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Nonparametric Identification of First-Price Auction with Unobserved Competition: A Density Discontinuity Framework

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Abstract

We consider nonparametric identification of independent private value first-price auction models, in which the analyst only observes winning bids. Our benchmark model assumes an exogenous number of bidders N. We show that, if the bidders observe N, the resulting discontinuities in the winning bid density can be used to identify the distribution of N. The private value distribution can be nonparametrically identified in a second step. This extends, under testable identification conditions, to the case where N is a number of potential buyers, who bid with some unknown probability. Identification also holds in presence of additive unobserved heterogeneity drawn from some parametric distributions.

Keywords: Auction models, unobserved competition, nonparametric identification, density discontinuities, bidder uncertainty, unobserved heterogeneity.

JEL classification: C14, C57, D44

1 Introduction

Motivations. There exists a large literature on nonparametric identification of auction models; see, e.g., Athey and Haile (2007) or Hendricks and Porter (2007) for a review. In the case of sealed-bid first-price auctions, a vast majority of work assumes that the analyst can observe all of the bids, or both the winning bid and the number of competitors. This may not always be observed. In French timber auctions, for example, only the winning bid may be available to researchers to preserve bidder anonymity (Lamy, 2012). Indeed, it is common practice in many markets that can be treated as auctions for only the winning bid (i.e. the transaction price) to be recorded. For instance, a company soliciting price quotes for a task to be completed is implicitly organizing a first-price auction. While the company may not record all quotes or the number of responses, the price paid to the winning bidder is likely to appear in accounting records. "Bidding wars" are becoming commonplace in housing markets, where houses are sold through a competitive bidding process resembling an informal first-price auction, as noted in Han and Strange (2015). Governments may offer subsidies to attract firms, as recently considered by Kim and Yu (2024) and Slattery (2020) using an auction framework. Observing all the subsidy offers may be difficult, because states or firms may both have some interest in confidentiality. Hence, in many economic situations of interest, the records may only contain the final winning bid. Therefore, the ability to identify auction primitives solely from winning bid data may enlarge the scope of auction theory applications.

A second motivation stems from misspecification considerations. Indeed, structural estimation procedures of auctions crucially depend on the number of active buyers, which may differ from the observed number. For instance, Laffont, Ossard and Vuong (1995) consider an application where bidders are agents of several retail sellers, in which case the number of bids underestimates the level of competition.

Given 11 observed bids, these authors estimate the number of active buyers to be 18, causing important changes in the estimated structural parameters. Alternatively, some buyers may enter an auction simply to gain information, in which case their bids would be dominated and never impact the winning bid. Some bidders may collude and place phantom bids in an attempt to hide their cartel memberships; counting these bids would be misleading as it does not account for collusion. For instance, in many procurement settings, sellers can be legally constrained to stop auctions with less than three bidders in attendance. In this case, two buyers may be tempted to contact a third to submit a cover bid but ultimately allow the auction to go through. Econometric methods that check the effectiveness of bidder attendance can be a useful tool for regulators before proceeding to further possibly costly investigations. More generally, using only winning bids provides a robust approach for identifying primitives of interest when the truthfulness of observed bids is dubious.

Last, participation is a parameter of interest in itself. As noted by Bulow and Klemperer (1996), increasing competitive participation would yield higher seller expected revenues than choosing an optimal reserve price under the symmetric independent private value paradigm. In our setup, the number of active buyers is viewed as a latent random variable which can vary across auctions. Estimating its distribution and comparing it with the observed number of bids when available may be useful to detect participation anomalies.

Baseline model. Under the independent private value paradigm, buyers bid below their private values. The mechanism of the Bayesian Nash equilibrium implies in turn that the bid probability density function (pdf hereafter) is positive throughout its support, illustrating the attractiveness of such profitable bids. As a consequence, the bid pdf is positive at its upper bound and zero thereafter, resulting in a discontinuity at this point.

We develop a new approach for identification of first-price auction models that exploits associated discontinuities in the winning bid pdf. First, we identify the distribution of unknown competition. In particular, we build on an important restriction that first-price auction models impose on the data: the bid quantile function must be strictly increasing with respect to the number of bidders. Therefore, the upper boundary of the bid distribution, conditioning on the number of bidders, is strictly increasing, as well. As bid densities are discontinuous at these locations, this creates jumps at these upper boundaries in the winning bid pdf. A novel result of the paper is that these jumps identify the distribution of the number of bidders.

Second, we identify the value distribution function by iteratively exploiting two equilibrium mappings that relate the value and bid quantile functions. Based on the location of the discontinuities in the winning bid density function, we create a sequence of expanding quantile intervals over which the private value quantile is identified. For every iteration, we start by identifying the bid quantile function in the most competitive auction, which has the largest number of bidders. This information can then be used to identify the value quantile function in the same quantile interval and further calculate the corresponding bid quantile for other competition levels.

Buyer uncertainty and unobserved heterogeneity. The paper considers several extensions of the baseline model. Section 3 focuses on buyer uncertainty and auction heterogeneity. As the econometrician in the baseline model, buyers may also face unknown competition, as considered in Harstad, Kagel and Levin (1990) and Kong (2020), among others. Such uncertainty may arise due to the presence of a reserve price, as considered in Guerre, Perrigne and Vuong (2000), or entry costs, as in Li and Zheng (2009) or Marmer, Shneyerov and Xu (2013). We consider a setup where the number of potential bidders, N, is known by the buyers but not the effective one. We allow N to vary across auctions, and give conditions on its support

ensuring identification of its distribution together with the private value one and the probability that a potential buyer bids.

First-price auctions with unobserved heterogeneity affecting the auctioned good are considered in Krasnokutskaya (2011), and are especially challenging in our framework. Indeed, the presence of a continuous unobserved heterogeneity component washes discontinuities out of the winning bid density. Fortunately, considering its first and second derivatives allows for the identification of the participation distribution. We also show that some features of the unobserved component distribution can be recovered, so that it can be parametrically identified, raising hope for nonparametric identification of the private value distribution.

Related literature. Allowing for unknown competition started early in the empirical auction literature. Laffont et al. (1995) estimate the number of buyers N as a parameter that they take to be constant across auctions. Paarsch (1997) treats unknown competition as a nuisance parameter, which is eliminated using conditional likelihood estimation. For ascending eBay auctions, Song (2004) shows that the private value distribution and a constant number of buyers are identified from winning and second-highest bids, but not from winning bids alone when N is random. More pertinent to our paper is the misclassification approach of An, Hu and Shum (2010), who study identification from the winning bid using a proxy $N^* \leq N$ for the number of buyers and an instrument that can be a discretized second bid. Shneyerov and Wong (2011) suppose that only winning bids and the number of active bidders are observed. Some recent work on ascending auctions with unobserved heterogeneity uses the winning bid plus additional information for identification when the number of bidders is imperfectly observed or unobserved. More specifically, Freyberger and Larsen (2022) and Luo and Xiao (2023) use additional bids, while Hernández, Quint and Turansick (2020) assumes that a "participation shifter" is available.

The present paper contributes to the literature on nonparametric identification of finite mixtures; see for instance the review of Compiani and Kitamura (2016). Existing identification results require either exclusion restrictions or multiple independent measurements. A first-price auction example of the latter can be found in D'Haultfœuille and Février (2015), who recover the distribution of an unobserved continuous auction characteristic from three bids. Our approach only uses the winning bid.

The discontinuity design (DD) literature has expanded rapidly in recent years; interested readers are encouraged to refer to review papers by Imbens and Lemieux (2008), Kleven (2016) and Jales and Yu (2017). Recent auction applications include Coviello and Marinello (2014), Choi, Neisheim and Rasul (2016) and Kawai, Nakabayashi, Ortner and Chassang (2023). As in the DD literature, this paper employs jump sizes for identification purposes — more specifically, to identify the participation distribution. However the structural nature of our paper departs from this literature.

The remainder of the paper is organized as follows. Section 2 presents the identification results for the baseline model. Section 3 extends the analysis to account for buyer uncertainty and unobserved heterogeneity. Section 4 concludes. Appendix A collects the remaining proofs.

2 The benchmark model

In this section, we start by describing the benchmark auction model and introduce two equilibrium mappings that are convenient for describing our discontinuity identification strategy. Next, we derive the restrictions that the model imposes on the observed winning bids, especially with respect to the formation of discontinuities. Finally, we describe our identification strategy in two steps. First, we identify the distribution

of the number of buyers from the discontinuities in the winning bid density function. Second, we identify the value distribution function using the two equilibrium mappings iteratively.

2.1 The symmetric independent private values paradigm

Suppose there is a single item for sale with N active symmetric buyers bidding for the item. All buyers observe N. In contrast, the analyst does not observe N, which causes auction-specific unobserved heterogeneity. Each buyer i also observes her private valuation V_i, which is unknown to other buyers. The private values V_i are *i.i.d.* draws from a distribution $F(\cdot)$, which is known to all the buyers and is independent of N. The buyers are risk neutral and their bids B_i are formed according to a symmetric best-response strategy. In sum, the primitives are the distribution of the number of buyers N and the private value distribution.

We assume that the analyst only observes the winning bid W, i.e., the maximum bid among the N buyers in the set $\mathcal{N}$ of active buyers

$$W = \max_{i \in \mathcal{N}} B_i.$$

Hence, the analyst observes draws from the unconditional cumulative probability distribution of the winning bid $G(\cdot)$, which is a mixture of the conditional winning bid distributions given N:

$$G(b) = \sum_{n=2}^{+\infty} \mathbb{P}\left(\max_{1 \leq i \leq n} B_i \leq b\right) \times \mathbb{P}(N = n) = \sum_{n=2}^{+\infty} G_n^n(b)\, \mathbb{P}(N = n), \qquad (1)$$

where $G_n(\cdot)$ is the conditional bid distribution given $N = n$.

The following two assumptions introduce some additional conditions for the distribution of N and for the private value distribution $F(\cdot)$.

Assumption N. *The number of active buyers N is a discrete random variable*

with support $\{\underline{n}, \ldots, \overline{n}\}$ *for some integers* $2 \leq \underline{n} \leq \overline{n} < \infty$, *i.e.,* $p_n = \mathbb{P}(N = n) > 0$ *for* $n = \underline{n}, \underline{n}+1, \ldots, \overline{n}$ *with* $\sum_{n=\underline{n}}^{\overline{n}} p_n = 1$.

Assumption IPV. *Buyers' private values* V_i *are i.i.d. draws from a common knowledge distribution* $F(\cdot)$ *and are unknown to competitors. The cumulative distribution function* $F(\cdot)$ *has a compact support* $[\underline{v}, \overline{v}]$. *Its probability density function* $f(\cdot)$ *is continuous and strictly positive over* $[\underline{v}, \overline{v}]$.

Both the theoretical and empirical literature adopt the assumption of a private value distribution with compact support. In particular, it rules out multiple asymmetric equilibria; see Maskin and Riley (1984, Remark 2.3), who also establish that symmetric Bayesian Nash Equilibrium bids are given by a strictly increasing and continuously differentiable function of private values.

For our discontinuity approach, the compact support assumption ensures the existence of discontinuities in the density of unconditional winning bids that we exploit in this paper. In particular, the winning bid densities $g_n(\cdot)$ given $N = n$ stay bounded away from 0 at the upper boundary of its support; see (7) below.

That the private value pdf $f(\cdot)$ is positive over $[\underline{v}, \overline{v}]$ is a current assumption in the auction literature, as seen in Maskin and Riley (1984), Lebrun (1999), Guerre et al. (2000), among others. It is used here to identify the lowest number $\underline{n}$ of active buyers; see Lemma 2.1-(iii) below. The alternative identification method of Section 3.1 allows us to relax this condition as noted in Footnote 4.

2.2 Bid and value quantile equilibrium mappings

In this subsection, we describe two equilibrium mappings that are repeatedly used in our identification procedure. Specifically, there is an equilibrium mapping from the value distribution to the bid distribution, and vice versa. Our discontinuity

identification strategy is conveniently described using the quantile framework as in Guerre, Perrigne and Vuong (2009), Liu and Luo (2017), and Guerre and Gimenes (2022), that we recount below.

Let $V(\alpha) = F^{-1}(\alpha)$ represent the private value quantile function, where $\alpha \in [0,1]$ is the quantile level. Let $B_n(\alpha)$ denote the bid quantile function given that n buyers participate in the auction, and set $B_n^{(1)}(\alpha) = \frac{\partial B_n(\alpha)}{\partial \alpha}$. Following Milgrom (2001)'s exposition of the identification strategy of Guerre, Perrigne and Vuong (2000), the private value quantile function $V(\cdot)$ can be viewed as the common valuation function of buyers who receive independent uniform private signals

$$A_i = F(V_i),$$

which determines their private values $V_i = V(A_i)$. By Assumption IPV, $B_i = \beta_n(A_i)$ for all i, where $\beta_n(\cdot)$ is strictly increasing and continuously differentiable. It follows that for any b in the range of $\beta_n(\cdot)$,

$$G_n(b) = \mathbb{P}(\beta_n(A_i) \leq b) = \mathbb{P}\left(A_i \leq \beta_n^{-1}(b)\right) = \beta_n^{-1}(b),$$

because A_i is uniformly distributed over $[0,1]$. Hence, the best-response strategy is the bid quantile function

$$\beta_n(\alpha) = B_n(\alpha) \text{ for all } \alpha \in [0,1].$$

Now, let us relate the bid and private value quantile functions. Suppose that buyer i receives signal α but makes a generic bid $B_n(a)$ for some $a \in [0,1]$. Since her opponents bid $B_n(A_j)$, the probability that her bid $B_n(a)$ wins the auction is given by $\mathbb{P}(\max_{j \neq i} A_j \leq a)$, which is equal to a^{n-1} as the signals of the $n-1$ opponents A_j, where $j \neq i$, are independent and uniform. It follows that the expected payoff of

buyer i is $\left(V\left(\alpha\right)-B_{n}\left(a\right)\right)a^{n-1}$, which is maximized when $a=\alpha$. Since

$$\begin{aligned}\frac{\partial}{\partial a}\left[\left(V\left(\alpha\right)-B_{n}\left(a\right)\right)a^{n-1}\right]\Big|_{a=\alpha} &= V\left(\alpha\right)\left(n-1\right)\alpha^{n-2}-\frac{\partial\left[B_{n}\left(\alpha\right)\alpha^{n-1}\right]}{\partial\alpha}\\ &= \left(n-1\right)\alpha^{n-2}\left(V\left(\alpha\right)-B_{n}\left(\alpha\right)-\frac{\alpha B_{n}^{(1)}\left(\alpha\right)}{n-1}\right),\end{aligned}$$

setting this derivative to 0 gives [1]

$$V\left(\alpha\right)=B_{n}\left(\alpha\right)+\frac{\alpha B_{n}^{(1)}\left(\alpha\right)}{n-1}. \tag{2}$$

This constitutes the equilibrium mapping from the bid quantile function to the value quantile function, which is the basis of the identification of $V(\cdot)$ with knowledge of $B_n(\cdot)$.

Now, let us consider the inverse of the mapping (2). Indeed, (2) is equivalent to $\frac{\partial\left[B_{n}(\alpha)\alpha^{n-1}\right]}{\partial\alpha}=V\left(\alpha\right)\left(n-1\right)\alpha^{n-2}$, and it follows

$$B_{n}\left(\alpha\right)=\frac{n-1}{\alpha^{n-1}}\int_{0}^{\alpha}t^{n-2}V(t)dt. \tag{3}$$

For convenience of identification that will be clarified later on, let us introduce the conditional bid upper bound

$$\overline{b}_{n}=B_{n}(1)=\left(n-1\right)\int_{0}^{1}t^{n-2}V(t)dt,$$

which gives

$$B_{n}(\alpha)=\frac{n-1}{\alpha^{n-1}}\left[\frac{\overline{b}_{n}}{n-1}-\int_{\alpha}^{1}t^{n-2}V\left(t\right)dt\right]. \tag{4}$$

This constitutes the equilibrium mapping from the value quantile to the bid quantile function. The two mappings represented in (2) and (4) are repeatedly used in our identification procedure.

[1] It gives equivalently that $B_{n}^{(1)}(\alpha)=(V(\alpha)-B_{n}(\alpha))\cdot\frac{n-1}{\alpha}<\infty$. As $g_{n}(b)=1/B_{n}^{(1)}\left[G_{n}(b)\right]$, it follows that a Bayesian Nash equilibrium bid pdf $g_n(\cdot)$ cannot vanish on $(\underline{v},\overline{b}_{n}]$, as mentioned in the introduction.

2.3 Structure of the winning bid distribution

The structure of winning bid distributions compatible with a first-price auction where buyers observe N follows from the mixture expression of $G(\cdot)$ in Equation (1) and the best-response differential equation (2).

Proposition 2.1 *A cdf $G(\cdot)$ is rationalized by a first-price auction model satisfying Assumptions N, IPV, and observability of N by buyers if and only if*

(i). The cdf $G(\cdot)$ has a mixture structure

$$G(\cdot) = \sum_{n=\underline{n}}^{\overline{n}} p_n G_n^n(\cdot), \tag{5}$$

where the $G_n(\cdot)$ are cdf, $2 \leq \underline{n} \leq \overline{n}$, and the positive p_n satisfy $\sum_{n=\underline{n}}^{\overline{n}} p_n = 1$.

(ii). The quantile functions $B_n(\cdot) = G_n^{-1}(\cdot)$ are continuously differentiable over $[0,1]$ and satisfy the compatibility conditions

$$B_n(\alpha) + \frac{\alpha B_n^{(1)}(\alpha)}{n-1} = B_m(\alpha) + \frac{\alpha B_m^{(1)}(\alpha)}{m-1}$$

for all $\underline{n} \leq n, m \leq \overline{n}$ and all $\alpha \in [0,1]$. Moreover, the function $V(\alpha) = B_n(\alpha) + \frac{\alpha B_n^{(1)}(\alpha)}{n-1}$ is continuously differentiable over $[0,1]$ with $V^{(1)}(\cdot) > 0$.

In short, a cdf $G(\cdot)$ as in Proposition 2.1 is a mixture with components constrained by compatibility conditions driven by the best response differential equation (2). The compatibility conditions of Proposition 2.1-(ii) reflect that the mixture components $G_n(\cdot)$ are generated by the same private value distribution, an important feature for identification. In particular, our identification results rely on the constraints it imposes on the extremities of the conditional bid pdf $g_n(\cdot)$, as illustrated in the next corollary. Recall $\overline{b}_n = B_n(1)$, $\underline{v} = V(0) = \underline{b}_n$, and $\overline{v} = V(1)$.

Corollary 2.1 *Suppose that the compatibility conditions of Proposition 2.1-(ii) hold. Then, for all* $n = \underline{n}, \ldots, \overline{n}$, *we have* $\overline{b}_n < \overline{v}$ *together with:*

$$g_n(\underline{v}) = \frac{n}{n-1} f(\underline{v}), \text{ with} \tag{6}$$

$$g_n\left(\overline{b}_n\right) = \frac{1}{(n-1)(\overline{v} - \overline{b}_n)}. \tag{7}$$

Equation (7) implies that $g_n\left(\overline{b}_n\right)$ is strictly positive. It turns out from (1) that this causes discontinuities in the winning bid pdf $g(\cdot)$ at each $\overline{b}_n$, as studied in the next section. As it follows that $\overline{b}_n$ is identified, (7) shows that $g_n\left(\overline{b}_n\right)$, where $n \in \{\underline{n}, \ldots, \overline{n}\}$, are determined by the common unknown parameter $\overline{v}$. We employ this consequence of the compatibility conditions of Proposition 2.1-(ii) later on to identify the distribution of N.

2.4 Winning bid density discontinuities

In this subsection, we introduce a numerical example to illustrate the discontinuity features of the winning bid pdf that follows from Corollary 2.1. This example will also be useful for introducing our identification procedure. A general lemma completes the example.

2.4.1 Numerical example

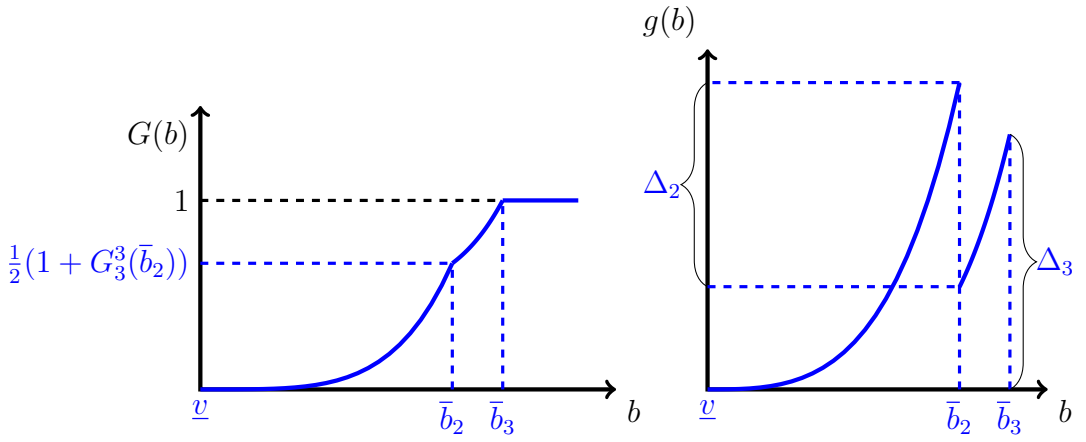
Consider the private value cdf $F(v) = v^2$ for all v in $[0,1]$ and a number of buyers $N = \{2,3\}$ with equal probability. As $V(\alpha) = \alpha^{1/2}$, it follows that

$$B_n(\alpha) = \frac{n-1}{\alpha^{n-1}} \int_0^\alpha t^{n-2+\frac{1}{2}} dt = \frac{n-1}{n-\frac{1}{2}} \alpha^{1/2}.$$

Hence, $\overline{b}_n = \frac{n-1}{n-\frac{1}{2}}$ yields the conditional bid pdf. $g_n(b)$, given $N = n$, is equal to $2b/\overline{b}_n^2$ on $[0, \overline{b}_n]$ and vanishes outside this interval. Note that the support of the conditional

density function increases with the number of buyers. As expected from (7), the pdf $g_2(\cdot)$ and $g_3(\cdot)$ are positive below their respective upper boundaries $\overline{b}_2$ and $\overline{b}_3$, and vanish beyond.

Let us now turn to the winning bid, the observation of the analyst. As expected, the unconditional pdf $g(b) = \frac{1}{2} \cdot 2G_2(b)g_2(b) + \frac{1}{2} \cdot 3G_3^2(b)g_3(b)$ displayed in Figure 1b is discontinuous at $\overline{b}_2$ and $\overline{b}_3$, with jump sizes Δ_2 and Δ_3, respectively. The resulting winning bid cdf exhibits kinks at these values, as illustrated in Figure 1a. In this example, Figure 1b exhibits two discontinuities (and Figure 1a exhibits two kinks) because N takes two potential values here.



(a) cdf $G(\cdot)$ (b) pdf $g(\cdot)$

Figure 1: Winning bid distribution ($V(\alpha) = \sqrt{\alpha}$ and $\mathbb{P}(N = 2) = \mathbb{P}(N = 3) = 1/2$)

2.4.2 The general case

The increasing support property of the conditional bid pdf and the winning bid pdf discontinuities in Figure 1b are generic, as shown in the upcoming lemma. Lemma 2.1-(i) states more generally that bids increase with competition — a key feature of first-price auctions that does not hold in ascending ones, or when buyers do not

observe N. Lemma 2.1-(ii) focuses on the winning bid pdf discontinuities and its jumps. The identification of $\underline{n}$ in Lemma 2.1-(iii) uses that the lower tail index of $G(b)$ is the one of $(G_{\underline{n}}(b))^{\frac{n}{\underline{n}}}$, which is $\underline{n}$.

Lemma 2.1 *Suppose Assumptions N and IPV hold. Then, all of the following hold.*

(i). For all α in $(0,1]$, $B_{\underline{n}}(\alpha) < \cdots < B_{\overline{n}}(\alpha) < V(\alpha)$ with $B_n(0) = V(0)$ for all n. In particular, for $\overline{b}_n = B_n(1)$, $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}} < \overline{v}$.

(ii). The cdf $G(\cdot)$ has a pdf $g(\cdot)$ with $g(\underline{v}) = 0$, which is continuous on the real line with the exception of the $\overline{n} - \underline{n} + 1$ discontinuity points $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}}$, with the interval $\left[\underline{v}, \overline{b}_{\overline{n}}\right]$ being the support of $G(\cdot)$. For $\underline{n} \leq n \leq \overline{n}$, the jumps $\Delta_n = \lim_{t \downarrow 0} \left(g(\overline{b}_n - t) - g(\overline{b}_n + t)\right)$ satisfy

$$\Delta_n = \frac{np_n}{(n-1)(\overline{v} - \overline{b}_n)}. \tag{8}$$

(iii). It holds that $\underline{n} = \lim_{t \downarrow 0} \frac{\log G(\underline{v}+t)}{\log t}$.

Lemma 2.1 is an important building block for identifying the competition distribution. Part (iii) is a tail identification result for $\underline{n}$ similar to Hill and Shneyerov (2013) who considered a common value framework. Lemma 2.1-(ii) shows that the jumps in the winning bid pdf identify $\mathbb{P}(N = n)$ up to the unknown $\overline{v}$.

2.5 Identification

Here, we first focus on identification of the participation distribution and then turn to the private values.

2.5.1 Identification of the distribution of N

In this subsection, we describe the identification of the support of N and its distribution using the discontinuity points and jump sizes. To identify the support, we exploit

two implications of Lemma 2.1: (a) the minimum number of buyers $\underline{n}$ is identified from the winning bid distribution tail near the lower boundary; (b) each number of buyers n generates a discontinuity in the winning bid distribution, which identifies the difference $\overline{n} - \underline{n}$. More specifically, Lemma 2.1-(ii) identifies $\underline{n}$ and $\overline{n}$ through $\underline{n} = \lim_{t \downarrow 0} \frac{\log G(\underline{v}+t)}{\log t}$ and

$$\overline{n} = \underline{n} + \mathsf{Card}\left\{b; g\left(\cdot\right) \text{ is discontinuous at } b\right\} - 1.$$

This also identifies the support of the distribution of N as $\mathbb{P}(N = n) > 0$ for all n with $\underline{n} \leq n \leq \overline{n}$ by Assumption N.

Next, we exploit the jumps in the pdf to identify $p_n = \mathbb{P}(N = n)$. Recall that Equation (8) identifies p_n up to the private value upper bound $\overline{v}$,

$$p_n = \frac{n-1}{n} \Delta_n \left(\overline{v} - \overline{b}_n\right).$$

But $\sum_{\underline{n}}^{\overline{n}} p_n = 1$ implies

$$\overline{v} = \frac{1 + \sum_{n=\underline{n}}^{\overline{n}} \frac{n-1}{n} \Delta_n \overline{b}_n}{\sum_{\underline{n}}^{n=\overline{n}} \frac{n-1}{n} \Delta_n}. \tag{9}$$

Hence, p_n satisfies

$$p_n = \frac{\frac{n-1}{n}\Delta_n}{\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k}\Delta_k} + \frac{n-1}{n}\Delta_n \left(\frac{\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k}\Delta_k \overline{b}_k}{\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k}\Delta_k} - \overline{b}_n \right), \quad n = \underline{n}, \ldots, \overline{n}, \tag{10}$$

and is identified because the discontinuity points $\overline{b}_k$ and jump sizes Δ_k are identified. We summarize these identification results in the next lemma.

Lemma 2.2 *Suppose Assumptions N and IPV hold. Then $\underline{v}$, $\overline{n}$, $\underline{n}$, $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}}$, $\overline{v}$, and the probabilities p_n, $n = \underline{n}, \ldots, \overline{n}$, are identified.*

The identifying equations (9) and (10) can also be used to derive inequality constraints satisfied by the jumps sizes Δ_n, discontinuity locations $\overline{b}_n$, and the lowest and

largest numbers of bidders $\underline{n}$ and $\overline{n}$. Indeed $\overline{v} > \overline{b}_{\overline{n}}$ and $0 \leq p_n \leq 1$ are equivalent to the following inequalities

$$\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k} \Delta_k \left(\overline{b}_{\overline{n}} - \overline{b}_k\right) \leq 1,$$

$$1 + \sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k} \Delta_k \left(\overline{b}_k - \overline{b}_n\right) \leq \frac{\sum_{k=\underline{n}}^{\overline{n}} \frac{k-1}{k} \Delta_k}{\frac{n-1}{n} \Delta_n}, \quad n = \underline{n}, \ldots, \overline{n},$$

given that $\Delta_n > 0$ must also hold by Lemma 2.1-(ii). A violation of any of these inequalities indicate that the model is misspecified.

2.5.2 Identification of the private value quantile function

We first return to the numerical example to illustrate our iterative identification procedure for the private value distribution.

Numerical example (cont'd). By Lemma 2.2, $\underline{n} = 2$, $\overline{n} = 3$ and $p_1 = p_2 = \frac{1}{2}$ are identified. Let us now turn to the identification of the private value distribution, which is based on the winning bid cdf

$$G(b) = \frac{1}{2}\left(G_2^2(b) + G_3^3(b)\right)$$

displayed in Figure 1a. Since $G_2^2(b) = 1$ on $[\overline{b}_2, \overline{b}_3]$,

$$G_3(b) = (2G(b) - 1)^{\frac{1}{3}}, \quad b \in [\overline{b}_2, \overline{b}_3].$$

It follows that $B_3(\cdot)$ is identified on $[\alpha_1, 1]$, where $\alpha_1 = G_3(\overline{b}_2)$, using the top portion of the winning bid distribution; see Figure 1a when $G(b) \in [\frac{1}{2}(1 + G_3^3(\overline{b}_2)), 1]$. Using the mapping (2) from the bid quantile function to the private value one gives

$$V(\alpha) = B_3(\alpha) + \frac{1}{2}\alpha B_3^{(1)}(\alpha),$$

and $V(\cdot)$ is identified on $[\alpha_1, 1]$. As private values do not depend upon N, using the mapping (4) from the private value quantile function gives

$$B_2(\alpha) = \frac{1}{\alpha}\left[\bar{b}_2 - \int_\alpha^1 V(t)dt\right]$$

so that $B_2(\cdot)$ is also identified on $[\alpha_1, 1]$. The identified $B_2(\alpha)$, $B_3(\alpha)$, and $V(\alpha)$, where $\alpha \in [\alpha_1, 1]$, are displayed in blue in Figure 2.

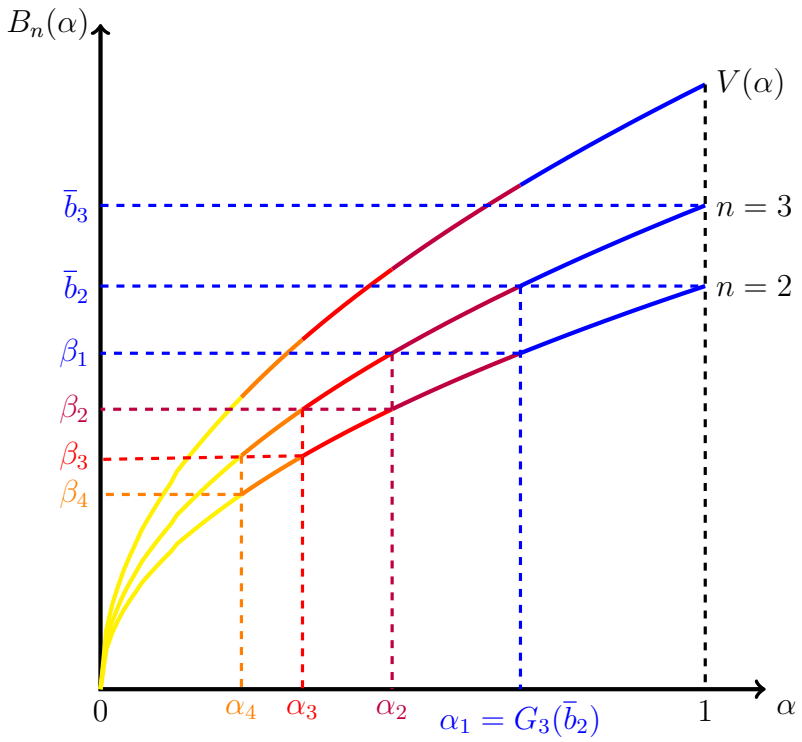


Figure 2: Iterative identification of $B_n(\alpha)$ and $V(\alpha)$ from $G(\cdot)$, as in Figure 1.

Next, we enlarge the interval $[\alpha_1, 1]$ over which $V(\cdot)$ is identified. For this purpose, let $\beta_1 = B_2(\alpha_1) < \bar{b}_2$ and observe that $G_2(b)$ is identified for $b \geq \beta_1$. Given that

$$G_3(b) = \left(G_2^2(b) - 2G(b)\right)^{\frac{1}{3}},$$

$G_3(b)$ is identified for $b \geq \beta_1$, as $B_3(\alpha)$ is identified for $\alpha \geq \alpha_2 = G_3(\beta_1)$. Figure 2 shows that $\alpha_2 < \alpha_1$ and arguing as above gives us identification of $V(\cdot)$ and $B_2(\cdot)$ on

$[\alpha_2, 1]$. Three portions of $V(\cdot)$, $B_3(\cdot)$, and $B_2(\cdot)$ are identified through three iterations and plotted in Figure 2 in purple, red, and orange, respectively. Furthermore, Figure 2 suggests that additional iterations of this identification procedure should allow us to recover any $V(\alpha)$.

The general case. The iterative identification described above can be easily generalized. As in Guerre et al (2009), competition (i.e., that the bid quantile function increases with n and only cross at 0) ensures that the quantile-level sequence $\{\alpha_k\}$ decreases to 0, allowing identification of the full private value distribution. This implies identification of the private value distribution when only observing the winning bid, as stated in the next general result.

Theorem 2.1 *Suppose Assumptions N and IPV hold and that the buyers observe the number of active buyers N. Then $F(\cdot)$ and the distribution of N are identified.*

2.6 Auction covariate and estimation strategy

Auction covariate. Many auctioned goods depend upon observed covariate that should also affect the private values. A popular specification from Haile, Hong and Shum (2003), Rezende (2008), considers a regression specification

$$\widetilde{V}_i = X'\theta + V_i.$$

It follows that

$$\widetilde{W} = X'\theta + W, \text{ where } W = \max_{i \in \mathcal{N}} B_i,$$

where the distribution of the bid B_i is identical to the one of the baseline model, provided the private values and the set $\mathcal{N}$, and in particular the number of active buyers N, are independent of X. Under this assumption, θ is identified regressing $\widetilde{W}$

on X and an intercept, allowing for the recovery of W and the identification approach of the baseline model to proceed.

It is however crucial here that the number of potential buyers, N, is independent of X. Otherwise, the regression above may identify a biased version $\theta^\star$ of θ which captures the possible correlation of $\mathbb{E}[W]$, which depends upon N, and X. The corresponding regression error is then $W^\star = X'(\theta - \theta^\star) + W$. If X has a continuous distribution, so will $X'(\theta - \theta^\star)$ when $\theta \neq \theta^\star$. The expression (19) of the density of the winning bid contaminated with continuous unobserved heterogeneity then suggests that the pdf of $W^\star$ will have no discontinuities even if N varies, so that the proposed identification approach does not apply.

Estimation strategy. In the presence of covariate, a two-step estimation method as above may blur discontinuities as described above, and therefore leads to further inconsistencies. Building upon our identification strategy seems unrealistic: while estimating discontinuity locations and sizes is theoretically feasible (see, among many others, Chu and Cheng (1996), who deal with both when the number of jump points is unknown), the iterative nature of the private value distribution identification renders its implementation difficult.

One-shot estimation in the fashion of Guerre et al (2000) cannot be applied directly due to the lack of a known number of bidders and a proper bid distribution. An alternative option is to rely on a likelihood estimation approach, such as the Bayesian ones implemented empirically to auctions by Li and Zheng (2009) and Aryal, Charankevich, Jeong and Kim (2023) among others. Theoretical studies in Hirano and Porter (2003) or Chernozhukov and Hong (2004) have illustrated efficiency and feasibility of the parametric Bayesian approach for nonregular models, including first-price auctions under the independent private value paradigm with a known number of bidders. In a parametric framework, the latter model induces bids which upper support

bounds depend upon the parameter of interest, with a positive pdf jumping to 0 at this location.

These works do not consider however the case of a pdf exhibiting several jumps as for our winning bid distribution. This case was addressed in Ibragimov and Has'minskii (1981, chapter 5) in the absence of covariate, while Ghosal and Samanta (1995) allowed for covariate. However, the number of discontinuity must be constant across parameters, meaning that in our framework, the difference $\overline{n}-\underline{n}$ of the maximal and minimal number of bidders should be known.

3 Buyer uncertainty and auction heterogeneity

3.1 Buyer uncertainty

Setup and assumptions. Consider an environment where there are N potentially active buyers, who submit a bid with probability d, an additional parameter to be identified. Buyer uncertainty then arises from the fact that the total number of active buyers is not known but follows a binomial distribution of parameter (N, d) given N, assuming from now on that the bidding decisions and N are independent. This auction setup is summarized in the next assumption.

Assumption BU. *There are N potentially active buyers, observable to the buyers but not the econometrician. Given N, each buyer decides to participate in the auction with probability d in $(0,1)$, privately and independently of the other buyers. Each active buyer draws a private value from the common knowledge distribution $F(\cdot)$ and the seller reserve price is $\underline{v} > 0$. The econometrician observes the winning bid W or that the auction fails if none of the buyers attend.*

In addition, the private value distribution $F(\cdot)$ satisfies Assumption IPV, and N

can vary across auctions with $p_n = \mathbb{P}(N = n)$, $n = \underline{n}, \ldots, \overline{n}$ as in Assumption N. This setup covers in particular the case where there is a reserve price r constant across auction, $F(\cdot)$ denoting the truncated private value distribution $F(\cdot|V \geq r)$ and $d = P(V \geq r)$ being the probability of entering the auction.

Model primitives. Under Assumption BU, the minimal bid is $\underline{b} = \underline{v} > 0$ when at least one bidder attends the auction. Let $G_n(\cdot|d)$ be the bid distribution given $N = n > 0$ with support $[\underline{b}, \overline{b}_n]$.

As a convention, the winning bid is set to 0 when there is no bid. It follows that the winning bid cdf, given that no buyers attend the auction and $N = n$, can be written as $G_n^0(\cdot|d)$ over $[0, \infty)$. Since the winning bid distribution given $N = n$ and $0 < m \leq n$ buyers attend the auction is $G_n^m(\cdot|d)$, the unconditional winning bid distribution is

$$\begin{aligned} G(b|d) &= \sum_{n=\underline{n}}^{\overline{n}} p_n \sum_{m=0}^{n} \left(\begin{array}{c} m \\ n \end{array} \right) d^m (1-d)^{m-n} G_n^m(b|d) \\ &= \sum_{n=\underline{n}}^{\overline{n}} p_n \left(1 - d + d \cdot G_n(b|d)\right)^n . \end{aligned} \tag{11}$$

A key difference with the baseline model is that

$$G(\underline{b}|d) = \sum_{n=\underline{n}}^{+\overline{n}} p_n (1-d)^n$$

is now the probability that no buyers attend the auction, which is positive. The lowest number of bidders $\underline{n}$ cannot be identified from the lower tail of $G(\cdot|d)$ as done in the baseline model, where $G(\cdot|d)$ was vanishing and behaving locally as a power $(b - \underline{b})^{\underline{n}}$ near $\underline{b}$.

Some closed-form expressions can be easily obtained for the equilibrium bid quantile function $B_n(\cdot|d) = G_n^{-1}(\cdot|d)$. Given $N = n$ and assuming that a buyer with

private value $V(\alpha) = F^{-1}(\alpha)$ attends the auction, the expected profit generated by a bid $B_n(a|d)$ is

$$\begin{aligned}
&(V(\alpha) - B_n(a|d)) \sum_{m=0}^{n-1} \left(\begin{array}{c} m \\ n-1 \end{array} \right) d^m (1-d)^{n-1-m} a^m \\
&\quad = (V(\alpha) - B_n(a|d)) \left(1 - d + d \cdot a\right)^{n-1}.
\end{aligned}$$

This expression is obtained noting that, when n bidders are potentially active and at least one is, the distribution of the remaining number of bids is a binomial with parameter $(n-1, d)$. The first-order condition characterizing the Bayesian Nash Equilibrium is then

$$\frac{d}{d\alpha} \left[\left(1 - d + d \cdot \alpha\right)^{n-1} B_n(\alpha|d) \right] = (n-1) d \left(1 - d + d \cdot \alpha\right)^{n-2} V(\alpha),$$

implying, given $B_n(0|d) = \underline{v}$,

$$V(\alpha) = B_n(\alpha|d) + \frac{\left(1 - d + d \cdot \alpha\right) B_n^{(1)}(\alpha|d)}{(n-1) \cdot d}, \tag{12}$$

$$B_n(\alpha|d) = \frac{(1-d)^{n-1} \underline{v} + (n-1) \cdot d \cdot \int_0^\alpha \left(1 - d + d \cdot t\right)^{n-2} V(t) dt}{\left(1 - d + d \cdot \alpha\right)^{n-1}}, \tag{13}$$

which are the counterparts of (2) and (3). In particular, integrating by parts in (13) shows that

$$B_n(\alpha|d) = V(\alpha) - \int_0^\alpha \left(\frac{1 - d + d \cdot t}{1 - d + d \cdot \alpha} \right)^{n-1} V^{(1)}(t) dt,$$

which implies that $B_n(\alpha|d)$ is strictly increasing with respect to n for all quantile levels $\alpha > 0$. Hence the largest bids $\overline{b}_n = B_n(1|d)$ satisfy $\overline{b}_{\underline{n}} < \overline{b}_{\underline{n}+1} < \cdots < \overline{b}_{\overline{n}} < \overline{v}$ as in the baseline model.

The next lemma describes implications of (12) and (13) for the conditional bid pdf $g_n(b|d) = \frac{d}{db} G_n(b|d)$, which parallels Corollary 2.1. Henceforth, we shall consider the additional parameter

$$\overline{v}^{(1)} = V^{(1)}(1) = \frac{1}{f(\overline{v})}.$$

Lemma 3.1 *Suppose Assumptions BU, N, and IPV hold. Then,*

$$g_n(b|d) = \left(\frac{2f(\underline{v})(1-d)}{(n-1)\cdot d\cdot(b-\underline{b})}\right)^{\frac{1}{2}}(1+o(1)) \text{ when } b\downarrow\underline{b}, \tag{14}$$

$$g_n(\overline{b}_n|d) = \frac{1}{(n-1)\cdot d\cdot(\overline{v}-\overline{b}_n)}, \tag{15}$$

$$g_n^{(1)}(\overline{b}_n|d) = \frac{n\cdot d\cdot\left(\overline{v}-\overline{b}_n\right)-\overline{v}^{(1)}}{((n-1)\cdot d)^2\left(\overline{v}-\overline{b}_n\right)^3}. \tag{16}$$

As the expression of the winning bid distribution (11) gives a pdf of

$$g(b|d) = \sum_{n=\underline{n}}^{\overline{n}} np_n\cdot d\cdot(1-d+d\cdot G_n(b|d))^{n-1}\, g_n(b|d),$$

(15) shows that $g(\cdot|d)$ exhibits downward jumps Δ_n at each $\overline{b}_n$, $n=\underline{n},\ldots,\overline{n}$ with

$$\Delta_n = \lim_{t\downarrow 0}\left(g(\overline{b}_n-t|d)-g(\overline{b}_n+t|d)\right) = \frac{np_n}{(n-1)(\overline{v}-\overline{b}_n)},$$

an expression identical to the jumps of the baseline model (8). As (13) ensures that $g(\cdot|d)$ is continuous at other points of $(\underline{b},\overline{b}_{\overline{n}})$, this can be used to identify $\overline{n}-\underline{n}$ and $p_n = \frac{n-1}{n}\Delta_n\left(\overline{v}-\overline{b}_n\right)$ up to n and $\overline{v}$.

Equation (14) shows that the conditional bid pdf $g_n(\cdot|d)$ and the winning bid pdf $g(\cdot|d)$ both diverge with a $-\frac{1}{2}$ power rate at the lowest bid $\underline{b}=\underline{v}$. This is intuitively due to buyer uncertainty, as a bidder can win with a very low bid if no others attend the auction. As a consequence, this can be used to check the presence of bidder uncertainty.[2] On the other hand, it does not allow for the identification of $\underline{n}$ using the lower tail behavior of $G(\cdot|d)$ as simply as in the baseline model.

[2]This can be tested using an asymptotically normal estimation of the lower tail-index exponent of the winning bid distribution, such as those reviewed in Beirlant, Herrmann and Teugels (2016). See also Hill and Shneyerov (2013) who also rely on tail-index estimation to test common against private values in an auction context.

Discontinuities and identification strategies. A possible way to recover the minimal number $\underline{n}$ of potentially active bidders relies on the derivative pdf discontinuities, as permitted by (16). The winning bid derivative pdf is

$$g^{(1)}(b|d) = \sum_{n=\underline{n}}^{\overline{n}} np_n \cdot d \cdot (1 - d + d \cdot G_n(b|d))^{n-2} \\ \times \left[(1 - d + d \cdot G_n(b|d))\, g_n^{(1)}(b|d) + (n-1) \cdot d \cdot g_n^2(b|d) \right],$$

which, by (13), is continuous over $(\underline{b}, \overline{b}_{\overline{n}})$, with the possible exception of the identified $\overline{b}_n$, $n = \underline{n}, \ldots, \overline{n}$, where it may exhibit jumps

$$\Delta_n^{(1)} = \lim_{t \downarrow 0} \left(g^{(1)}(\overline{b}_n - t|d) - g^{(1)}(\overline{b}_n + t|d) \right) = n d g_n^{(1)}(\overline{b}_n|d) + n(n-1) d^2 g_n^2(\overline{b}_n|d) \\ = p_n \frac{n}{n-1} \frac{1}{(\overline{v} - \overline{b}_n)^2} \left(2 + \frac{1}{n-1} - \frac{1}{n-1} \frac{\overline{v}^{(1)}}{d} \frac{1}{\overline{v} - \overline{b}_n} \right),$$

by (15), (16), and a little algebra. To take advantage of the fact that $n - \underline{n}$ is identified as the rank of $\overline{b}_n$,[3], set

$$m = n - \underline{n}, \quad \overline{b}(m) = \overline{b}_{\underline{n}+m}, \quad \varrho(m) = \frac{\Delta_{\underline{n}+m}^{(1)}}{\Delta_{\underline{n}+m}},$$

which are all identified. It then follows from the expression of $\Delta_n^{(1)}$ and Δ_n given the above that

$$\varrho(m)(\underline{n} + m - 1)(\overline{v} - \overline{b}(m))^2 - (2\underline{n} + 2m - 1)\,(\overline{v} - \overline{b}(m)) - \frac{\overline{v}^{(1)}}{d} = 0. \qquad (17)$$

As this equation includes the three unknowns $\underline{n}$, $\overline{v}$, and $\overline{v}^{(1)}/d$, three of these equations are, in principle, needed for identification from (17). However, implementing this strategy may be hard due to the nonlinear nature of (17).

A simple overidentification strategy introduces as additional unknowns some well chosen nonlinear functions, such as $\underline{n}\left(\overline{v} - \overline{b}(0)\right)^2$, $\left(\overline{v} - \overline{b}(0)\right)^2$, and $\underline{n}\left(\overline{v} - \overline{b}(0)\right)$, to

[3]This follows from $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}}$, defining the rank of $\overline{b}_{\underline{n}}$ as 0, the one of $\overline{b}_{\overline{n}}$ being $\overline{n} - \underline{n}$.

transform (17) into a linear equation. This allows us to obtain a condition ensuring identification of the considered auction specification as stated in the next proposition.

Proposition 3.1 *Suppose Assumptions BU, N, and IPV hold, and that* $\overline{n} - \underline{n} \geq 5$. *Assume there is a subset* $\mathcal{M}$ *of* $\{0, \ldots, \overline{n} - \underline{n}\}$ *with six elements, and let* $I_{\mathcal{M}}$ *be the* 6×6 *matrix with the following row entries, for* m *in* $\mathcal{M}$,

$$\Big[1,\ \varrho(m),\ m\varrho(m),\ \overline{b}(m) - \overline{b}(0),\ 2m + (m-1)\varrho(m)\left(\overline{b}(m) - \overline{b}(0)\right),$$
$$\left(\overline{b}(m) - \overline{b}(0)\right)\left(\varrho(m)\left(\overline{b}(m) - \overline{b}(0)\right) + 2\right)\Big].$$

Then, if $\det(I_{\mathcal{M}}) \neq 0$, *the uncertainty probability* d, $\underline{n}$, $\overline{n}$, *and* p_n *for* $n = \underline{n}, \ldots, \overline{n}$ *together with the private value distribution* $F(\cdot)$ *are identified.*

As $\varrho(m)$ and $\overline{b}_n$ can be estimated from the data, the overidentification condition $\det(I_{\mathcal{M}}) \neq 0$ is testable. Proposition 3.1 holds under the condition $\overline{n} - \underline{n} \geq 5$, a condition that can be weakened by introducing fewer additional unknowns in (17) and by taking into account that $\underline{n}$ is an integer number.[4]

Buyer uncertainty and number of buyers. To some extent, the entry probability d in (17) can depend upon the number of potential bidders n. For instance, the parameters β_0 and β_1 in $d_n = 1/(\beta_0 + \beta_1 \cdot n)$ can be identified using a straightforward modification of Proposition 3.1.

Entry cost models as considered in Li and Zheng (2009) also generates buyer uncertainty with a parameter d depending upon n and an entry cost c. If in addition the private value distribution depends upon a finite-dimensional θ, then the buyer uncertainty parameter is determined by an indifference condition which ensures

[4]Note that Proposition 3.1 also applies when buyers are certain about participation (i.e. $d = 1$), in which case $\underline{n}$ is identified without relying on the tail argument used in Lemma 2.1-(iii).

$d = d(n, c, \theta)$, assuming c is constant across auctions. Equation (17) can then be used to identified c and θ, using only that the largest bid $\overline{b}_n$ increases with n. Indeed, a difficulty there is that, as noted by Li and Zheng (2009), bidding strategies may not increase with the number of potential bidders. If c varies across auction, nonparametric identification can be restored if the entry cost can be small enough to allow entry of all bidders, suggesting that cost variation can be useful. Indeed, if $d = 1$ for small enough entry costs, identification follows from the baseline case.

3.2 Unobserved auction heterogeneity

Setup and assumptions. Consider now a setup with auction heterogeneity, where for each buyer i,

$$\widetilde{V}_i = \chi + V_i, \tag{18}$$

χ being an auction-specific variable which is not observed by the econometrician but common knowledge to buyers, and V_i is an i.i.d. private value component drawn from $F(\cdot)$ satisfying Assumption IPV.

Assumption UAH. *The unobserved auction heterogeneity component χ is independent of N and all the private values V_i. The pdf $\varphi(\cdot)$ of χ has a compact support $[0, \overline{\chi}] \subset [0, \infty)$, over which it is strictly positive and twice continuously differentiable. $\overline{\chi} \neq \overline{b}_n - \overline{b}_m$ for all $\underline{n} \leq n, m \leq \overline{n}$.*

The restriction on $\overline{\chi}$ shortens some proofs but can be easily removed.

Model primitives. Under (18), the bids $\widetilde{B}_i$ are equal to $\chi + B_i$, where the conditional quantile function $B_n(\cdot)$ of the i.i.d. B_i given $N = n$ is given by (3) and satisfies (2). It follows that the winning bid $\widetilde{W}$ is now

$$\widetilde{W} = \chi + W, \text{ where } W = \max_{i \in \mathcal{N}} B_i.$$

The conditional cdf of W is the one of the baseline model, so that, $\Phi(\cdot)$ being the cdf of χ,

$$\widetilde{G}_n^n(b) = \mathbb{P}\left(\widetilde{W} \leq b \middle| N = n\right) = \mathbb{P}\left(\chi \leq b - W \middle| N = n\right) = \int_{\underline{b}}^{\overline{b}} \Phi(b-t) n G_n^{n-1}(t) g_n(t) dt,$$

as the pdf of W is $nG_n^{n-1}(b)g_n(b)$. It follows that the pdf of $\widetilde{W}$ is, by Assumption N and recalling that χ belongs to $[0, \overline{\chi}]$,

$$\begin{aligned}\widetilde{g}(b) &= \sum_{n=\underline{n}}^{\overline{n}} p_n \int_{b-\overline{\chi}}^{b} \varphi(b-t) n G_n^{n-1}(t) g_n(t) dt \\ &= \int_{b-\overline{\chi}}^{b} \varphi(b-t) g(t) dt \text{ where } g(t) = \sum_{n=\underline{n}}^{\overline{n}} p_n n G_n^{n-1}(t) g_n(t), \end{aligned} \tag{19}$$

noting that $g(\cdot)$ is the winning bid pdf of the baseline model.

Winning bid pdf derivatives discontinuities. Integrating out $g(\cdot)$ in (19) gives a smooth pdf $\widetilde{g}(\cdot)$. However discontinuities arise when differentiating $\widetilde{g}(\cdot)$. It indeed holds that applying the Leibniz rule for integral differentiation to (19) yields

$$\begin{aligned}\widetilde{g}^{(1)}(b) = &\sum_{n=\underline{n}}^{\overline{n}} p_n n \left[\varphi(0) G_n^{n-1}(b) g_n(b) - \varphi(\overline{\chi}) G_n^{n-1}(b-\overline{\chi}) g_n(b-\overline{\chi})\right] \\ &+ \sum_{n=\underline{n}}^{\overline{n}} p_n \int_{b-\overline{\chi}}^{b} \varphi^{(1)}(b-t) n G_n^{n-1}(t) g_n(t) dt. \end{aligned} \tag{20}$$

As the integral expression above remains continuous, (20) implies that the discontinuities of $\widetilde{g}^{(1)}(\cdot)$ arise at each $\overline{b}_n$ and $\overline{b}_n + \overline{\chi}$, with jumps that are opposite in sign but of proportional magnitude.[5] The next lemma summarizes some properties of $\widetilde{g}(\cdot)$ and its first and second derivatives. Recall that $\overline{v}^{(1)} = V^{(1)}(0) = 1/f(\underline{v})$.

Lemma 3.2 *Suppose Assumptions BU, N, and IPV hold. Then:*

[5]Hence, if $\overline{b}_n = \overline{b}_m + \overline{\chi}$ for some n and m, some upward or downward jumps can be set to 0 and then washed out. Assumption UAH simplifies identification here by ruling out this possibility.

(i). $\widetilde{g}(\cdot)$ *is continuous over* $[0,\infty)$ *with* $\underline{n} = \lim_{t\downarrow 0} \frac{\log \widetilde{g}(\underline{b}+t)}{\log t}$*;*

(ii). $\widetilde{g}^{(1)}(\cdot)$ *is continuous over* $[0,\infty)$*, except at* $\overline{b}_n$ *and* $\overline{b}_n + \overline{\chi}$*,* $n = \underline{n}, \ldots, \overline{n}$*. It has a downward jump at* $\overline{b}_n$ *of size*

$$\widetilde{\Delta}_n = \varphi(0) p_n \frac{n}{n-1} \frac{1}{\overline{v} - \overline{b}_n}$$

and an upward jump at $\overline{b}_n + \overline{\chi}$ *of size* $\frac{\varphi(\overline{\chi})}{\varphi(0)} \widetilde{\Delta}_n$*;*

(iii). $\widetilde{g}^{(2)}(\cdot)$ *is continuous over* $(0,\infty)$*, except at* $\overline{b}_n$ *and* $\overline{b}_n + \overline{\chi}$*,* $n = \underline{n}, \ldots, \overline{n}$*. It has a jump at each* $\overline{b}_n$*, of size*

$$\widetilde{\Delta}_n^{(1)} = p_n \left[\varphi(0) n \frac{(2n-1)\left(\overline{v} - \overline{b}_n\right) - \overline{v}^{(1)}}{(n-1)^2 \left(\overline{v} - \overline{b}_n\right)^3} + \varphi^{(1)}(0) \frac{n}{n-1} \frac{1}{\overline{v} - \overline{b}_n} \right]$$

and a jump at each $\overline{b}_n + \overline{\chi}$*.*

Identification of the participation distribution p_n**.** Lemma 3.2-(i) ensures that $\underline{n}$ is identified while (ii) implies that the number of jumps of $\widetilde{g}^{(1)}(\cdot)$ above $\underline{b}$ is $2(\overline{n} - \underline{n})$ under the restriction on $\overline{\chi}$ of Assumption AUH, so that $\overline{n}$ can also be recovered. It also shows that $\overline{b}_{\underline{n}} < \cdots < \overline{b}_{\overline{n}}$ are identified as locations of downward jumps. As upward jumps are located at $\overline{b}_{\underline{n}} + \overline{\chi} < \cdots < \overline{b}_{\overline{n}} + \overline{\chi}$, the unobserved heterogeneity upper bound support $\overline{\chi}$ is also identified. This may help identify parametric heterogeneity distributions that depend upon a unique parameter.

Identifying the participation distribution is more difficult than in the baseline model because the downward jumps $\widetilde{\Delta}_n$ in Lemma 3.2-(ii) now depend upon two unknown parameters, $\overline{v}$ and $\varphi(0)$, unless the latter is identified. Introducing the upward jumps do not help, as they depend on the unknown $\overline{v}$ and $\varphi(\overline{\chi})$. To address this issue, one can try to identify $\overline{v}$ using the discontinuity ratio $\widetilde{\varrho}_n = \frac{\widetilde{\Delta}_n^{(1)}}{\widetilde{\Delta}_n}$, which, by Lemma 3.2-(ii,iii), satisfies

$$(n-1)\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v} - \overline{b}_n\right)^2 - (2n-1)\left(\overline{v} - \overline{b}_n\right) - \overline{v}^{(1)} = 0. \tag{21}$$

This equation involves three unknowns, $\overline{v}$, $\overline{v}^{(1)}$, and the ratio $\varphi^{(1)}(0)/\varphi(0)$. Three of these equations are, in principle, needed for identification. Similar to equation (17), it can be used as for (over)identification purposes, considering several values of n and introducing extra variables that are nonlinear functions of the initial unknowns to back out $\overline{v}$, $\overline{v}^{(1)}$, and $\varphi^{(1)}(0)/\varphi(0)$ as the unique solution of an extended linear system.

Proposition 3.2 *Suppose Assumptions BU, N, and IPV hold, and that* $\overline{n} - \underline{n} \geq 5$. *Assume there is a subset* $\widetilde{\mathcal{N}}$ *of* $\{\underline{n}, \ldots, \overline{n}\}$ *with six elements, and let* $I_{\widetilde{\mathcal{N}}}$ *be the* 6×6 *matrix with the following row entries, for* n *in* $\widetilde{\mathcal{N}}$,

$$\left[1,\ n,\ (n-1)\widetilde{\varrho}_n\ , (n-1)\left(\overline{b}_n - \overline{b}_{\underline{n}}\right),\ (n-1)\left(\overline{b}_n - \overline{b}_{\underline{n}}\right)^2,\ (n-1)\left(\overline{b}_n - \overline{b}_{\underline{n}}\right)\widetilde{\varrho}_n\right].$$

Then, if $\det(I_{\widetilde{\mathcal{N}}}) \neq 0$, *the participation distribution* $\{p_n, n = \underline{n}, \ldots, \overline{n}\}$ *is identified, as* $\underline{v}$, $\overline{v}$, *and* $\overline{v}^{(1)}$.

In addition, $\overline{\chi}$, $\varphi(0)$, $\varphi^{(1)}(0)$, $\varphi(\overline{\chi})$, *and* $\varphi^{(1)}(\overline{\chi})$ *are also identified.*

Proposition 3.2 gives a testable condition, ensuring that the participation distribution can be identified. It also leaves the door open for identification of parametric private value distributions with a parameter in a one-to-one correspondence with $\left(\underline{v}, \overline{v}, \overline{v}^{(1)}\right)$, as the latter can be identified. As explained before the Proposition, the participation distribution is identified using only downward jumps at each $\overline{b}_n$. Hence using also upward jumps at $\overline{b}_n + \overline{\chi}$ as well as differences between upward and downward jump locations yield overidentification of the participation distribution and $\overline{\chi}$, which can be used in principle for specification testing.

Similarly, the parametric unobserved heterogeneity distribution that can be uniquely recovered from $\left(\overline{\chi}, \varphi(0), \varphi^{(1)}(0), \varphi(\overline{\chi}), \varphi^{(1)}(\overline{\chi}),\right)$ can also be identified. As the expression of the winning bid pdf $\widetilde{g}(\cdot)$ in (19) shows that it is the convolution of the baseline winning bid pdf $g(\cdot)$ by $\varphi(\cdot)$, the deconvolution technique of Krasnokutskaya (2011)

using the identified $\varphi(\cdot)$ allows for the recovery of $g(\cdot)$. If so, the identification procedure developed for the baseline model can be applied to nonparametrically recover the private value distribution.

4 Final remarks

This paper shows that, under the independent symmetric private value paradigm, the first-price winning bid is sufficient to identify model primitives when competition is observed by the buyers but not the econometrician. This is suitable in the presence of phantom bids or when the number of observed bids does not reflect participation. To some extent, buyers can be uncertain about their competitors and auction-specific unobserved heterogeneity can be present.

Our theoretical results shed light on new identification arguments for discrete mixture models, which are widely used in economic applications, in particular when unobserved heterogeneity is plausible. In our model, the mixture components are generated by the same function. The components are ordered according first-order stochastic dominance and their supports are nested. These two features may appear in other relevant economic mixtures. Investigating how essential these features are for identification can also be of interest for future extensions.

Building on our constructive identification strategy, a natural next step is to develop an estimation procedure. Although the identification approach is constructive, its direct implementation in estimation poses challenges due to the iterative nature of the procedure, which can amplify errors. As discussed in Section 2.6, there is apparently no available methodology that could tackle the estimation problem at hand, even in a parametric framework. In ongoing work, we are developing an estimation framework to address these issues. In particular, we explore the use of Bayesian methods, which offer a coherent way to incorporate prior information, impose structural

restrictions, and enhance computational tractability.

References

An, Y., Y. Hu & M. Shum (2010). Estimating first-price auctions with an unknown number of bidders: a misclassification approach. *Journal of Econometrics* **157**, 328–341.

Aryal, G., H. Charankevich, S. Jeong & D.-H. Kim (2023). Procurements with bidder asymmetry in cost and risk-aversion. *Journal of Business and Economic Statistics* **41**, 1143-1156.

Athey, S. & P.A. Haile (2007). Nonparametric approaches to auctions. *Handbook of Econometrics* **6A**, 3847–3965.

Beirlant, J., K. Herrmann & J.L. Teugels (2016). Estimation of the extreme value index. Chapter 5 in *Extreme events in finance: a handbook of extreme value theory and its applications*, edited by F. Longin, 154–176.

Bulow, J. & P. Klemperer (1996). Auctions versus negotiations. *American Economic Review* **86**, 180–194.

Chernozhukov, V. & H. Hong (2004). Likelihood estimation and inference in a class of nonregular econometric models. *Econometrica* **72**, 1445-1480.

Choi, S., L. Nesheim & I. Rasul (2016). Reserve price effects in auctions: estimates from multiple regression-discontinuity designs. *Economic Inquiry* **54**, 294–314.

Chu, C.K. & P.E. Cheng (1996). Estimation of a jump points and jump values of a density function. *Statistica Sinica*

Compiani, G. & Y. Kitamura (2016). Using mixtures in econometrics models: a brief review and some new results. *Econometrics Journal* **19**, C95–C127.

Coviello, D. & M. Mariniello (2014). Public requirements in public procurement: evidence from a regression discontinuity design. *Journal of Public Economics* **109**, 76–100.

D'Haultfœuille, X. & P. Février Identification of mixture models using support variations. *Journal of Econometrics* **189**, 70–82.

Freyberger, J. & B.L. Larsen (2022). Identification in ascending auctions with application to digital rights management. *Quantitative Economics* **13**, 505-543.

Ghosal, S. & T. Samanta (1995). Asymptotic behaviour of Bayes estimates and posterior-distributions in multiparameter nonregular cases. *Mathematical Methods of Statistics* **4**, 361–388.

Gimenes, N. & E. Guerre (2022). Quantile regression methods for first-price auctions. Forthcoming, *Journal of Econometrics* **226**, 224–247.

Guerre, E. & Y. Luo (2019). Nonparametric identification of first-price auction with unobserved competition: a density discontinuity framework. Document paper, `arXiv.1908.05476v1`.

Guerre, E., I. Perrigne & Q. Vuong (2000). Optimal nonparametric estimation of first-price auctions. *Econometrica* **68**, 525–574.

Guerre, E., I. Perrigne & Q. Vuong (2009). Nonparametric identification of risk-aversion in first-price auctions under exclusion restrictions. *Econometrica* **77**, 1193–1227.

Grundl, S. & Y. Zhu (2019). Identification and estimation of risk aversion in first-price auctions with unobserved auction heterogeneity. *Journal of Econometrics* **210**, 363–378.

Haile, P.A., H. Hong & M. Shum (2003). Nonparametric tests for common values in First-price sealed-bid auctions. Document paper, Cowles Foundation.

Han, L. & W.C. Strange (2015). The microstructure of housing markets: search, bargaining, and brokerage. *Handbook of Urban and Regional Economics* **5B**, 813-886.

Harstad, R.M., J.H. Kagel & D. Levin (1990). Equilibrium bid functions for auctions with an uncertain number of bidders. *Economic Letters* **33**, 35–40.

Hendricks, K. & R.H. Porter (2007). An empirical perspective on auctions. *Handbook of Industrial Organization* **3**, 2073–2143.

Hernández, C., D. Quint & C. Turansick (2020). Estimation in English auctions with unobserved heterogeneity. *RAND Journal of Economics* **51**, 868–904.

Hill, J.B. & A. Shneyerov (2013). Are there common values in first-price auctions? A tail-index nonparametric test. *Journal of Econometrics* **174**, 144–164.

Hirano, K. & J.R. Porter (2003). Asymptotic efficiency in parametric structural models with parameter-dependent support. *Econometrica* **71**, 1307–1338.

Ibragimov, I.A. & R.Z. Has'minskii (1981). *Statistical estimation. Asymptotic theory.* Springer.

Imbens, G.W. & T. Lemieux (2008). Regression discontinuity designs: a guide to practice. *Journal of Econometrics* **142**, 615–635.

Jales , H. & Z. Yu (2017). Identification and estimation using a density discontinuity approach. *Advances in Econometrics* **38**, 29–72, special issue on "Regression Discontinuity Designs", Matias D. Cattaneo & Juan Carlos Escanciano Eds.

Kawai, K. J. Nakabayashi, J. Ortner & S. Chassang (2023). Using bid rotation and incumbency to detect collusion: a regression discontinuity approach. *The Review of Economic Studies* **90**, 376–403.

Kim, D. & B. Yu (2024). Government incentives and firm location choices. *Public Choice* **2024**, 1–27.

Kleven, H.J. (2016). Bunching. *Annual Review of Economics* **8**, 435–464.

Krasnokutskaya, E. (2011). Identification and estimation of auctions models with unobserved heterogeneity. *The Review of Economic Studies* **78**, 293–327.

Kong, Y. (2020). Not knowing the competition: evidence and implications for auction design. *The RAND Journal of Economics* **51**, 840-867.

Korostelev, A.P. & A.B. Tsybakov (1993). *Minimax Theory of Image Reconstruction*. Lecture Notes in Statistics (LNS vol. 82). Springer-Verlag.

Laffont, J.J., H. Ossard & Q. Vuong (1995). Econometrics of first-price auctions. *Econometrica* **63**, 953–980.

Lamy, L. (2012). The econometrics of auctions with asymmetric anonymous bidders. *Journal of Econometrics* **167**, 113–132.

Lebrun, B (1999). First price auctions in the asymmetric N buyer case. *International Economic Review* **40**, 125–142.

Lebrun, B (2006). Uniqueness of the equilibrium in first-price auctions. *Games and Economic Behavior* **55**, 131–151.

Li, T. & X. Zheng (2009). Entry and Competition Effects in First-Price Auctions: Theory and Evidence from Procurement Auctions. *Review of Economic Studies* **76**, 1397–1429.

Liu, N. & Y. Luo (2017). A Nonparametric Test for Comparing Valuation Distributions in First-Price Auctions. *International Economic Review* **58**, 857–888.

Luo, Y. & R. Xiao (2023). Identification of Auction Models Using Order Statistics. *Journal of Econometrics* **236**, 105457.

Marmer, V., A. Shneyerov & P. Xu (2013). What model for entry in first-price auctions? A nonparametric approach. *Journal of Econometrics* **176**, 46–58.

Maskin, E. & J. Riley (1984). Optimal auction with risk averse buyers. *Econometrica* **52**, 1473–1518.

Milgrom, P. (2001). *Putting auction theory to work.* Cambridge University Press.

Paarsch, H.J. (1997). Deriving an estimate of the optimal reserve price: an application to British Columbian timber sales. *Journal of Econometrics* **78**, 333–357.

Rezende, L. (2008). Econometrics of auctions by least squares. *Journal of Applied Econometrics* **23**, 925–948.

Frühwirth-Schnatter, S. (2006). *Finite mixture and Markov switching models.* Springer.

Slattery, C.R. (2020). Bidding for firms: subsidy competition in the U.S.. Job market paper, University of Virginia.

Shneyerov, A. & A.C.L. Wong (2011). Identification in first-price and Dutch auctions when the number of potential bidders is unobservable. *Games and Economic Behavior* 72, 574–582.

Song, U. (2004). Nonparametric estimation of an eBay auction model with an unknown number of bidders. Working paper, University of British Columbia

Appendix A: proofs

A.1 Proof of Proposition 2.1

It remains to be shown that (i) and (ii) are sufficient. The function $V(\cdot)$ in (ii) is a quantile function associated with a cdf $F(\cdot)$ satisfying the requirements of Assumption IPV, while the mixture weights p_n define a distribution for N, as in Assumption N. These $\{p_n, \underline{n} \leq n \leq \overline{n}\}$ and private value quantile function $V(\cdot)$ generate a distribution for N and best response bidding strategy functions $B_n(\cdot)$ by (3), with $G(\cdot)$ as a winning bid cdf. □

A.2 Proof of Corollary 2.1

The compatibility conditions imply that (3) holds, and integrating by parts gives

$$B_n(\alpha) = \frac{1}{\alpha^{n-1}} \int_0^\alpha V(t)\, d\left[t^{n-1}\right] = V(\alpha) - \int_0^\alpha \left(\frac{t}{\alpha}\right)^{n-1} V^{(1)}(t)\, dt.$$

Hence, $\overline{b}_n = \overline{v} - \int_0^1 t^{n-1} V^{(1)}(t)\, dt < \overline{v}$ as $V^{(1)}(\cdot) > 0$. Note that this also gives $B_n(\alpha) < V(\alpha)$ for all $\alpha > 0$, and then $B^{(1)}(\alpha) > 0$ by (2). When α goes to 0, the following holds

$$\begin{aligned} B_n(\alpha) &= V(0) + V^{(1)}(0)\alpha + o(\alpha) - \int_0^\alpha \left(\frac{t}{\alpha}\right)^{n-1} \left(V^{(1)}(0) + o(1)\right) dt \\ &= V(0) + \frac{n-1}{n} V^{(1)}(0)\alpha + o(\alpha), \end{aligned}$$

which shows that $B_n^{(1)}(0) = \frac{n-1}{n} V^{(1)}(0)$. As $B_n^{(1)}(\cdot) > 0$, the conditional bid pdf $g_n(\cdot)$ satisfies

$$g_n(b) = \frac{1}{B_n^{(1)}(G_n(b))} \text{ for all } b \in \left[\underline{v}, \overline{b}_n, \right]. \tag{22}$$

Hence, $g_n(\underline{v}) = 1/B_n^{(1)}(0) = \frac{n}{n-1} 1/V^{(1)}(0) = \frac{n}{n-1} f(\underline{v})$, which is (6). For (7), (2) and (22) give

$$g_n\left(\overline{b}_n\right) = \frac{G_n\left(\overline{b}_n\right)}{(n-1)\left(V\left(G_n\left(\overline{b}_n\right)\right) - \overline{b}_n\right)} = \frac{1}{(n-1)\left(\overline{v} - \overline{b}_n\right)}$$

as $G_n\left(\overline{b}_n\right) = 1$, so that (7) holds. □

A.3 Proof of Lemma 2.1

As

$$B_n(\alpha) = \frac{1}{\alpha^{n-1}} \int_0^\alpha V(t)\, d\left[t^{n-1}\right] = V(\alpha) - \int_0^\alpha \left(\frac{t}{\alpha}\right)^{n-1} V^{(1)}(t)\, dt,$$

differentiating with respect to n gives

$$\frac{\partial B_n(\alpha)}{\partial n} = -\int_0^\alpha \left(\frac{t}{\alpha}\right)^{n-1} \log\left(\frac{t}{\alpha}\right) V^{(1)}(t)\, dt \geq 0.$$

The inequality is strict except when $\alpha = 0$, in which case $B_n(0) = \underline{v}$ for all n. It follows that the bid cdf given that $N = n$, $G_n(\cdot)$, has a support $[\underline{v}, \overline{b}_n]$, with an upper bound $\overline{b}_n = B_n(1)$, which is strictly increasing with respect to n and strictly smaller than $\overline{v} = \lim_{n\uparrow\infty} \overline{b}_n$. Hence, this proves Part (i). For part (ii), the expression for jumps (8) follows from (5), which shows that the winning bid pdf is

$$g(b) = \sum_{k=\underline{n}}^{\overline{n}} p_k k G_k^{k-1}(b)\, g_k(b), \tag{23}$$

with $g_k(b) = 0$ for $b > \overline{b}_n$ when $k \leq n$ by Lemma 2.1-(i). This gives

$$\begin{aligned}
&g(\overline{b}_n - t) - g(\overline{b}_n + t) \\
&= \sum_{k=n}^{\overline{n}} p_k k G_k^{k-1}\left(\overline{b}_n - t\right) g_k\left(\overline{b}_n - t\right) - \sum_{k=n+1}^{\overline{n}} p_k k G_k^{k-1}\left(\overline{b}_n + t\right) g_k\left(\overline{b}_n + t\right) \\
&\to n p_n g_n\left(\overline{b}_n\right) = \Delta_n,
\end{aligned}$$

when t goes to 0. The equality (7) for $g_n\left(\overline{b}_n\right)$ then gives (8). For part (iii), continuity of $B_n^{(1)}(\cdot)$, which is bounded away from 0 and infinity, and (22) shows that $g_n(\cdot)$ is continuous with $g_n(\underline{v}) > 0$ by (6). When t goes to 0, this gives

$$\begin{aligned}
G(\underline{v} + t) &= \sum_{n=\underline{n}}^{\overline{n}} p_n \left(\int_{\underline{v}}^{\underline{v}+t} g_n(u) du\right)^n = \sum_{n=\underline{n}}^{\overline{n}} p_n g_n^n(\underline{v}) t^n (1 + o(1)) \\
&= p_{\underline{n}} g_{\underline{n}}^{\underline{n}}(\underline{v}) t^{\underline{n}} (1 + o(1)),
\end{aligned}$$

as $p_{\underline{n}} g^{\underline{n}}_{\underline{n}}(\underline{v}) > 0$, which implies $\underline{n} = \lim_{t \downarrow 0} \frac{\log G(\underline{v}+t)}{\log t}$. □

A.4 Proof of Theorem 2.1

We now obtain identification of the value quantile function by iteratively exploiting the two equilibrium mappings in (2) and (4). We proceed in three steps:

Step 1. Note that the winning bid distribution satisfies

$$G(b) = 1 - p_{\overline{n}} + p_{\overline{n}} G^{\overline{n}}_{\overline{n}}(b) \text{ for all } b \text{ in } \left[\overline{b}_{\overline{n}-1}, \overline{b}_{\overline{n}}\right]$$

so that $G_{\overline{n}}(\cdot)$ is identified over $\left[\overline{b}_{\overline{n}-1}, \overline{b}_{\overline{n}}\right]$ as follows:

$$G_{\overline{n}}(b) = \left(\frac{G(b) - (1 - p_{\overline{n}})}{p_{\overline{n}}}\right)^{1/\overline{n}} \text{ for } b \text{ in } \left[\overline{b}_{\overline{n}-1}, \overline{b}_{\overline{n}}\right].$$

Set

$$\alpha_1 = G_{\overline{n}}\left(\overline{b}_{\overline{n}-1}\right).$$

It follows that $B_{\overline{n}}(\cdot)$ is identified on $[\alpha_1, 1]$, i.e.,

$$B_{\overline{n}}(\alpha) = W\left[(1 - p_{\overline{n}}) + p_{\overline{n}} \alpha^{\overline{n}}\right],$$

where $W(\cdot) = G^{-1}(\cdot)$ is the winning bid quantile function.

Using the mapping from the bid quantile function to the value quantile function (2) shows that the private value quantile function satisfies, for all $\alpha \in [\alpha_1, 1]$,

$$\begin{aligned} V(\alpha) &= B_{\overline{n}}(\alpha) + \frac{1}{\overline{n} - 1} \alpha B^{(1)}_{\overline{n}}(\alpha) \\ &= W\left[(1 - p_{\overline{n}}) + p_{\overline{n}} \alpha^{\overline{n}}\right] + \frac{\overline{n} p_{\overline{n}}}{\overline{n} - 1} \alpha^{\overline{n}} W^{(1)}\left[(1 - p_{\overline{n}}) + p_{\overline{n}} \alpha^{\overline{n}}\right], \end{aligned}$$

and $V(\cdot)$ is identified over $[\alpha_1, 1]$.

Using the mapping from the value quantile function to the bid quantile function (4) shows that the bid quantile functions $B_n(\cdot)$, $n = \underline{n}, \ldots, \overline{n} - 1$ are also identified over $[\alpha_1, 1]$. Hence, $\{B_n(\alpha), \alpha \in [\alpha_1, 1]\}$ and $\left\{G_n(b), b \in [B_n(\alpha_1), \overline{b}_n]\right\}$ are identified, for all $n = \underline{n}, \ldots, \overline{n}$.

Step 2. We now expand the interval over which $G_{\overline{n}}(\cdot)$ is identified using an iterative argument. Define

$$\beta_1 = B_{\overline{n}-1}(\alpha_1),$$

which is identified from the last step. Note that $\beta_1 < \overline{b}_{\overline{n}-1}$ whenever $\alpha_1 > 0$ because, by Lemma 2.1-(i),

$$\beta_1 = B_{\overline{n}-1}\left[G_{\overline{n}}\left(\overline{b}_{\overline{n}-1}\right)\right] < B_{\overline{n}}\left[G_{\overline{n}}\left(\overline{b}_{\overline{n}-1}\right)\right] = \overline{b}_{\overline{n}-1}.$$

The definition of $G(\cdot)$ implies that

$$G_{\overline{n}}(b) = \left(\frac{G(b) - \sum_{n=\underline{n}}^{\overline{n}-1} p_n G_n^n(b)}{p_{\overline{n}}}\right)^{1/\overline{n}}, \tag{24}$$

where $G(\cdot)$ and p_n are identified, and $G_n(\cdot)$ are identified on $[B_n(\alpha_1), \overline{b}_{\overline{n}}]$ for all $n = \underline{n}, \ldots, \overline{n}-1$. Since $B_{\underline{n}}(\alpha_1) < \ldots < B_{\overline{n}-1}(\alpha_1) = \beta_1$, $[\beta_1, \overline{b}_{\overline{n}}] \subseteq [B_n(\alpha_1), \overline{b}_{\overline{n}}]$ for all n. Therefore, the conditional bid distribution $G_{\overline{n}}(b)$ is identified on $[\beta_1, \overline{b}_{\overline{n}}]$.

Step 3. We now identify $V(\cdot)$ on a growing interval $[\alpha_k, 1]$ using an induction argument and the identified $V(\cdot)$ on $[\alpha_1, 1]$. For an integer $k \geq 2$, define

$$\alpha_k = G_{\overline{n}}(\beta_{k-1}) = G_{\overline{n}}\left[B_{\overline{n}-1}(\alpha_{k-1})\right], \quad \beta_k = B_{\overline{n}-1}(\alpha_k).$$

Identification of $V(\cdot)$ on the growing interval $[\alpha_k, 1]$ is established in Lemma 4.1 below.

Lemma 4.1 *Suppose Assumptions N and IPV hold. Then,*

(i). the sequences $\{\alpha_k, k \geq 1\}$ *and* $\{\beta_k, k \geq 1\}$ *are decreasing sequences with*

$$\lim_{k\to\infty} \alpha_k = 0.$$

(ii). $\{\alpha_k, k \geq 1\}$ *is identified. For any integer number* $k \geq 2$, $\{V(\alpha), \alpha \in [\alpha_k, 1]\}$ *is identified if* $\{V(\alpha), \alpha \in [\alpha_{k-1}, 1]\}$ *is identified.*

The proof of Lemma 4.1 is given at the end of this section. Let us now return to the identification of $V(\alpha)$ for any arbitrary $\alpha > 0$. By Lemma 4.1-(i), there exists k such that $\alpha > \alpha_k$ and Lemma 4.1-(ii) yields identification of $V(\alpha)$. Given that $V(0) = \underline{v}$ is identified by Lemma 2.2, the theorem is proven. □

Proof of Lemma 4.1. Consider (i) first. As $\alpha_k = G_{\overline{n}}[B_{\overline{n}-1}(\alpha_{k-1})]$ with $B_{\overline{n}-1}(\alpha) \leq B_{\overline{n}}(\alpha)$,

$$\alpha_k = G_{\overline{n}}[B_{\overline{n}-1}(\alpha_{k-1})] \leq G_{\overline{n}}[B_{\overline{n}}(\alpha_{k-1})] = \alpha_{k-1},$$

which implies that α_k decreases. Moreover, $\beta_k = B_{\overline{n}-1}(\alpha_k)$ decreases because $B_{\overline{n}-1}(\cdot)$ is strictly increasing. Since $\alpha_k \geq 0$, α_k converges to a limit α, which satisfies $\alpha = G_{\overline{n}}[B_{\overline{n}-1}(\alpha)]$ under Assumption IPV. In other words, the limit α satisfies $B_{\overline{n}}(\alpha) = B_{\overline{n}-1}(\alpha)$. This gives $\alpha = 0$ as $B_{\overline{n}}(\alpha) > B_{\overline{n}-1}(\alpha)$, except for $\alpha = 0$.

Now, consider (ii). That α_k is identified for all k follows from an induction argument, observing α_1 is identified. Suppose then that α_k and $\{V(\alpha), \alpha \in [\alpha_k, 1]\}$ are identified. Recall

$$\alpha_{k+1} = G_{\overline{n}}(\beta_k) = G_{\overline{n}}[B_{\overline{n}-1}(\alpha_k)], \quad \beta_{k+1} = B_{\overline{n}-1}(\alpha_{k+1}).$$

Then, (4) and Lemma 2.2 give that $\{B_n(\alpha); \alpha \in [\alpha_k, 1]\}$, for all $n = \underline{n}, \ldots, \overline{n}-1$ are identified, as β_k. Now (24) and Lemma 2.2 show that $G_{\overline{n}}(b)$ is identified for all $b \geq \beta_k$, and then $\alpha_{k+1} = G_{\overline{n}}(\beta_k)$ is identified. (2) then gives that $\{V(\alpha); \alpha \in [\alpha_{k+1}, 1]\}$ is identified. This ends the proof of the lemma. □

A.5 Proof of Lemma 3.1

Consider (14) first. Note that (13) shows that $B_n(\cdot|d)$ is continuously differentiable over $[0, 1]$. Expanding (12) gives, when α goes to 0,

$$B_n^{(1)}(\alpha|d) = (n-1) \cdot d \frac{V(\alpha) - B_n(\alpha|d)}{1 - d + d \cdot \alpha} = (n-1) \cdot d \cdot \alpha \frac{V^{(1)}(0) - B_n^{(1)}(0|d)}{1-d} + o(\alpha),$$

which implies $B^{(1)}(0|d) = 0$ and then

$$B_n^{(1)}(\alpha|d) = \frac{(n-1)dV^{(1)}(0)}{1-d}\alpha + o(\alpha) = \frac{(n-1)d}{(1-d)f(\underline{v})}\alpha + o(\alpha),$$
$$B_n(\alpha) = \underline{b} + \frac{(n-1)d}{(1-d)f(\underline{v})}\frac{\alpha^2}{2} + o(\alpha^2) \text{ so that when } b \downarrow \underline{b}$$
$$G_n(b|d) = \left(\frac{2f(\underline{v})(1-d)}{(n-1)d}(b-\underline{b})\right)^{\frac{1}{2}}(1+o(1)).$$

This gives (14), noting

$$g_n(b|d) = \frac{1}{B_n^{(1)}\left(G_n(b|d)|\, d\right)} = \left(\frac{2f(\underline{v})(1-d)}{(n-1)d(b-\underline{b})}\right)^{\frac{1}{2}}(1+o(1)).$$

(15) also follows from $g_n(b|d) = 1/B_n^{(1)}\left(G_n(b|d)|\, d\right)$ and (12), which gives

$$g_n(\bar{b}_n|d) = \frac{1}{(n-1)d}\frac{1-d+d\cdot G_n(b|d)}{V\left(G_n(b|d)\right)-b}\bigg|_{b=\bar{b}_n} = \frac{1}{(n-1)d(\overline{v}-\bar{b}_n)}.$$

For (16), first observe that

$$g_n^{(1)}(b|d) = \frac{d}{db}\left[\frac{1}{B_n^{(1)}\left(G_n(b|d)|\, d\right)}\right] = -\frac{B_n^{(2)}\left(G_n(b|d)|\, d\right)g_n(b|d)}{\left(B_n^{(1)}\left(G_n(b|d)\right)|\, d\right)^2}$$
$$= -\frac{B_n^{(2)}\left(G_n(b|d)|\, d\right)}{\left(B_n^{(1)}\left(G_n(b|d)\right)|\, d\right)^3},$$

where

$$B_n^{(2)}(\alpha|d) = \frac{d}{d\alpha}\left[(n-1)\cdot d\frac{V(\alpha)-B_n(\alpha|d))}{1-d+d\cdot\alpha}\right]$$
$$= -(n-1)\cdot d^2\frac{V(\alpha)-B_n(\alpha|d)}{(1-d+d\cdot\alpha)^2} + (n-1)\cdot d\frac{V^{(1)}(\alpha)-B_n^{(1)}(\alpha|d)}{1-d+d\cdot\alpha}$$
$$= -\frac{n(n-1)\cdot d^2\left(V(\alpha)-B_n(\alpha|d)\right)}{(1-d+d\cdot\alpha)^2} + (n-1)\cdot d\frac{V^{(1)}(\alpha)}{1-d+d\cdot\alpha}.$$

Hence,

$$g_n^{(1)}(\bar{b}_n|d) = -\frac{B_n^{(2)}(1|d)}{\left(B_n^{(1)}(1|d)\right)^3} = \frac{(n-1)\cdot d\cdot\left[n\cdot d\cdot\left(\overline{v}-\bar{b}_n\right)-\overline{v}^{(1)}\right]}{\left((n-1)\cdot d\cdot(\overline{v}-\bar{b}_n)\right)^3}$$
$$= \frac{n\cdot d\cdot\left(\overline{v}-\bar{b}_n\right)-\overline{v}^{(1)}}{\left((n-1)\cdot d\right)^2\left(\overline{v}-\bar{b}_n\right)^3}.$$

This ends the proof of the lemma. □

A.6 Proof of Proposition 3.1

Set $x_1 = \overline{v}-\bar{b}(0)$, $x_2 = \underline{n}$, and $x_3 = \frac{\overline{v}^{(1)}}{d}$. Additionally, define the extra unknowns

$$x_4 = \underline{n}\left(\overline{v}-\bar{b}(0)\right)^2, \qquad x_5 = \left(\overline{v}-\bar{b}(0)\right)^2, \qquad x_6 = \underline{n}\left(\overline{v}-\bar{b}(0)\right),$$

and set $y_m = -(m-1)\varrho(m)(\bar{b}(m)-\bar{b}(0))^2 - (2m-1)(\bar{b}(m)-\bar{b}(0))$.

It follows that

$$\begin{aligned}
&(\underline{n}+m-1)(\overline{v}-\bar{b}(m))^2 = (\underline{n}+m-1)\left(\overline{v}-\bar{b}(0)-(\bar{b}(m)-\bar{b}(0))\right)^2\\
&= \underline{n}\left(\overline{v}-\bar{b}(0)\right)^2 + (\underline{n}+m-1)\left(\bar{b}(m)-\bar{b}(0)\right)^2\\
&\quad + (m-1)\left(\overline{v}-\bar{b}(0)\right)^2 - 2(\bar{b}(m)-\bar{b}(0))\left[\underline{n}\left(\overline{v}-\bar{b}(0)\right)+(m-1)\left(\overline{v}-\bar{b}(0)\right)\right]\\
&= -2(\bar{b}(m)-\bar{b}(0))(m-1)\cdot x_1 + (\bar{b}(m)-\bar{b}(0))^2\cdot x_2 + x_4 + (m-1)\cdot x_5\\
&\quad - 2(\bar{b}(m)-\bar{b}(0))\cdot x_6 + (m-1)(\bar{b}(m)-\bar{b}(0))^2,\\
&(2\underline{n}+2m-1)\left(\overline{v}-\bar{b}(m)\right) = (2\underline{n}+2m-1)\left(\overline{v}-\bar{b}(0)-(\bar{b}(m)-\bar{b}(0))\right)\\
&\quad = (2m-1)\cdot x_1 - 2(\bar{b}(m)-\bar{b}(0))\cdot x_2 + 2\cdot x_6 - (2m-1)(\bar{b}(m)-\bar{b}(0)).
\end{aligned}$$

Hence, using these new notations shows that (17) is equivalent to

$$\begin{aligned}
&-\left(2m-1+(m-1)\varrho(m)(\bar{b}(m)-\bar{b}(0))\right)\cdot x_1\\
&+(\bar{b}(m)-\bar{b}(0))\left(\varrho(m)(\bar{b}(m)-\bar{b}(0))+2\right)\cdot x_2\\
-x_3 + \varrho(m)\cdot x_4 + (m-1)\varrho(m)\cdot x_5 &- 2\left(\bar{b}(m)-\bar{b}(0)+1\right)\cdot x_6 = y_m.
\end{aligned}$$

Elementary determinant algebra shows that, when $\det(I_{\mathcal{M}}) \neq 0$, the corresponding linear system obtained for m varying across $\mathcal{M}$ uniquely determines $x_1, \ldots, x_6$, and then $\underline{n}$, $\overline{v}$.

Hence $\overline{n}$ is identified using the number of pdf discontinuities. $p_n = \frac{n-1}{n}\Delta_n\left(\overline{v} - \overline{b}_n\right)$ then identifies p_n. As $\underline{b}$ and $G(\underline{b}|d) = \sum_{n=\underline{n}}^{+\overline{n}} p_n\,(1-d)^n$ are identified, d is identified, since $\sum_{n=\underline{n}}^{+\overline{n}} p_n x^n$ is an identified polynomial function which is strictly increasing in x over $[0,1]$.

Identification of $F(\cdot)$ can then be established as in the baseline model, using that $B_n(\alpha|d)$ strictly increases with n for α in $(0,1]$ with $B_n(0|d) = \underline{v}$ for all n. □

A.7 Proof of Lemma 3.2

Continuity of $\widetilde{g}(\cdot)$ follows from (19). As $g_n(0) > 0$ by (6), (19) implies, for b sufficiently close to $\underline{b}$,

$$\begin{aligned}\widetilde{g}(b) &= \int_{\underline{b}}^{b} (\varphi(0) + o(1)) \left(\sum_{n=\underline{n}}^{n=\overline{n}} g_n^n(0)(t-\underline{b})^{n-1}(1+o(1)) \right) dt \\ &= \varphi(0) g_{\underline{n}}^{\underline{n}}(0)(b-\underline{b})^{\underline{n}}(1+o(1)),\end{aligned}$$

which implies $\underline{n} = \lim_{t \downarrow 0} \frac{\log \widetilde{g}(\underline{b}+t)}{\log t}$.

(ii) follows from (20) and (7), which states that $g_n(\overline{b}_n) = 1/\left((n-1)(\overline{v} - \overline{b}_n)\right)$. For

(iii), differentiating (20) gives

$$\begin{aligned}\widetilde{g}^{(2)}(b) &= \sum_{n=\underline{n}}^{\overline{n}} p_n n \left[\varphi(0) G_n^{n-1}(b) g_n^{(1)}(b) - \varphi(\overline{\chi}) G_n^{n-1}(b-\overline{\chi}) g_n^{(1)}(b-\overline{\chi})\right] \\ &+ \sum_{n=\underline{n}}^{\overline{n}} p_n n(n-1) \left[\varphi(0) G_n^{n-2}(b) g_n^2(b) - \varphi(\overline{\chi}) G_n^{n-2}(b-\overline{\chi}) g_n^2(b-\overline{\chi})\right] \\ &+ \sum_{n=\underline{n}}^{\overline{n}} p_n n \left[\varphi^{(1)}(0) G_n^{n-1}(b) g_n(b) - \varphi^{(1)}(\overline{\chi}) G_n^{n-1}(b-\overline{\chi}) g_n(b-\overline{\chi})\right] \\ &+ \sum_{n=\underline{n}}^{\overline{n}} p_n \int_{b-\overline{\chi}}^{b} \varphi^{(2)}(b-t) n G_n^{n-1}(t) g_n(t) dt.\end{aligned}$$

Hence,

$$\widetilde{\Delta}_n^{(1)} = p_n \left[n\varphi(0) \left(g_n^{(1)}(\overline{b}_n) + (n-1) g_n^2(\overline{b}_n)\right) + n\varphi^{(1)}(0) g_n(\overline{b}_n)\right].$$

Let us now compute $g_n^{(1)}(\overline{b}_n) = -B_n^{(2)}(1) / \left(B_n^{(1)}(1)\right)^3$. (2) implies

$$B_n^{(2)}(\alpha) = -n(n-1)\frac{(V(\alpha) - B_n(\alpha))}{\alpha^2} + (n-1)\frac{V^{(1)}(\alpha)}{\alpha}$$

so that

$$B_n^{(2)}(1) = -n(n-1)\left(\overline{v} - \overline{b}_n\right) + (n-1)\overline{v}^{(1)}.$$

Hence,

$$g_n^{(1)}(\overline{b}_n) = -\frac{B_n^{(2)}(1)}{\left(B_n^{(1)}(1)\right)^3} = \frac{n\left(\overline{v} - \overline{b}_n\right) - \overline{v}^{(1)}}{(n-1)^2\left(\overline{v} - \overline{b}_n\right)^3}$$

by computations similar to the ones at the end of Section A.5. This gives

$$\begin{aligned}\widetilde{\Delta}_n^{(1)} &= \varphi(0) p_n \left(n\frac{n\left(\overline{v} - \overline{b}_n\right) - \overline{v}^{(1)}}{(n-1)^2\left(\overline{v} - \overline{b}_n\right)^3} + n(n-1)\left(\frac{1}{(n-1)\left(\overline{v} - \overline{b}_n\right)}\right)^2\right) \\ &+ \varphi^{(1)}(0) p_n \frac{n}{n-1}\frac{1}{\overline{v} - \overline{b}_n} \\ &= p_n \left[\varphi(0)\frac{n(2n-1)\left(\overline{v} - \overline{b}_n\right) - n\overline{v}^{(1)}}{(n-1)^2\left(\overline{v} - \overline{b}_n\right)^3} + \varphi^{(1)}(0)\frac{n}{n-1}\frac{1}{\overline{v} - \overline{b}_n}\right].\end{aligned}$$

This ends the proof of the lemma. □

A.8 Proof of Proposition 3.2

Set $x_1 = \overline{v} - \overline{b}_{\underline{n}}$, $x_2 = \frac{\varphi^{(1)}(0)}{\varphi(0)}$, $x_3 = \overline{v}^{(1)}$ and

$$x_4 = \left(\overline{v} - \overline{b}_{\underline{n}}\right)^2, \qquad x_5 = \frac{\varphi^{(1)}(0)}{\varphi(0)}\left(\overline{v} - \overline{b}_{\underline{n}}\right)^2, \qquad x_6 = \frac{\varphi^{(1)}(0)}{\varphi(0)}\left(\overline{v} - \overline{b}_{\underline{n}}\right).$$

Set $y_n = -(n-1)\widetilde{\varrho}_n(\overline{b}_n - \overline{b}_{\underline{n}})^2 - (2n-1)(\overline{b}_n - \overline{b}_{\underline{n}})$.

As

$$\begin{aligned}
&(n-1)\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v} - \overline{b}_n\right)^2 = (n-1)\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v} - \overline{b}_{\underline{n}} - (\overline{b}_n - \overline{b}_{\underline{n}})\right)^2 \\
&\quad = (n-1)\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v} - \overline{b}_{\underline{n}}\right)^2 - 2(n-1)(\overline{b}_n - \overline{b}_{\underline{n}})\left(\widetilde{\varrho}_n - \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)\left(\overline{v} - \overline{b}_{\underline{n}}\right) \\
&\qquad - (n-1)(\overline{b}_n - \overline{b}_{\underline{n}})^2\frac{\varphi^{(1)}(0)}{\varphi(0)} + (n-1)\widetilde{\varrho}_n(\overline{b}_n - \overline{b}_{\underline{n}})^2 \\
&\quad = -2(n-1)(\overline{b}_n - \overline{b}_{\underline{n}})\widetilde{\varrho}_n \cdot x_1 - (n-1)(\overline{b}_n - \overline{b}_{\underline{n}})^2 \cdot x_2 + (n-1)\widetilde{\varrho}_n \cdot x_4 \\
&\qquad - (n-1)\cdot x_5 + 2(n-1)(\overline{b}_n - \overline{b}_{\underline{n}})\cdot x_6 + (n-1)\widetilde{\varrho}_n(\overline{b}_n - \overline{b}_{\underline{n}})^2, \\
&-(2n-1)\left(\overline{v} - \overline{b}_n\right) = -(2n-1)\cdot x_1 + (2n-1)(\overline{b}_n - \overline{b}_{\underline{n}}),
\end{aligned}$$

(21) is equivalent to

$$\begin{aligned}
-\left[2(n-1)(\overline{b}_n - \overline{b}_{\underline{n}})\widetilde{\varrho}_n + (2n-1)\right]\cdot x_1 - (n-1)(\overline{b}_n - \overline{b}_{\underline{n}})^2 \cdot x_2 + x_3 \\
(n-1)\widetilde{\varrho}_n \cdot x_4 - (n-1)\cdot x_5 + 2(n-1)(\overline{b}_n - \overline{b}_{\underline{n}})\cdot x_6 = y_n.
\end{aligned}$$

Stacking these equations for n in $\widetilde{M}$ gives a linear system with a unique solution when $\det(I_{\widetilde{\mathcal{M}}}) \neq 0$. Hence, the initial parameters are $\left(\overline{v}, \overline{v}^{(1)}, \frac{\varphi^{(1)}(0)}{\varphi(0)}\right)$. The identity $\widetilde{\Delta}_n = \varphi(0)p_n\frac{n}{n-1}\frac{1}{\overline{v}-\overline{b}_n}$ in Lemma 3.2 and $\sum_{n=\underline{n}}^{\overline{n}} p_n = 1$ allow for the recovery of $\varphi(0)$ and p_n, $n = \underline{n}, \ldots, \overline{n}$. $\overline{\chi}$ has already been identified, and $\varphi(\overline{\chi})$ can be recovered from the upward jump size $\widetilde{\Delta}_n = \varphi(\overline{\chi})p_n\frac{n}{n-1}\frac{1}{\overline{v}-\overline{b}_n}$. As $\varphi^{(1)}(\overline{\chi})/\varphi(\overline{\chi})$ satisfies an equation similar to (21), $\varphi^{(1)}(\overline{\chi})$ is also identified. □