}\left(\log\frac{3}{2} + \log\frac{5}{2}\right) = 0.29$ and $U(m_U') = \frac{1}{2}\left(\log\frac{4}{2} + \log\frac{4}{2}\right) = 0.30$. Therefore, $U(m_U') > U(m_U)$ as in Axiom 8. Under an Upward Parent Disequalizing Transfer of 1, we create $m_U''$ where $x'' = (1,3)$. In this case, $U(m_U'') = \frac{1}{2}\left(\log\frac{3}{1} + \log\frac{5}{3}\right) = 0.35$, and $U(m_U'') > U(m_U)$ as in Axiom 9.

**Theorem 4.** An upward mobility measure that satisfies Symmetry, Replication Invariance, Normalization, Decomposability, Simple Child Increment, Simple Parent Decrement, Upward Switch Independence, Upward Child Equalizing Transfer, and Upward Parent Disequalizing Transfer is consistent with second order upward mobility dominance.

**Proof.** Given mobility pairs $m_U^A$ and $m_U^B$, where $n_A = n(m^A)$, $n_B = n(m^B)$, and $n_A = n_B$, then $B \succ_{M1}^U A$ if $m_U^A$ is obtained from $m_U^B$ by a series of steps which satisfy these axioms. This can be shown by adding one additional step to the proof of Theorem 2 between steps 3 and 4.

3b. For any pair of children where $\delta > 0$, $y_{U,j}^A = y_{U,j}^B + \delta$, $y_{U,i}^A + \delta \leq y_{U,i}^B$, and $x_j^A \leq y_{U,j}^A - \delta$, transfer $\delta$ from child $j$ to child $i$ in society $A$ to set $y_{U,j}^{A\prime} = y_{U,j}^B$. By Upward Child Equalizing Transfer, $A' \succ_{M2}^U A$.[13]

This completes the proof.

For downward mobility, the second order dominance axioms are flipped.

**Axiom 8a: Downward Child Disequalizing Transfer.** If $m_D'$ is obtained from $m_D$ by a disequalizing transfer of $\delta$ between children $i$ and $j$ after which both children are downwardly mobile or immobile where $y_i \leq y_j$, $y_i' = y_i - \delta$, $y_j' = y_j + \alpha$, and $y_j' \leq x_j$, then $U(m_D) > U(m_D')$.

**Axiom 9a: Downward Parent Equalizing Transfer.** If $m_U'$ is obtained from $m_U$ by a Pigou-Dalton transfer of $\delta$ between parents $i$ and $j$ after which both children are downwardly mobile or immobile where $x_i < x_j$, $x_i + \delta \leq x_j - \delta$, $x_i' = x_i + \delta$, $x_j' = x_j - \delta$, and $x_j' \leq y_j$, then $U(m_U) > U(m_U')$.

Again, the change in each axiom causes more of the downward mobility to occur at lower income levels where the utility loss is greater given a concave utility function. The proof to Theorem 4 can be amended to hold for second order downward mobility dominance by amending step 3b to 3c.

3c. For any pair of children where $\delta > 0$, $y_{D,j}^A = y_{D,j}^B + \delta$, $y_{D,i}^A - \delta \leq y_{D,i}^B$, and $x_i^A \geq y_{D,i}^A + \delta$, transfer $\delta$ from child $j$ to child $i$ in society $A$ to set $y_{D,j}^{A\prime} = y_{D,j}^B$. By Downward Child Disequalizing Transfer, $A' \succ_{M2}^D A$.[14]

In addition, comparing the flip vectors for second order stochastic dominance is equivalent to testing for second order mobility dominance. If we integrate (3.2.1) by parts we get that given the same parent distribution, the change in utility due to upward mobility is

[13] As before, if Step 1 of the proof had equalized the child income distributions, this step would utilize the parent axiom, in this case Upward Parent Disequalizing Transfer.
[14] As before, if Step 1 of the proof had equalized the child income distributions, this step would utilize the parent axiom, in this case Downward Parent Equalizing Transfer.

$$
\begin{aligned}
&U(m_U^B) - U(m_U^A) \\
&\qquad\qquad = V'(b)\int_0^b \left(F_{UY}^A(\hat{c}) - F_{UY}^B(\hat{c})\right)d\hat{c} \\
&\qquad\qquad - \int_0^b \left(\int_0^b \left(F_{UY}^A(\hat{c}) - F_{UY}^B(\hat{c})\right)d\hat{c}\right)V''(c)dc.
\end{aligned} \tag{4.3.1}
$$

This is the condition for second order stochastic dominance in Bawa (1975). Therefore if the upward mobility flip vector for $B$ $(y_U^B, x_A)$ second order stochastically dominates the flip vector for $A$ $(y_U^A, x_B)$, then $B$ second order upward mobility dominates $A$. For downward mobility, as before, the comparison is reversed. If the flip vector for $A$ $(y_D^A, x_B)$ second order stochastically dominates the flip vector for $B$ $(y_D^B, x_A)$, then $B$ second order downward mobility dominates $A$.

In Figure 3, we show how second order mobility dominance allows us to make unambiguous comparisons in the cases from Panel B in Figure 1 and Figure 2 where there is no first order dominance. In this figure, we compare $A$ and $B$ where $x^A = x^B = (1{,}5)$, $y^A = (0{,}6)$, and $y^B = (2{,}4)$. Each society has one child who experienced 1 unit of upward mobility and one child who experienced 1 unit of downward mobility. Figure 3 shows that $B$ second order upward mobility dominates $A$ because the poor child experienced the upward mobility in $B$ $(1 \rightarrow 2)$ compared to the rich child in $A$ $(5 \rightarrow 6)$. The figure also shows that $A$ second order downward mobility dominates $B$ because the poor child experienced the downward mobility in $A$ $(1 \rightarrow 0)$ compared to the rich child in $B$ $(5 \rightarrow 4)$.

# 5 Extensions and Other Considerations

## 5.1 Aggregating Upward and Downward Mobility

Some authors combine upward and downward mobility to obtain an aggregate measure of mobility. We have focused on measuring each separately. However, it is useful and illustrative to examine different ways of aggregating upward and downward mobility from mobility curves to explore how they relate to other measures.

In the distance-based mobility literature, upward and downward mobility are added together to measure total change without focusing on the direction of the mobility. With mobility curves, the sum of the area under the upward and downward mobility curves is equal to the Fields and Ok (1996) distance-based measure.[15]

On the other hand, it would be reasonable to argue that because downward mobility is a net loss, it should be subtracted from upward mobility. By looking at the difference between the upward and downward mobility curves and second order mobility curves, we can see the relationship between mobility measurement, poverty measurement, and stochastic dominance between the parent and child income distributions.

The mobility curve approach is closely related to the FGT poverty measures (J. Foster, Greer, and Thorbecke 1984). These poverty measures define poverty (given the notation used in this paper) as follows:

$$P_\alpha(x,c) = \frac{1}{n}\sum_{i=1}^{n} I(x_i < c)\left(\frac{c - x_i}{c}\right)^\alpha \tag{5.1.1}$$

The FGT measure when $\alpha = 0$ is the headcount index, and when $\alpha = 1$, it is the average (per capita) poverty gap. Changes in poverty between two periods can therefore be defined as $\Delta_{P_\alpha}$:

$$\Delta_{P_\alpha}(m,c) = \frac{1}{nc^\alpha}\left(\sum_{i=1}^{n} I(x_i < c)(c - x_i)^\alpha - \sum_{i=1}^{n} I(x_i < c)(c - y_i)^\alpha\right) \tag{5.1.2}$$

In Appendix 1, we show that the change in the headcount ratio ($\Delta_{p_0}$) is equal to

$$\Delta_{P_0}(m,c) = M_D(c) - M_U(c) \tag{5.1.3}$$

We also show that for the average poverty gap

$$\Delta_{P_1}(m,c) = \frac{\widehat{M}_D(c) - \widehat{M}_U(c)}{c} \tag{5.1.4}$$

The mobility curves and second order mobility curves are a decomposition of changes in headcount poverty and the poverty gap.

If (5.1.3) or (5.1.4) are $\geq 0$ or $\leq 0$ at all cutoffs (and $>$ or $<$ at some cutoffs), then there is first order (5.1.3) or second order (5.1.4) stochastic dominance between the parent and child income distributions (or equivalently, first order and second order poverty dominance as in Foster and Shorrocks (1988)). For example, if $\Delta_{P_0}(m,c) \leq 0\ \forall c$, and for some $c$, $\Delta_{P_0}(m,c) <$

[15] The Fields and Ok (1996) measure can also be calculated easily from the second order mobility curve, which is the area under the mobility curve up to each cutoff $c$. Therefore at the maximum income $b$, the second order mobility curve is equal to the area under the entire mobility curve in each direction. So the sum of second order upward and downward mobility at $b$, $\widehat{M}_U(b) + \widehat{M}_D(b)$, is equal to Fields and Ok's distance-based mobility.

0, then the child achievement distribution first order stochastically and poverty dominates the parent achievement distribution.

## 5.2 Selection of Mobility Variable

Many evaluations of mobility are conducted using transformations of the basic domain variables that fundamentally alter the levels of child achievements that are seen as equivalent to levels of parent achievements. For example, the transformation of income into ranks entails application of the inverse distribution function of the children's distribution to each child income and the inverse distribution function of the parents' distribution to each parent income, where the two distributions can be very different or they can change differently over time. Depending on other incomes in each distribution, a child with more income than their parents can be upwardly mobile, downwardly mobile, or immobile. This type of transformation may be appropriate for certain exercises, particularly if the resulting notion of advantage is seen as being relevant despite its potential arbitrariness, lack of clarity, or lack of grounding in how individuals themselves might value the underlying achievement in the context being evaluated. Moreover, one must be careful in drawing conclusions when several distinct transformations are being utilized at once to obtain the derived variables. The question is whether the mobility is fundamental and inherent to the changing achievement or does it arise primarily due to the variations in the transformations over time. Assuming the transformations are justified, the approach developed in this paper applies and can provide information about upward and downward mobility levels in the resulting transformed variables.

# 6 Intergenerational Mobility in the United States

# 7 Conclusion

## References


Bawa, Vijay S. 1975. "Optimal Rules for Ordering Uncertain Prospects." *Journal of Financial Economics* 2: 95–121.

Fields, Gary S. 2010. "Does Income Mobility Equalize Longer-Term Incomes? New Measures of an Old Concept." *Journal of Economic Inequality* 8 (4) (May 29): 409–427. doi:10.1007/s10888-009-9115-6.

Fields, Gary S., and Efe Ok. 1996. "The Meaning and Measurement of Income Mobility." *Journal of Economic Theory* 71 (2): 349–377.

———. 1999. "Measuring Movement of Incomes." *Economica* 66 (264): 455–471.

Foster, James E., and Anthony Shorrocks. 1988. "Poverty Orderings and Welfare Dominance." *Social Choice and Welfare* 5 (2): 179–198.

Foster, James, J Greer, and E Thorbecke. 1984. "A Class of Decomposable Poverty Measures." *Econometrica* 52: 761–65.

Maasoumi, Esfandiar, and Sourushe Zandvakili. 1986. "A Class of Generalized Measures of Mobility with Applications." *Economics Letters* 22 (1): 97–102.

Mitra, Tapan, and Efe Ok. 1998. "The Measurement of Income Mobility: A Partial Ordering Approach." *Economic Theory* 12 (1): 77–102.

Shorrocks, Anthony. 1978. "Income Inequality and Income Mobility." *Journal of Economic Theory* 19 (2): 376–393.

Yitzhaki, Shlomo, and Quentin Wodon. 2004. "Mobility, Inequality, and Horizontal Equity." In *Studies on Economic Well-Being: Essays in the Honor of John P. Formby (Research on Economic Inequality, Volume 12)*, edited by John A. Bishop and Yoram Amiel, 12:177–199. Emerald Group Publishing Limited.

## Figures

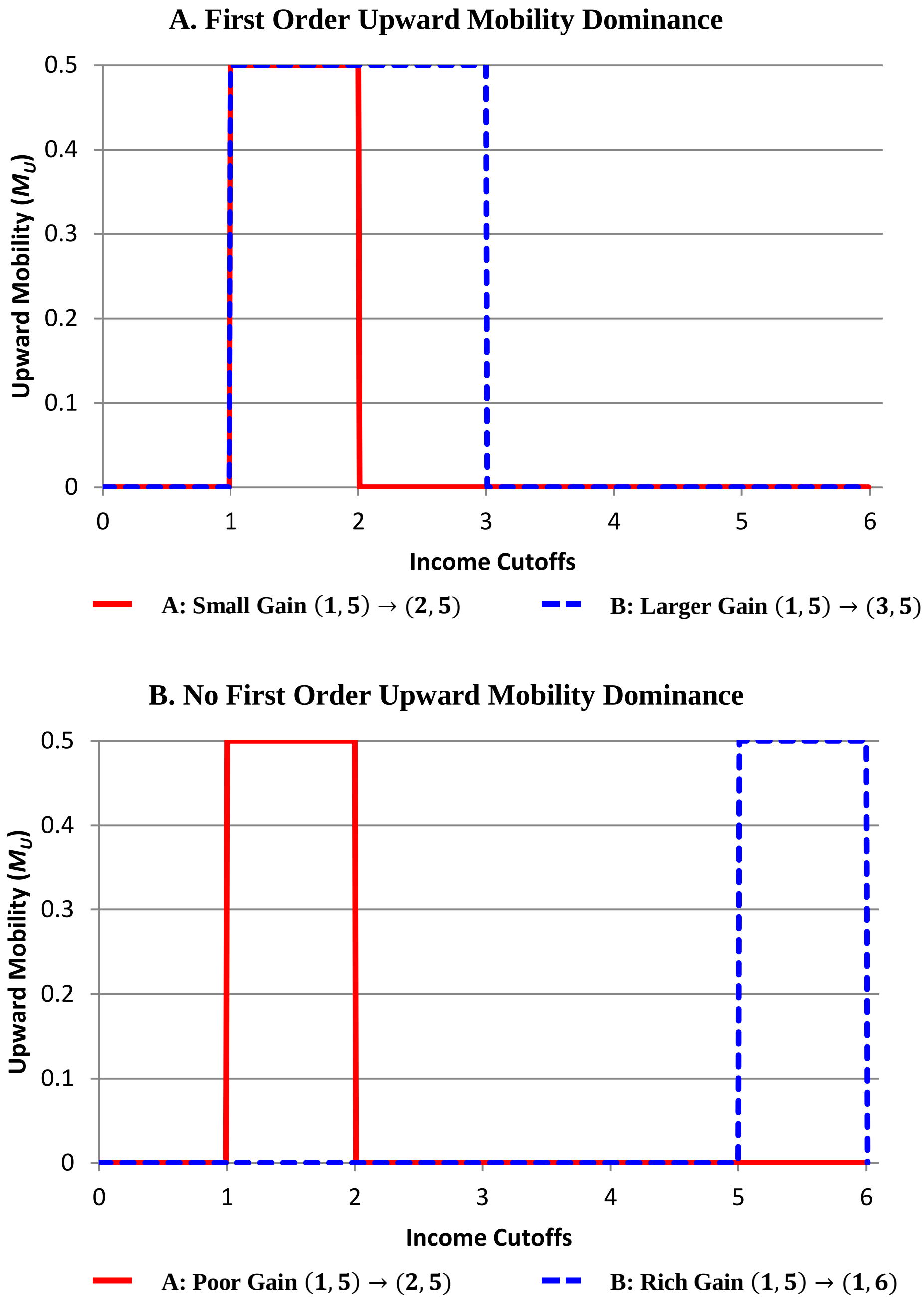


**Figure 1: Upward Mobility Curves and First Order Mobility Dominance**

This figure shows the upward mobility curves for two examples. In Panel A, the curve compares a case where upward mobility unambiguously increased utility by more in society $B$ than $A$, and there is first order upward mobility dominance. In both societies, the child from the poor parent household ($x_1 = 1$) is upwardly mobile, but in $B$ the child experienced greater upward mobility ($y_1^A = 2$ and $y_1^B = 3$). Panel B shows a case where, in both societies, one child is upwardly mobile by 1 unit of income. However as the curves show, there is no first order mobility dominance because it is the child of the poor parents in $A$ and the child of the rich parents in $B$ that are upwardly mobile.

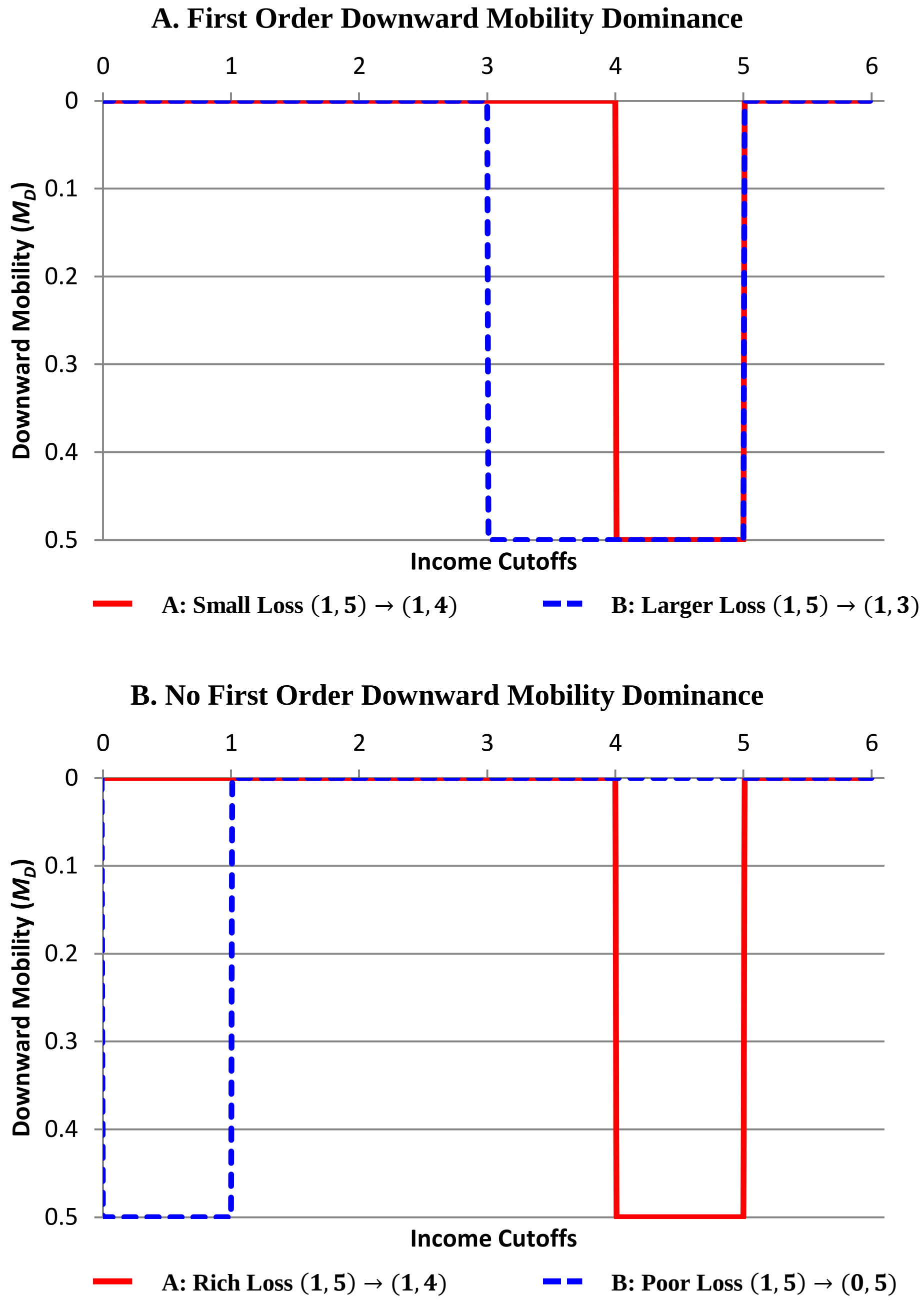


**Figure 2: Downward Mobility Curves and First Order Mobility Dominance**

This figure shows the downward mobility curves for two examples. In Panel A, the curve compares a case where downward mobility unambiguously decreased utility by more in society $B$ than $A$, and there is first order downward mobility dominance. In both societies, the child from the rich parent household ($x_2 = 5$) is downwardly mobile, but in $B$ the child experienced greater downward mobility ($y_2^A = 4$ and $y_2^B = 3$). Panel B shows a case where, in both societies, one child is downwardly mobile by 1 unit of income. However as the curves show, there is no first order mobility dominance because it is the child of the rich parents in $A$ and the child of the poor parents in $B$ that are downwardly mobile.


**Figure 3: Second Order Mobility Curves**

In this figure, we show how second order mobility dominance allows us to make unambiguous comparisons in the cases from Panel B in Figure 1 and Figure 2 where there is no first order dominance. We compare $A$ and $B$ where $x^A = x^B = (1,5)$, $y^A = (0,6)$, and $y^B = (2,4)$. Each society has one child who experienced 1 unit of upward mobility and one child who experienced 1 unit of downward mobility. This figure shows that $B$ second order upward mobility dominates $A$ because the poor child experienced the upward mobility in $B$ $(1 \to 2)$ compared to the rich child in $A$ $(5 \to 6)$. It also shows that $A$ second order downward mobility dominates $B$ because the poor child experienced the downward mobility in $A$ $(1 \to 0)$ compared to the rich child in $B$ $(5 \to 4)$.

# Appendix 1: Mobility Curves and Poverty

## A1.1. Headcount Poverty

For the headcount poverty index, equation (5.1.2) reduces to:

$$\Delta_{P_0}(m,c) = \frac{1}{n}\left(\sum_{i=1}^{n} I(x_i < c) - \sum_{i=1}^{n} I(y_i < c)\right) \tag{A1.1}$$

Separating the poor parent-child pairs into those who experienced mobility and those who did not yields:

$$\begin{aligned}\Delta_{P_0}(m,c) = \frac{1}{n}\Bigg(\sum_{i=1}^{n}\big[&\big(I(x_i < c)I(y_i < c) + I(x_i \geq c)I(y_i < c)\big)\\ &- \big(I(x_i < c)I(y_i < c) + I(x_i < c)I(y_i \geq c)\big)\big]\Bigg)\\ &= remain\ poor + mobile\ down - (remain\ poor + mobile\ up)\end{aligned} \tag{A1.2}$$

After simplifying, we get the change in the headcount index to be a function of upward and downward mobility across the poverty line $c$:

$$\begin{aligned}\Delta_{P_0}(m,c) &= \frac{1}{n}\left(\sum_{i=1}^{n} I(x_i > c)I(y_i \leq c) - I(x_i \leq c)I(y_i > c)\right)\\ &= M_D(c) - M_U(c)\end{aligned} \tag{A1.3}$$

## A1.2. The Poverty Gap

For the poverty gap, it is simpler to look at second order mobility for a parent-child pair $i$ given the poverty line $c$. The possible outcomes for any parent-child pair are summarized in Table A1.**Error! Reference source not found.**. From the last column in the table, in each possible situation either $\Delta_{P_1,i} = -\widehat{M}_{U,i}$ or $\Delta_{P_1,i} = \widehat{M}_{D,i}$. Therefore, over all parent-child pairs $i$:

$$\Delta_{P_1}(m,c) = \frac{\left(\widehat{M}_D(c) - \widehat{M}_U(c)\right)}{c} \tag{A1.4}$$

## A1.3. The Squared Gap

A decomposition of the simple forms above is not possible for the squared gap, because the decomposition for children who experienced mobility and where both the parents and child were poor would contain a residual term. This is due to the fact that the difference between $(c-y_i)^2$ and $(c-x_i)^2$ is equal to $y_i^2+2c(y_i-x_i)-x_i^2$ which is not a function of $(y_i-x_i)^\alpha$, $(c-y_i)^\alpha$, or $(c-x_i)^\alpha$ as the corresponding differences are for the headcount and the poverty gap, and the squared gap contribution would be for parent-child pair $i$ if only one of the parent and child were poor, but not both.

**Table A1.1: Aggregate Mobility and the Poverty Gap Contribution for Parent-Child Pair $i$**

| Description | Income Relative to the Poverty Line | Poverty Gap ($P_{1,i}$) in Periods 1 and 2 | Contribution to Aggregate Mobility ($M_U^j$ and $M_D^j$) | Relationship Between $\Delta_{P_1^j}$ and $M_i^j$ |
|---|---|---|---|---|
| Upward mobility, parent and child poor | $c \geq y_i \geq x_i$ | $P_{1,i}^{Child}=\frac{1}{nc}(c-y_i)$<br>$P_{1,i}^{Parent}=\frac{1}{nc}(c-x_i)$<br>$\Delta_{P_{1,i}}=\frac{1}{nc}(x_i-y_i)$ | $\hat{M}_{U,i}(c)=\frac{1}{n}(y_i-x_i)$<br>$\hat{M}_{D,i}(c)=0$ | $\Delta_{P_{1,i}}=-\frac{\hat{M}_{U,i}(c)}{c}$ |
| Downward mobility, parent and child poor | $c \geq x_i \geq y_i$ | $P_{1,i}^{Child}=\frac{1}{nc}(c-y_i)$<br>$P_{1,i}^{Parent}=\frac{1}{nc}(c-x_i)$<br>$\Delta_{P_{1,i}}=\frac{1}{nc}(y_i-x_i)$ | $\hat{M}_{U,i}(c)=0$<br>$\hat{M}_{D,i}(c)=\frac{1}{n}(x_i-y_i)$ | $\Delta_{P_{1,i}}=\frac{\hat{M}_{D,i}(c)}{c}$ |
| Downward mobility, child poor, parent non-poor | $x_i \geq c \geq y_i$ | $P_{1,i}^{Child}=0$<br>$P_{1,i}^{Parent}=\frac{1}{nc}(c-x_i)$<br>$\Delta_{P_{1,i}}=\frac{1}{nc}(x_i-c)$ | $\hat{M}_{U,i}(c)=\frac{1}{n}(c-x_i)$<br>$\hat{M}_{D,i}(c)=0$ | $\Delta_{P_{1,i}}=-\frac{\hat{M}_{U,i}(c)}{c}$ |
| Upward mobility, parent poor, child non-poor | $x_i \geq c \geq y_i$ | $P_{1,i}^{Child}=\frac{1}{nc}(c-y_i)$<br>$P_{1,i}^{Parent}=0$<br>$\Delta_{P_{1,i}}=\frac{1}{nc}(c-y_i)$ | $\hat{M}_{U,i}(c)=0$<br>$\hat{M}_{D,i}(c)=\frac{1}{n}(c-y_i)$ | $\Delta_{P_{1,i}}=\frac{\hat{M}_{D,i}(c)}{c}$ |
| Any mobility, parent and child non-poor | $c \leq x_i$<br>$c \leq y_i$<br>$y_i \geq x_i$ or $y_i \leq x_i$ | $P_{1,i}^{Child}=0$<br>$P_{1,i}^{Parent}=0$<br>$\Delta_{P_{1,i}}=0$ | $\hat{M}_{U,i}(c)=0$<br>$\hat{M}_{D,i}(c)=0$ | $\Delta_{P_{1,i}}=0$ |