



Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Country portfolio dynamics

Michael B. Devereux^{a,*}, Alan Sutherland^{b,1}^a CEPR, NBER and Department of Economics, University of British Columbia, 997-1873 East Mall, Vancouver, BC, Canada V6T 1Z1^b CEPR and School of Economics and Finance, University of St Andrews, St Andrews, Fife KY16 9AL, UK

ARTICLE INFO

Article history:

Received 12 April 2009

Accepted 23 February 2010

Available online 24 March 2010

JEL classification:

E52

E58

F41

Keywords:

Country portfolios

Solution methods

ABSTRACT

This paper presents a general approximation method for characterizing time-varying equilibrium portfolios in a two-country dynamic general equilibrium model. The method can be easily adapted to most dynamic general equilibrium models, it applies to environments in which markets are complete or incomplete, and it can be used for models of any dimension. Moreover, the approximation provides simple, easily interpretable closed-form solutions for the dynamics of equilibrium portfolios.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

The process of financial globalization has led to an unprecedented increase in the size and complexity of gross financial positions and gross financial flows among countries. Lane and Milesi-Ferretti (2007a) argue that this increase in cross-border asset holdings may have significant implications for understanding the international transmission mechanism, the resolution of external imbalances, and the effects of macroeconomic policy.² Until quite recently, however, most open economy macroeconomic models have not included the analysis of the composition of gross country portfolios and gross capital flows, focusing instead on net foreign assets as a measure of a country's external position and the current account as a measure of financial flows. Probably the main reason for this neglect has been the technical difficulties faced in deriving optimal portfolio positions for general equilibrium models with incomplete markets, while at the same time retaining enough tractability to explore the responses to macroeconomic shocks and the effects of economic policy.³

This paper presents a general approximation method for characterizing time-varying equilibrium portfolios in a two-country dynamic general equilibrium model. The method can be easily adapted to most dynamic general equilibrium models, it applies to environments in which markets are complete or incomplete, and it can be used for models of any dimension. Moreover, the approximation provides simple, easily interpretable closed-form solutions for the dynamics of equilibrium portfolios.

* Corresponding author. Tel.: +1 604 822 2542; fax: +1 604 822 5915.

E-mail addresses: devm@interchange.ubc.ca (M.B. Devereux), ajs10@st-and.ac.uk (A. Sutherland).

¹ Tel.: +44 1334 462446; fax: +44 1334 462444.

² See also Lane and Milesi-Ferretti (2001, 2007b) and the subsequent work of Ghironi et al. (2009), Gourinchas and Rey (2007), and Tille (2003, 2008).

³ Engel and Matsumoto (2009) and Kollmann (2006) show how portfolio allocation problems can be analysed in open economy models with complete international financial markets. While this provides a valuable starting point for analysis, it is not a fully satisfactory approach, given the extensive evidence of incompleteness in international financial markets.

The approach presented in this paper follows the fundamental contribution of Samuelson (1970) in recognizing that successively higher-order aspects of portfolio behaviour may be captured by a higher degree of approximation of an investors objective function. We modify and adapt this approach to a dynamic stochastic general equilibrium (DSGE) environment, and derive simple formulae for equilibrium asset holdings which can be applied to any DSGE model that can be solved by standard first or second order approximation methods. Building on Devereux and Sutherland (2009), which shows how to obtain the zero-order (or steady state) portfolio holdings, we obtain expressions which fully characterize the way in which portfolio holdings evolve over time at the first order. For simple models, optimal portfolios may be derived analytically. For more complex models, the paper provides a simple, one step, computationally efficient approach to generating numerical results.⁴

The approach to characterizing portfolio dynamics here is based on Taylor-series approximation of a model's equilibrium conditions. The standard log-linear approximation procedures used in macroeconomics can not be directly applied to portfolio problems. This is for two reasons. First, the equilibrium portfolio is not determined by a first order approximation of a DSGE model. Second, the equilibrium portfolio is indeterminate in a non-stochastic steady state, a fact which appears to rule out the most natural choice of approximation point.⁵ The first problem can be overcome by considering higher-order approximations of the portfolio problem. The second problem can be overcome by treating the value of portfolio holdings at the approximation point as an unknown, to be determined endogenously as part of the solution. The procedure described in Devereux and Sutherland (2009) solves for portfolio holdings at the approximation point by looking at the first-order optimality conditions of the portfolio problem in the (stochastic) neighbourhood of the non-stochastic steady state.⁶

In general, a second-order approximation of the portfolio problem is sufficient to capture the different risk characteristics of assets. It is therefore sufficient to tie down a solution for steady-state portfolio holdings. However, in order to solve for the dynamic behaviour of asset holdings around the steady state portfolio, it is necessary to know how variations in state variables affect the risk characteristics of assets. This, in turn, requires consideration of a *third-order* approximation of the portfolio problem. A third-order approximation captures the first-order effect of state variables on second moments and thus makes it possible to understand how portfolios should be adjusted as state variables evolve. We show that a third-order approximation of the portfolio optimality conditions (used in combination with first and second-order approximations of the non-portfolio parts of the model) can be solved to yield an analytical formula which captures the dynamics of optimal country portfolios. We show that, even in its general form, this formula provides valuable insights into the fundamental factors that determine portfolio dynamics.

The general principles underlying the derivation of approximate solutions to portfolio problems were stated by Samuelson (1970). Using a static model of a portfolio problem for a single agent and exogenous returns, he showed that, in general, to derive the solution for portfolio holdings up to n th order accuracy, one has to approximate the portfolio problem up to order $n+2$. It is easy to see that our solution procedure follows this general principle. Our solution for the steady-state (or zero-order accurate) portfolio is derived using a second-order approximation of the portfolio optimality conditions, and our solution for the first-order accurate portfolio is derived using a third-order approximation of the portfolio optimality conditions. An important innovation of our procedure, relative to the principle established by Samuelson, is that, to derive n th-order accurate solutions for portfolios, only the portfolio optimality conditions need be approximated up to order $n+2$. The other optimality and equilibrium conditions of the model need only be approximated up to order $n+1$. This leads to a considerable simplification of the solution procedure.⁷

In a recent paper, Tille and van Wincoop (2009) use this same general set of principles to solve for the steady-state and first-order behaviour of country portfolios in an open economy model. The Tille and van Wincoop approach is identical to ours to the extent that, for any given model, the methods are based on solving the same set of equations. While we focus on an analytical approach, Tille and van Wincoop (2009) describe an iterative numerical algorithm which can be used to solve for the coefficients of the Taylor-series approximation for portfolio behaviour. It is straightforward to show that, for any

⁴ In the existing literature, a number of alternative approaches have been developed for analysing incomplete-markets models. Judd et al. (2002) and Evans and Hnatkovska (2005) present numerical algorithms for solving dynamic portfolio problems in general equilibrium. These methods are, however, very complex compared to our approach and represent a significant departure from standard DSGE solution methods. Devereux and Saito (2007) use a continuous time framework which allows some analytical solutions to be derived in a restricted class of models.

⁵ It is important to understand that these are two distinct problems. The first problem arises in the approximated form of the model *with* stochastic shocks, while the second arises in the non-approximated form of the model *without* stochastic shocks. In both cases the portfolio is indeterminate because all assets are identical. This arises in a first-order approximation because certainty equivalence holds. And it arises in the non-stochastic steady state because of the absence of stochastic shocks.

⁶ Judd (1998) and Judd and Guu (2001) show how the problem of portfolio indeterminacy in the non-stochastic steady state can be overcome by using a Bifurcation theorem in conjunction with the Implicit Function Theorem. The solution approach presented here relies on first and second-order approximations of the model, rather than the Implicit Function and Bifurcation Theorems, but the steady-state gross portfolio holdings derived using our technique correspond to the approximation point derived by the Judd and Guu method.

⁷ Samuelson focused on a simple partial equilibrium static model where time-variation in portfolio holdings is irrelevant (because there is only one time period). Samuelson's approach derives successively higher-order components of the solution for portfolio holdings using higher-order approximations of the model. We are considering a dynamic framework where portfolio holdings are potentially time varying and we show that time variation can be captured in the approximate solution by considering higher orders of approximation of the model. The logic of our approach is the same as Samuelson's despite the lack of time variation in Samuelson's model.

given model, the steady-state and dynamic portfolio behaviour generated using the Tille and van Wincoop approach is identical to the analytical solution supplied by our approach.

An advantage of the analytical approach is that it provides a formula which can be applied to a wide range of DSGE models.⁸ In many cases this formula may yield closed-form analytical solutions for equilibrium portfolios. Such solutions can provide important insights and intuitions which are not available from numerical solutions. In addition, the formula can be used to generate numerical results for more complex models without the need for iterative algorithms. Finally, by employing the formula for portfolio holdings derived below, the user does not actually have to undertake higher order approximations. That is, the solution for the zero order portfolio solution requires only a first order approximation of the model, and the first order solution requires only a second order approximation of the model.

The paper proceeds as follows. Section 2 describes the structure of a basic two-country two-asset model. Section 3 briefly reviews the Devereux and Sutherland (2009) derivation of the steady-state portfolio for this model. Section 4 describes the solution for the first-order dynamic behaviour of portfolio holdings around this steady state. Section 5 derives the solution for expected excess returns. Section 6 applies the method to a simple endowment economy with trade in nominal bonds. Section 7 concludes the paper.

2. A two-asset open-economy model

The solution procedure is developed in the context of a simple two-country dynamic general equilibrium model. To make the steps as transparent as possible, the model here is restricted to a case where only two assets are internationally traded. In addition, we assume that agents in each country consume an identical composite consumption good, so that purchasing power parity holds. Generalising the analysis to cases with many assets, many countries and asymmetries between countries, and cases where PPP does not hold, is straightforward.⁹ In order to develop the solution procedure, it is not necessary to set out the details of the whole model. Only the features necessary for portfolio choice need to be directly included. Other aspects of the model, such as the production structure and labour supply, can be neglected since they are not directly relevant for deriving the expressions for steady-state or first-order properties of portfolios.

It is assumed that the world consists of two countries, which will be referred to as the home country and the foreign country. The home country is assumed to produce a good (or a bundle of goods) with aggregate quantity denoted Y_H (which can be endogenous) and aggregate price P_H . Similarly the foreign country produces quantity Y_F of a foreign good (or bundle of goods) at price P_F^* . In what follows foreign currency prices are denoted with an asterisk.

Agents in the home country have a utility function of the form

$$U_t = E_t \sum_{\tau=t}^{\infty} \theta_{\tau} [u(C_{\tau}) + v(\cdot)] \quad (1)$$

where C is consumption and $u(C) = (C^{1-\rho})/(1-\rho)$. θ_{τ} is the discount factor, which is determined as follows:

$$\theta_{\tau+1} = \theta_{\tau} \beta(C_{A\tau}), \quad \theta_0 = 1$$

where C_A is aggregate home consumption and $0 < \beta(C_A) < 1$, $\beta'(C_A) \leq 0$. If $\beta(C_A)$ is a constant (i.e. $\beta'(C_A) = 0$) then the discount factor is exogenous. In this case, if international financial markets are incomplete, there is a unit root in the first-order approximated model. Although our solution approach works perfectly well in this case, there may be occasions where it is useful to eliminate the unit root. This can be achieved by allowing $\beta'(C_A) < 0$. Endogenising the discount factor in this way has no impact on the applicability of our solution approach. The function $v(\cdot)$ captures those parts of the preference function which are not relevant for the portfolio problem.¹⁰ The consumer price index for home agents is denoted P .

It is assumed that there are two assets and a vector of two gross returns (for holdings of assets from period $t-1$ to t) given by

$$r'_t = [r_{1,t} \ r_{2,t}]$$

Asset payoffs and asset prices are measured in terms of the aggregate consumption good (i.e. in units of C). Returns are defined to be the sum of the payoff of the asset and capital gains divided by the asset price. It is assumed that the vector of available assets is exogenous and predefined.

The budget constraint for home agents is given by

$$W_t = \alpha_{1,t-1} r_{1,t} + \alpha_{2,t-1} r_{2,t} + Y_t - C_t \quad (2)$$

⁸ We emphasise that the method is designed for use with *general equilibrium* models. The method relies on some of the equilibrium first-order properties of expected asset returns. Additional restrictions may need to be imposed before the method can be applied to a partial equilibrium setting where asset returns are specified exogenously.

⁹ Devereux and Sutherland (2009) develop the procedure for solving for the steady state portfolio in a much more general environment. Dedola and Lombardo (2009) show how to apply the method to models with many countries.

¹⁰ For convenience we adopt the CRRA functional form for $u(C)$ and assume that utility is additively separable in $u(C)$ and $v(\cdot)$. Generalising our approach to deal with alternative functional forms is straightforward.

where $\alpha_{1,t-1}$ and $\alpha_{2,t-1}$ are the real holdings of the two assets purchased at the end of period $t-1$ and brought into period t . It follows that

$$\alpha_{1,t-1} + \alpha_{2,t-1} = W_{t-1} \quad (3)$$

where W_{t-1} is net wealth at the end of period $t-1$.¹¹ In (2) Y is the total disposable income of home agents expressed in terms of the consumption good. Thus, Y may be given by $Y_H P_H / P + T$ where T is a fiscal transfer (or tax if negative).

The budget constraint can be re-written as

$$W_t = \alpha_{1,t-1} r_{x,t} + r_{2,t} W_{t-1} + Y_t - C_t \quad (4)$$

where

$$r_{x,t} = r_{1,t} - r_{2,t}$$

Here asset 2 is used as a numeraire and $r_{x,t}$ measures the “excess return” on asset 1.

At the end of each period agents select a portfolio of assets to hold into the following period. Thus, for instance, at the end of period t home agents select $\alpha_{1,t}$ to hold into period $t+1$. The first-order condition for the choice of $\alpha_{1,t}$ can be written in the following form:

$$E_t[u'(C_{t+1})r_{1,t+1}] = E_t[u'(C_{t+1})r_{2,t+1}] \quad (5)$$

Foreign agents face a similar portfolio allocation problem with a budget constraint given by

$$W_t^* = \alpha_{1,t-1}^* r_{x,t} + r_{2,t} W_{t-1}^* + Y_t^* - C_t^* \quad (6)$$

Foreign agents are assumed to have preferences similar to (1) so the first-order condition for foreign-country agents' choice of $\alpha_{1,t}^*$ is

$$E_t[u'(C_{t+1}^*)r_{1,t+1}] = E_t[u'(C_{t+1}^*)r_{2,t+1}] \quad (7)$$

To simplify notation, in what follows we will drop the subscript from $\alpha_{1,t}$ and simply refer to α_t . It should be understood, therefore, that $\alpha_{1,t} = \alpha_t$ and $\alpha_{2,t} = W_t - \alpha_t$.

In any given DSGE model, there will be a set of first-order conditions relating to intertemporal choice of consumption and labour supply for the home and foreign consumers and a set of first-order conditions for profit maximisation and factor demands for home and foreign producers. Taken as a whole, and combined with an appropriate set of equilibrium conditions for goods and factor markets, this full set of equations will define the general equilibrium of the model. As already explained, the details of these non-portfolio parts of the model are not necessary for the exposition of the solution method, so they are not shown explicitly at this stage. In what follows these omitted equations are simply referred to as the “non-portfolio equations” or the “non-portfolio equilibrium conditions” of the model.

The non-portfolio equations of the model will normally include some exogenous forcing variables. In the typical macroeconomic model these take the form of AR(1) processes which are driven by zero-mean innovations. In what follows, the matrix of second moments of the innovations is denoted Σ . As is the usual practice in the macroeconomic literature, the innovations are assumed to be i.i.d. Therefore, Σ is assumed to be non-time-varying. We further assume (although this is not necessary for our solution method to work) that all third moments of the vector of innovations are zero.

It is assumed that the innovations are symmetrically distributed in the interval $[-\varepsilon, \varepsilon]$. This is convenient because, given the general equilibrium nature of the models we are dealing with, it ensures that all exogenous and endogenous variables have a maximum deviation from their non-stochastic steady state values which is proportional to ε . ε can therefore be treated as an index of the size of deviations of any variable from its value at the point of approximation, and any residual in an equation approximated up to order n can be captured by a term denoted $O(\varepsilon^{n+1})$. This greatly simplifies notation without having any significant loss of generality.¹²

The solution procedure is based on a Taylor-series approximation of the model. The approximation is based around a point where the vector of non-portfolio variables is given by \bar{X} and portfolio holdings are given by $\bar{\alpha}$. In what follows a bar over a variable indicates its value at the approximation point and a hat indicates the log-deviation from the approximation point (except in the case of $\hat{\alpha}$, \hat{W} and \hat{r}_x , which are defined below).

3. Steady-state portfolios

This section briefly reviews our approach to solving for the steady-state portfolio, $\bar{\alpha}$.¹³ As already explained, a second-order approximation of the portfolio problem is sufficient to capture the different risk characteristics of assets and is

¹¹ We interpret W_t as the home country's net wealth, which represents its total net claims on the foreign country. Assets in this set-up are defined to be in zero net supply. Hence any income on durable assets, such as the income on (home) capital, is included as part of income, Y_t . Claims to capital may be traded indirectly, however, since the asset menu can include a security with the identical rate of return to the home capital stock. Our method for deriving portfolio dynamics works equally in the alternative approach, where wealth is defined in gross terms and some assets are in positive net supply. The present approach makes our derivations easier, however.

¹² If, alternatively, the innovations are assumed to have infinite support it is possible to use the concept of “order in probability” to denote residuals. See Kim et al. (2008) for further discussion of “order in probability”.

¹³ A more comprehensive coverage is contained in Devereux and Sutherland (2009).

therefore sufficient to tie down a solution for $\bar{\alpha}$. The solution for $\bar{\alpha}$ is defined to be the one which ensures that the second-order approximations of the portfolio optimality conditions (5) and (7) are satisfied within a neighbourhood of \bar{X} and $\bar{\alpha}$. We use the symmetric non-stochastic steady state of the model as the approximation point for non-portfolio variables. Thus $\bar{W} = 0$, $\bar{Y} = \bar{C}$ and $\bar{r}_1 = \bar{r}_2 = 1/\beta$. Note that this implies $\bar{r}_x = 0$. Since $\bar{W} = 0$, it also follows that $\bar{\alpha}_2 = -\bar{\alpha}_1 = -\bar{\alpha}$.

Taking a second-order approximation of the home-country portfolio first-order conditions yields

$$E_t[\hat{r}_{x,t+1} + \frac{1}{2}(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) - \rho \hat{C}_{t+1} \hat{r}_{x,t+1}] = O(\varepsilon^3) \quad (8)$$

where $\hat{r}_{x,t+1} = \hat{r}_{1,t+1} - \hat{r}_{2,t+1}$. Applying a similar procedure to the foreign first-order conditions yields

$$E_t[\hat{r}_{x,t+1} + \frac{1}{2}(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) - \rho \hat{C}_{t+1}^* \hat{r}_{x,t+1}] = O(\varepsilon^3) \quad (9)$$

The home and foreign optimality conditions, (8) and (9), can be combined to show that, in equilibrium, the following equations must hold:

$$E_t[(\hat{C}_{t+1} - \hat{C}_{t+1}^*) \hat{r}_{x,t+1}] = 0 + O(\varepsilon^3) \quad (10)$$

and

$$E[\hat{r}_{x,t+1}] = -\frac{1}{2} E[\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2] + \rho \frac{1}{2} E_t[(\hat{C}_{t+1} + \hat{C}_{t+1}^*) \hat{r}_{x,t+1}] + O(\varepsilon^3) \quad (11)$$

These two equations express the portfolio optimality conditions in a form which is particularly convenient for deriving equilibrium portfolio holdings and excess returns. Eq. (10) provides a set of equations which must be satisfied by equilibrium portfolio holdings. And Eq. (11) shows the corresponding set of equilibrium expected excess returns. Note that these equations define the behaviour of portfolios and expected excess returns in equilibrium. The behaviour of expected excess returns is therefore endogenously determined.

In order to evaluate the left hand side of Eq. (10) it is sufficient to derive expressions for the first-order behaviour of consumption and excess returns. This requires a first-order accurate solution for the non-portfolio parts of the model. Portfolio decisions affect the first-order solution of the non-portfolio parts of the model in a particularly simple way. This is for three reasons. First, portfolio decisions only enter the non-portfolio parts of the model via budget constraints. Second, the only aspect of the portfolio decision that enters a first-order approximation of the budget constraints is $\bar{\alpha}$, the steady-state portfolio. And third, to a first-order approximation, the portfolio excess return is a zero mean i.i.d. random variable.¹⁴

The fact that only the steady-state portfolio enters the first-order model can be illustrated by considering a first-order approximation of the home budget constraint.¹⁵ For period $t+1$ this is given by

$$\hat{W}_{t+1} = \frac{1}{\beta} \hat{W}_t + \hat{Y}_{t+1} - \hat{C}_{t+1} + \frac{\bar{\alpha}}{\beta \bar{Y}} \hat{r}_{x,t+1} + O(\varepsilon^2) \quad (12)$$

where $\hat{W}_t = (W_t - \bar{W})/\bar{C}$. Notice that the deviation of α from its steady-state value does not enter this equation because excess returns are zero in the steady state, i.e. $\bar{r}_x = 0$.

The fact that the portfolio excess return, $\bar{\alpha} \hat{r}_{x,t+1}$, is a zero-mean i.i.d. random variable follows from Eq. (11). This equation shows that, up to a first order, the expected excess return is zero. Deviations of the realised excess return, $\hat{r}_{x,t+1}$, from the expected excess return are therefore i.i.d. (and are a linear function of the i.i.d. innovations affecting the exogenous forcing processes of the model). The fact that the expected excess return is zero (up to a first order) arises endogenously from equilibrium in asset markets. This can be explained in economic terms by noting that, up to a first order, agents are indifferent about the risk characteristics of assets. Arbitrage therefore requires that, in equilibrium, all asset pay the same expected rate of return.¹⁶

These properties can now be used to derive a solution for $\bar{\alpha}$. In what follows, it proves convenient to define $\tilde{\alpha} \equiv \bar{\alpha}/(\beta \bar{Y})$ and to describe the solution procedure in terms of the solution for $\tilde{\alpha}$. The corresponding solution for $\bar{\alpha}$ is simply given by $\bar{\alpha} = \tilde{\alpha} \beta \bar{Y}$.

To derive a solution for $\tilde{\alpha}$ it is useful initially to treat the realised excess return on the portfolio as an exogenous independent mean-zero i.i.d. random variable denoted ξ_t . Thus, in (12), replace $(\bar{\alpha}/\beta \bar{Y}) \hat{r}_{x,t+1}$ by ξ_t . We can then incorporate (12) with $(\bar{\alpha}/\beta \bar{Y}) \hat{r}_{x,t+1}$ replaced by ξ_t , into the linear approximation to the rest of the non-portfolio equations of the model. As in any standard dynamic rational expectations model, we may summarise the entire first-order approximation (of the

¹⁴ In fact, the first of these properties is not critical for the implementation of our solution method. Our method can easily handle cases where the portfolio excess return (i.e. $\alpha_{1,t-1} r_{x,t}$) enters equations other than the budget constraint.

¹⁵ From Walras's law it follows that it is only necessary to consider one budget constraint.

¹⁶ The equality of expected returns (at the first order) is thus a natural endogenous feature of a very wide class of general equilibrium macro models. Our solution method can be adapted to handle cases where small trading frictions prevent arbitrage from ensuring equality of first-order expected returns. In such cases it would be necessary to work with returns appropriately adjusted for the costs of trading frictions. Note that, if our solution method is to be applied to a partial equilibrium model, where excess returns are specified exogenously, it would be necessary to impose (by assumption) the equality of expected excess returns at the first order level.

non-portfolio equations) as follows:

$$A_1 \begin{bmatrix} s_{t+1} \\ E_t[c_{t+1}] \end{bmatrix} = A_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + A_3 x_t + B \xi_t + O(\varepsilon^2) \quad (13)$$

$$x_t = N x_{t-1} + \varepsilon_t$$

where s is a vector of predetermined variables (including \hat{W}), c is a vector of jump variables (including \hat{C} , \hat{C}^* , and \hat{r}_x), x is a vector of exogenous forcing processes, ε is a vector of i.i.d. shocks, and B is a column vector with unity in the row corresponding to the equation for the evolution of net wealth (12) and zero in all other rows.¹⁷ The state-space solution to (13) can be derived using any standard solution method for linear rational expectations models and can be written as follows:

$$\begin{aligned} s_{t+1} &= F_1 x_t + F_2 s_t + F_3 \xi_t + O(\varepsilon^2) \\ c_t &= P_1 x_t + P_2 s_t + P_3 \xi_t + O(\varepsilon^2) \end{aligned} \quad (14)$$

This form of the solution shows explicitly, via the F_3 and P_3 matrices, how the first-order accurate behaviour of all the model's variables depend on exogenous i.i.d. innovations to net wealth.¹⁸

By extracting the appropriate rows from (14) it is possible to write the following expression for the first-order accurate relationship between excess returns, $\hat{r}_{x,t+1}$, and ε_{t+1} and ξ_{t+1}

$$\hat{r}_{x,t+1} = [R_1] \xi_{t+1} + [R_2]_i [\varepsilon_{t+1}]^i + O(\varepsilon^2) \quad (15)$$

where the matrices R_1 and R_2 are formed from the appropriate rows of (14).¹⁹ Similarly extracting the appropriate rows from (14) yields the following expression for the first-order behaviour of $(\hat{C}_{t+1} - \hat{C}_{t+1}^*)$:

$$(\hat{C}_{t+1} - \hat{C}_{t+1}^*) = [D_1] \xi_{t+1} + [D_2]_i [\varepsilon_{t+1}]^i + [D_3]_k [z_{t+1}]^k + O(\varepsilon^2) \quad (16)$$

where $z'_{t+1} = [x_t \ s_{t+1}]$ is a vector formed from the exogenous driving processes and the endogenous state variables. Expressions (15) and (16) are written using tensor notation (in the form described, for instance, by Judd, 1998; Juillard, 2003).²⁰ This notation will prove particularly useful in the next section, where higher-order approximations are considered.

Now recognize that the term ξ_{t+1} represents the home country's return on its portfolio, which depends on asset holdings and excess returns, i.e.

$$\xi_{t+1} = \tilde{\alpha} \hat{r}_{x,t+1}$$

Substituting into (15) and (16), we get

$$\hat{r}_{x,t+1} = [\tilde{R}_2]_i [\varepsilon_{t+1}]^i + O(\varepsilon^2) \quad (17)$$

$$(\hat{C}_{t+1} - \hat{C}_{t+1}^*) = [\tilde{D}_2]_i [\varepsilon_{t+1}]^i + [D_3]_k [z_{t+1}]^k + O(\varepsilon^2) \quad (18)$$

where

$$[\tilde{R}_2]_i = \frac{1}{1 - [R_1] \tilde{\alpha}} [R_2]_i \quad (19)$$

$$[\tilde{D}_2]_i = \left(\frac{[D_1] \tilde{\alpha}}{1 - [R_1] \tilde{\alpha}} [R_2]_i + [D_2]_i \right) \quad (20)$$

¹⁷ In more general cases, where the portfolio excess return enters the model in equations other than the budget constraint, the portfolio excess return should be replaced by ξ_t wherever it appears in the model. The vector B would then contain non-zero elements in all the rows corresponding to the equations where ξ_t appears.

¹⁸ When writing a model in the form of (13) we are following the convention that s_t contains the value of the s variables prior to the realisation of ε_t , while c_t and x_t contain the values of the c and x variables after the realisation of ε_t .

¹⁹ Note that, because $\hat{r}_{x,t+1}$ is a zero-mean i.i.d. variable up to first-order accuracy, (15) does not depend on the vector of state variables.

²⁰ For instance, a subscript or superscript i refers to the i th element of a vector. When a letter appears in a term, first as a subscript on one vector, and then as a superscript on another vector, it denotes the sum of the products of the respective terms in the two vectors. Thus $[A]_i [B]^i$ denotes the inner product of vectors A and B . (Note that we enclose the matrix names in square brackets in order to separate subscripts which are part of the matrix name from subscripts which refer to matrix elements.) When manipulating equations written in tensor notation one simply needs to keep in mind that a term such as $[A]_i [B]^i$ is just a convenient way of writing $\sum_n a_i b_i$ where a_i and b_i are the i th elements of A and B and n is the number of elements in A and B . Manipulations of tensor equations therefore obey all the rules of equations containing such summation terms. See Chapter 14 of Judd (1998) for more discussion.

Eqs. (17) and (18) now show how, for any given value of $\tilde{\alpha}$, consumption and excess returns depend on the vector of exogenous innovations, ε . Therefore, these expressions can be used to evaluate the left-hand side of (10) and thus to derive an expression for $\tilde{\alpha}$.

Note that, as shown in Devereux and Sutherland (2009), the second-order approximation of the portfolio problem is time invariant. Thus the time subscripts can be dropped in (10). Substituting (17) and (18) into (10) implies²¹

$$[\tilde{D}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij} = 0 \quad (21)$$

Finally substituting for $[\tilde{D}_2]_i$ and $[\tilde{R}_2]_j$ using (19) and (20) and solving for $\tilde{\alpha}$ yields

$$\tilde{\alpha} = \frac{[D_2]_i[R_2]_j[\Sigma]^{ij}}{([R_1][D_2]_i[R_2]_j - [D_1][R_2]_i[R_2]_j)[\Sigma]^{ij}} + O(\varepsilon) \quad (22)$$

This is the tensor-notation equivalent of the expression for $\tilde{\alpha}$ derived in Devereux and Sutherland (2009).

4. First-order time-variation in portfolios

The portfolio solution given in (22) is non-time-varying. This is because time variation in the true portfolio, α_t , has no effect on the properties of consumption, excess returns, or any other variable in the vector $[s, c]$, when evaluated up to first-order accuracy. But because we are modelling a dynamic environment where the portfolio choice decision is not identical in every period, the true portfolio will in general vary across periods. Thus, α_t will in general vary around $\bar{\alpha}$. In order to solve for the behaviour of asset holdings around $\bar{\alpha}$ it is necessary to know how the risk characteristics of assets are affected by the predictable evolution of state variables such as wealth, or persistent movements in output. To capture these effects, it is necessary to determine how these state variables affect the second moments that govern the optimal portfolio choice. This in turn requires consideration of a third-order approximation of the portfolio problem. A third-order approximation of the portfolio problem captures the effect of state variables on second moments and thus makes it possible to understand how portfolios should be adjusted as state variables evolve.

Taking a third-order approximation of the home and foreign country portfolio first-order conditions yields

$$E_t \left[\begin{aligned} &\hat{r}_{x,t+1} + \frac{1}{2}(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) + \frac{1}{6}(\hat{r}_{1,t+1}^3 - \hat{r}_{2,t+1}^3) \\ &- \rho \hat{C}_{t+1} \hat{r}_{x,t+1} + \frac{\rho^2}{2} \hat{C}_{t+1}^2 \hat{r}_{x,t+1} - \frac{\rho}{2} \hat{C}_{t+1} (\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) \end{aligned} \right] = 0 + O(\varepsilon^4) \quad (23)$$

$$E_t \left[\begin{aligned} &\hat{r}_{x,t+1} + \frac{1}{2}(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) + \frac{1}{6}(\hat{r}_{1,t+1}^3 - \hat{r}_{2,t+1}^3) \\ &- \rho \hat{C}_{t+1}^* \hat{r}_{x,t+1} + \frac{\rho^2}{2} \hat{C}_{t+1}^{*2} \hat{r}_{x,t+1} - \frac{\rho}{2} \hat{C}_{t+1}^* (\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) \end{aligned} \right] = 0 + O(\varepsilon^4) \quad (24)$$

Combining these two conditions implies that portfolio holdings must ensure that the following holds:

$$E_t \left[\begin{aligned} &-\rho(\hat{C}_{t+1} - \hat{C}_{t+1}^*) \hat{r}_{x,t+1} + \frac{\rho^2}{2} (\hat{C}_{t+1}^2 - \hat{C}_{t+1}^{*2}) \hat{r}_{x,t+1} \\ &-\frac{\rho}{2} (\hat{C}_{t+1} - \hat{C}_{t+1}^*) (\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) \end{aligned} \right] = 0 + O(\varepsilon^4) \quad (25)$$

while expected returns are given by

$$E_t[\hat{r}_{x,t+1}] = E_t \left[\begin{aligned} &-\frac{1}{2}(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) - \frac{1}{6}(\hat{r}_{1,t+1}^3 - \hat{r}_{2,t+1}^3) \\ &+ \frac{\rho}{2} (\hat{C}_{t+1} + \hat{C}_{t+1}^*) \hat{r}_{x,t+1} - \frac{\rho^2}{4} (\hat{C}_{t+1}^2 + \hat{C}_{t+1}^{*2}) \hat{r}_{x,t+1} \\ &+ \frac{\rho}{4} (\hat{C}_{t+1} + \hat{C}_{t+1}^*) (\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) \end{aligned} \right] + O(\varepsilon^4) \quad (26)$$

These are the third-order equivalents of (10) and (11).

Notice that (25) contains only second and third-order terms. Thus it is possible to evaluate the left-hand side of (25) using first and second-order accurate solutions for consumption and excess returns from the rest of the model. Second-order accurate solutions for the behaviour of consumption and excess returns can be obtained by solving a second-order approximation of the non-portfolio parts of the model.

As in the first-order case, it is possible to show that portfolio decisions affect the second-order solution of the non-portfolio parts of the model in a particularly simple way. In particular, as before, portfolio decisions only enter the

²¹ Here the tensor notation $[\tilde{D}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij}$ denotes the sum across all i and j of the product of the i th element of \tilde{D}_2 , the j th element of \tilde{R}_2 and the (ij) th element of Σ .

non-portfolio parts of the model via the portfolio excess return in budget constraints.²² Furthermore, the portfolio excess return (as it relates to the time varying element of the portfolio) is a zero mean i.i.d. random variable.

To see this, first take a second-order approximation of the home budget constraint as follows²³:

$$\hat{W}_{t+1} = \frac{1}{\beta} \hat{W}_t + \hat{Y}_{t+1} - \hat{C}_{t+1} + \tilde{\alpha} \hat{r}_{x,t+1} + \frac{1}{2} \hat{Y}_{t+1}^2 - \frac{1}{2} \hat{C}_{t+1}^2 + \frac{1}{2} \tilde{\alpha} (\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) + \hat{\alpha}_t \hat{r}_{x,t+1} + \frac{1}{\beta} \hat{W}_t \hat{r}_{2,t+1} + O(\varepsilon^3) \quad (27)$$

where

$$\hat{\alpha}_t = \frac{1}{\beta \bar{Y}} (\alpha_t - \bar{\alpha}) = \frac{\alpha_t}{\beta \bar{Y}} - \tilde{\alpha}$$

Here $\hat{\alpha}_t$ represents the (level) deviation in the portfolio holding from its steady state value (adjusted by $1/\beta \bar{Y}$). Note that the value of $\tilde{\alpha}$ in this equation is given by (22) (i.e. the steady-state portfolio calculated in the previous section), so it is not necessary to solve again for $\tilde{\alpha}$. Recall that, $\alpha_{1,t} = \alpha_t$ and that $\alpha_{1,t} + \alpha_{2,t} = W_t$ so

$$\hat{\alpha}_{1,t} = \hat{\alpha}_t, \quad \hat{\alpha}_{2,t} = (1/\beta) \hat{W}_t - \hat{\alpha}_t \quad (28)$$

The objective in this section is to solve for the behaviour of $\hat{\alpha}_t$. Movements in the optimal portfolio are determined by time-variation in the economic environment. It therefore follows that, up to a first-order approximation, movements in $\hat{\alpha}_t$ will be a linear function of the state variables of the model. We thus postulate that $\hat{\alpha}_t$ has the following functional form:

$$\hat{\alpha}_t = \gamma' z_{t+1} = [\gamma]_k [z_{t+1}]^k \quad (29)$$

where $z'_{t+1} = [x_t \ s_{t+1}]$.²⁴ Our objective is to solve for the vector of coefficients in this expression, i.e. γ .

This postulated functional form for the determination of $\hat{\alpha}_t$ implies that, from the point of view of period t , the value of z_{t+1} is known and thus $\hat{\alpha}_t$ is known. In turn, this implies that (as in the derivation of the steady-state portfolio) the realised excess return on (the time-varying element of) the portfolio, $\hat{\alpha}_t \hat{r}_{x,t+1}$, in period $t+1$ is a zero-mean i.i.d. random variable (up to second-order accuracy).²⁵ Bearing this in mind, the solution for γ can now be derived using a procedure which is very similar to the solution procedure for the steady-state portfolio.

As in the previous section, initially assume that the realised excess return on the time-varying part of the portfolio is an exogenous independent mean-zero i.i.d. random variable denoted ζ_t . The second-order approximation of the home country budget constraint in period t can therefore be written in the form

$$\hat{W}_t = \frac{1}{\beta} \hat{W}_{t-1} + \hat{Y}_t - \hat{C}_t + \tilde{\alpha} \hat{r}_{x,t} + \frac{1}{2} \hat{Y}_t^2 - \frac{1}{2} \hat{C}_t^2 + \frac{1}{2} \tilde{\alpha} (\hat{r}_{1,t}^2 - \hat{r}_{2,t}^2) + \zeta_t + \frac{1}{\beta} \hat{W}_{t-1} \hat{r}_{2,t} + O(\varepsilon^3) \quad (30)$$

where, again, the value of $\tilde{\alpha}$ in this equation is given by (22).²⁶ Now assume that the entire second-order approximation of the non-portfolio equations of the model can be summarised in a matrix system of the form

$$\tilde{A}_1 \begin{bmatrix} s_{t+1} \\ E_t[c_{t+1}] \end{bmatrix} = \tilde{A}_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + \tilde{A}_3 x_t + \tilde{A}_4 A_t + \tilde{A}_5 E_t[A_{t+1}] + B \zeta_t + O(\varepsilon^3) \quad (31)$$

$$x_t = N x_{t-1} + \varepsilon_t \quad (32)$$

$$A_t = \text{vech} \left(\begin{bmatrix} x_t \\ s_t \\ c_t \end{bmatrix} [x_t \ s_t \ c_t] \right) \quad (33)$$

where B is a column vector with unity in the row corresponding to the equation for the evolution of net wealth (30) and zero in all other rows.²⁷ This is the second-order analogue of (13), which was used in the derivation of the solution for the steady-state portfolio. However, note that in this case the coefficient matrices on the first-order terms differ from (13) because (31) incorporates the effects of the steady-state portfolio. This is indicated by the tildes over the matrices A_1, A_2, A_3, A_4 and A_5 .

²² Again, this particular property is not crucial for our procedure to work. It is simple to generalise our method to handle cases where the portfolio excess return enters other equations of the model.

²³ As before, Walras's law implies that we need only consider one budget constraint.

²⁴ Given that $\hat{\alpha}_t$ represents portfolio decisions made at the end of period t for holdings of assets into period $t+1$, it follows that $\hat{\alpha}_t$ will depend on the value of state variables observable at time t . In terms of the notational convention we follow, the relevant vector is therefore $[x_t \ s_{t+1}]$, i.e. the values of x and s prior to the realisation of ε_{t+1} .

²⁵ To see why this is the case, note that we are approximating $\hat{\alpha}_t \hat{r}_{x,t+1}$ in (27) only up to second-order accuracy. Because $\hat{\alpha}_t$ is a first-order variable, $\hat{r}_{x,t+1}$ is also measured up to first order. We have already shown that up to a first order, $\hat{r}_{x,t+1}$ is a mean zero i.i.d. variable.

²⁶ To clarify, Eq. (30) is formed by replacing $\hat{\alpha}_{t-1} \hat{r}_{x,t}$ with ζ_t .

²⁷ Note that A_t is a vectorised form of the matrix of cross products. The matrix of cross products is symmetric, so (33) uses the $\text{vech}(\cdot)$ operator, which converts a matrix into a vector by stacking the columns of its upper triangle. Note also that the form of Eq. (31) may not be general enough to encompass all dynamic general equilibrium models. For instance, some models may contain terms in the lagged value of A_t . Such terms can easily be incorporated into (31) without affecting our solution approach.

The state-space solution to this set of equations can be derived using any second-order solution method (see for instance [Lombardo and Sutherland, 2007](#)). By extracting the appropriate rows and columns from the state-space solution it is possible to write expressions for the second-order behaviour of $(\hat{C}-\hat{C}^*)$ and \hat{r}_x in the following form²⁸:

$$(\hat{C}-\hat{C}^*) = [\tilde{D}_0] + [\tilde{D}_1]\zeta + [\tilde{D}_2]_i[\varepsilon]^i + [\tilde{D}_3]_k[(z^f)^k + (z^s)^k] + [\tilde{D}_4]_{ij}[\varepsilon]^i[\varepsilon]^j + [\tilde{D}_5]_{ki}[\varepsilon]^i[z^f]^k + [\tilde{D}_6]_{ij}[\varepsilon]^i[z^f]^j + O(\varepsilon^3) \quad (34)$$

$$\hat{r}_x = [\tilde{R}_0] + [\tilde{R}_1]\zeta + [\tilde{R}_2]_i[\varepsilon]^i + [\tilde{R}_3]_k[(z^f)^k + (z^s)^k] + [\tilde{R}_4]_{ij}[\varepsilon]^i[\varepsilon]^j + [\tilde{R}_5]_{ki}[\varepsilon]^i[z^f]^k + [\tilde{R}_6]_{ij}[\varepsilon]^i[z^f]^j + O(\varepsilon^3) \quad (35)$$

where time subscripts have been omitted to simplify notation and z^f and z^s are, respectively, the first and second-order parts of the solution for z . These expressions are the second-order analogues of (15) and (16) (but note again that they incorporate the effects of the steady-state portfolio).²⁹ These expressions show how the second-order behaviour of $(\hat{C}-\hat{C}^*)$ and \hat{r}_x depend on the excess returns on the time-varying element of portfolios (represented by ζ) and the state variables and exogenous i.i.d. innovations.

As we noted above, up to first-order accuracy, the expected excess return is zero and, up to second-order accuracy, it is a constant with a value given by (11). This implies that $[\tilde{R}_3]_k[z^f]^k = 0$ and that the terms $[\tilde{R}_3]_k[z^s]^k$ and $[\tilde{R}_6]_{ij}[\varepsilon]^i[\varepsilon]^j$ are constants. It also follows that

$$[\tilde{R}_0] = E[\hat{r}_x] - [\tilde{R}_3]_k[z^s]^k - [\tilde{R}_4]_{ij}[\Sigma]^{ij} - [\tilde{R}_6]_{ij}[\varepsilon]^i[\varepsilon]^j$$

so

$$\hat{r}_x = E[\hat{r}_x] - [\tilde{R}_4]_{ij}[\Sigma]^{ij} + [\tilde{R}_1]\zeta + [\tilde{R}_2]_i[\varepsilon]^i + [\tilde{R}_4]_{ij}[\varepsilon]^i[\varepsilon]^j + [\tilde{R}_5]_{ki}[\varepsilon]^i[z^f]^k + O(\varepsilon^3) \quad (36)$$

Now recognize that ζ is endogenous and given by

$$\zeta = \hat{\alpha}\hat{r}_x = [\gamma]_k[z^f]^k\hat{r}_x$$

This is a second-order term, so \hat{r}_x can be replaced by the first-order parts of (36), that is, by the term $[\tilde{R}_2]_i[\varepsilon]^i$. This implies that

$$\zeta = [\gamma]_k[z^f]^k\hat{r}_x = [\tilde{R}_2]_i[\gamma]_k[\varepsilon]^i[z^f]^k \quad (37)$$

so (34) and (36) can be rewritten as follows:

$$(\hat{C}-\hat{C}^*) = [\tilde{D}_0] + [\tilde{D}_2]_i[\varepsilon]^i + [\tilde{D}_3]_k[(z^f)^k + (z^s)^k] + [\tilde{D}_4]_{ij}[\varepsilon]^i[\varepsilon]^j + ([\tilde{D}_5]_{ki} + [\tilde{D}_1][\tilde{R}_2]_i[\gamma]_k)[\varepsilon]^i[z^f]^k + [\tilde{D}_6]_{ij}[\varepsilon]^i[\varepsilon]^j + O(\varepsilon^3) \quad (38)$$

$$\hat{r}_x = E[\hat{r}_x] - [\tilde{R}_4]_{ij}[\Sigma]^{ij} + [\tilde{R}_2]_i[\varepsilon]^i + [\tilde{R}_4]_{ij}[\varepsilon]^i[\varepsilon]^j + ([\tilde{R}_5]_{ki} + [\tilde{R}_1][\tilde{R}_2]_i[\gamma]_k)[\varepsilon]^i[z^f]^k + O(\varepsilon^3) \quad (39)$$

These two expressions provide some of the components necessary to evaluate the left hand side of (25). The following expressions for the first-order behaviour of home and foreign consumption and the two asset returns are also required

$$\hat{C} = [\tilde{C}_2^H]_i[\varepsilon]^i + [\tilde{C}_3^H]_k[z^f]^k + O(\varepsilon^2), \quad \hat{C}^* = [\tilde{C}_2^F]_i[\varepsilon]^i + [\tilde{C}_3^F]_k[z^f]^k + O(\varepsilon^2) \quad (40)$$

$$\hat{r}_1 = [\tilde{R}_2^1]_i[\varepsilon]^i + [\tilde{R}_3^1]_k[z^f]^k + O(\varepsilon^2), \quad \hat{r}_2 = [\tilde{R}_2^2]_i[\varepsilon]^i + [\tilde{R}_3^2]_k[z^f]^k + O(\varepsilon^2) \quad (41)$$

where it should be noted that $[\tilde{R}_3^1]_k = [\tilde{R}_3^2]_k$. The coefficient matrices for these expressions can be formed by extracting the appropriate elements from the first-order parts of the solution to (31).

Substituting (38)–(41) into (25) and deleting terms of order higher than three yields

$$\begin{aligned} & [\tilde{D}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij} + (E[\hat{r}_x] - [\tilde{R}_4]_{ij}[\Sigma]^{ij})[\tilde{D}_3]_k[z^f]^k + [\tilde{R}_4]_{ij}[\tilde{D}_3]_k[\Sigma]^{ij}[z^f]^k + [\tilde{R}_2]_i([\tilde{D}_5]_{kj} + [\tilde{D}_1][\tilde{R}_2]_j[\gamma]_k)[\Sigma]^{ij}[z^f]^k + [\tilde{D}_2]_i([\tilde{R}_5]_{kj} \\ & + [\tilde{R}_1][\tilde{R}_2]_j[\gamma]_k)[\Sigma]^{ij}[z^f]^k - \rho[\tilde{R}_2]_i([\tilde{C}_2^H]_j[\tilde{C}_3^H]_k - [\tilde{C}_2^F]_j[\tilde{C}_3^F]_k)[\Sigma]^{ij}[z^f]^k + \frac{1}{2}([\tilde{R}_2^1]_i[\tilde{R}_2^1]_j - [\tilde{R}_2^2]_i[\tilde{R}_2^2]_j)[\tilde{D}_3]_k[\Sigma]^{ij}[z^f]^k \\ & + [\tilde{D}_2]_i[\tilde{R}_2]_j[\tilde{R}_3^1]_k[\Sigma]^{ij}[z^f]^k = 0 + O(\varepsilon^4) \end{aligned} \quad (42)$$

where use has been made of the fact that $[\tilde{D}_0]$ is a second-order term and that all third moments of ε are assumed to be zero.³⁰

The fact that solutions (34) and (35) are based on an approximation where the steady-state portfolio is given by (22) by definition implies that

$$[\tilde{D}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij} = 0 \quad (43)$$

²⁸ The appendix discusses the steps necessary to derive these equations from a state-space solution based on [Lombardo and Sutherland \(2007\)](#).

²⁹ Note that the matrices \tilde{R}_2 and \tilde{D}_2 in (34) and (35) will, in fact, be identical to the matrices defined by Eqs. (19) and (20) (which were derived in the process of solving for the steady state portfolio).

³⁰ The generalisation of the solution procedure to handle non-zero third moments is simply a matter of allowing for a constant term in the expression for \hat{z} . This constant term captures the first-order component of the steady state portfolio (which can be added to the zero-order component derived in the previous section to obtain a first-order accurate expression for the steady-state portfolio). It is this first-order constant term that would be identified by [Samuelsons \(1970\)](#) approach to approximating portfolio behaviour.

This implies that (42) is homogeneous in $[z]$. Thus, the following equation must be satisfied for all k :

$$\begin{aligned} & (E[\hat{r}_x] - [\tilde{R}_4]_{ij}[\Sigma]^{ij})[\tilde{D}_3]_k + [\tilde{R}_4]_{ij}[\tilde{D}_3]_k[\Sigma]^{ij} + [\tilde{R}_2]_i([\tilde{D}_5]_{kj} + [\tilde{D}_1][\tilde{R}_2]_j[\gamma]_k)[\Sigma]^{ij} + [\tilde{D}_2]_i([\tilde{R}_5]_{kj} + [\tilde{R}_1][\tilde{R}_2]_j[\gamma]_k)[\Sigma]^{ij} \\ & - \rho[\tilde{R}_2]_i([\tilde{C}_2^H]_{kj} - [\tilde{C}_2^F]_{kj})[\Sigma]^{ij} + \frac{1}{2}([\tilde{R}_2]_i[\tilde{R}_2]_j - [\tilde{R}_2]_i[\tilde{R}_2]_j)[\tilde{D}_3]_k[\Sigma]^{ij} + [\tilde{D}_2]_i[\tilde{R}_2]_j[\tilde{R}_3]_k[\Sigma]^{ij} = 0 + O(\varepsilon^3) \end{aligned} \quad (44)$$

Using (40) and (41) it is possible to write the following expression for expected excess returns:

$$E[\hat{r}_x] = \frac{1}{2}([\tilde{R}_2]_i[\tilde{R}_2]_j - [\tilde{R}_2]_i[\tilde{R}_2]_j) + \rho[\tilde{C}_2^H]_{ij}[\tilde{R}_2]_j + \rho[\tilde{C}_2^F]_{ij}[\tilde{R}_2]_j)[\Sigma]^{ij} + O(\varepsilon^3) \quad (45)$$

Substituting this into (44), using the fact that from (38) and (40), it must be that $[\tilde{D}_2] = [\tilde{C}_2^H] - [\tilde{C}_2^F]$, $[\tilde{D}_3] = [\tilde{C}_3^H] - [\tilde{C}_3^F]$, and simplifying yields

$$\begin{aligned} & -\frac{\rho}{2}[\tilde{D}_2]_i[\tilde{R}_2]_j([\tilde{C}_3^H]_{kj} + [\tilde{C}_3^F]_{kj})[\Sigma]^{ij} + [\tilde{R}_2]_i([\tilde{D}_5]_{kj} + [\tilde{D}_1][\tilde{R}_2]_j[\gamma]_k)[\Sigma]^{ij} + [\tilde{D}_2]_i([\tilde{R}_5]_{kj} \\ & + [\tilde{R}_1][\tilde{R}_2]_j[\gamma]_k)[\Sigma]^{ij} + [\tilde{D}_2]_i[\tilde{R}_2]_j[\tilde{R}_3]_k[\Sigma]^{ij} = 0 + O(\varepsilon^3) \end{aligned} \quad (46)$$

which, by applying (43), simplifies to

$$[\tilde{R}_2]_i([\tilde{D}_5]_{kj} + [\tilde{D}_1][\tilde{R}_2]_j[\gamma]_k)[\Sigma]^{ij} + [\tilde{D}_2]_i[\tilde{R}_5]_{kj}[\Sigma]^{ij} = 0 + O(\varepsilon^3) \quad (47)$$

which implies, for all k , that

$$\gamma_k = -\frac{([\tilde{R}_2]_i[\tilde{D}_5]_{kj}[\Sigma]^{ij} + [\tilde{D}_2]_i[\tilde{R}_5]_{kj}[\Sigma]^{ij})}{[\tilde{D}_1][\tilde{R}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij}} + O(\varepsilon) \quad (48)$$

which is our solution for γ .³¹ Eq. (48) expresses the solution for γ in terms of tensor notation. It can equivalently be stated in the form of a matrix expression, as follows:

$$\gamma' = -(\tilde{D}_1 \tilde{R}_2 \Sigma \tilde{R}_2')^{-1}(\tilde{R}_2 \Sigma \tilde{D}_5' + \tilde{D}_2 \Sigma \tilde{R}_5') + O(\varepsilon) \quad (49)$$

It should be emphasized that implementing this solution procedure requires *only* that the user apply (48), which needs only information from the second-order approximation of the model in order to construct the D and R matrices. So long as the model satisfies the general properties described in Section 2, the other details of the model, such as production, labour supply, and price setting can be varied without affecting the implementation. The derivations used to obtain (48) do not need to be repeated. In summary, the solution for equilibrium γ has three steps:

1. Solve the non-portfolio equations of the model in the form of (31) to yield a state-space solution.
2. Extract the appropriate rows from this solution to form Eqs. (34) and (35) and thus derive \tilde{D}_1 , \tilde{R}_2 , \tilde{D}_2 , \tilde{R}_5 and \tilde{D}_5 .
3. Calculate γ using (48) or (49).

What is the intuition behind expression (48)? When we evaluate the portfolio selection equation up to a third order, we can no longer describe the optimal portfolio choice as being determined by a constant covariance between $(\hat{C} - \hat{C}^*)$ and \hat{r}_x . Predictable movements in state variables will lead to time-variation in this covariance, and this requires changes in the optimal portfolio composition. Take for instance the first term in the numerator of (48), given by $[\tilde{R}_2]_i[\tilde{D}_5]_{kj}[\Sigma]^{ij}$. Looking at (34), we see that $[\tilde{D}_5]$ captures the way in which movements in state variables affect the response of the consumption difference to stochastic shocks. Since this leads to a predictable change in the covariance between the $(\hat{C} - \hat{C}^*)$ and \hat{r}_x so long as $[\tilde{R}_2]$ is non-zero, a compensating adjustment of the optimal portfolio is required. The other term in the numerator has a similar interpretation; predictable movements in the state variables affect the response of \hat{r}_x to stochastic shocks at the second order, and so long as $[\tilde{D}_2]$ is non-zero, this changes the covariance between $(\hat{C} - \hat{C}^*)$ and \hat{r}_x , and requires a change in the optimal portfolio.

5. Expected excess returns

Having derived an expression for γ , and thus an expression for $\hat{\alpha}$, it is now relatively simple also to solve for the dynamics of expected excess returns, $E[\hat{r}_x]$ using (26). Notice that (26) can be written as follows:

$$E_t[\hat{r}_{x,t+1} + \frac{1}{2}(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) + \frac{1}{6}(\hat{r}_{1,t+1}^3 - \hat{r}_{2,t+1}^3)] = \Gamma_t + O(\varepsilon^4) \quad (50)$$

where

$$\Gamma_t = E_t \left[\frac{\rho}{2}(\hat{C}_{t+1} + \hat{C}_{t+1}^*)\hat{r}_{x,t+1} - \frac{\rho^2}{4}(\hat{C}_{t+1}^2 + \hat{C}_{t+1}^{*2})\hat{r}_{x,t+1} + \frac{\rho}{4}(\hat{C}_{t+1} + \hat{C}_{t+1}^*)(\hat{r}_{1,t+1}^2 - \hat{r}_{2,t+1}^2) \right] \quad (51)$$

³¹ The error term in (48) is of order $O(\varepsilon)$. Thus the solution for γ is of the same order of approximation as the solution for $\hat{\alpha}$ (the steady state portfolio). Note, however, that the solution for $\hat{\alpha}$ will, nevertheless, be of first-order accuracy because $\hat{\alpha}$ depends on the (inner) product of γ and z , where the latter is evaluated up to first order accuracy.

In what follows we present a solution for Γ_t .³² We postulate that Γ_t is a linear function of the state variables, z , as follows:

$$\Gamma_t = \delta_0 + \delta' z_{t+1} \quad (52)$$

Notice that third-order evaluation of Γ requires first and second-order approximate expressions for \hat{r}_x and $\hat{C} + \hat{C}^*$. The second-order solution for \hat{r}_x is given in (39). The second-order solution for $\hat{C} + \hat{C}^*$ can be written in the following form:

$$(\hat{C} + \hat{C}^*) = [\tilde{G}_0] + [\tilde{G}_1]\xi + [\tilde{G}_2]_i[\varepsilon]^i + [\tilde{G}_3]_k([\mathcal{Z}^f]^k + [\mathcal{Z}^s]^k) + [\tilde{G}_4]_{ij}[\varepsilon]^i[\varepsilon]^j + [\tilde{G}_5]_{ki}[\varepsilon]^i[\mathcal{Z}^f]^k + [\tilde{G}_6]_{ij}[\mathcal{Z}^f]^i[\mathcal{Z}^f]^j + O(\varepsilon^3) \quad (53)$$

where, to simplify notation, time subscripts have been omitted. After substituting for ξ using (37), this becomes

$$(\hat{C} + \hat{C}^*) = [\tilde{G}_0] + [\tilde{G}_2]_i[\varepsilon]^i + [\tilde{G}_3]_k([\mathcal{Z}^f]^k + [\mathcal{Z}^s]^k) + [\tilde{G}_4]_{ij}[\varepsilon]^i[\varepsilon]^j + ([\tilde{G}_5]_{ki} + [\tilde{G}_1][\tilde{R}_2]_i[\gamma]_k)[\varepsilon]^i[\mathcal{Z}^f]^k + [\tilde{G}_6]_{ij}[\mathcal{Z}^f]^i[\mathcal{Z}^f]^j + O(\varepsilon^3) \quad (54)$$

For convenience we define

$$[\tilde{G}_5^A]_{ki} = [\tilde{G}_5]_{ki} + [\tilde{G}_1][\tilde{R}_2]_i[\gamma]_k$$

$$[\tilde{R}_5^A]_{ki} = [\tilde{R}_5]_{ki} + [\tilde{R}_1][\tilde{R}_2]_i[\gamma]_k$$

After substitution from (39)–(41) and (54), (51) becomes

$$\Gamma = \frac{\rho}{2} \left[\begin{aligned} & [\tilde{G}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij} + [\tilde{R}_5^A]_{ki}[\tilde{G}_2]_j[\Sigma]^{ij}[\mathcal{Z}^f]^k \\ & + [\tilde{G}_5^A]_{ki}[\tilde{R}_2]_j[\Sigma]^{ij}[\mathcal{Z}^f]^k + \frac{1}{2}[\tilde{G}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij}[\tilde{R}_3^1]_k[\mathcal{Z}^f]^k \\ & + [\tilde{G}_3]_k[\mathcal{Z}^f]^k E[\hat{r}_x] \\ & - \frac{\rho}{2}(2[\tilde{C}_2^H]_k[\tilde{C}_3^H]_k + 2[\tilde{C}_2^F]_i[\tilde{C}_3^F]_k)[\tilde{R}_2]_j[\Sigma]^{ij}[\mathcal{Z}^f]^k \\ & + \frac{1}{2}[\tilde{G}_3]_k[\mathcal{Z}^f]^k([\tilde{R}_2^1]_i[\tilde{R}_2^1]_j - [\tilde{R}_2^2]_i[\tilde{R}_2^2]_j)[\Sigma]^{ij} \end{aligned} \right] \quad (55)$$

where use has been made of the fact that $[\tilde{G}_0]$ is a second-order term and that all third moments of ε are assumed to be zero. Making use of (45) this can be simplified to

$$\Gamma = \frac{\rho}{2} \left[\begin{aligned} & [\tilde{G}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij} + [\tilde{R}_5^A]_{ki}[\tilde{G}_2]_j[\Sigma]^{ij}[\mathcal{Z}^f]^k \\ & + [\tilde{G}_5^A]_{ki}[\tilde{R}_2]_j[\Sigma]^{ij}[\mathcal{Z}^f]^k + [\tilde{G}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij}[\tilde{R}_3^1]_k[\mathcal{Z}^f]^k \\ & + \frac{\rho}{2}[\tilde{G}_3]_k[\tilde{G}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij}[\mathcal{Z}^f]^k \\ & - \frac{\rho}{2}(2[\tilde{C}_3^H]_k[\tilde{C}_2^H]_i + 2[\tilde{C}_3^F]_k[\tilde{C}_2^F]_i)[\tilde{R}_2]_j[\Sigma]^{ij}[\mathcal{Z}^f]^k \end{aligned} \right] \quad (56)$$

This expression can be further simplified by noting that, in equilibrium, $[\tilde{D}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij} = 0$ and

$$2[\tilde{C}_3^H]_k[\tilde{C}_2^H]_i + 2[\tilde{C}_3^F]_k[\tilde{C}_2^F]_i = [\tilde{D}_3]_k[\tilde{D}_2]_i + [\tilde{G}_3]_k[\tilde{G}_2]_i$$

hence

$$\Gamma = \frac{\rho}{2} \left[\begin{aligned} & [\tilde{G}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij} + [\tilde{R}_5^A]_{ki}[\tilde{G}_2]_j[\Sigma]^{ij}[\mathcal{Z}^f]^k \\ & + [\tilde{G}_5^A]_{ki}[\tilde{R}_2]_j[\Sigma]^{ij}[\mathcal{Z}^f]^k + [\tilde{G}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij}[\tilde{R}_3^1]_k[\mathcal{Z}^f]^k \end{aligned} \right] \quad (57)$$

It thus follows that

$$\delta_0 = \frac{\rho}{2}[\tilde{G}_2]_i[\tilde{R}_2]_j[\Sigma]^{ij} \quad (58)$$

$$\delta_k = \frac{\rho}{2}[\tilde{R}_5^A]_{ki}[\tilde{G}_2]_j[\Sigma]^{ij} + \frac{\rho}{2}[\tilde{G}_5^A]_{ki}[\tilde{R}_2]_j[\Sigma]^{ij} + \delta_0[\tilde{R}_3^1]_k \quad (59)$$

As before, it is not necessary to derive these expressions for each model. Having obtained a solution for γ via (49) it is simple to evaluate $[\tilde{R}_5^A]$ and $[\tilde{G}_5^A]$ and thus apply (58) and (59) to obtain δ_0 and δ .

Eqs. (58) and (59) show that expected excess returns, while zero at the first order, are not in general zero at higher orders. Eq. (58) shows that expected excess returns will contain a constant term which is second order and which depends on the second moments of model variables. This effectively captures the steady state risk premium in expected excess returns. Eq. (59) defines the coefficients which show how the risk premium varies through time in response to changes in state variables. The resulting time variation in expected excess returns occurs at the third order level because the coefficients, δ_k , are second order while the time variation in state variables in (52) is measured at the first order level.

³² It is straightforward to derive expressions for the terms $(\hat{r}_1^2 - \hat{r}_2^2)$ and $(\hat{r}_1^3 - \hat{r}_2^3)$ so, for the sake of brevity, we focus on the term Γ .

The steady state and time varying element in risk premia identified by these calculations are likely to be quite small in many standard macro models, unless one assumes that second moments are implausibly large and/or risk aversion is very strong (as measured by the parameter ρ). [Gourinchas and Rey \(2007\)](#) claim that excess returns on US assets have been quite large and vary systematically in response to shocks to the US current account. [Devereux and Sutherland \(2010\)](#) show that risk premia in a simple dynamic endowment model, calculated using the solution method described in this paper, are unlikely to be as large as suggested by the [Gourinchas and Rey \(2007\)](#) data. However, risk premia may be larger in models which allow for more general specifications of consumer preferences. Alternatively large measured excess returns may reflect trading frictions in financial markets. As previously noted, our solution method can be adapted to handle such frictions.

6. Example

The solution procedure is illustrated using a simple dynamic endowment model. This is a one-good, two-country economy where the utility of home households is given by

$$U_t = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{C_t^{1-\rho}}{1-\rho} \quad (60)$$

where C is consumption of the single good.³³ There is a similar utility function for foreign households. The home and foreign endowments of the single good are auto-regressive processes of the form

$$\log Y_t = \zeta_Y \log Y_{t-1} + \varepsilon_{Y,t}, \quad \log Y_t^* = \zeta_Y \log Y_{t-1}^* + \varepsilon_{Y^*,t} \quad (61)$$

where $0 \leq \zeta_Y \leq 1$ and ε_Y and ε_{Y^*} are i.i.d. shocks symmetrically distributed over the interval $[-\varepsilon, \varepsilon]$ with $\text{Var}[\varepsilon_Y] = \text{Var}[\varepsilon_{Y^*}] = \sigma_Y^2$. Asset trade is restricted to home and foreign nominal bonds. The budget constraint of home agents is given by

$$W_t = \alpha_{B,t-1} r_{B,t} + \alpha_{B^*,t-1} r_{B^*,t} + Y_t - C_t \quad (62)$$

where W is net wealth, α_B and α_{B^*} are holdings of home and foreign bonds and $r_{B,t}$ and $r_{B^*,t}$ are the real returns on bonds. Net wealth is the sum of bond holdings, i.e. $W_t = \alpha_{B,t} + \alpha_{B^*,t}$. Real returns on bonds are given by

$$r_{B,t} = R_{B,t} \frac{P_{t-1}}{P_t}, \quad r_{B^*,t} = R_{B^*,t} \frac{P_{t-1}^*}{P_t^*} \quad (63)$$

where P and P^* are home and foreign currency prices for the single tradeable good and R_B and R_{B^*} are the nominal returns on bonds. The law of one price holds so $P = SP^*$ where S is the nominal exchange rate (defined as the home currency price of foreign currency).

Consumer prices are assumed to be determined by simple quantity theory relations of the following form:

$$M_t = P_t Y_t, \quad M_t^* = P_t^* Y_t^* \quad (64)$$

where home and foreign money supplies, M and M^* , are assumed to be exogenous auto-regressive processes of the following form:

$$\log M_t = \log M_{t-1} + \varepsilon_{M,t}, \quad \log M_t^* = \log M_{t-1}^* + \varepsilon_{M^*,t} \quad (65)$$

where ε_M and ε_{M^*} are i.i.d. shocks symmetrically distributed over the interval $[-\varepsilon, \varepsilon]$ with $\text{Var}[\varepsilon_M] = \text{Var}[\varepsilon_{M^*}] = \sigma_M^2$.³⁴

The first-order conditions for home and foreign consumption and bond holdings are

$$C_t^{-\rho} = \beta E_t[C_{t+1}^{-\rho} r_{B^*,t+1}], \quad C_t^{*- \rho} = \beta E_t[C_{t+1}^{*- \rho} r_{B^*,t+1}] \quad (66)$$

$$E_t[C_{t+1}^{-\rho} r_{B,t+1}] = E_t[C_{t+1}^{-\rho} r_{B^*,t+1}], \quad E_t[C_{t+1}^{*- \rho} r_{B,t+1}] = E_t[C_{t+1}^{*- \rho} r_{B^*,t+1}] \quad (67)$$

Finally, equilibrium consumption plans must satisfy the resource constraint

$$C_t + C_t^* = Y_t + Y_t^* \quad (68)$$

To make the example easy, the shock processes are assumed to be independent from each other. There are four sources of shocks in this model and only two independent assets. Hence, assets markets are incomplete.

³³ In this example, we assume a constant time discount factor so as to allow for explicit algebraic solutions for portfolios.

³⁴ It would be more conventional to assume that the demand for real balances depends on consumption rather than output (as assumed in (64)). Our solution method can easily handle either assumption, but the formulation in (64) yields particularly simple results which are useful for demonstrating the application of the method.

6.1. Solution for steady-state bond holdings

Solving the model in linearised form and applying (22) yields the following expression for steady state bond holdings:

$$\tilde{\alpha}_B = -\tilde{\alpha}_{B^*} = -\frac{\sigma_Y^2}{2(\sigma_M^2 + \sigma_Y^2)(1 - \beta\zeta_Y)}$$

Home residents hold a gross negative position in home-currency bonds, because their real return (inversely related to the home price level) is positively correlated with home consumption.

6.2. Solution for first-order time-variation in bond holdings

Solving the model up to the second order, and applying the procedures described in Section 4, we obtain the following expressions:

$$\tilde{D}_1 = [2(1 - \beta)]$$

$$\tilde{R}_2 = [1 \quad -1 \quad -1 \quad 1], \quad \tilde{D}_2 = [\Theta - 2\Delta \quad 2\Delta - \Theta \quad 2\Delta \quad -2\Delta]$$

$$\tilde{R}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{D}_5 = \begin{bmatrix} \Delta & -\Delta & -\Delta & \Delta \\ \Delta & -\Delta & -\Delta & \Delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\Phi & \Phi & 0 & -2\Phi \end{bmatrix}$$

where $\Delta = (\beta - 1)\tilde{\alpha}_B$, $\Theta = (1 - \beta)/(1 - \beta\zeta_Y)$, $\Phi = (1 - \beta)/\beta$ and, for simplicity, we set $\rho = 1$. The vectors z_t and ε_t are defined as follows:

$$z_t = [\hat{Y}_{t-1} \quad \hat{Y}_{t-1}^* \quad \hat{M}_{t-1} \quad \hat{M}_{t-1}^* \quad \hat{W}_{t-1}]'$$

$$\varepsilon_t = [\varepsilon_{Y,t} \quad \varepsilon_{Y^*,t} \quad \varepsilon_{M,t} \quad \varepsilon_{M^*,t}]'$$

The solution for $\hat{\alpha}_{B,t}$ is

$$\hat{\alpha}_{B,t} = \gamma_1 \hat{Y}_t + \gamma_2 \hat{Y}_t^* + \gamma_3 \hat{M}_t + \gamma_4 \hat{M}_t^* + \gamma_5 \hat{W}_t / \beta \quad (69)$$

where

$$\gamma_1 = \gamma_2 = \frac{1}{2} \tilde{\alpha}_B, \quad \gamma_3 = \gamma_4 = 0, \quad \gamma_5 = \frac{1}{2}$$

Note that, from (28), it follows that the solution for $\hat{\alpha}_{B^*,t}$ is

$$\hat{\alpha}_{B^*,t} = -\gamma_1 \hat{Y}_t - \gamma_2 \hat{Y}_t^* - \gamma_3 \hat{M}_t - \gamma_4 \hat{M}_t^* + (1 - \gamma_5) \hat{W}_t / \beta \quad (70)$$

The intuition behind the time variation in portfolios in this example follows the logic of the previous section. Predictable movements in home income make the consumption difference $(\hat{C} - \hat{C}^*)$ more sensitive to stochastic shocks to home or foreign income, when evaluated up to a second order. This means that consumers in each country must increase the degree to which nominal bonds hedge consumption risk. So, for instance, in response to a predictable rise in home income, home consumption becomes more sensitive to home output shocks, at the second order. As a result home consumers increase their short position in home currency bonds. For the same reason, they increase their long position in foreign bonds. A predictable rise in foreign income has the same effect.

In this example, movements in net wealth are distributed equally among home and foreign currency bonds. Hence, as the home country's wealth increases, beginning in the symmetric steady state, it increases its holdings of both bonds, becoming less short in home currency bonds, and more long in foreign currency bonds. Of course the foreign country experiences exactly the opposite movement.

The expressions for $\hat{\alpha}_{B,t}$ and $\hat{\alpha}_{B^*,t}$ given in (69) and (70) can be used to study the dynamic response of bond holdings to shocks. Fig. 1 shows the response of home-country gross and net asset holdings to a persistent fall in home income.³⁵ Fig. 1 shows that the short-run impact of a persistent fall in Y is a large one-time *increase* in home-country net wealth. This comes from an (unanticipated) capital gain on the home portfolio, caused by a jump in P , given that home currency bonds are a liability for the home country.³⁶ But since the home endowment is persistently lower, net wealth subsequently falls and converges to a new (lower) steady state. The extent of the initial rise and subsequent fall in net wealth depends on the scale of the initial portfolio positions $\tilde{\alpha}_B$ and $\tilde{\alpha}_{B^*}$. As σ_M^2 falls relative to σ_Y^2 , steady state gross asset and liability positions are

³⁵ The figure is based on the following parameter values: $\beta = 0.98$, $\rho = 1.0$, $\zeta_Y = 0.9$, and $\sigma_Y^2 = \sigma_M^2$. Bond holdings are measured in terms of the deviation from steady-state value expressed as a percentage of steady-state income.

³⁶ An equivalent interpretation is that the home country gains from an exchange rate depreciation.

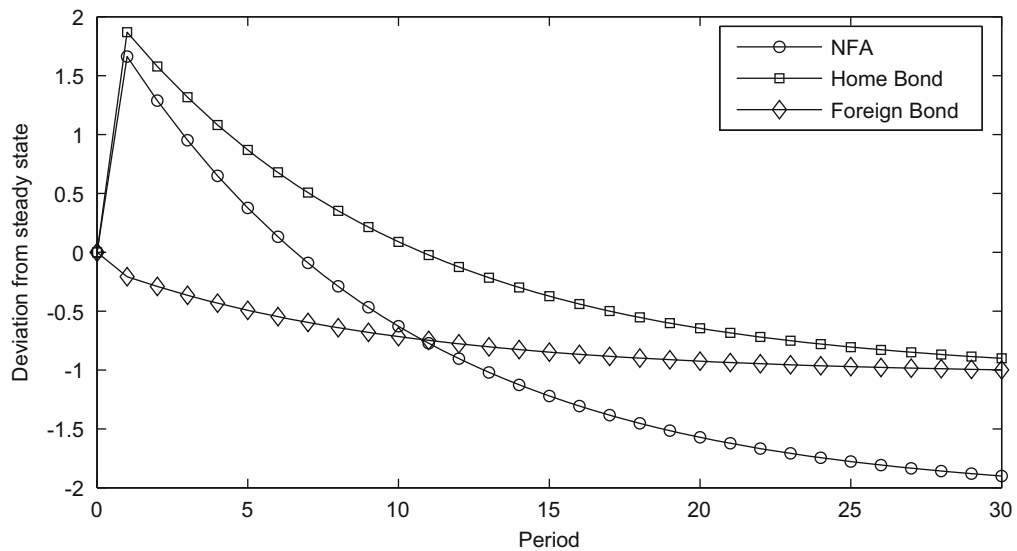


Fig. 1. Home country gross and net asset holdings.

higher. With greater leverage, the initial rise in net wealth then becomes larger, and the subsequent decline smaller, so that the response to a shock tends towards that under complete markets.

The movement in gross asset and liability positions are illustrated by the other plots in Fig. 1, which show how the time path for net wealth is divided between holdings of home and foreign bonds. The short run effect of the fall in Y is to cause a rise in the holdings of the home bond which is roughly equal in magnitude to the fall in net wealth. This can be understood by considering Eq. (69) which shows that the fall in Y and the rise in \hat{W} both imply that it is optimal for home agents to increase their holdings of home bonds. On the other hand, the shock to income has a much smaller short-run effect on home country holdings of the foreign bond because the fall in Y and the rise in W have offsetting effects on $\hat{\alpha}_{B^*}$, as can be seen from (70). After the initial shock, as net wealth gradually falls, the holdings of home bonds and foreign bonds both decline to new lower levels.

6.3. Solution for expected excess returns

The solution for expected excess returns can be obtained by applying the method described in Section 5. The second order solution of the model yields the following expressions for the matrices \tilde{R}_5^A , \tilde{G}_2 and \tilde{G}_5^A :

$$\tilde{R}_5^A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{G}_2 = [1 \ 1 \ 0 \ 0]$$

$$\tilde{G}_5^A = \begin{bmatrix} \zeta_Y \Delta \Theta + \Psi & -\zeta_Y \Delta \Theta - \Psi & -\zeta_Y \Delta \Theta & \zeta_Y \Delta \Theta \\ -\zeta_Y \Delta \Theta - \Psi & \zeta_Y \Delta \Theta + \Psi & \zeta_Y \Delta \Theta & -\zeta_Y \Delta \Theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (2\Delta - \Theta)\Phi & (\Theta - 2\Delta)\Phi & -2\Delta\Phi & 2\Delta\Phi \end{bmatrix}$$

where $\Psi = \zeta_Y(1 - \zeta_Y)\Theta(1 + \Theta)/2\Phi$.

Given the symmetric structure of this example model it is simple to show that

$$E_t[\frac{1}{2}(\hat{r}_{B,t+1}^2 - \hat{r}_{B^*,t+1}^2) + \frac{1}{6}(\hat{r}_{B,t+1}^3 - \hat{r}_{B^*,t+1}^3)] = 0 \quad (71)$$

so $E_t[\hat{r}_{B,t+1} - \hat{r}_{B^*,t+1}] = \Gamma_t + O(\varepsilon^4)$.

The expressions for \tilde{R}_2 , \tilde{R}_5^A , \tilde{G}_2 and \tilde{G}_5^A given above can be used to derive an expression for Γ_t in the following form:

$$\Gamma_t = \delta_0 + \delta_1 \hat{Y}_t + \delta_2 \hat{Y}_t^* + \delta_3 \hat{M}_t + \delta_4 \hat{M}_t^* + \delta_5 \hat{W}_t \quad (72)$$

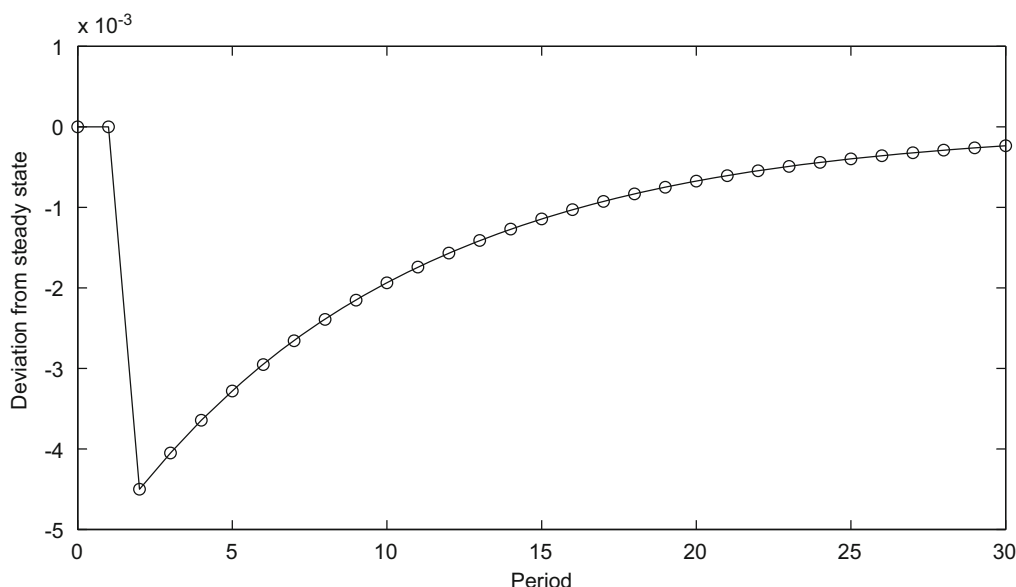


Fig. 2. Expected excess return on home bonds.

Applying (58) it is found that

$$\delta_0 = 0 \quad (73)$$

While applying (59) shows that

$$\delta_1 = -\delta_2 = \frac{1}{2}\zeta_Y\sigma_Y^2, \quad \delta_3 = \delta_4 = \delta_5 = 0 \quad (74)$$

The solutions (73) and (74) indicate that the unconditional expected excess return on home currency bonds, evaluated up to the third order, is zero. This is natural, since the model in this example is completely symmetric across countries. Shocks to either country's endowment, however, will lead to time variation in expected excess returns. In response to a persistent shock to the home (foreign) endowment, the expected excess return on home currency bonds increases (decreases).

Fig. 2 illustrates the impact of a fall in the home country endowment (the same shock as in Fig. 1) on expected excess returns on home currency bonds. In response to the fall in the home endowment, there is a shift in demand towards home currency bonds, and away from foreign currency bonds (as illustrated in Fig. 1). This is associated with a persistent fall in the expected return on home currency bonds relative to foreign currency bonds.³⁷ Since the home country has a negative position in home bonds, this anticipated fall in excess returns facilitates a continued portfolio transfer towards the home agent.³⁸ But the movement of expected excess returns in this example is very small, as shown in Fig. 2. Thus, following the negative shock to the home endowment, the anticipated excess return on the home country portfolio is negligible in comparison to the initial, unanticipated portfolio gain immediately following the shock.

7. Conclusion

This paper develops a simple analytical method for characterizing optimal equilibrium portfolios up to a first order in stochastic dynamic general equilibrium models. In addition to obtaining time-varying optimal portfolio holdings, the approach also gives a solution for time varying excess returns (or risk-premiums). There are a number of advantages of our approach relative to previous models of portfolio choice. First, the method is not restricted to situations of low dimensionality—we can use (49) to characterize portfolio holdings in any dynamic economic model in which it is practical to employ second-order solution methods. Second, as we have shown, the method applies equally to contexts where financial markets are either complete or incomplete. Third, the application of the formula does not actually require the user to go beyond a second-order solution to the underlying model. While, as we have shown, capturing first order aspects of portfolio behaviour requires a third-order approximation of the portfolio selection equations, all implications of that approximation are already contained in the derived expressions for the response of portfolio holdings to predictable state

³⁷ Using (23), (24) and (71) it is possible to show that third-order movements in $E_t[r_{x,t+1}]$ are equal (but opposite in sign) to third-order movements in the covariance between the stochastic discount factor (of both countries) and the excess return. Thus the decline in the excess return shown in Fig. 2 is matched by a rise (of equal magnitude) in the correlations between the home and foreign stochastic discount factors and the excess return.

³⁸ This is consistent with the empirical findings of Gourinchas and Rey (2007). They find that a decline in the US trade surplus is associated with a persistent increase in the expected excess return on the US external portfolio.

variables. The ingredients on the right hand side of (49) can all be obtained from a second-order approximation of the non-portfolio parts of the model.

More generally, an advantage of our general formula is that it can provide simple and clear insights into the factors which determine the dynamic evolution of portfolios and returns in general equilibrium. These insights may not always be easy to obtain using a purely numerical solution procedure.

Acknowledgements

We are grateful for comments and suggestions from the editor and two anonymous referees. This research is supported by the ESRC World Economy and Finance Programme, award number 156-25-0027. Devereux also thanks SSHRC, the Bank of Canada, and the Royal Bank of Canada for financial support. The views expressed in this paper are those of the authors, and do not represent the view of the Bank of Canada.

Appendix

A number of alternative solution algorithms are now available for obtaining second-order accurate solutions to DSGE models. See, for instance, Judd (1998), Jin and Judd (2002), Sims (2000), Kim et al. (2008), Schmitt-Grohé and Uribe (2004) and Lombardo and Sutherland (2007). For the purposes of implementing our solution procedure for portfolio dynamics, any of the methods described in this literature can be used to derive second-order accurate solutions to the non-portfolio parts of a model. Care must be taken, however, to ensure that the solution thus obtained is transformed into the correct format. As an example of the steps required to accomplish this, in this appendix we show how the Lombardo and Sutherland (2007) solution can be transformed into the required format. Similar steps can be used to transform the second-order solutions obtained by other methods.

It is assumed that the entire second-order approximation of the non-portfolio equations of the model can be summarised in a matrix system of the form

$$\tilde{A}_1 \begin{bmatrix} s_{t+1} \\ c_{t+1} \end{bmatrix} = \tilde{A}_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + \tilde{A}_3 x_t + \tilde{A}_4 A_t + \tilde{A}_5 E_t[A_{t+1}] + B \zeta_t + O(\varepsilon^3) \quad (75)$$

$$x_t = N x_{t-1} + \varepsilon_t \quad (76)$$

$$A_t = \text{vech} \left(\begin{bmatrix} x_t \\ s_t \\ c_t \end{bmatrix} \begin{bmatrix} x_t & s_t & c_t \end{bmatrix} \right) \quad (77)$$

Lombardo and Sutherland (2007) show that the solution to a system of this form can be written as follows:

$$s_{t+1} = \tilde{F}_1 x_t + \tilde{F}_2 s_t + \tilde{F}_3 \zeta_t + \tilde{F}_4 V_t + \tilde{F}_5 \text{vech}(\Sigma) + O(\varepsilon^3) \quad (78)$$

$$c_t = \tilde{P}_1 x_t + \tilde{P}_2 s_t + \tilde{P}_3 \zeta_t + \tilde{P}_4 V_t + \tilde{P}_5 \text{vech}(\Sigma) + O(\varepsilon^3) \quad (79)$$

where

$$\Sigma = E_t \varepsilon_{t+1} \varepsilon'_{t+1} \quad (80)$$

$$V_t \equiv \text{vech} \left(\begin{bmatrix} x_t \\ s_t^f \end{bmatrix} \begin{bmatrix} x_t & s_t^f \end{bmatrix} \right) \quad (81)$$

$$s_{t+1}^f = \tilde{F}_1 x_t + \tilde{F}_2 s_t^f + O(\varepsilon^2) \quad (82)$$

where the superscript f indicates the first-order part of the solution.

When written in this form, the solutions for s_{t+1} and c_t depend on x_t , s_t and the cross product of the vector $[x_t \ s_t^f]'$. And thus the solution for c_{t+1} depends on x_{t+1} , s_{t+1} and the cross product of the vector $[x_{t+1} \ s_{t+1}^f]'$. Notice, however, that the solutions for $\hat{c}_{t+1} - \hat{c}_{t+1}^*$ and $\hat{r}_{x,t+1}$, given in Eqs. (34) and (35), are expressed in terms of z_{t+1} and ε_{t+1} (where $z'_{t+1} = [x_t \ s_{t+1}]$) and cross products of z_{t+1}^f and ε_{t+1} . We show here how the solutions given in (78) and (79) can be re-written in the appropriate form.

First note that $[x_t \ s_t^f]'$ and z_t^f are related via the following equation:

$$\begin{bmatrix} x_t \\ s_t^f \end{bmatrix} = U_1 z_t^f + U_2 \varepsilon_t$$

where

$$U_1 = \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}, \quad U_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

It is thus possible to derive the following expression for V_t (where V_t is defined in (81)):

$$V_t = X_1 \text{vech}(\varepsilon_t \varepsilon_t') + X_2 \text{vec}(z_t^f \varepsilon_t') + X_3 \text{vech}(z_t^f z_t^{f'}) \quad (83)$$

where

$$X_1 = L^c U_2 \otimes U_2 L^h$$

$$X_2 = L^c [U_2 \otimes U_1 + U_1 \otimes U_2 P']$$

$$X_3 = L^c U_1 \otimes U_1 L^h$$

where the matrices L^c and L^h are conversion matrices such that

$$\text{vech}(\cdot) = L^c \text{vec}(\cdot)$$

$$L^h \text{vech}(\cdot) = \text{vec}(\cdot)$$

and P is a 'permutation matrix' such that, for any matrix Z ,³⁹

$$\text{vec}(Z) = P \text{vec}(Z')$$

Eqs. (76) and (83) can now be used to write (78) and (79) in the following form:

$$s_{t+1} = \tilde{F}_1 \varepsilon_t + [\tilde{F}_1 N, \tilde{F}_2] z_t + \tilde{F}_3 \zeta_t + \tilde{F}_4 X_1 \text{vech}(\varepsilon_t \varepsilon_t') + \tilde{F}_4 X_2 \text{vec}(z_t^f \varepsilon_t') + \tilde{F}_4 X_3 \text{vech}(z_t^f z_t^{f'}) + \tilde{F}_5 \text{vech}(\Sigma) + O(\varepsilon^3) \quad (84)$$

$$c_t = \tilde{P}_1 \varepsilon_t + [\tilde{P}_1 N, \tilde{P}_2] z_t + \tilde{P}_3 \zeta_t + \tilde{P}_4 X_1 \text{vech}(\varepsilon_t \varepsilon_t') + \tilde{P}_4 X_2 \text{vec}(z_t^f \varepsilon_t') + \tilde{P}_4 X_3 \text{vech}(z_t^f z_t^{f'}) + \tilde{P}_5 \text{vech}(\Sigma) + O(\varepsilon^3) \quad (85)$$

and thus

$$c_{t+1} = \tilde{P}_1 \varepsilon_{t+1} + [\tilde{P}_1 N, \tilde{P}_2] z_{t+1} + \tilde{P}_3 \zeta_{t+1} + \tilde{P}_4 X_1 \text{vech}(\varepsilon_{t+1} \varepsilon_{t+1}') + \tilde{P}_4 X_2 \text{vec}(z_{t+1}^f \varepsilon_{t+1}') + \tilde{P}_4 X_3 \text{vech}(z_{t+1}^f z_{t+1}^{f'}) + \tilde{P}_5 \text{vech}(\Sigma) + O(\varepsilon^3) \quad (86)$$

These expressions now express the solution to the non-portfolio parts of the model in a form which is appropriate for constructing Eqs. (34) and (35). So, for instance, if \hat{c} and \hat{c}^* are, respectively, the i th and j th elements of the vector c , then \tilde{D}_2 is formed from the difference between i th and j th rows of \tilde{P}_1 , while \tilde{D}_5 is formed from the difference between i th and j th rows of $\tilde{P}_4 X_2$. In the latter case, the row vector is transformed into the matrix \tilde{D}_5 using the $\text{vec}^{-1}(\cdot)$ operator.

References

- Devereux, M., Saito, M., 2007. A portfolio theory of international capital flows. Unpublished manuscript, UBC.
- Devereux, M., Sutherland, A., 2009. Country portfolios in open economy macro models. *Journal of the European Economic Association*, forthcoming.
- Devereux, M., Sutherland, A., 2010. Valuation effects and the dynamics of net external assets. *Journal of International Economics* 80, 129–143.
- Dedola, L., Lombardo, G., 2009. Solving for optimal portfolio dynamic choices with multiple agents and assets. Unpublished manuscript, European Central Bank.
- Engel, C., Matsumoto, A., 2009. The international diversification puzzle when goods prices are sticky: it's really about exchange-rate hedging, not equity portfolios. *American Economic Journal: Macroeconomics* 1, 155–188.
- Evans, M., Hnatkovska, V., 2005. International capital flows, returns and world financial integration. NBER Working Paper 11701.
- Ghironi, F., Lee, J., Rebucci, A., 2009. The valuation channel of external adjustment. IMF Working Paper No. 09/275.
- Gourinchas, P., Rey, H., 2007. International financial adjustment. *Journal of Political Economy* 115, 665–703.
- Jin, H.-H., Judd, K., 2002. Perturbation methods for general dynamic stochastic models. Unpublished manuscript, Stanford University.
- Judd, K., 1998. *Numerical Methods in Economics*. MIT Press, Cambridge.
- Judd, K., Guu, S.-M., 2001. Asymptotic methods for asset market equilibrium analysis. *Economic Theory* 18, 127–157.
- Judd, K., Kubler, F., Schmiedders, K., 2002. A solution method for incomplete asset markets with heterogeneous agents. Unpublished manuscript, Stanford University.
- Juillard, M., 2003. Solving stochastic dynamic equilibrium models: a k-order perturbation approach. Unpublished manuscript, University of Paris VIII.
- Kim, J., Kim, S., Schaumburg, E., Sims, C., 2008. Calculating and using second order accurate solutions of discrete time dynamic equilibrium models. *Journal of Economic Dynamics and Control* 32, 3397–3414.
- Kollmann, R., 2006. International portfolio equilibrium and the current account. CEPR Discussion Paper No 5512.
- Lombardo, G., Sutherland, A., 2007. Computing second-order-accurate solutions for rational expectation models using linear solution methods. *Journal of Economic Dynamics and Control* 31, 515–530.
- Lane, P., Milesi-Ferretti, G.-M., 2001. The external wealth of nations: measures of foreign assets and liabilities for industrial and developing countries. *Journal of International Economics* 55, 263–294.
- Lane, P., Milesi-Ferretti, G.-M., 2007a. A global perspective on external positions. In: Clarida, R. (Ed.), *G7 Current Account Imbalances: Sustainability and Adjustment*. University of Chicago Press, Chicago, pp. 67–102.

³⁹ Here the $\text{vec}(\cdot)$ operator converts a matrix into a vector by stacking its columns. See the Appendix to Lombardo and Sutherland (2007) for further discussion of the construction of these matrices.

- Lane, P., Milesi-Ferretti, G.-M., 2007b. The external wealth of nations mark II: revised and extended estimates of foreign assets and liabilities, 1970–2004. *Journal of International Economics* 73, 223–250.
- Samuelson, P., 1970. The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments. *Review of Economic Studies* 37, 537–542.
- Schmitt-Grohé, S., Uribe, M., 2004. Solving dynamic general equilibrium models using a second-order approximation to the policy function. *Journal of Economic Dynamics and Control* 28, 755–775.
- Sims, C., 2000. Second-order accurate solutions of discrete time dynamic equilibrium models. Unpublished manuscript, Princeton University.
- Tille, C., 2003. The impact of exchange rate movements on U.S. foreign debt. *Current Issues in Economics and Finance* 9, 1–7.
- Tille, C., 2008. Financial integration and the wealth effect of exchange rate fluctuations. *Journal of International Economics* 75, 283–294.
- Tille, C., van Wincoop, E., 2009. International capital flows. *Journal of International Economics*, forthcoming.