Solving for Country Portfolios: Notes

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These notes refer to the two-country endowment model used as an example in Devereux and Sutherland (2006, 2007). The model equations are summarised below in their exact and approximated forms.

Model Equations

- The budget constraint of home agents:

$$W_t = W_{t-1} r_{B^*,t} + Y_t - C_t + \alpha_{B,t-1}(r_{B,t} - r_{B^*,t}) \tag{1}$$

- Consumption Euler equations:

$$C_t^{-\rho} = \beta E_t \left[C_{t+1}^{-\rho} r_{B^*,t+1} \right] \qquad C_t^{*-\rho} = \beta E_t \left[C_{t+1}^{*-\rho} r_{B^*,t+1} \right] \tag{2}$$

- Money demand:

$$M_t = P_t Y_t, \qquad M_t^* = P_t^* Y_t^* \tag{3}$$

- Resource constraint:

$$C_t + C_t^* = Y_t + Y_t^* \tag{4}$$

- Return on bonds:

$$r_{B,t} = \frac{1}{Z_{t-1} P_t} \qquad r_{B^*,t} = \frac{1}{Z_{t-1}^* P_t^*} \tag{5}$$

- Bond prices:

$$Z_t = \beta E_t \left[C_{t+1}^{-\rho} P_{t+1}^{-1} \right] C_t^{\rho} \qquad Z_t^* = \beta E_t \left[C_{t+1}^{-\rho} P_{t+1}^{*-1} \right] C_t^{\rho} \tag{6}$$

- Endowments:

$$\log Y_t = \zeta_Y \log Y_{t-1} + \varepsilon_{Y,t} \qquad \log Y_t^* = \zeta_Y \log Y_{t-1}^* + \varepsilon_{Y^*,t} \tag{7}$$

- Money supplies:

$$\log M_t = \zeta_M \log M_{t-1} + \varepsilon_{M,t} \qquad \log M_t^* = \zeta_M \log M_{t-1}^* + \varepsilon_{M^*,t} \tag{8}$$

Approximated model equations

- The budget constraint of home agents:

$$\begin{aligned}\hat{W}_t = {} & \tfrac{1}{\beta}\hat{W}_{t-1} + \hat{Y}_t - \hat{C}_t + \tilde{\alpha}\hat{r}_{x,t} + \\ & \tfrac{1}{2}\left[\hat{Y}_t^2 - \hat{C}_t^2 + \tilde{\alpha}(\hat{r}_{B,t}^2 - \hat{r}_{B^*,t}^2)\right] + \hat{\alpha}_{t-1}\hat{r}_{x,t} + \tfrac{1}{\beta}\hat{W}_{t-1}\hat{r}_{B^*,t}\end{aligned} \tag{9}$$

- Consumption Euler equations (combined):

$$\begin{aligned}-\rho\hat{C}_t + \rho\hat{C}_t^* = {} & E_t\left[-\rho\hat{C}_{t+1} + \rho\hat{C}_{t+1}^*\right] + \\ & \tfrac{1}{2}E_t\left[\rho^2\hat{C}_{t+1}^2 + \hat{r}_{B^*,t+1}^2 - 2\rho\hat{C}_{t+1}\hat{r}_{B^*,t+1}\right] - \tfrac{1}{2}\rho^2\hat{C}_t^2 - \\ & \tfrac{1}{2}E_t\left[\rho^2\hat{C}_{t+1}^{*2} + \hat{r}_{B^*,t+1}^2 - 2\rho\hat{C}_{t+1}^*\hat{r}_{B^*,t+1}\right] + \tfrac{1}{2}\rho^2\hat{C}_t^{*2}\end{aligned} \tag{10}$$

- Money demand:

$$\hat{M}_t = \hat{P}_t + \hat{Y}_t \qquad \hat{M}_t^* = \hat{P}_t^* + \hat{Y}_t^* \tag{11}$$

- Resource constraint:

$$\hat{C}_t + \hat{C}_t^* = \hat{Y}_t + \hat{Y}_t^* + \frac{1}{2}(\hat{C}_t^2 + \hat{C}_t^{*2} - \hat{Y}_t^2 - \hat{Y}_t^{*2}) \tag{12}$$

- Return on bonds:

$$\hat{r}_{B,t} = -\hat{Z}_{t-1}^d - \hat{P}_t \qquad \hat{r}_{B^*,t} = -\hat{Z}_{t-1}^{d*} - \hat{P}_t^* \tag{13}$$

- Bond prices:

$$\begin{aligned}\hat{Z}_t = {} & E_t\left[-\rho\hat{C}_{t+1} - \hat{P}_{t+1}\right] + \rho\hat{C}_t + \\ & \tfrac{1}{2}E_t\left[\rho^2\hat{C}_{t+1}^2 + \hat{P}_{t+1}^2 - 2\rho\hat{C}_{t+1}\hat{P}_{t+1}\right] - \tfrac{1}{2}\left[\rho^2\hat{C}_t^2 + \hat{Z}_t^2 - 2\rho\hat{C}_t\hat{Z}_t\right]\end{aligned} \tag{14}$$

$$\begin{aligned}\hat{Z}_t^* = {} & E_t\left[-\rho\hat{C}_{t+1} - \hat{P}_{t+1}^*\right] + \rho\hat{C}_t + \\ & \tfrac{1}{2}E_t\left[\rho^2\hat{C}_{t+1}^2 + \hat{P}_{t+1}^{*2} - 2\rho\hat{C}_{t+1}\hat{P}_{t+1}^*\right] - \tfrac{1}{2}\left[\rho^2\hat{C}_t^2 + \hat{Z}_t^{*2} - 2\rho\hat{C}_t^*\hat{Z}_t^*\right]\end{aligned} \tag{15}$$

- Excess return and consumption difference:

$$\hat{r}_{x,t} = \hat{r}_{B,t} - \hat{r}_{B^*,t} \qquad \hat{C}_t^D = \hat{C}_t - \hat{C}_t^* \tag{16}$$

- Dummy variables:

$$\hat{Z}_t^d = \hat{Z}_t \qquad \hat{Z}_t^{d*} = \hat{Z}_t^* \tag{17}$$

- Endowments:

$$\hat{Y}_t = \zeta_Y\hat{Y}_{t-1} + \varepsilon_{Y,t} \qquad \hat{Y}_t^* = \zeta_Y\hat{Y}_{t-1}^* + \varepsilon_{Y^*,t} \tag{18}$$

- Money supplies:

$$\hat{M}_t = \zeta_M\hat{M}_{t-1} + \varepsilon_{M,t} \qquad \hat{M}_t^* = \zeta_M\hat{M}_{t-1}^* + \varepsilon_{M^*,t} \tag{19}$$

Zero-order portfolio

Devereux and Sutherland (2006) show that the solution for the zero-order component of portfolios, $\tilde{\alpha}$, can be derived using a first-order approximation of the model. This can be written in the form

$$A_1 \begin{bmatrix} s_{t+1} \\ E_t [c_{t+1}] \end{bmatrix} = A_2 \begin{bmatrix} s_t \\ c_t \end{bmatrix} + A_3 x_t + B\xi_t + O\left(\epsilon^2\right) \tag{20}$$

$$x_t = N x_{t-1} + \varepsilon_t \tag{21}$$

where the vectors s, c and x are defined as

$$s_t = \begin{bmatrix} \hat{W}_{t-1} & \hat{Z}^d_{t-1} & \hat{Z}^{d*}_{t-1} \end{bmatrix}' \tag{22}$$

$$c_t = \begin{bmatrix} \hat{C}_t & \hat{C}^*_t & \hat{P}_t & \hat{P}^*_t & \hat{r}_{B,t} & \hat{r}_{B^*,t} & \hat{Z}_t & \hat{Z}^*_t & \hat{r}_{x,t} & \hat{C}^D_t \end{bmatrix}' \tag{23}$$

$$x_t = \begin{bmatrix} \hat{Y}_t & \hat{Y}^*_t & \hat{M}_t & \hat{M}^*_t \end{bmatrix}' \tag{24}$$

and the coefficient matrices are constructed from the first-order parts of equations (9) to (17). The term $\tilde{\alpha}\hat{r}_{x,t}$ in the budget constraint is replaced with ξ_t. The solution to this system can be written in the form:

$$s_{t+1} = F_1 x_t + F_2 s_t + F_3 \xi_t + O\left(\epsilon^2\right) \tag{25}$$

$$c_t = P_1 x_t + P_2 s_t + P_3 \xi_t + O\left(\epsilon^2\right) \tag{26}$$

The solution procedure for $\tilde{\alpha}$ requires expressions for $\hat{r}_{x,t+1}$ and $\hat{C}^D_{t+1}$ of the following form

$$\hat{r}_{xt+1} = R_1 \xi_{t+1} + R_2 \varepsilon_{t+1} + O\left(\epsilon^2\right) \tag{27}$$

$$\hat{C}^D_{t+1} = D_1 \xi_{t+1} + D_2 \varepsilon_{t+1} + D_3 z_t + O\left(\epsilon^2\right) \tag{28}$$

where $z_t = \begin{bmatrix} x_t & s_{t+1} \end{bmatrix}'$. These expressions can be obtained by combining (21) and (26) to yield

$$c_{t+1} = Q_1 \xi_{t+1} + Q_2 \varepsilon_{t+1} + Q_3 z_t + O\left(\epsilon^2\right) \tag{29}$$

where

$$Q_1 = P_3 \qquad Q_2 = P_1 \qquad Q_3 = [P_1 N,\ P_2] \tag{30}$$

$\hat{r}_{x,t+1}$ is the ninth element of c_{t+1}, so R_1 and R_2 are the ninth rows of Q_1 and Q_2 respectively. Likewise, $\hat{C}^D_{t+1}$ is the tenth element of c_{t+1}, so D_1, D_2 and D_3 are the tenth rows of Q_1, Q_2 and Q_3 respectively.

First-order portfolio

Devereux and Sutherland (2007) show that the solution for the first-order component of portfolios, $\hat{\alpha}$, can be derived using a second-order approximation of the model. This can be written in the form

$$\tilde{A}_1 \left[\begin{array}{c} s_{t+1} \\ E_t\left[c_{t+1}\right] \end{array} \right] = \tilde{A}_2 \left[\begin{array}{c} s_t \\ c_t \end{array} \right] + \tilde{A}_3 x_t + \tilde{A}_4 \Lambda_t + \tilde{A}_5 E_t[\Lambda_{t+1}] + B\xi_t + O\left(\epsilon^3\right) \tag{31}$$

$$x_t = N x_{t-1} + \varepsilon_t \tag{32}$$

$$\Lambda_t = \mathrm{vech}\left(\left[\begin{array}{c} x_t \\ s_t \\ c_t \end{array} \right] \left[\begin{array}{ccc} x_t & s_t & c_t \end{array} \right] \right) \tag{33}$$

where again the vectors s, c and x are defined as above and the coefficient matrices are constructed from equations (9) to (17). The term $\hat{\alpha}_{t-1}\hat{r}_{x,t}$ in the budget constraint is replaced with ξ_t and $\tilde{\alpha}$ is set at the value derived using the first-order approximation of the model. The solution to this system can be written in the form:

$$s_{t+1} = \tilde{F}_1 x_t + \tilde{F}_2 s_t + \tilde{F}_3 \xi_t + \tilde{F}_4 V_t + \tilde{F}_5 \Sigma + O\left(\epsilon^3\right) \tag{34}$$

$$c_t = \tilde{P}_1 x_t + \tilde{P}_2 s_t + \tilde{P}_3 \xi_t + \tilde{P}_4 V_t + \tilde{P}_5 \Sigma + O\left(\epsilon^3\right) \tag{35}$$

The solution procedure for $\hat{\alpha}$ requires expressions for $\hat{r}_{x,t+1}$ and $\hat{C}^D_{t+1}$ of the following form

$$\begin{aligned} \hat{r}_x \quad = \quad & [\tilde{R}_0] + [\tilde{R}_1]\xi + [\tilde{R}_2]_i[\varepsilon]^i + [\tilde{R}_3]_k([z^f]^k + [z^s]^k) \\ & +[\tilde{R}_4]_{i,j}[\varepsilon]^i[\varepsilon]^j + [\tilde{R}_5]_{k,i}[\varepsilon]^i[z^f]^k + [\tilde{R}_6]_{i,j}[z^f]^i[z^f]^j + O\left(\epsilon^3\right) \end{aligned} \tag{36}$$

$$\begin{aligned} \hat{C}^D_{t+1} \quad = \quad & [\tilde{D}_0] + [\tilde{D}_1]\xi + [\tilde{D}_2]_i[\varepsilon]^i + [\tilde{D}_3]_k([z^f]^k + [z^s]^k) \\ & +[\tilde{D}_4]_{i,j}[\varepsilon]^i[\varepsilon]^j + [\tilde{D}_5]_{k,i}[\varepsilon]^i[z^f]^k + [\tilde{D}_6]_{i,j}[z^f]^i[z^f]^j + O\left(\epsilon^3\right) \end{aligned} \tag{37}$$

The appendix to Devereux and Sutherland (2007) shows that these expressions can be obtained by rewriting (35) as follows

$$\begin{aligned} & c = \tilde{Q}_0 + \tilde{Q}_1\xi + \tilde{Q}_2\varepsilon + \tilde{Q}_3(z^f + z^s) + \\ & \tilde{Q}_4 \mathrm{vech}\left(\varepsilon\varepsilon'\right) + \tilde{Q}_5 \mathrm{vec}\left(\varepsilon z^{f\prime}\right) + \tilde{Q}_6 \mathrm{vech}\left(z^f z^{f\prime}\right) + O\left(\epsilon^3\right) \end{aligned} \tag{38}$$

where time subscripts have been omitted and

$$\tilde{Q}_0 = \tilde{P}_5 \mathrm{vech}(\Sigma) \qquad \tilde{Q}_1 = \tilde{P}_3 \qquad \tilde{Q}_3 = [\tilde{P}_1 N,\ \tilde{P}_2] \qquad \tilde{Q}_4 = \tilde{P}_4 X_1 \qquad \tilde{Q}_5 = \tilde{P}_4 X_2 \qquad \tilde{Q}_6 = \tilde{P}_4 X_3 \tag{39}$$

$$X_1 = L^c U_2 \otimes U_2 L^h \qquad X_2 = L^c \left[U_2 \otimes U_1 + U_1 \otimes U_2 P' \right] \qquad X_3 = L^c U_1 \otimes U_1 L^h$$

$$U_1 = \left[\begin{array}{cc} N & 0 \\ 0 & I \end{array} \right], \qquad U_2 = \left[\begin{array}{c} I \\ 0 \end{array} \right]$$

The matrices L^c and L^h are conversion matrices such that

$$\mathrm{vech}(\cdot) = L^c\, \mathrm{vec}(\cdot), \qquad L^h \mathrm{vech}(\cdot) = \mathrm{vec}(\cdot)$$

and P is a 'permutation matrix' such that, for any matrix Z,

$$\mathrm{vec}(Z) = P \mathrm{vec}(Z')$$

$\hat{r}_x$ is the ninth element of c, so $\tilde{R}_1$, $\tilde{R}_2$, $\tilde{R}_3$, $\tilde{R}_4$, $\tilde{R}_5$ and $\tilde{R}_6$ are formed from the ninth rows of $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{Q}_3$, $\tilde{Q}_4$, $\tilde{Q}_5$ and $\tilde{Q}_6$ respectively. $\hat{C}^D_{t+1}$ is the tenth element of c, so $\tilde{D}_1$, $\tilde{D}_2$, $\tilde{D}_3$, $\tilde{D}_4$, $\tilde{D}_5$ and $\tilde{D}_6$ are formed from the tenth rows of $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{Q}_3$, $\tilde{Q}_4$, $\tilde{Q}_5$ and $\tilde{Q}_6$ respectively.

References

[1] Devereux, M and A Sutherland (2006) "Solving for Country Portfolios in Open Economy Macro Models" CEPR Discussion Paper No 5966.

[2] Devereux, M and A Sutherland (2007) "Country Portfolio Dynamics" CEPR Discussion Paper No 6208.

[3] Lombardo, G and A. Sutherland (2007) "Computing Second-Order-Accurate Solutions for Rational Expectation Models using Linear Solution Methods" *Journal of Economic Dynamics and Control*, 31, 515-530.