The long-run relationship between market risk and return

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Abstract

Many finance applications require an annual measure of the market premium for equity. Using a long sample combined with a very parsimonious conditional variance function, we find a positive relationship between market risk and expected excess returns. Unlike traditional exponential-smoothing filters, our specification has a well-defined unconditional variance and allows for mean reverting volatility forecasts. Although total volatility is significantly priced, the smooth long-run component in volatility is more important for capturing the dynamics of the premium. This parameterization produces realistic time-varying market equity premium estimates over the entire 1840-2003 period. For example, our results show that the premium was relatively low in the mid-1990s but has recently increased. Results are robust to univariate specifications that condition on either levels or logs of past realized volatility (RV), as well as to a new bivariate risk-return model of returns and RV for which the conditional variance of excess returns is the conditional expectation of the realized volatility process.

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1 Introduction

The expected return on the market portfolio is an important input for many decisions in finance. For example, accurate measures or forecasts of the equity premium are important for computing risk-adjusted discount rates, capital budgeting decisions involving the cost-of-equity capital, as well as optimal investment allocations.

The simplest approach to measuring the market premium is to use the historical average market excess return. Unfortunately, this assumes that the premium is constant over time. If the premium is time varying, as modern asset pricing theory suggests, then an historical average will be sensitive to the time period used. For example, if the level of market risk were higher in some subperiods than others, then the average excess return will be sensitive to the subsample chosen.

A better approach to estimating the premium is to directly incorporate the information governing changes in risk. For example, the Merton (1980) model implies that the market equity premium is a positive function of market risk, where risk is measured by the variance of the premium. Under certain conditions discussed in the next section, intertemporal asset pricing models reduce to a conditional version of Merton (1980). That is, if the conditional variance of the market portfolio return is larger, investors will demand a higher premium to compensate for the increase in risk. This positive risk-return relationship for the market portfolio has generated a large literature which investigates the empirical evidence.

Historically, authors have found mixed evidence concerning the relationship between the expected return on the market and its conditional variance. In some cases a significant positive relationship is found, in others it is insignificant, and still others report it as being significantly negative. Examples include Campbell (1987), Engle, Lilien, and Robins (1987), French, Schwert, and Stambaugh (1987), Chou (1988), Harvey (1989), Turner, Startz, and Nelson (1989), Baillie and DeGennaro (1990), Glosten, Jagannathan, and Runkle (1993) and Whitelaw (1994). These papers explicitly model the dynamics of the conditional variance process using a GARCH model for which the conditional variance is a function of the past squared innovations to returns, or using a Markov-switching model, or using instrumental variables.

Recent empirical work investigating the relationship between market risk and return offers some resolution to the conflicting results in the early literature. Scruggs (1998) includes an additional risk factor implied by the model of Merton (1973), arguing that ignoring it in an empirical test of the risk-return relationship results in a misspecified model. Including a second factor, measured by long-term government bond returns,

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1Recent papers by Claus and Thomas (2001), Fama and French (2002), and Donaldson, Kamstra, and Kramer (2004) focus on alternative models, such as earnings and dividend growth, to measure the market premium.

2There are many other models of asset premiums, many of which can be thought of as versions of a multi-factor approach, such as the three-factor model of Fama and French (1992), or the arbitrage pricing theory of Ross (1976).

3Examples of intertemporal models that do not require consumption data are the intertemporal asset pricing models proposed by Merton (1973) and Campbell (1993), and the conditional capital asset pricing model, for example, Campbell (1987) and references therein.

4Table 1 of Scruggs (1998) summarizes the empirical evidence.

This paper revisits the Merton (1980) model to further explore a conditional version of that risk-return specification motivated by the intertemporal asset pricing models of Campbell (1993) and Merton (1973). Firstly, we utilize a long sample of data, concentrating on the length of the time period rather than the number of data points or frequency of sampling.6 Secondly, we investigate the empirical importance of imposing a zero intercept (proportional model) in the conditional mean.7 Thirdly, Merton highlighted the significance of changes in the level of market risk (heteroskedasticity) and its potential effect on inference about the equity premium. Similar to the papers mentioned above, we build directly on Merton’s approach by exploiting conditional heteroskedasticity, and thus forecastability of the volatility of market excess returns, in order to implement a time-varying risk model of the equity premium. However, concentrating on the market risk-return relationship,8 we extend this literature by improving the signal-to-noise ratio in our volatility forecasts, and pricing those components of volatility that are most important for capturing the dynamics of the relationship.

How do we achieve these improved estimates of the market premium? Firstly, we use a nonparametric measure of ex post variance, referred to as realized volatility (RV), which has been extensively developed in a series of papers by Andersen, Bollerslev, Diebold and co-authors, and Barndorff-Nielsen and Shephard.9 Of particular relevance for this paper, Andersen and Bollerslev (1998) show that RV is considerably more accurate than traditional measures of ex post latent variance. Due to data constraints for our long historical sample, in this paper we construct annual RV using monthly squared excess returns.10

Secondly, as in Andersen, Bollerslev, Diebold, and Labys (2003), French, Schwert, Pagan and Hong (1990), Backus and Gregory (1993), Whitelaw (2000) and Linton and Perron (2003). As discussed in the Appendix of Merton (1980) for i.i.d. returns, a longer time span, rather than a higher sampling frequency, is required to improve the efficiency of the estimator of the mean. In contrast, higher frequency returns, which we use in this paper, do improve the accuracy of the variance estimate.

Campbell and Thompson (2004) also argue for 'sensible restrictions' on 'the signs of coefficients and return forecasts'.

Throughout this paper, unless otherwise indicated, the phrase 'market risk-return relationship' will refer to the case in which the conditional market premium is a function of the conditional variance of the market return.

For example, Andersen, Bollerslev, Diebold, and Labys (2001) and Barndorff-Nielsen and Shephard (2002a).

and Stambaugh (1987), and Maheu and McCurdy (2002), our volatility forecasts condition on past realized volatility. Since RV is less noisy than traditional proxies for latent volatility, it is also a better information variable with which to forecast future volatility.

Thirdly, we propose a new volatility model based on exponential smoothing. Our model inherits the good forecasting properties of the popular exponential smoothing model but avoids degeneracy in the asymptotic limit and allows for mean reversion. Given the relatively low frequency data necessary to allow a long time sample, a very parsimonious volatility function is critical. Our volatility specification, which only requires one parameter per volatility component, not only produces more precise estimates of the risk-return relationship but also delivers realistic estimates of the market premium in contrast, for example, to those from a GARCH-in-mean model based on squared excess returns.

We also investigate whether or not total market risk or just some component of it is priced. We achieve this by extending the conditional variance specification to a component forecasting model. This more flexible conditioning function allows different decay rates for different volatility components and allows our risk-return model to determine which components of the volatility best explain the dynamics of the equity risk premium.

Finally, the most important contribution of this paper is to generalize the risk-return model for the market equity premium by estimating a joint stochastic specification of annual excess returns and the logarithm of RV. In this case, the conditional variance of excess returns is obtained as the conditional expectation of the RV process.

In summary, we use improved measures of volatility in a parsimonious forecasting model which allows components of volatility with different decay rates to be priced in a conditional risk-return model. Our empirical results show that, for 164 years of the U.S. equity market, there is a significant positive relationship between market risk and the market-wide equity premium, whether or not one includes an intercept term in the conditional mean. The equity premium varies considerably over time and confirms that the average excess return associated with subperiods can be misleading as a forecast. Nevertheless, long samples of historical information are useful as conditioning information and contribute to improved estimates of the time-varying market premium.

In our 2-component specifications of the conditional variance, one component tracks long-run moves in volatility while another captures the short-run dynamics. The 2-component conditional variance specification provides a superior variance forecast. Furthermore, it is the long-run component in the variance that provides a stronger risk-return relationship.

The paper is organized as follows. Section 2 introduces the models that motivate our empirical study, and discusses the importance of the measurement and modeling of the variance of market returns. Section 3 details our results on the significance of the risk-return relationship for several model specifications. We discuss the importance of volatility components, and the range of implied premiums that the models produce. Finally, Section 4 summarizes the results and future work.
2 The Risk-Return Model

Both static and intertemporal models of asset pricing imply a risk-return relationship. Examples of intertemporal models which do not require consumption data are the intertemporal asset pricing models proposed by Merton (1973) and Campbell (1993), and also the conditional CAPM.

The IAPM of Merton (1973) relates the expected market return and variance through a representative agent’s coefficient of relative risk aversion and also allows sensitivity of the market premium to a vector of state variables (or hedge portfolios) which capture changing investment opportunities. Under some assumptions, the intertemporal model implies a market risk-return relationship with no additional factors, that is, market risk is captured by the variance of the market portfolio. Merton (1980) argues that this case will be a close approximation to the intertemporal asset pricing model in Merton (1973) if either the variance of the change in wealth is much larger than the variance of the change in the other factor(s), or if the change in consumption in response to a change in wealth is much larger than that associated with a change in other state variable(s). Sufficient conditions are if the investment opportunity set is essentially constant, or if the representative investor has logarithmic utility.

Campbell (1993) provides a discrete-time intertemporal model which substitutes-out consumption. In this case, the expected market premium is a function of its variance as well as its covariance with news (revisions in expectations) about future returns on the market. As in Merton (1973), if the coefficient of relative risk aversion is equal to 1 or if the investment opportunity set is constant or uncorrelated with news about future market returns, the expected market premium will only be a function of the market return variance. However, the Campbell (1993) derivation provides an alternative, empirically plausible, condition under which that market risk-return relationship obtains. If the covariance of the market return with news about future investment opportunities is proportional to the variance of the market return, then the latter will be a sufficient statistic for market risk.\footnote{Campbell (1996) reports empirical evidence in support of an analogous condition in a cross-sectional application for which the restriction is that covariances of all asset returns with news about future returns on invested wealth are proportional to their covariances with the current return on wealth. In this case, most of the explained cross-sectional variation in returns is explained by cross-sectional variation in the assets’ covariances with the market return.}

Section III of Campbell (1993) provides alternative conditions that produce a conditional market risk-return relationship.

This motivates our first test equation in which we focus on the conditional market risk-return relationship,

\[ r_{M,t} = \gamma_1 \sigma^2_{M,t} + \epsilon_t, \tag{2.1} \]

where \( r_{M,t} \) is the excess return on the market, \( \sigma^2_{M,t} \) is the conditional variance of the market excess return (that is, the forecast of the variance conditional on time \( t-1 \) information), and \( \epsilon_t \) is the time \( t \) innovation. This model implies a proportional relationship between the market equity premium and its conditional variance. As discussed in Section 1 above, Merton (1980) argues that one should impose the prior that the \textit{expected} equity premium is non-negative. Using test equation (2.1), this will be satisfied as long as \( \hat{\gamma}_1 > 0 \).
The conventional test equation is a linear model,

\[ r_{M,t} = \gamma_0 + \gamma_1 \sigma^2_{M,t} + \epsilon_t. \]  

(2.2)

For example, Scruggs (1998) motivates the addition of an intercept to account for market imperfections, such as differential tax treatment of equity versus T-bill returns, which might account for a constant equity premium unrelated to risk.

We have two alternative test equations: the linear model (2.2); and the proportional model (2.1). These empirical models are motivated as special cases of an IAPM. Each of the empirical models implies a time-varying equity premium which is a function of its own conditional second moment, that is, a forecast of the equity premium’s time \( t \) variance conditional on time \( t - 1 \) information.

### 2.1 Measuring and Forecasting Volatility

In this section, we discuss how we measure and then forecast the volatility which drives the time-varying risk premiums. Note that, throughout the paper, we use the term volatility to refer generically to either the variance or standard deviation. Where necessary for clarity, we refer specifically to whether it is an ex post (realized) measure or a conditional estimate (forecast); and whether we are referring to a variance or a standard deviation. For ease of notation, we also drop the subscript \( M \) on the market excess return and its conditional variance so that henceforth \( r_t \equiv r_{M,t} \) and \( \sigma^2_t \equiv \sigma^2_{M,t} \).

#### 2.1.1 Measuring Volatility

In this paper, we employ a nonparametric measure of volatility. A traditional proxy for ex post latent volatility has been squared returns or squared residuals from a regression model. As shown by Andersen and Bollerslev (1998), this measure of volatility is very noisy and of limited use in assessing features of volatility such as its time-series properties.

Better measures of ex post latent volatility are available. In this paper, we use a measure of ex post variance, termed realized volatility (RV), developed in a series of papers by Andersen, Bollerslev, Diebold and co-authors, and Barndorff-Nielsen and Shephard. The increment of quadratic variation is a natural measure of ex post variance over a time interval. RV is computed as the sum of squared returns over this time interval. As shown by Andersen, Bollerslev, Diebold, and Labys (2001), as the sampling frequency is increased, the sum of squared returns converges to the quadratic variation over a fixed time interval for a broad class of models. Thus RV is a consistent estimate of ex post variance for that period. The asymptotic distribution of RV has been studied by Barndorff-Nielsen and Shephard (2002a) who provide conditions under which RV is also an unbiased estimate. Recent reviews of this growing literature are Andersen, Bollerslev, and Diebold (2004) and Barndorff-Nielsen, Graversen, and Shephard (2004).

The highest frequency data available for parts of our time period of study is monthly. Therefore, in this paper we use squared monthly excess returns to compute annual RV. Defining the RV of the market excess return for year \( t \) as \( RV_t \), we construct annual RV
data as

\[ RV_t = \sum_{j=1}^{12} r_{t,j}^2 \]

(2.3)

where \( r_{t,j} \) is the continuously compounded excess return in the \( j^{th} \) month in year \( t \).

### 2.1.2 Forecasting Volatility

Our time-varying risk model of the equity premium is forward looking. That is, the expected market equity premium is a function of market equity risk. According to our test equations, the latter is measured by the conditional variance of market excess returns. Therefore, we need a forecast of the time \( t \) volatility, conditional on information at time \( t - 1 \). Our volatility forecasts condition on past RV. Given that RV has a superior signal-to-noise ratio for measuring latent volatility, it should be a superior conditioning variable for forecasting future volatility.

What functional form should be used to summarize information in past RV? Given our limited data measured at annual frequencies, parsimony is an obvious objective. One candidate is the infinite exponential smoothing function. In this case, forecasts can be derived from the recursion

\[ \sigma_t^2 = (1 - \alpha)RV_{t-1} + \alpha \sigma_{t-1}^2, \]

(2.4)

in which \( 0 \leq \alpha < 1 \), \( \alpha \) is the smoothing parameter, and \( \sigma_t^2 \) is the conditional variance. A small value of \( \alpha \) puts more weight on the the most recent observable value of RV, that is \( RV_{t-1} \), and less weight on the past forecast \( \sigma_{t-1}^2 \). Conversely, an \( \alpha \) close to 1 puts less weight on recent observations and more weight on past forecasts which smooths the data. This model is analogous to the popular RiskMetrics filter, except for the fact that the standard practice is to smooth on lagged squared returns rather than on lagged RV. The recursion in equation (2.4) implies the following weighting function on past RV,

\[ \sigma_t^2 = (1 - \alpha) \sum_{j=0}^{\infty} \alpha^j RV_{t-j-1}, \]

(2.5)

in which the weights sum to one.

Although infinite exponential smoothing provides parsimonious forecasting, it possesses several drawbacks. For instance, it does not allow for mean reversion in volatility; and, as Nelson (1990) has shown in the case of squared returns or squared innovations to returns, the model is degenerate in its asymptotic limit. The same problem applies to integrated GARCH (IGARCH) models without an intercept. To circumvent these problems, but still retain the parsimony and accuracy of exponential smoothing, we propose the following new specification motivated from equation (2.5),

\[ \sigma_t^2 = \omega + (1 - \alpha) \sum_{j=0}^{\tau-1} \alpha^j RV_{t-j-1}, \]

(2.6)
where we truncate the expansion at the finite number τ. In this specification the weights sum to less than one allowing mean reversion in volatility.\footnote{In the empirical applications below, we fixed τ = 40 due to presample data requirements. Results were robust to other choices for τ as long as it is reasonably large, for example, τ > 35.}

Corollary 1 of Andersen, Bollerslev, Diebold, and Labys (2003) shows that, under empirically realistic conditions, the conditional expectation of quadratic variation (QV_t) is equal to the conditional variance of returns,\footnote{Meddahi (2002) and Barndorff-Nielsen and Shephard (2002b) also discuss the theoretical relationship between integrated volatility and RV.} that is, $E_{t-1}(QV_t) = \text{Var}_{t-1}(r_t) \equiv \sigma_t^2$.\footnote{We assume that any stochastic component in the intraperiod conditional mean is negligible compared to the total conditional variance.} Assuming that RV is a unbiased estimator of quadratic variation it follows that $E_{t-1}(RV_t) = \sigma_t^2$, and we can derive the unconditional variance as

$$E(\sigma_t^2) = \frac{\omega}{1 - (1 - \alpha) \sum_{j=0}^{\tau-1} \alpha^j}. \quad (2.7)$$

This unconditional variance leads to our strategy of variance targeting by setting $\sigma^2$ from the data and using

$$\omega = \sigma^2 [1 - (1 - \alpha) \sum_{j=0}^{\tau-1} \alpha^j]. \quad (2.8)$$

In summary, this new specification is similar in spirit to exponential smoothing but allows for mean reversion in volatility and is not degenerate in the asymptotic limit. In addition, the finite unconditional variance allows for variance targeting which means that only one parameter needs to be estimated. Our specification is also more parsimonious than the covariance-stationary GARCH(1,1) model.\footnote{The covariance-stationary GARCH(1,1) model, $\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$, with $\epsilon_{t-1} \equiv r_{t-1} - E_{t-1}r_{t-1}$, can be written as $\sigma_t^2 = \omega/(1 - \beta) + \alpha \sum_{i=0}^{\infty} \beta^i \epsilon_{t-1-i}^2$ which requires an extra parameter compared to (2.6).} As discussed further below, at least for our sample of annual data the more parsimonious specification is critical for precision of the estimates of the risk-return relationship and also for generating reasonable premium estimates.

As is evident from equation (2.6), past data receive an exponentially declining weight. As we will see below, this is not flexible enough to capture the time-series dynamics of RV. A simple approach to providing a more flexible model is to allow different components of volatility to decay at different rates.\footnote{Several literatures, for example, long memory, mixtures of regimes, mixtures of jumps and stochastic volatility, etc., have highlighted that a single exponential decay rate is inadequate to capture volatility dynamics over time. Examples include Engle and Lee (1999), Maheu and McCurdy (2000), Alizadeh, Brandt, and Diebold (2002), Bollerslev and Zhou (2002), Chernov, Gallant, Ghysels, and Tauchen (2003), Chacko and Viceira (2003) and Ghysels, Santa-Clara, and Valkanov (2004).} This can be achieved with a component volatility function which estimates the conditional variance as the average of two or more components. Formally, define the $k$-component volatility model as

$$\sigma_{t,(k)}^2 = \omega + \frac{1}{k} \sum_{i=1}^{k} \sigma_{t,i}^2, \quad (2.9)$$
where

$$\sigma_{t,i}^2 = (1 - \alpha_i) \sum_{j=0}^{\tau-1} \alpha_j^i RV_{t-j}, \quad i = 1, \ldots, k.$$ 

Note that the conditional variance components are projections on past RV. We do not specify the high-frequency dynamics of spot volatility. Indeed, one of the attractions of using RV, besides being an efficient estimate of ex post volatility, is that it will be a consistent estimate of volatility for a very large class of empirically realistic models. Therefore, our modeling assumptions are on the annual conditional variance process given observations on RV.

Related work on volatility modeling includes Engle and Lee (1999) and Ghysels, Santa-Clara, and Valkanov (2005). Relative to component GARCH models, our parameterization only requires 1 parameter per component rather than two. Ghysels, Santa-Clara, and Valkanov (2005) use a mixed data sampling (MIDAS) approach to estimate volatility. In that paper, the monthly conditional variance of returns is modelled using a flexible functional form to estimate the weight given to each lagged daily squared return. They find that a parsimonious weighting scheme with two parameters works well. Our conditional variance specification maintains the parsimony but, unlike Ghysels, Santa-Clara, and Valkanov (2005), allows mean reversion and has a non-degenerate asymptotic limit. This allows us to use variance targeting which may be important to gain precision in our application. In addition, we extend the existing literature to investigate a bivariate risk-return specification introduced in the next section.

Our objective is to have a parsimonious and flexible function that summarizes information in past RV that might be useful for forecasting changes in the market equity risk premium. We allow for alternative components of volatility with different decay rates. Not only is this a more flexible way to capture the time-series dynamics of volatility, but it also allows us to investigate whether or not a particular component, rather than the full conditional variance, is more important in driving the market premium.

### 2.2 Equity Premium Model

In this subsection we introduce two alternative empirical specifications of the risk-return relationship. Each of our models jointly estimate the conditional mean and conditional variance parameters using maximum likelihood. We label the first specification univariate since it fits the stochastic excess return process by conditioning on variance forecasts which are estimated using a projection on past RV as in equation (2.9). The second specification is bivariate since we estimate a bivariate stochastic specification of annual excess returns and log(RV). In that case, the conditional variance of excess returns is obtained as the conditional expectation of the RV process.

#### 2.2.1 Univariate Risk-Return Specifications

The conditional mean is,

$$r_t = \gamma_0 + \gamma_1 \sigma_{t,(q)}^2 + \epsilon_t, \quad \epsilon_t = \sigma_{t,(k)} z_t, \quad z_t \sim N(0, 1); \quad \gamma\text{ (2.10)}$$

and the conditional variance is either equation (2.9) applied to levels of RV, or applied to log(RV) as in

\[
\log \sigma^2_{t,(k)} = \omega + \frac{1}{k} \sum_{i=1}^{k} \log \sigma^2_{t,i}, \tag{2.11}
\]

\[
\log \sigma^2_{t,i} = (1 - \alpha_i) \sum_{j=0}^{r-1} \alpha^j_i \log RV_{t-j-1}, \quad i = 1, \ldots, k,
\]

where \(k\) indexes the total number of variance components and \(q\) indexes the number of variance components that affect the conditional mean. That is, we specify \(\sigma^2_{t,(q)}\) in the conditional mean, in which \(q\) denotes the number of volatility components that affect the market premium when the conditional variance follows a \(k\)-component model. Note that \(q \leq k\). For example, where we want the total variance to enter the mean, we set \(q = k\). On the other hand, for models in which \(q = 1, k = 2\), we let the maximum likelihood estimator determine which component of the variance is optimal in the mean specification.

The \(t\)-th contribution to the loglikelihood is

\[
l_t = -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log(\sigma^2_{t,(k)}) - \frac{(r_t - \gamma_0 - \gamma_1 \sigma^2_{t,(q)})^2}{2\sigma^2_{t,(k)}}, \tag{2.12}
\]

where \(\sigma^2_{t,(q)}\) depends on the levels or log specification. The loglikelihood \(\sum_{t=1}^{T} l_t\) is maximized with respect to the parameters \(\gamma_0, \gamma_1, \alpha_1, \ldots, \alpha_k\).

### 2.2.2 Bivariate Risk-Return Specification

In this case, we estimate a bivariate stochastic specification of annual excess returns and log(RV). The parameterization has conditional mean

\[
r_t = \gamma_0 + \gamma_1 \sigma^2_{t,(q)} + \epsilon_t, \quad \epsilon_t = \sigma_{t,(k)} z_t, \quad z_t \sim N(0,1) \tag{2.13}
\]

and the following conditional variance specification

\[
\sigma^2_{t,(k)} \equiv E_{t-1}(RV_t|k) = \exp(E_{t-1}(\log RV_t|k) + 0.5 \text{Var}_{t-1}(\log RV_t|k))
\]

\[
\log RV_t = \omega + \frac{1}{k} \sum_{i=1}^{k} \log \sigma^2_{t,i} + \eta_t, \quad \eta_t \sim N(0, \phi^2)
\]

\[
\log \sigma^2_{t,i} = (1 - \alpha_i) \sum_{j=0}^{r-1} \alpha^j_i \log RV_{t-j-1}, \quad i = 1, \ldots, k. \tag{2.14}
\]

Again, \(\sigma^2_{t,(q)} \equiv E_{t-1}(RV_t|q), \quad q \leq k\), represents the conditional variance component(s) that affect the conditional mean. \(l_t\) consists of contributions from the return and volatility equation,

\[
l_t = -\log(2\pi) - \frac{1}{2} \log(\sigma^2_{t,(k)}) - \frac{1}{2} \log(\phi^2) - \frac{(r_t - \gamma_0 - \gamma_1 \sigma^2_{t,(q)})^2}{2\sigma^2_{t,(k)}} - \frac{(\log RV_t - \omega - \frac{1}{k} \sum_{i=1}^{k} \log \sigma^2_{t,i})^2}{2\phi^2}. \tag{2.15}
\]
In this case, by parameterizing the joint density of annual excess returns and log(RV), we explicitly model the annual log(RV) process as stochastic given the most recent information. However, making a conditional log-normal assumption for RV allows for a convenient calculation of the conditional variance as in equation (2.14).

3 Results

3.1 Data and Descriptive Statistics

We are evaluating the risk-return relationship associated with equity for the market as a whole. Therefore, we require data on a broad equity market index and on a risk-free security so that we can construct returns on the equity index in excess of the risk-free rate.

There are two considerations. First, as shown by Merton (1980) for i.i.d. returns, we can only increase the precision of expected return estimates by increasing the length of the time period (calendar span) covered by our historical sample. In other words, sampling more frequently (for example, collecting daily instead of monthly data) will not improve the precision of the market premium estimates. For this reason we want our historical sample to be as long as possible. Second, since we use a nonparametric measure of realized volatility which is constructed as the sum of squared intra-period returns, we will need data at a higher frequency than our estimation frequency in order to compute realized volatility (RV) as in equation (2.3).

Monthly U.S. equity returns for the 1802-1925 period are from Schwert (1990). The Schwert index was constructed from various sources: railway and financial companies up to 1862 from Smith and Cole (1935); the Macaulay (1938) railway index (1863-1870); the value-weighted market index published by the Cowles Commission for the time period 1871-1885; and the Dow Jones index of industrial and railway stocks for 1885-1925.¹⁷ For the 1926-2003 period, we use monthly returns (including distributions) for the value-weighted portfolio of NYSE, NASDAQ and AMEX stocks compiled by the Center for Research in Security Prices (CRSP).

We use annual bill yield data from Jeremy Siegel for the risk-free rate for the 1802-1925 period. Siegel (1992) describes how these annual yields were constructed to remove the very variable risk premiums on commercial paper rates in the 19th century.¹⁸ Monthly yields were interpolated from the annual yields for this time period. For the period 1926-2003, we use bid yields from the Fama riskfree file, that is, monthly observations on U.S. 3 month T-Bills, provided by CRSP.

¹⁷Schwert adjusts for the time averaging in the original series for 1863-1885 and adds an estimate of dividends for the period 1802-1870.

¹⁸The market excess return is often measured with respect to a long-term (for example, 30-year) Treasury yield rather than the T-Bill yield used in this paper. Although this choice depends on investment horizon and the purpose of the calculation, for example, capital budgeting versus asset pricing, our focus is on forecasting the market equity premium over a (approximately) riskfree rate rather than over an alternative risky asset. Booth (1999) compares several approaches and shows that for certain periods the risk associated with long-term Treasuries was almost as large as that associated with market equity.
The above data are converted to continuously compounded monthly excess returns by taking the natural logarithm of one plus the monthly equity return and subtracting one-twelve of the natural logarithm of one plus the annual T-Bill yield.\textsuperscript{19} As expected for indices of equity returns, due to stale prices there is positive first-order serial correlation in the monthly excess returns which are required to compute annual RV. One can adapt the RV estimator, as in French, Schwert, and Stambaugh (1987), Bai, Russell, and Tiao (2004) or Hansen and Lunde (2004), to take account of this serial dependence in the intraperiod returns. Alternatively, as in Andersen, Bollerslev, Diebold, and Ebens (2001) and Maheu and McCurdy (2002), we use an AR(1) filter to remove the serial correlation from the monthly excess returns prior to computing annual RV as in equation (2.3).\textsuperscript{20} For the conditional mean, we use the unfiltered annual continuously compounded excess returns computed as the sum of the corresponding monthly continuously compounded excess returns. Henceforth, unless otherwise indicated, returns and excess returns refer to continuously compounded rates.\textsuperscript{21}

Figure 1 plots annual market excess returns for the entire sample from 1803-2003. Figures 2 and 3 plot two alternative measures of \textit{ex post} volatility, the absolute value of the excess returns and the square root of RV, respectively. Notice how much smoother the square root of RV is than the absolute value of annual excess returns. It is also clear from these plots that the data for the period 1803-1834 has a very different structure than that for the remainder of the sample. As noted in Schwert (1990), data from only a small number of companies was available for that subperiod. Both Schwert (1989) and Pastor and Stambaugh (2001) drop this period. For the same reason, our analyses focus on the time period 1840-2003. Starting in 1840 provides presample values to condition our time-varying volatility model.

Table 1 reports summary statistics for the time period, 1840-2003. The average excess return is 4.25%.

### 3.2 Univariate Risk-Return Results

As discussed in Section 2.2.1, the univariate specifications fit the stochastic excess return process by conditioning on variance forecasts which are estimated using a projection on past RV. We report results using both the levels of RV, as in equations (2.9) and (2.10), and the natural logarithm of RV, as in equations (2.11) and (2.10).\textsuperscript{22} Due to the importance of the length of the time period necessary for efficient conditional mean forecasts, we use all of our data and thus forecasts are in-sample. See Inoue and Kilian

\textsuperscript{19}For 1926-2003, the monthly T-Bill yields were already 365-day continuously compounded annual yields in which case they were just converted to monthly rates.

\textsuperscript{20}The AR(1) process is fit to our estimation sample 1840-2003 but the filter is applied to the entire sample 1802-2003 since the presample returns are used for conditioning the volatility model. Results using unfiltered monthly excess returns to compute RV are very similar. They are available from the authors on request.

\textsuperscript{21}As noted below, the relationship between market risk and return is stronger using simple (compounded monthly) returns. However, we use continuously compounded returns to conform with our RV estimator.

\textsuperscript{22}Andersen, Bollerslev, Diebold, and Labys (2001) show that for foreign exchange rates the natural logarithm of RV is more normally distributed than the level of RV.

3.2.1 Using RV Levels

Table 2 reports parameter estimates and likelihood ratio test (LRT) results associated with the risk-return relationship estimated using the alternative component volatility models applied to levels of RV. Note that $\omega$ is computed, as in equation (2.8), where $\sigma^2$ is set to the sample average of volatility. For each model, U(1), U(2) and U(2$s$), we report both the linear parameterization of the risk-return relationship, equation (2.2) and the proportional parameterization with $\gamma_0 = 0$, as in equation (2.1). Models U(2) and U(2$s$) have the same volatility specification but U(2$s$) allows the risk-return model to determine which volatility component has the most explanatory power for the dynamics of the equity premium. We find that, when given the choice, the maximum likelihood estimator always chooses to price the long-run or smooth component.

It is clear from Figure 4 that the 2-component volatility specification tracks RV better than the 1-component version. Figure 5 plots the individual components of volatility from the U(2) specification. This shows that the smooth component is very persistent, as expected from the smoothing coefficient estimate $\hat{\alpha}_1 = .923$ reported in Table 2. Note in particular how long it takes the smooth component to decay from its high level in the 1930s. On the other hand, the second component has much lower persistence, $\hat{\alpha}_2 = .273$, which implies that it is more influenced by the most recent RV.

Table 2 reveals that the risk-return relationship is positive and statistically significant for all specifications. The LRT statistics in the final column of the table all reject the null hypothesis that $\gamma_1 = 0$. This result is particularly strong for the proportional parameterization of the risk premium in which case the test statistics are all greater than 10 so that the p-value associated with the null hypothesis is very small. The t-statistic for $\hat{\gamma}_1$ also supports this conclusion in that it is greater than 3.2 for all of the alternative volatility specifications for the proportional model.

For linear parameterizations of the premium, the LRT results also reveal a significantly positive relationship between excess market returns and their conditional variance, although, with the exception of the U(2$s$) model, the significance is less strong when there is an intercept in the conditional mean. For these linear parameterizations, the t-statistic on the risk-return slope coefficient, $\gamma_1$, is highest (2.761) for the U(2$s$) model. That version also has the highest log-likelihood, 43.16, and a LRT statistic of 14.44 which strongly rejects the hypothesis that $\gamma_1 = 0$.

For all volatility specifications, the more parsimonious proportional premium model is not rejected by the data. That is, the intercept in the conditional mean is statistically insignificant, both from the perspective of t-tests on $\gamma_0$ and from the LRT reported in the 2nd-last column of Table 2. In addition, by restricting the intercept to be zero the precision of the estimate $\hat{\gamma}_1$ improves. For instance, as noted above, the t-statistics associated with the hypothesis $\gamma_1 = 0$ increase, as do the LRT statistics which are about twice as high for the proportional as opposed to the linear model. The one exception is

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23Note that these volatility plots are square roots of the volatility forecasts versus $\sqrt{RV_t}$.

24The same application to annual excess returns compounded from simple monthly returns resulted in LRT statistics greater than 21 and t-statistics ranging from 4.4 to 4.6.
again the U(2s) model which reveals a strongly positive market risk-return relationship for both the proportional and linear parameterization of the premium.

Figure 6, plots the market equity premium forecast for the proportional risk premium parameterization. When total volatility from the 2-component volatility specification is priced (the U(2) model), there is a high peak in the forecasted market equity premium which lasts for 2 years, 1933 and 1934. In contrast, pricing the smooth component of volatility, the U(2s) model, results in a lower peak which decays more slowly. In this case, the premium forecast ranges from 1.5% to 10.5% with the average forecasted premium over the 1840-2003 sample equal to 4.25 percent.

Table 3 reports model diagnostics and statistics for volatility and premium fit. These include mean absolute error (MAE) and root mean squared error (RMSE), as well as the $R^2$ from Mincer and Zarnowitz (1969) forecast regressions for a particular model’s premium and volatility forecasts. The results in Table 3 support the 2-component models U(2) and U(2s) over the more restrictive 1-component version U(1). For example, as suggested by Figure 4, the $R^2$ increases from about 10% to 30% by adding a second volatility component. Note that the U(2s) model, which had the highest log-likelihood, also has the best premium fit whether measured from the perspective of MAE, RMSE, or $R^2$.

3.2.2 Using Log(RV)

We also fit the univariate risk-return model using variance forecasts which are estimated from a projection on the natural logarithm (rather than levels) of past RV, that is, equations (2.10) and (2.11). Tables 4 and 5 report these results.

The parameter estimates using log(RV), reported in Table 4, reveal that the risk-return relationship is again positive for all specifications and, with one exception, is very significant — even more so than when projecting on the levels of past RV. The exception, with a $p$-value of 0.08 for the test of the restriction $\gamma_1 = 0$, is the linear parameterization of the premium with a 1-component volatility specification. Nevertheless, even in that case, the proportional parameterization of the premium does result in a statistically significant positive risk-return relationship. For the proportional risk premium parameterization, the LRT statistics in the final column are even larger than in Table 2. The $t$-statistics associated with the risk-return slope parameter estimates $\hat{\gamma}_1$ range from 3.7 to 3.8 in this case.

Table 5 shows that the 2-component parameterizations of volatility dominate the 1-component version from the perspective of all three criteria, MAE, RMSE, or $R^2$, for volatility fit. As in the levels case, the $U_{log}(2s)$ model, which prices the smooth component of volatility, has the best overall fit.

Figure 9 shows that the smooth proportional premium forecasts using the univariate model that conditions on log(RV), that is, the $U_{log}(2s)$ model, range from 1.5% to 11.6% with an average of 4.27% over the period 1840 to 2003. This Figure also shows that these premium forecasts are similar to the univariate case in which volatility forecasts

---

25For example for the volatility forecasts, the regression is $RV_t^5 = a + b\hat{\sigma}_t + u_t$ in which $\hat{\sigma}_t$ is the square root of the particular model’s volatility forecast for time $t$ given information up to time $t - 1$.

26Again, using simple returns produced higher $t$-statistics, up to 5.2.
were estimated from past levels of RV rather than log(RV), except that in the log(RV) case the peaks are slightly higher with faster decay.\(^{27}\)

### 3.3 Bivariate Risk-Return Results

In this section, we generalize our risk-return model for the market equity premium by estimating a joint stochastic specification of the conditional mean and annual log(RV) of market excess returns. The logarithmic specification of RV ensures that volatility is non-negative during estimation of the joint stochastic process. The system of test equations is (2.13) and (2.14) and the results are summarized in Tables 6 and 7, as well as Figures 7 and Figure 9.

The results are very comparable to the univariate results reported above. Table 7 and Figure 7 show that the 2-component volatility specification fits log(RV) better. For example, the \(R^2\) for volatility fit increases from about 20% to 27% by adding the 2nd volatility component.

Table 6 reports that the risk-return relationship is positive for all models and statistically significant for all of the models except one. As in the log(RV) univariate case, the \(t\)-statistic associated with \(\gamma_1\) is smaller for the linear risk premium parameterization using a 1-component volatility specification. However, since we are unable to reject that \(\gamma_0 = 0\), either by \(t\)-tests or by the LR tests reported in the 2nd-last column of the table, the data support a proportional parameterization of the risk premium. In that case, the risk-return relationship is significantly positive for all volatility specifications. Again, as in Tables 2 and 4, the LRT results in the final column indicate that the relationship between excess market returns and their conditional variance is very strong for proportional parameterizations of the risk premium.

Table 7 reports that the preferred model is, once again, the \(2_s\) specification, that is, 2 volatility components with the smooth component being priced in the conditional mean.\(^{28}\) As in the univariate cases, all of the premium fit statistics, MAE, RMSE, or \(R^2\), support this conclusion; as does the log likelihood which is \(-144.45\) for the \(B(2_s)\) model, as opposed to \(-146\) for the \(B(2)\) model, and \(-149.82\) for the \(B(1)\) model which has a 1-component volatility specification.

As displayed in Figure 9, the proportional premium for the preferred \(B(2_s)\) specification is very similar to that for the univariate case which conditions on past log(RV). The premium forecasts from this bivariate stochastic model range from 1.4% to 10.6% with an average of 4.2%.

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\(^{27}\)The plots of the premium forecasts for all specifications show that the premiums are increasing from a low level over the first few years of the estimation period. This is partly due to the low volatility of the presample data from 1802-1834 which the initial part of our estimation period require as conditioning information when \(\tau = 40\).

\(^{28}\)Given our relatively small number of annual observations, we investigated potential small-sample bias for the maximum-likelihood estimator by simulating 1000 bootstrap samples using the parameter estimates for the \(B(2_s)\) model in Table 6 as the DGP. The bias for all of the parameters was small; for example, the mean MLE for \(\gamma_1\) was 3.712 as opposed to the true parameter of 3.758. This suggests that the maximum-likelihood estimator is reliable for our sample. This addresses one of the issues raised by Stambaugh (1999).
Finally, note that all of our specifications deliver reasonable estimates of the market premium. This is in direct contrast to conventional models. For instance, we investigated several GARCH-in-mean models which all produce unrealistic estimates that can range as high as 38% for the annual premium.\(^\text{29}\)

4 Summary and Future Directions

This paper evaluates the market risk-return relationship for U.S. equity over the period 1840-2003 using a time-varying market premium for equity risk. We begin with a univariate specification of the risk-return relationship. This application models the stochastic market excess returns by conditioning on variance forecasts which are estimated by projecting onto past RV. We assess the robustness of those results by also estimating a univariate version which projects onto past log(RV).

We propose a new parsimonious and flexible function that summarizes information in past RV that might be useful for forecasting changes in the market equity risk premium. We allow for alternative components of volatility with different decay rates. Not only is this a more flexible way to capture the time-series dynamics of volatility, but it also allows us to investigate whether or not a particular component, rather than the full conditional variance, is more important in driving the market premium.

Our conditional variance specification maintains the parsimony of exponential smoothing functions but avoids degeneracy in the asymptotic limit. As a result, our specification allows mean reversion and targets the implied long-run variance. In addition, we allow for increased flexibility by explicitly modeling more than one volatility component.

Finally, we generalize the risk-return model for the market equity premium by estimating a joint stochastic specification of annual excess returns and log(RV). In this case, the conditional variance of excess returns is obtained as the conditional expectation of the RV process.

In summary, in addition to our new volatility specification, we consider two new contributions to the literature. First, unlike the univariate specifications that dominate the literature, we extend the analysis to a bivariate risk-return model of returns and RV. Secondly, we investigate whether or not one volatility component is more important than total volatility in driving the dynamics of the equity premium.

All of the empirical specifications support a conclusion that the relationship between risk and return for the market equity premium is positive and statistically significant. The higher is the expected market risk, the higher the market equity premium. This relationship is strongest for a two component specification of volatility with the long-run smooth component being priced in the conditional mean.\(^\text{30}\) In fact, the preferred model from the perspective of overall as well as equity premium fit, for both the univariate and bivariate specifications, is the \(2_s\) model which uses a 2-component volatility specification.

\(^{29}\)In addition, the conditional variance was insignificant in the mean equation with an estimate of 2.39 and a standard error of 1.50 for the levels specification.

\(^{30}\)A persistent component is also shown to be important for asset pricing in Bansal and Yaron (2004). Also, see Bansal, Dittmar, and Kiku (2005) who investigate the implications for risk and return of risks in the long run versus the short run.
and prices the smooth component of volatility in the conditional mean.

Figure 10 illustrates the premium for the 1990-2003 period. The time-varying premium estimates for the period 1993 to 2001 were below the long-run average of 4.25%, reaching a low of 3.14% in 1997. In addition, they are considerably lower than the 1980-2003 average continuously compounded excess return of 5.9% reported in the last row of Table 1. This graphically illustrates the point, discussed in Section 1, that the average realized excess return is not a very reliable forecast of the market equity premium since it will be sensitive to the subsample chosen. Finally, the forecasted continuously compounded premium has been increasing from the recent low of 3.14% in 1997 to 4.7% in 2003.
Table 1: Summary Statistics

<table>
<thead>
<tr>
<th>Sample</th>
<th>Excess Return</th>
<th>Realized Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>StdDev</td>
</tr>
<tr>
<td>1840-2003</td>
<td>0.0425</td>
<td>0.1838</td>
</tr>
</tbody>
</table>
Table 2: Parameter Estimates: Univariate Risk-Return using RV Levels

\[ r_t = \gamma_0 + \gamma_1 \sigma_{t,(q)}^2 + \epsilon_t, \quad \epsilon_t = \sigma_{t,(k)} z_t, \quad z_t \sim N(0, 1) \]

\[ \sigma_{t,(k)}^2 = \omega + \frac{1}{k} \sum_{i=1}^{k} \sigma_{t,i}^2; \quad \sigma_{t,i}^2 = (1 - \alpha_i) \sum_{j=0}^{\tau-1} \alpha_i^j RV_{t-j-1}, \quad i = 1, \ldots, k \]

Model Labels: \( U(k) \equiv \) univariate risk-return model with \( k \)-component model of \( RV \); \( U(2s) \equiv \) univariate risk-return model with 2-component volatility but the conditional mean prices the smooth volatility component.

<table>
<thead>
<tr>
<th></th>
<th>( \gamma_0 )</th>
<th>( \gamma_1 )</th>
<th>( \omega )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( L )</th>
<th>( LRT_{\gamma_0=0} )</th>
<th>( LRT_{\gamma_1=0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U(1) ) ( (q = k = 1) )</td>
<td>-0.029</td>
<td>2.536</td>
<td>0.0001</td>
<td>.892</td>
<td>.942</td>
<td>39.39</td>
<td>4.62</td>
<td>(0.03)</td>
</tr>
<tr>
<td>Proportional Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( U(2) ) ( (q = k = 2) )</td>
<td>-0.024</td>
<td>2.367</td>
<td>0.0001</td>
<td>.942</td>
<td>.273</td>
<td>41.49</td>
<td>5.10</td>
<td>(0.02)</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( U(2s) ) ( (q = 1, k = 2) )</td>
<td>-0.055</td>
<td>3.711</td>
<td>0.0001</td>
<td>.945</td>
<td>.319</td>
<td>43.16</td>
<td>14.44</td>
<td>(0.00)</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Notes: Conditional mean coefficient estimates of \( \gamma_1 \) with t-statistics in brackets; \( \omega \) is the volatility function intercept which is consistent with targeting the sample average of \( RV \); volatility function coefficient estimates \( \alpha_i \) have standard errors in parenthesis; \( L \) is the log-likelihood function; and \( LRT \) are likelihood ratio test statistics with p-values in parenthesis.
Table 3: Model Comparisons and Diagnostics: Univariate Risk-Return using RV Levels

\[ r_t = \gamma_0 + \gamma_1 \sigma^2_{t,(q)} + \epsilon_t, \quad \epsilon_t = \sigma_{t,(k)} z_t, \quad z_t \sim N(0,1) \]

\[ \sigma^2_{t,(k)} = \omega + \frac{1}{k} \sum_{i=1}^{k} \sigma^2_{t,i} ; \quad \sigma^2_{t,i} = (1 - \alpha_i) \sum_{j=0}^{\tau-1} \alpha^j RV_{t-j-1}, \quad i = 1, ..., k \]

Model Labels: \( U(k) \equiv \) univariate risk-return model with \( k \)-component model of \( RV \); \( U(2_s) \equiv \) univariate risk-return model with 2-component volatility but the conditional mean prices the smooth volatility component.

<table>
<thead>
<tr>
<th>Model ((q = k = 1))</th>
<th>Diagnostics</th>
<th>Volatility Fit</th>
<th>Premium Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U(1) )</td>
<td>L</td>
<td>( LB^2(10) )</td>
<td>( LB(10) )</td>
</tr>
<tr>
<td>( (q = k = 1) )</td>
<td>39.39</td>
<td>(0.78)</td>
<td>(0.29)</td>
</tr>
<tr>
<td>( U(2) )</td>
<td>L</td>
<td>( LB^2(10) )</td>
<td>( LB(10) )</td>
</tr>
<tr>
<td>( (q = k = 2) )</td>
<td>41.49</td>
<td>(0.86)</td>
<td>(0.46)</td>
</tr>
<tr>
<td>( U(2_s) )</td>
<td>L</td>
<td>( LB^2(10) )</td>
<td>( LB(10) )</td>
</tr>
<tr>
<td>( (q = 1, k = 2) )</td>
<td>43.16</td>
<td>(0.99)</td>
<td>(0.21)</td>
</tr>
</tbody>
</table>

\( L \) is the log-likelihood function. \( LB^2(10) \) is the Ljung and Box (1978) portmanteau test for serial correlation in the squared standardized residuals up to 10 lags; \( LB(10) \) is the same for the levels of standardized residuals; and \( KS \) is the Kiefer and Salmon (1983) test for departures from the Normal distribution for standardized residuals. We report p-values for these diagnostic tests. MAE \( \equiv \) mean absolute error; RMSE \( \equiv \) square root of the mean-squared error; \( R^2 \) from Mincer-Zarnowitz regressions applied to the variance and premium forecasts.
Table 4: Parameter Estimates: Univariate Risk-Return using Log(RV)

\[ r_t = \gamma_0 + \gamma_1 \sigma^2_{t,(q)} + \epsilon_t, \quad \epsilon_t = \sigma_t(z_t), \quad z_t \sim N(0, 1) \]
\[ \log \sigma^2_{t,(k)} = \omega + \frac{1}{k} \sum_{i=1}^{k} \log \sigma^2_{t,i}; \quad \log \sigma^2_{t,i} = (1 - \alpha_i) \sum_{j=0}^{t-1} \alpha_i^j \log RV_{t-j-1}, \quad i = 1, \ldots, k \]

Model Labels: \( U_{\log}(k) \equiv \) univariate risk-return model with \( k \)-component model of the natural logarithm of \( RV \); \( U_{\log}(2s) \equiv \) univariate risk-return model with 2-component volatility but the conditional mean prices the smooth volatility component.

<table>
<thead>
<tr>
<th>Model</th>
<th>( \gamma_0 )</th>
<th>( \gamma_1 )</th>
<th>( \omega )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( L )</th>
<th>( LRT_{\gamma_0=0} )</th>
<th>( LRT_{\gamma_1=0} )</th>
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</thead>
<tbody>
<tr>
<td>( U_{\log}(1) )</td>
<td>-0.005</td>
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<td>( )</td>
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<td>( )</td>
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<td></td>
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<td></td>
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<td>(0.00)</td>
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<tr>
<td>( U_{\log}(2s) )</td>
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<td>-0.0807</td>
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<td>(0.00)</td>
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Notes: Conditional mean coefficient estimates of \( \gamma_i \) with t-statistics in brackets; \( \omega \) is the volatility function intercept which is consistent with targeting the mean of \( \log(RV) \); volatility function coefficient estimates \( \alpha_i \) have standard errors in parenthesis; \( L \) is the log-likelihood function; and \( LRT \) are likelihood ratio test statistics with p-values in parenthesis.
Table 5: Model Comparisons and Diagnostics: Univariate Risk-Return using Log(RV)

\[ r_t = \gamma_0 + \gamma_1 \sigma^2_{t(q)} + \epsilon_t, \quad \epsilon_t = \sigma_{t(k)} z_t, \quad z_t \sim N(0,1) \]

\[ \log \sigma^2_{t(k)} = \omega + \frac{1}{k} \sum_{i=1}^{k} \log \sigma^2_{t,i}; \quad \log \sigma^2_{t,i} = (1 - \alpha_i) \sum_{j=0}^{\tau-1} \alpha_i^j \log RV_{t-j-1}, \quad i = 1, \ldots, k \]

Model Labels: \( U_{\log}(k) \equiv \) univariate risk-return model with \( k \)-component model of the natural logarithm of RV; \( U_{\log}(2s) \equiv \) univariate risk-return model with 2-component volatility but the conditional mean prices the smooth volatility component.

<table>
<thead>
<tr>
<th>Model</th>
<th>Diagnostics</th>
<th>Volatility Fit</th>
<th>Premium Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LB^2(10)</td>
<td>LB(10)</td>
<td>KS</td>
</tr>
<tr>
<td>( U_{\log}(1) ) (q = k = 1)</td>
<td>27.82</td>
<td>(0.25)</td>
<td>(0.27)</td>
</tr>
<tr>
<td>( U_{\log}(2) ) (q = k = 2)</td>
<td>28.92</td>
<td>(0.46)</td>
<td>(0.34)</td>
</tr>
<tr>
<td>( U_{\log}(2s) ) (q = 1, k = 2)</td>
<td>29.20</td>
<td>(0.28)</td>
<td>(0.18)</td>
</tr>
</tbody>
</table>

\( L \) is the log-likelihood function. \( LB^2(10) \) is the Ljung and Box (1978) portmanteau test for serial correlation in the squared standardized residuals up to 10 lags; \( LB(10) \) is the same for the levels of standardized residuals; and \( KS \) is the Kiefer and Salmon (1983) test for departures from the Normal distribution for standardized residuals. We report p-values for these diagnostic tests. MAE \( \equiv \) mean absolute error; RMSE \( \equiv \) square root of the mean-squared error; \( R^2 \) from Mincer-Zarnowitz regressions applied to the variance and premium forecasts.
Table 6: Parameter Estimates: Bivariate Return and Log(RV) Model

\[ r_t = \gamma_0 + \gamma_1 \sigma^2_{t(q)} + \epsilon_t, \quad \epsilon_t = \sigma_{t(k)} z_t, \quad z_t \sim N(0, 1) \]

\[ \sigma^2_{t(k)} \equiv E_{t-1}(RV_t|k) = \exp(E_{t-1}(\log RV_t|k) + .5\text{Var}_{t-1}(\log RV_t|k)); \quad \sigma^2_{t(q)} = E_{t-1}(RV_t|q), \quad q \leq k \]

\[ \log RV_t = \omega + \frac{1}{k} \sum_{i=1}^{k} \log \sigma^2_{t,i} + \eta_t, \quad \eta_t \sim N(0, \phi^2) \]

\[ \log \sigma^2_{t,i} = (1 - \alpha_i) \sum_{j=0}^{\tau-1} \alpha^j_i \log RV_{t-j-1}, \quad i = 1, ..., k \]

Model Labels: B(k) \equiv \text{bivariate model of returns and the natural logarithm of } RV \text{ with } k\text{-component volatility. } B(2s) \equiv \text{bivariate model with 2-component volatility but the conditional mean prices the smooth volatility component.}

<table>
<thead>
<tr>
<th>Model</th>
<th>( \gamma_0 )</th>
<th>( \gamma_1 )</th>
<th>( \omega )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( \phi )</th>
<th>( L )</th>
<th>( LRT_{\gamma_0=0} )</th>
<th>( LRT_{\gamma_1=0} )</th>
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</thead>
<tbody>
<tr>
<td>B(1)</td>
<td>0.005</td>
<td>1.170</td>
<td>-0.2e-5</td>
<td>0.811</td>
<td>-149.82</td>
<td>1.76</td>
<td>(0.18)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((q = k = 1))</td>
<td>0.221</td>
<td>1.306</td>
<td>(0.063)</td>
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<tr>
<td>(\nu)</td>
<td>1.342</td>
<td>-0.3e-5</td>
<td>0.699</td>
<td>0.812</td>
<td>-149.85</td>
<td>0.06</td>
<td>(0.00)</td>
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<tr>
<td>(\nu)</td>
<td>3.028</td>
<td>(0.062)</td>
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<tr>
<td>Proportional Model</td>
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<td></td>
</tr>
<tr>
<td>B(2)</td>
<td>-0.022</td>
<td>2.205</td>
<td>-0.0655</td>
<td>0.917</td>
<td>0.316</td>
<td>0.803</td>
<td>-146.00</td>
<td>4.28</td>
<td>0.04</td>
</tr>
<tr>
<td>((q = k = 2))</td>
<td>-0.763</td>
<td>1.973</td>
<td>(0.041)</td>
<td>(0.149)</td>
<td>(0.046)</td>
<td></td>
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<td></td>
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<tr>
<td>(\nu)</td>
<td>1.443</td>
<td>-0.0420</td>
<td>0.906</td>
<td>0.803</td>
<td>-146.30</td>
<td>0.60</td>
<td>(0.00)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\nu)</td>
<td>3.125</td>
<td>(0.040)</td>
<td></td>
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<tr>
<td>Proportional Model</td>
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</tr>
<tr>
<td>B(2*)</td>
<td>-0.068</td>
<td>3.758</td>
<td>-0.1117</td>
<td>0.929</td>
<td>0.312</td>
<td>0.802</td>
<td>-144.45</td>
<td>7.38</td>
<td>0.01</td>
</tr>
<tr>
<td>((q = 1, k = 2))</td>
<td>-1.730</td>
<td>2.821</td>
<td>(0.031)</td>
<td>(0.149)</td>
<td>(0.046)</td>
<td></td>
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</tr>
<tr>
<td>(\nu)</td>
<td>1.496</td>
<td>-0.0420</td>
<td>0.906</td>
<td>0.802</td>
<td>-145.99</td>
<td>3.08</td>
<td>(0.08)</td>
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<tr>
<td>(\nu)</td>
<td>3.227</td>
<td>(0.038)</td>
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<tr>
<td>Proportional Model</td>
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</tbody>
</table>

Notes: Conditional mean coefficient estimates of \( \gamma_i \) with t-statistics in brackets; \( \omega \) is the volatility function intercept which is consistent with targeting the mean of \( \log(RV) \); volatility function coefficient estimates \( \alpha_i \) have standard errors in parenthesis; \( L \) is the log-likelihood function; and \( LRT \) are likelihood ratio test statistics with p-values in parenthesis.
Table 7: Model Comparisons and Diagnostics: Bivariate Return and Log(RV) Model

\[ r_t = \gamma_0 + \gamma_1 \sigma^2_{t(k)} + \epsilon_t, \quad \epsilon_t = \sigma_t(k) z_t, \quad z_t \sim N(0,1) \]

\[ \sigma^2_{t(k)} \equiv \text{E}_{t-1}(RV_t|k) = \exp(\text{E}_{t-1}(\log RV_t|k) + 0.5\text{Var}_{t-1}(\log RV_t|k)); \quad \sigma^2_{t(q)} \equiv \text{E}_{t-1}(RV_t|q), \quad q \leq k \]

\[ \log RV_t = \omega + \frac{1}{k} \sum_{i=1}^{k} \log \sigma^2_{i,t} + \eta_t, \quad \eta_t \sim N(0,\phi^2) \]

\[ \log \sigma^2_{i,t} = (1 - \alpha_i) \sum_{j=0}^{\tau-1} \alpha_i^j \log RV_{t-j-1}, \quad i = 1, \ldots, k \]

Model Labels: B(k) \equiv bivariate model of returns and the natural logarithm of RV with k-component volatility. B(2_s) \equiv bivariate model with 2-component volatility but the conditional mean prices the smooth volatility component.

<table>
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<tr>
<th>Model</th>
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<th>Volatility Fit</th>
<th>Premium Fit</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>LB^2(10)</td>
<td>LB(10)</td>
<td>KS</td>
</tr>
<tr>
<td>B(1)</td>
<td>-149.82</td>
<td>(0.16)</td>
<td>(0.30)</td>
<td>(0.29)</td>
</tr>
<tr>
<td>(q = k = 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B(2)</td>
<td>-146.00</td>
<td>(0.55)</td>
<td>(0.34)</td>
<td>(0.21)</td>
</tr>
<tr>
<td>(q = k = 2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B(2_s)</td>
<td>-144.45</td>
<td>(0.88)</td>
<td>(0.18)</td>
<td>(0.29)</td>
</tr>
<tr>
<td>(q = 1, k = 2)</td>
<td></td>
<td></td>
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L is the log-likelihood function. LB^2(10) is the Ljung and Box (1978) portmanteau test for serial correlation in the squared standardized residuals up to 10 lags; LB(10) is the same for the levels of standardized residuals; and KS is the Kiefer and Salmon (1983) test for departures from the Normal distribution for standardized residuals. We report p-values for these diagnostic tests. MAE \equiv mean absolute error; RMSE \equiv square root of the mean-squared error; R^2 from Mincer-Zarnowitz regressions applied to the variance and premium forecasts.
Figure 1: Annual Realized Equity Premium: 1803-2003

Figure 2: Annual Absolute Value Measure of Ex Post Volatility

Figure 3: Realized Volatility Measure of Ex Post Volatility
Figure 7: Volatility Forecast vs Realized: B(1) and B(2)

Figure 8: Volatility Component Forecasts: B(2) Model

Figure 9: Proportional Premium Comparisons
Figure 10: Proportional Premium Forecasts: 1990-2003
References


Campbell, J. Y., and S. B. Thompson (2004): “Predicting the Equity Premium Out of Sample: Can Anything Beat the Historical Average?,” manuscript, Harvard University.


