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Multiproduct Multimarket Price Competition under Capacity Constraints

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Abstract

Capacity constraints are central to oligopoly competition in many industries, yet existing multiproduct Bertrand theory does not characterize equilibrium when capacity binds across markets. We establish existence and uniqueness of Bertrand–Nash equilibrium in a multimarket, multiproduct oligopoly under multinomial logit demand, with both linear and convex costs. Capacity creates cross-market spillovers: pricing in one market affects the shadow value of capacity in others. Methodologically, we extend the aggregative-games framework to a nested fixed-point structure separating across-market capacity allocation from within-market pricing, using tools from nonsmooth analysis to handle kinks from binding constraints. The framework yields new insights for merger analysis: binding capacity dampens merger-induced price increases through shadow-cost relief, while post-merger reallocation of scarce capacity can raise consumer surplus. However, the merged firm's privately optimal reallocation generally differs from the social optimum, creating a role for merger remedies.

Keywords: Differentiated Products; Capacity Constraints; Mergers; Aggregative Games; Cross-Market Spillovers; Equilibrium Uniqueness

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1 Introduction

Capacity constraints are a ubiquitous feature of competition in many industries. Production facilities are typically built through lumpy investment, so short-run output is bounded by physical limits—such as kiln capacity in cement, generation capacity in electricity, refining capacity in fuels, furnace capacity in steel, and assembly capacity in automobiles. In such environments, firms face product-specific bottlenecks that must be allocated across multiple markets. When capacity constraints bind, changes in demand, supply, or policy in one market can alter incentives and outcomes in others. As a result, these constraints are central to the counterfactual analyses that motivate much empirical industrial organization, including studies of mergers, trade policy, supply disruptions, and command-and-control policies such as production quotas. Such cross-market spillover effects are central to evaluating welfare and distributional impacts across regions, firms, and consumers.

Empirical industrial organization has long recognized the importance of capacity constraints.[1] At the same time, the theoretical foundation for these studies remains incomplete: existing theory does not deliver existence and uniqueness results for multimarket Bertrand competition by multiproduct firms under binding capacity constraints. Standard sufficient conditions that justify global first-order or Karush–Kuhn–Tucker (KKT) characterizations, such as (quasi-)concavity or (log-)supermodularity, typically fail in multiproduct settings (Spady (1984); Hanson and Martin (1996); Nocke and Schutz (2018)). Moreover, capacity constraints generate cross-market price interdependencies and introduce kinks and non-smoothness in firms' objective functions, undermining the regularity conditions that support familiar existence and uniqueness arguments.

This paper fills that gap by establishing existence and uniqueness of the Bertrand–Nash equilibrium in a multimarket, multiproduct oligopoly with capacity constraints under multinomial logit (MNL) demand, and by characterizing the implications for merger analysis. Our approach extends the aggregative-game framework to environments with binding capacity constraints, yielding a nested equilibrium characterization that captures both within-market pricing interactions and across-market capacity allocation. Conceptually, we isolate mechanisms that arise only when capacity scarcity is modeled explicitly, namely cross-market reallocation effects and the shadow cost of capacity. Practically, we show how binding constraints reshape merger incentives and outcomes: allowing a merged firm to reallocate scarce capacity across products and markets can increase total surplus relative to a no-reallocation benchmark, but such reallocation is profit-driven and need not be socially efficient.

[1] See, e.g., Ryan (2012); Miller and Osborne (2014); Bodéré (2023); Weiss (2023); Zahur (2024); Miller et al. (2025); Brandt et al. (2026).

Methodologically, we handle cross-market dependence by extending the aggregative game approach, in which each firm's payoff depends on rivals' actions only through a low-dimensional aggregator (Selten, 1970). It replaces a high-dimensional best-response system with a fixed-point problem in a small set of sufficient statistics, which greatly simplifies computation and comparative statics. Recent applications use aggregative structure in oligopoly with single-product firms (Anderson et al., 2020), multiproduct firms (Nocke and Schutz (2018); Garrido (2024)), and merger analysis that exploits aggregative properties (Caradonna et al. (2025); Nocke and Schutz (2025)). Existing multiproduct Bertrand results use an aggregator that operates within a market (scalar in Nocke and Schutz (2018), and vector-valued under richer substitution patterns in Garrido (2024)). However, in our setting, capacity constraints add an additional cross-market, within-product layer, because for each product we need another sufficient statistic to summarize the shadow cost of scarce capacity allocated across markets. This yields a nested fixed-point structure with two interacting vector-valued fixed-point problems, consisting of an across-market layer induced by capacity constraints and a within-market layer governing pricing. The resulting problem is substantially more complex than the single-layer, vector-valued fixed-point problem studied in prior work.

Moreover, we study both a benchmark setting with linear cost functions and an alternative setting with convex costs. In the benchmark case, we use tools from nonsmooth analysis to handle the kinks introduced by binding capacity constraints, offering an approach that may be useful for related problems. In particular, our uniqueness argument uses the Clarke generalized Jacobian and Pourciau (1983) extension of the global univalence theorem to non-smooth mappings. Intuitively, the Clarke generalized Jacobian collects limiting Jacobians near non-differentiable points; if these generalized derivatives preserve the sign structure needed for univalence, global uniqueness extends beyond smooth cases. We show that under MNL demand with capacity constraints, own-market effects dominate cross-market spillovers in the relevant sense, keeping the generalized Jacobian nonsingular even at kinks. Combined with our nested fixed-point construction, this delivers equilibrium existence and uniqueness.

Our work builds on and contributes to the multiproduct Bertrand oligopoly literature by incorporating capacity constraints that link incentives, decisions and outcomes across markets, a feature that is largely absent from existing theoretical results. Earlier work addresses single-product firms under a variety of demand systems (Caplin and Nalebuff (1991); Aksoy-Pierson et al. (2013)). For logit and CES demand, Konovalov and Sándor (2010) use lattice-theoretic arguments (building on Kellogg (1976)) to prove existence and uniqueness of multiproduct firms' price equilibria. For nested logit with multiproduct firms, existence has been shown under additional restrictions such as symmetric costs (Anderson and de Palma (1992)), firm-specific nests (Liu (2006)), and product-specific price sensitivities (Gallego and Wang (2014); Jiang et al. (2017)). More recently, Nocke and Schutz (2018) establish existence and uniqueness of equilibrium in multiproduct Bertrand

competition under any demand with IIA property. Garrido (2024) extends these results to nested logit and nested CES demand with arbitrary firm–nest overlap. However, none of these contributions study capacity constraints that induce interdependence in pricing decisions across multiple markets.

There is also a related literature that studies dynamic pricing with capacity constraints and intertemporal trade-offs (e.g., Dudey (1992); Gale and Holmes (1993); Martínez-de Albéniz and Talluri (2011)). More recently, Betancourt et al. (2024) develop a general dynamic multiproduct pricing game and provide conditions for equilibrium existence and uniqueness in airline markets, where the key mechanism is the stochastic arrival of consumers and the associated intertemporal trade-off between selling today and preserving capacity for future demand. Our model is related to, but fundamentally distinct from, this literature. In particular, these papers study stock constraints in which discrete inventory diminishes stochastically as consumers arrive sequentially over a finite horizon, a setting appropriate for perishable goods such as airline seats or concert tickets with a fixed deadline. In contrast, we abstract from demand uncertainty and focus on static pricing with capacity constraints that operate through production costs.

Lastly, our paper relates to the literature on oligopoly competition with capacity constraints. Much of the classic theoretical work in this literature focuses on homogeneous goods. For example, Osborne and Pitchik (1986) characterize the set of Bertrand-Nash equilibria in a duopoly model. Alsabah et al. (2021) establish uniqueness of the Cournot–Nash equilibrium in a multimarket model with homogeneous goods and capacity constraints, and Caldentey and Haugh (2024) characterize the unique equilibrium in a multimarket Cournot model with heterogeneous firm-level capacity constraints under independent linear inverse demand. In contrast, we study a setting with product differentiation, a key feature that is routinely incorporated in empirical applications.

The rest of the paper is organized as follows: In Section 2, we introduce the model and firm's constrained maximization problem. In Section 3, we present the main results by first showing that the KKT conditions are sufficient for profit maximization, and then proving that there is a unique Nash equilibrium that maximizes firms' profits, together with an equilibrium-finding algorithm. In Section 4, we show some comparative statics and merger analysis. Next, we discuss some extensions in Section 5, and Section 6 concludes. Additional proof details are provided in the **Online Appendix**.[2]

2 Model

Consider multiple firms competing to sell differentiated products to consumers across different geographic regions or segments. Each firm $f \in \mathscr{F}$ offers a set of products $\mathscr{J}^f$, and these products

[2]Available at https://github.com/luo8yao/mmc/blob/main/online_appendix_mmc.pdf.

are sold in multiple markets $m \in \mathscr{M}$, where each market m has L_m potential consumers. We assume that each product is owned by a single firm, and let $\mathscr{J}$ denote the set of all products [3]. Each consumer solves a discrete choice problem, selecting the product that yields the highest utility. Consumers may also choose an outside option, meaning they buy nothing from the set of firms we study. We model this trade-off using a standard MNL demand system. Let s_{jm} denote the market share of product j in market m, p_{jm} its price, and δ_{jm} a mean utility term capturing all non-price determinants of demand. The parameter $\alpha > 0$ governs consumers' price sensitivity. Market shares are then given by

$$s_{jm} = \frac{\exp\left(\delta_{jm} - \alpha p_{jm}\right)}{1 + \sum_{j'} \exp\left(\delta_{j'm} - \alpha p_{j'm}\right)}, \tag{1}$$

where the "1" in the denominator represents the outside option, whose utility is normalized to zero.

Each firm wants to maximize profits by choosing prices for all its products across all markets. However, firms have limited capacity κ_j on a product j. This creates strategic trade-offs, under which a firm may raise prices not only to increase margins, but also to manage demand when capacity is scarce. Formally, firm f solves:

$$\pi^f = \max_{p^f} \sum_m L_m \sum_{j \in \mathscr{J}^f} \left(p_{jm} - c_{jm}\right) s_{jm} \quad \text{s.t.} \quad \sum_m L_m s_{jm} \leq \kappa_j \text{ for all } j \in \mathscr{J}^f \tag{2}$$

We denote by $p_m = (p_{jm})_{j \in \mathscr{J}}$ the vector of prices of all products in market m, by $p^f = (p_{jm})_{j \in \mathscr{J}^f, m \in \mathscr{M}}$ the vector of prices of firm f's products across all markets, and by $p = (p^f)_{f \in \mathscr{F}}$ the full price allocation across all firms.

We let c_{jm} be the (constant) marginal cost of product j in market m, [4] and we will discuss extensions with non-constant marginal costs and finer geographic structure in Section 5.

Now consider this constrained maximization problem. As noted above, we cannot directly apply the Karush–Kuhn–Tucker (KKT) Theorem to establish that the first-order conditions are necessary and sufficient for global optimality. This is because the firm's profit function is neither concave nor quasi-concave with multiproduct firms (Spady (1984); Hanson and Martin (1996)). Although weaker conditions, such as invexity, could in principle justify applying the KKT theorem, they are not straightforward to verify in this setting. Furthermore, the payoff function may not be supermodular or log-supermodular due to the existence of multiproduct firms (Nocke and Schutz (2018)). Due to these difficulties, we take an alternative approach by constructing a firm f's best-

[3] Mathematically, $\mathscr{J}^f$ and $\mathscr{J}^{f'}$ are disjoint (i.e., $\mathscr{J}^f \cap \mathscr{J}^{f'} = \emptyset$ if $f' \neq f$), and $\cup_{f \in \mathscr{F}} \mathscr{J}^f = \mathscr{J}$.

[4] For example, we can interpret the total cost of supplying product j in market m as the sum of a product-specific cost c_j and a market-specific transportation cost τ_{jm}.

response operator and show that, after appropriately extending the domain to include infinite prices, the KKT conditions are both necessary and sufficient for global optimality.

We compactify the firm's choice set by allowing $p_{jm} = \infty$ using a similar approach to Nocke and Schutz (2018). At this price, the firm effectively withdraws product j from the market, resulting in zero demand and, consequently, zero profit from that product. We define the profit function for finite prices and extend it by continuity to price vectors with infinite components[5].

For notational convenience in what follows, let

$$H_m(p_m) \equiv 1 + \sum_{j'} \exp\left(\delta_{j'm} - \alpha p_{j'm}\right) \tag{3}$$

denote the denominator of the market-share equation (1), so that $s_{jm} = \exp\left(\delta_{jm} - \alpha p_{jm}\right)/H_m(p_m)$. Also, let $H^{-f} = (H_m^{-f})_{m \in \mathscr{M}}$ denote the aggregator summarizing all rivals' pricing strategies:

$$H^{-f} := \left(1 + \sum_{k \notin \mathscr{J}^f} \exp(\delta_{km} - \alpha p_{km})\right)_{m \in \mathscr{M}}.$$

We are now ready to state the KKT conditions. The Lagrangian of firm f's constrained problem is

$$\mathscr{L}^f(p^f, H^{-f}) = \pi^f(p^f, H^{-f}) - \sum_{j \in \mathscr{J}^f} \lambda_j \left(\sum_m L_m s_{jm} - \kappa_j\right),$$

where $\lambda_j \geq 0$ is the Lagrange multiplier on product j's capacity constraint.

Definition 1 (KKT Conditions)**.** *The KKT conditions of the multiproduct firm's profit maximization problem in multiple markets hold at price vector $p \in \mathscr{P}$, if for each firm f with price vector p^f and for any fixed H^{-f}, there exists a Lagrange multiplier $\lambda_j \geq 0$ such that for every $j \in \mathscr{J}^f$ the following hold.*

1. ***Interior case*** *($p_{jm} < \infty$ for all m): the firm sets a finite price in every market.*

   (a) Stationarity*: at the optimum, the marginal profit from raising p_{jm} is exactly offset by the marginal shadow cost of the capacity it consumes,*

$$\frac{\partial \mathscr{L}(p^f, H^{-f})}{\partial p_{jm}} = \frac{\partial \pi^f}{\partial p_{jm}} - L_m \sum_{j' \in \mathscr{J}^f} \lambda_{j'} \frac{\partial s_{j'm}}{\partial p_{jm}} = 0.$$

[5]However, the only exception occurs when $p_{jm} \to \infty$ for every j and there is no outside option; in this case, the limit of firm's profit may not even exist. This degenerate case does not arise in our model due to the presence of an outside option.

(b) Primal feasibility*: total quantity sold cannot exceed capacity,*

$$\sum_m L_m s_{jm} \leq \kappa_j \quad \textit{for all } j \in \mathscr{J}^f.$$

(c) Dual feasibility*: shadow costs are non-negative,*

$$\lambda_j \geq 0 \quad \textit{for all } j \in \mathscr{J}^f.$$

(d) Complementary slackness*: capacity has positive shadow value only when it binds,*

$$\lambda_j \left(\sum_m L_m s_{jm} - \kappa_j \right) = 0 \quad \textit{for all } j \in \mathscr{J}^f.$$

2. ***Corner case*** *($p_{jm} = \infty$ for some m): the firm optimally withdraws product j from market m rather than serving it at any finite price. This is optimal only if no feasible deviation to a finite price would raise profits, i.e.,*

$$\pi^f(p^f, H^{-f}) \geq \pi^f\left(\left(\tilde{p}_{jm}, (p_{jm'})_{m' \in \mathscr{M} \setminus \{m\}}, (p_{km})_{k \in \mathscr{J}^f \setminus \{j\}, m \in \mathscr{M}} \right), H^{-f} \right)$$

for all finite $\tilde{p}_{jm}$ such that the resulting allocation still satisfies the capacity constraint

$$\sum_m L_m s_{jm}\left(\left(\tilde{p}_{jm}, (p_{jm'})_{m' \in \mathscr{M} \setminus \{m\}}, (p_{km})_{k \in \mathscr{J}^f \setminus \{j\}, m \in \mathscr{M}} \right), H^{-f} \right) \leq \kappa_j.$$

The Lagrange multiplier λ_j represents the shadow cost, i.e., the marginal value of relaxing the capacity constraint of product j. Formally, λ_j measures the marginal increase in the firm's maximized profit from expanding the capacity κ_j of product j by one unit; under standard regularity conditions, $\lambda_j = \partial \pi^*(\kappa)/\partial \kappa_j$. Economically, λ_j captures the marginal value of scarce capacity of product j: when capacity binds, allocating one more unit to a given sale crowds out other profitable uses of that capacity, so the firm prices as if its marginal cost were higher by λ_j. In addition, the complementary slackness condition $\lambda_j \left(\sum_m L_m s_{jm} - \kappa_j \right) = 0$ implies that $\lambda_j = 0$ when the constraint is slack, because capacity is not valuable at the margin; and that $\lambda_j > 0$ only when the constraint binds, in which case capacity constraint is valuable and distorts the firm's pricing incentives.

3 Main Results

The proof consists of two parts. First, we prove that the KKT conditions are necessary and sufficient for each firm's global optimization problem. This step is essential: if the KKT conditions were not sufficient for global optimality, a profile satisfying every firm's KKT conditions could still fail to be a Nash equilibrium, since some firm might have a profitable deviation not ruled out by KKT. Second, using this KKT sufficiency result, we prove that there exists a unique strategy profile that satisfies all firms' KKT conditions simultaneously. Since at that profile each firm satisfies the sufficient conditions for a global best response, every firm is indeed best-responding to its rivals. Therefore, this profile is the unique Nash equilibrium.

3.1 KKT Sufficiency

Consider the problem defined in (2). The compactification to infinite prices guarantees that each firm's problem has at least one solution, as in Nocke and Schutz (2018). We can further restrict attention to interior solutions in part (1) of Definition 1 by showing directly that any profile with boundary prices is suboptimal (see Online Appendix A). Notice that the stationary condition 1(a) can be equivalently rewritten as:

$$p_{jm} - c_{jm} - \lambda_j = \underbrace{\frac{1}{\alpha} + \sum_{k \in \mathscr{J}^f} (p_{km} - c_{km} - \lambda_k)\, s_{km}}_{:= \nu_m^f \text{ (market power markup)}} \tag{4}$$

For any given vector of shadow costs λ^f, the right-hand side is independent of j, depending only on the summation over all products of firm f (i.e., $k \in \mathscr{J}^f$). This implies that the left-hand side, $p_{jm} - c_{jm} - \lambda_j$, takes a common value across all $j \in \mathscr{J}^f$ in market m. We denote this common value by ν_m^f.

This yields the price decomposition:

$$p_{jm}(\nu_m^f, \lambda_j) = \nu_m^f + c_{jm} + \lambda_j. \tag{5}$$

Replacing the prices in (4) using the decomposition in (5) and solving for ν_m^f yields the following important identity

$$\nu_m^f = \frac{1}{\alpha(1 - \sum_{k \in \mathscr{J}^f} s_{km})}. \tag{6}$$

Equation (6) implies that ν_m^f is directly related to firm f's total market share in market m, and therefore we refer to this quantity as the *market power markup*. As in Nocke and Schutz (2018),

the market-level uniformity of v_m^f arises due to the Independence of Irrelevant Alternatives (IIA) property of the multinomial logit model. However, unlike in prior work, in this setting the firm can have different v_m^f across markets. In addition, the price p_{jm} is decomposed into market power markup, marginal cost, and shadow cost via (5).

Specifically, holding rivals' strategies fixed, we show that for any given vector of shadow costs, the stationarity conditions in (4) pin down a unique value of v_m^f. The key is that these conditions can be reduced to a single equation in a single unknown, v_m^f. Consequently, the firm's best response $p_m^f = \left(p_{jm}(v_m^f, \lambda_j)\right)_{j \in \mathscr{J}^f}$ is also uniquely determined.

Lemma 1. *For firm f producing products $j \in \mathscr{J}^f$, fix an arbitrary shadow cost vector $\lambda^f = (\lambda_j)_{j \in \mathscr{J}^f} \in \mathbb{R}_+^{|\mathscr{J}^f|}$ and arbitrary rivals' strategies, represented through H_m^{-f}, there exists a unique vector of prices $p_m^f = (p_{jm})_{j \in \mathscr{J}^f}$ that satisfies the KKT conditions in Definition 1. Moreover, the unique mapping $v_m^f(\lambda^f) : \mathbb{R}_+^{|\mathscr{J}^f|} \to \mathbb{R}_+$ is continuously differentiable (of class $\mathscr{C}^1$).*

Proof. By the market power markup property implied by the stationary condition in (4) and (5), it follows that v_m^f must satisfy:

$$\hat{F}(v_m^f, \lambda^f) := H_m^{-f}\left(v_m^f - \frac{1}{\alpha}\right) - \frac{1}{\alpha}\sum_{j \in \mathscr{J}^f} \exp\left(\delta_{jm} - \alpha\left(v_m^f + c_{jm} + \lambda_j\right)\right) = 0.$$

Note that $\hat{F}(v_m^f, \lambda^f)$ is clearly continuous and $\mathscr{C}^1$ on $\mathbb{R}_+ \times \mathbb{R}_+^{|\mathscr{J}^f|}$, we consider its derivative:

$$\frac{d\hat{F}(v_m^f, \lambda^f)}{dv_m^f} = H_m^{-f} + \sum_{j \in \mathscr{J}^f} \exp\left(\delta_{jm} - \alpha\left(v_m^f + c_{jm} + \lambda_j\right)\right) > 0$$

Notice that $\hat{F}\left(\frac{1}{\alpha}, \lambda^f\right) < 0$, and $\lim_{v_m^f \to \infty} \hat{F}(v_m^f, \lambda^f) = \infty$. Hence, there exists a unique solution $v_m^f > \frac{1}{\alpha}$ by the Intermediate Value Theorem (IVT), and $\hat{F}$ is injective. We can then apply the Implicit Function Theorem (IFT) to $\hat{F}\left(v_m^f, \lambda^f\right) = 0$. Since $\partial \hat{F}/\partial v_m^f > 0$, there exists a unique continuously differentiable function $v_m^f(\lambda^f)$ such that $\hat{F}\left(v_m^f(\lambda^f), \lambda^f\right) = 0$.

By differentiating the equation $\hat{F}(v_m^f(\lambda^f), \lambda^f) = 0$, we have

$$\frac{\partial v_m^f(\lambda^f)}{\partial \lambda_j} = -\frac{\partial \hat{F}(v_m^f(\lambda^f), \lambda^f)/\partial \lambda_j}{\partial \hat{F}(v_m^f(\lambda^f), \lambda^f)/\partial v_m^f} = -s_{jm} < 0$$

□

We then define a best-response operator for the shadow cost of product $j \in \mathscr{J}^f$, conditional on the shadow costs of the firm's other products $k \in \mathscr{J}^f \backslash \{j\}$, given a fixed vector of $H^{-f} =$

$(H_m^{-f})_{m\in\mathscr{M}} = \left(1+\sum_{k\notin\mathscr{J}^f}\exp(\delta_{km}-\alpha p_{km})\right)_{m\in\mathscr{M}}$. We then show the existence of a unique solution vector λ^f. Specifically, we define the aggregate quantity of product j as $\Gamma_j(\lambda_j;\lambda_{-j}^f) := \sum_m L_m s_{jm}(\lambda^f)$ and the operator:

$$\Psi_j(\lambda^f) = \begin{cases} 0, & \text{if } \Gamma_j(0;\lambda_{-j}^f) \le \kappa_j \\ \Gamma_j^{-1}(\kappa_j;\lambda_{-j}^f), & \text{if } \Gamma_j(0;\lambda_{-j}^f) > \kappa_j \end{cases} \tag{7}$$

where $\Gamma_j^{-1}(\kappa_j;\lambda_{-j}^f)$ returns the unique positive λ_j required for the aggregate quantity of product j to match exactly the capacity limit κ_j. This definition of the operator reflects complementary slackness: the capacity constraint is slack (i.e., $\lambda_j = 0$) if and only if $\Gamma_j(0;\lambda_{-j}^f) \le \kappa_j$; otherwise, the multiplier adjusts to make the constraint active. By construction, the operator also satisfies primal and dual feasibility. We define the simultaneous best response operator of firm f as:

$$\Psi(\lambda^f) := \left(\Psi_j(\lambda^f)\right)_{\forall j\in\mathscr{J}^f}.$$

We handle cross-market dependence through a sufficient statistic: the shadow cost λ_j summarizes how the capacity constraint links product j's pricing across markets. Given λ_j, the optimal price in each market is determined independently. This reduces the firm's problem from jointly optimizing over all $\left(p_{jm}\right)_{j,m}$ to finding a fixed-point in the space of shadow costs $\left(\lambda_j\right)_{j\in\mathscr{J}^f}$. Therefore, a solution to the firm's profit maximization problem corresponds to a fixed-point of Ψ.

To prove existence, we show that the domain of λ is compact and then apply Brouwer fixed-point theorem. The lower bound $\lambda \ge 0$ holds trivially by construction. For an upper bound, fix $H_m^{-f} > 0$ for all markets m, and consider product j. If we send all other products' multipliers to $+\infty$, their market shares go to zero, leaving product j facing the least competitive environment. In this extreme case, j's demand is maximized, and the firm must assign the largest shadow cost λ_j, thereby reducing product j's market share until its capacity constraint binds with equality. This provides a finite uniform upper bound for λ_j, hence compactness of the domain of λ. Next, Lemma 2 shows that the solution satisfying the KKT conditions is unique.

Lemma 2. *The operator Ψ has a unique fixed-point in $\prod_{j\in\mathscr{J}^f}[0,\bar{\lambda}_j]$. Further, λ^* satisfies all KKT conditions if and only if λ^* is a fixed-point of Ψ.*

Proof sketch. See Online Appendix A for the complete proof. Here, we provide a brief summary of the main steps. Recall from the proof of Lemma 1 that $v_m^f(\lambda^f)$ is clearly bounded below. So we can show that $p_{jm}(v_m^f(\lambda^f),\lambda_j) \to \infty$ and $s_{jm}(\lambda^f) \to 0$ if and only if $\lambda_j \to \infty$, and $\frac{\partial s_{jm}}{\partial \lambda_j} < 0$, $\frac{\partial s_{jm}}{\partial \lambda_k} > 0$ for $k \in \mathscr{J}^f\backslash\{j\}$, and $\left|\frac{\partial s_{jm}}{\partial \lambda_j}\right| > \sum_{k\in\mathscr{J}^f\backslash\{j\}}\left|\frac{\partial s_{jm}}{\partial \lambda_k}\right|$ (proved in Online Appendix Lemma A.5). Since $\Gamma_j\left(\lambda_j;\lambda_{-j}^f\right) := \sum_m L_m s_{jm}$, Γ_j also has these properties.

Next, Online Appendix Lemma A.6 establishes that the operator Ψ maps a compact set continuously to itself. First, Ψ_j is well-defined and continuous by analyzing three cases based on whether initial aggregate quantity $\Gamma_j\left(0;\lambda_{-j}^f\right)$ exceeds, falls below, or equals capacity κ_j. In each case, strict monotonicity of Γ_j in λ_j ensures either a unique interior solution (via IVT) or a corner solution at zero, with continuity following from the joint continuity of Γ_j. Second, we consider the compactness. The operator Ψ_j is bounded below by zero by construction. For the upper bound, define $\bar{\lambda}_j := \sup_{\lambda_{-j}^f} \Psi_j(\lambda^f)$. Since Γ_j is increasing in λ_{-j}^f, this supremum is attained as $\lambda_{-j}^f \to \infty$. If $\Gamma_j(0,\infty) \leq \kappa_j$, then $\bar{\lambda}_j = 0$. Otherwise, $\bar{\lambda}_j = \Gamma_j^{-1}(\kappa_j,\infty) < \infty$, established via contradiction using monotonicity of Γ_j in λ_j. Hence $\Psi : \prod_j [0,\bar{\lambda}_j] \to \prod_j [0,\bar{\lambda}_j]$, and existence of λ^{f*} follows from Brouwer's theorem.

Then, the Univalence Lemma A.1 in Appendix A shows that the operator Ψ is globally Lipschitz on a compact domain. Lemma A.1 also shows that if two equilibria existed, the generalized Mean Value Theorem (Pourciau, 1983) would require a singular matrix in the Clarke Jacobian $\partial F([x^*,x^{**}])$. However, since all limiting Jacobians are diagonally dominant Z-matrices, which is a property preserved under convex combinations, every element of the convex hull ∂F is a non-singular M-matrix (Berman and Plemmons, 1994), establishing uniqueness.

□

To prove the uniqueness of zeros of the mapping $F\left(\lambda^f\right) = \lambda^f - \Psi\left(\lambda^f\right)$, we must address a technical challenge: at the boundary where $\Gamma_j\left(0;\lambda_{-j}^f\right) = \kappa_j$, the operator Ψ_j transitions between the smooth function $\Gamma_j^{-1}\left(\kappa_j;\lambda_{-j}^f\right)$ and the constant zero. This transition is continuous but not differentiable—the derivative from the slack side is zero, while the derivative from the binding side is generically nonzero by the Implicit Function Theorem. Standard global univalence theorems, such as the Gale-Nikaido theorem in Gale and Nikaido (1965), or Kellogg (1976) as applied via Konovalov and Sándor (2010), require the mapping to be differentiable everywhere and thus cannot be applied directly.

We resolve this by invoking the Clarke generalized Jacobian and Pourciau (1983)'s extension of global univalence to nonsmooth mappings. The Clarke generalized Jacobian at a non-differentiable point is defined as the convex hull of limiting Jacobians from nearby differentiable points. The key insight is that if the Jacobian J_F at all differentiable points is a strictly diagonally dominant Z-matrix, then every element of the Clarke generalized Jacobian inherits this structure because strict diagonal dominance is preserved under convex combinations. This ensures non-singularity of the generalized Jacobian throughout the domain, yielding global uniqueness [6].

Because the shadow cost solution vector $\lambda^f(p^{-f})$ is unique, its implied price vector is also unique by Lemma 1. Since we know that at least one maximizer exists and any maximizer must

[6]Details are provided in the appendix section "Univalence Lemma" (Appendix A).

satisfy the KKT conditions, this unique price vector derived from the KKT system must coincide with the maximizer. Therefore, the KKT conditions are necessary and sufficient for the firm's profit maximization problem, as stated in the following theorem.

Theorem 1. *The KKT conditions defined in Definition 1 are both necessary and sufficient for characterizing the solution of each firm's profit maximization problem. For any fixed rivals' aggregator vector $(H_m^{-f})_{m\in\mathscr{M}}$, there exists a unique solution satisfying these conditions, given by $p^*_{jm} = p_{jm}(v_m^f(\lambda^{*f}), \lambda_j^*)$ for all products of firm f (i.e., $j \in \mathscr{J}^f$).*

Proof. See Online Appendix A for the proofs of each Lemma. First, Online Appendix Lemma A.1 shows that setting all prices to infinity is suboptimal: a firm can always profitably deviate by setting a finite price above marginal cost while satisfying capacity constraints. It also shows that pricing below marginal cost is strictly dominated. The argument is not simply that money-losing products should be dropped because doing so naively could violate capacity constraints on other products if demand reallocates. The proof constructs a careful deviation by removing unprofitable products and adjusting the profitable products' prices to preserve their market shares exactly. Because removing products shrinks total firm presence in the market, the same shares are achievable at higher prices. Since quantities stay the same (preserving feasibility) but markups rise, profit strictly increases.

Second, each firm's profit maximization problem has solutions by Online Appendix Lemma A.2. Since any $p_{jm} < c_{jm}$ in any market is strictly dominated by Online Appendix Lemma A.1, we further allow $p_{jm} = \infty$ by a continuous extension compactifying the choice space. In particular, the ambient space, $\prod_{j,m}[c_{jm}, \infty]$, is compact by Tychonoff's theorem. Since the the action space $\mathscr{A}^f$ is a non-empty closed subset of a compact ambient space and $\pi^f(p^f, H^{-f})$ is continuous, firm f's profit maximization problem must have a solution. We also show that KKT conditions are necessary for these solutions by Online Appendix Lemma A.3.

To establish sufficiency, we need to show that any solution satisfying the KKT conditions is globally optimal. By Online Appendix Lemma A.4, it suffices to consider only finite prices, so we focus on condition 1 of Definition 1. To prove Online Appendix Lemma A.4, we first show that the optimality of infinite pricing $p_{jm} = \infty$ requires $\lambda_j = \infty$, which implies $p_{jm} = \infty$ for all markets. Then, suppose some products are priced at infinity in all markets. We construct an alternative price vector that sets these products at finite (but high) prices. The key difficulty in the price construction is to satisfy a three-way trade-off simultaneously: first, the new prices must exceed marginal cost; second, it must be profit-improving (i.e., the profit gain from selling previously unserved products exceeds the cannibalization loss on other products); third, new quantities must not violate capacity constraints. We show the existence of such an alternative pricing which finishes the proof.

Finally, for any fixed rival firms' aggregator vector $(H_m^{-f})_{m\in\mathscr{M}}$, the operator Ψ has a unique fixed-point λ^{*f}, and any λ^f satisfying the KKT conditions must be a fixed-point of Ψ by Lemma

2. By Lemma 1, this unique λ^{*f} determines a unique price vector $p^*_{jm} = p_{jm}(\nu^f_m(\lambda^{*f}), \lambda^*_j)$ satisfying the stationary conditions. Therefore, there exists exactly one multiplier vector λ^{*f} satisfying the KKT conditions. Since the KKT conditions are necessary for optimality and admit a unique solution, they must also be sufficient, which finishes the proof.

□

3.2 Nash Equilibrium

Next, we characterize the Nash equilibrium by solving all firms' KKT conditions simultaneously, without holding fixed rivals' prices. Recall, the market share of product j in market m is given by:

$$\hat{s}_{jm} = \frac{\exp\left(\delta_{jm} - \alpha p_{jm}\right)}{1 + \sum_{j' \in \mathcal{J}} \exp\left(\delta_{j'm} - \alpha p_{j'm}\right)} := \frac{\exp\left(\delta_{jm} - \alpha p_{jm}\right)}{H_m} \tag{8}$$

where H_m is a scalar aggregator summarizing the strategic behavior of all firms. The term H_m aggregates the attractiveness of all inside options (and the outside option) in market m. When H_m rises, holding p_{jm} fixed, product j's share falls. In other words, H_m is a natural index of competition intensity in market m. In equilibrium, H_m must be consistent with firms' prices: $H_m = 1 + \sum_{j' \in \mathcal{J}} \exp\left(\delta_{j'm} - \alpha p_{j'm}\right)$.

To prove the existence of a Nash equilibrium, we first show that for any fixed vector $\mathbf{H} = (H_m)_{m \in \mathcal{M}}$, there exists a unique vector of shadow costs, which in turn determines a unique vector of prices. We then demonstrate that there exists a vector $\mathbf{H}$ satisfying the equilibrium condition. By construction, the KKT conditions are satisfied simultaneously for all firms at this fixed-point in $\mathbf{H}$, implying a Nash Equilibrium because no firm has any incentive to deviate.

To begin, note that conditional on $\mathbf{H}$, the argument in Section 3.1 continues to apply. Fixing $\mathbf{H}$, each λ_j acts like an additional marginal cost: increasing λ_j lowers j's share and reduces the firm's capacity usage, while increasing another within-firm multiplier shifts demand toward j and raises j 's capacity usage. Products owned by other firms affect s_{jm} only through $\mathbf{H}$, so conditional on $\mathbf{H}$ they generate no direct cross-effects. These monotonicity properties ensure the shadow cost operator is well defined and yield a finite upper bound on λ_j by considering the least-competitive environment.

As for the uniqueness, because cross-effects across firms are zero when $\mathbf{H}$ is fixed, each firm's multipliers affect only its own capacity constraints and the fixed-point conditions separate across firms. The Jacobian is therefore block diagonal across firms, and uniqueness reduces to checking non-singularity within each firm's block. Within a firm, the same Jacobian properties holds as before, which guarantees the relevant mapping is one-to-one within each block. Thus, for any

fixed $\mathbf{H}$, the overall solution simultaneously satisfies the KKT conditions of all firms is unique. The result is summarized in the following Lemma:

Lemma 3. *For any fixed* $\mathbf{H}$*, the operator* $\hat{\Psi}$ *is well-defined and maps* $\prod_{i\in\mathscr{J}^f}[0,\bar{\lambda}_i]$ *into itself. Moreover, the fixed-point of* $\hat{\Psi}$ *is unique.*

Proof. This proof is analogous to Lemma 2. The proof of existence is analogous to Online Appendix Lemma A.6, with the key modification that $\hat{\Gamma}_j(\lambda;\mathbf{H}) = \sum_m L_m \hat{s}_{jm}(\lambda,\mathbf{H})$ now depends on $\mathbf{H}$. Specifically, Online Appendix B proves Online Appendix Lemma B.1 (the analogue of Lemma 1) and Online Appendix Lemma B.2 (the analogue of Online Appendix Lemma A.5), and the latter shows that $\hat{\Gamma}_j$ satisfies the same monotonicity properties: $\frac{\partial\hat{\Gamma}_j}{\partial\lambda_j} < 0$, $\frac{\partial\hat{\Gamma}_j}{\partial\lambda_k} > 0$ for $k \in \mathscr{J}^f\backslash\{j\}$, and $\frac{\partial\hat{\Gamma}_j}{\partial\lambda_k} = 0$ for $k \notin \mathscr{J}^f$. The well-definedness of $\hat{\Psi}_j$ and the bound $\bar{\lambda}_j(\mathbf{H}) := \hat{\Gamma}_j^{-1}(\kappa_j;\infty,\mathbf{H})$ follow by identical arguments.

For the uniqueness proof, the key modification is that now everything depends on $\mathbf{H}$ compared to Lemma 2. The key structural difference is that cross-effects are zero for products outside the firm, but this does not affect the argument; if anything, it strengthens diagonal dominance. In fact, $\hat{\Psi}_j$ depends only on $\lambda^f := (\lambda_k)_{k\in\mathscr{J}^f}$, and the Jacobian J_F is block-diagonal across firms, and uniqueness reduces to showing each firm's block is non-singular. Within each block, Online Appendix Lemma B.2 provides the identical sign structure and diagonal dominance as Online Appendix Lemma A.5. The remainder of the argument in Lemma 2 proceeds identically.

□

This highlights the advantage of the aggregative approach: once we express the game in terms of the aggregator, the arguments proceed as in Section 3.1, and the analysis reduces to an outer fixed-point problem in $\mathbf{H}$. Notice that $\mathbf{H}$ aggregates all firms' strategies, in other words, the complex cross-firm interactions are encoded in $\mathbf{H}$ and each firm responds independently to the change in $\mathbf{H}$. It also yields immediate welfare implications in Section 4.1, because $\mathbf{H}$ maps directly into consumer surplus.

To complete the proof, it remains to show that there exists a unique aggregator vector $\mathbf{H}$ that satisfies the fixed-point condition implied by its definition:

$$H_m = 1 + \sum_j \exp\big(\delta_{jm} - \alpha\hat{p}_{jm}(\mathbf{H})\big), \quad \forall m \in \mathscr{M}. \tag{9}$$

For any conjectured vector of market conditions $\mathbf{H}$, each firm optimally sets prices given its capacity constraints, yielding market shares that imply a new aggregate statistic $T_m(\mathbf{H}) = 1 + \sum_j \exp\big(\delta_{jm} - \alpha\hat{p}_{jm}(\mathbf{H})\big)$. Existence of equilibrium follows from three observations. First, aggregators are clearly bounded by $\underline{H}_m := 1$ and $\bar{H}_m := 1 + \sum_j \exp\big(\delta_{jm}\big)$ since prices are always

strictly positive. Second, small changes in market conditions induce small changes in firm behavior. The key economic insight is that shadow costs can kink when a firm switches between slack and binding regimes: the mapping is generally not differentiable at the threshold. However, at the boundary where the constraint just becomes binding, the slack and binding expressions coincide, so the shadow cost has no jump discontinuity and remains continuous. Third, the induced mapping $T(\mathbf{H})$ is continuous because the corresponding price vector is not set-valued by Lemma 3 and therefore cannot jump across multiple optimizers as $\mathbf{H}$ changes; we establish this formally using a graph-theoretic argument. Thus, by these three observations, there exists a fixed-point $\mathbf{H}^*$ such that conjectured and realized market conditions coincide.

Notice that (9) is equivalent to:

$$\underbrace{\sum_{j \in \mathscr{J}} s_{jm}(\mathbf{H})}_{\text{Actual total inside share}} = \underbrace{1 - H_m^{-1}}_{\text{Required inside share}}, \quad \forall m \in \mathscr{M}. \tag{10}$$

The left-hand side of (10) is the actual total inside share generated by firms' pricing decisions and reflects the supply side of the market. The right-hand side is the inside share required by the definition of H_m itself and reflects the demand side. Equilibrium requires these two sides to coincide in every market.

Uniqueness follows from a partial effect on the supply side, which we call the *spillover effect*. This effect ensures that the two sides of (10) move in opposite directions as competition changes. When market m becomes more competitive (i.e., H_m rises), on the demand side of (10), the "required" inside market share $1 - H_m^{-1}$ increases, but supply responds heterogeneously. For binding products, higher competition in market m makes it profitable to reallocate sales to other markets while keeping total capacity utilization fixed. For slack products, the direct competition effect dominates: although firms lower markups in response to competition, market shares still fall. Thus, total supply decreases in market m.

The spillover effect operates through the shared binding capacity constraint. When market m becomes more competitive, binding products spill over into market k and raise the supply of binding products there. Slack products in market k then face higher market power markups ν_k^f as binding products crowd in, lowering slack products' supply. However, the effect of binding products dominates, so total supply in market k rises.

Remark 1 (Spillover Effects and Cross-Market Linkages)**.** *Cross-market spillovers arise precisely because capacity is shared across markets, which is the central mechanism of our contribution. Importantly, these cross-market effects vanish when capacity constraints are absent or market-specific, highlighting that shared capacity is the source of strategic interdependence.*

The preceding analysis establishes the spillover effects at points where each product's constraint status is fixed. But as market conditions $\mathbf{H}$ vary, some products may switch from slack and binding regimes, creating kinks at the exact threshold where a constraint switches status in the excess demand system where standard derivatives do not exist. To establish uniqueness, we extend the analysis to these regime boundaries using the Clarke generalized Jacobian (Pourciau, 1983) again, which handles non-differentiable points by taking convex hulls of nearby limiting Jacobians. The spillover structure is preserved under this generalization, which leads to the following theorem:

Theorem 2. $\mathbf{H}^*$ *is unique.*

Proof sketch. See Online Appendix B for the complete proof. First, Online Appendix Lemma B.3 establishes the existence of fixed-points of the aggregator mapping. We construct a compact, convex set $[\underline{H},\bar{H}]^{|\mathscr{M}|}$ that is invariant under the mapping T, using the bounds $\underline{H}_m = 1$ and $\bar{H}_m = 1 + \sum_j \exp\left(\delta_{jm}\right)$. Second, we establish continuity of T. Online Appendix Lemma B.1 shows that $\hat{v}_m^f(\lambda, H_m)$ is of class $\mathscr{C}^1$ by the Implicit Function Theorem. Moreover, $\hat{\lambda}(\mathbf{H})$ is continuous because the correspondence $\Xi(\mathbf{H}) = \{\lambda : \lambda = \hat{\Psi}(\lambda;\mathbf{H})\}$ is single-valued (Lemma 3) and upper hemicontinuous. Existence then follows from Brouwer's fixed-point theorem.

Next, Online Appendix Lemma B.4 characterizes how the aggregator vector affects a firm's within-regime response in market shares. [7] The proof proceeds in two cases. In the all-slack regime (no binding capacity constraints), markets are decoupled. Increasing competition in market k reduces thw firm's market share in k but has no effect on other markets. In the binding regime, capacity constraints create cross-market linkages through the multipliers $\lambda_{\mathscr{B}^f}$, [8]which are implicitly defined by the system $\mathbf{G}(\lambda_{\mathscr{B}^f};\mathbf{H}) = \mathbf{0}$. The Jacobian $\mathbf{J} = \frac{\partial \mathbf{G}}{\partial \lambda_{\mathscr{B}^f}}$ is a strictly diagonally dominant Z-matrix and we can show that $\frac{\partial \mathbf{G}}{\partial H_k} < 0$. The Implicit Function Theorem then yields $\frac{\partial \lambda_i}{\partial H_k} < 0$ for all binding products $i \in \mathscr{B}^f$. This generates cross-market spillovers: for $m \neq k$, the total derivative $\frac{dv_m^f}{dH_k} = \sum_{i\in\mathscr{B}^f} \frac{\partial v_m^f}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial H_k} > 0$ (as a product of two negatives), implying $\frac{dS_m^f}{dH_k} > 0$. Decomposing by product type, binding products gain share in market m while slack products lose share, with the former effect dominating. For the own-market effect, total capacity utilization of binding products is fixed across markets, so gains in other markets imply $\frac{dS_k^{\mathscr{B}^f}}{dH_k} < 0$. Combined with $\frac{dS_k^{\mathscr{S}^f}}{dH_k} < 0$, which we also show, we obtain $\frac{dS_k^f}{dH_k} < 0$.

Unlike standard smooth settings, our model features kinks where capacity constraints switch between binding and slack. We analyze uniqueness in this nonsmooth environment by first establishing global Lipschitz continuity of the excess-demand system (10) via a piecewise argument

[7] We address the non-differentiability that arises when moving across regimes later.

[8] Let $\mathscr{B}^f$ denote the set of binding products and $\mathscr{S}^f$ the set of slack products of firm f.

over finitely many regimes. We then use Online Appendix Lemma B.4 to show that the Jacobian structure of this system satisfies the required properties. We can therefore apply the Univalence Lemma A.1 and use the Clarke generalized Jacobian together with the technique of Pourciau (1983) to conclude uniqueness.

□

It is worth noting an alternative route one might take to establish the Nash equilibrium results. One could fix the shadow cost vector λ, apply Konovalov and Sándor (2010) to obtain a unique price equilibrium $p(\lambda)$, and then study the induced mapping in λ to show it has a unique fixed-point. While this approach is feasible in principle, it is much harder to implement in our setting because capacity constraints create cross-market pricing linkages. Without the aggregator $\mathbf{H}$, working directly with λ tends to lose the clear monotonicity structure. Although λ summarizes firms' behavior across markets, firms' best responses still depend on the entire profile of rivals' strategies within each market, generating interdependencies that are difficult to order. The aggregator $\mathbf{H}$ resolves this difficulty by collapsing the within-market strategic environment to a low-dimensional object that summarizes the relevant competitive effects. This dimensionality reduction is exactly what restores tractability in the fixed-point argument. Moreover, Konovalov and Sándor (2010) is developed for unconstrained problems and does not, by itself, guarantee that the KKT conditions are sufficient to characterize equilibrium when constraints bind. In our constrained environment, establishing KKT sufficiency is therefore an essential additional step. Once we work with $\mathbf{H}$, however, we only need to solve a single outer fixed-point problem, so the aggregator does not add complexity; if anything, it simplifies the proof. Lastly, $\mathbf{H}$ also has a clear welfare interpretation, which makes it especially convenient for the experiments and welfare analysis in the Section 4.

Remark 2. *The aggregator framework provides a computationally tractable approach (Algorithm 1) for finding a Nash equilibrium.*

Finally, notice that the traditional methods solve for equilibrium prices directly, requiring a fixed-point in price space $\boldsymbol{p} \in \mathbb{R}^{|\mathscr{J}|\times|\mathscr{M}|}$, which is a system of $|\mathscr{J}| \times |\mathscr{M}|$ equations and unknowns. In contrast, our approach exploits the sufficient statistic property of the aggregators for every market: once the $|\mathscr{M}|$-dimensional vector $\mathbf{H}$ is determined, all equilibrium prices can be recovered explicitly. This reduces the core computational problem from $|\mathscr{J}| \times |\mathscr{M}|$ dimensions to just $|\mathscr{M}|$ dimensions—a substantial simplification when the number of products is large. Algorithm 1 implements this insight through a nested structure with three levels, each possessing distinct convergence properties.

First, for the inner loop that determines markups, given $(\boldsymbol{\lambda}, H_m)$ the markup ν_m^f solves a one-dimensional equation. By Online Appendix Lemma B.1, the problem is strictly monotone, so

Newton's method or bisection converges reliably from any starting point. This one-dimensional step is computationally trivial.

Second, in the middle loop, given $\mathbf{H}$, the shadow costs $\boldsymbol{\lambda}$ are determined by the best-response operator $\hat{\Psi}$ defined by:

$$\hat{\Psi}_j(\lambda_j;\lambda_{-j}) = \begin{cases} 0, & \text{if } \hat{\Gamma}_j(0;\lambda_{-j}) \le \kappa_j \\ \hat{\Gamma}_j^{-1}(\kappa_j;\lambda_{-j}), & \text{if } \hat{\Gamma}_j(0;\lambda_{-j}) > \kappa_j \end{cases} \tag{11}$$

where $\hat{\Gamma}_j(\lambda_j;\lambda_{-j}) := \sum_m L_m \hat{s}_{jm}(\lambda)$. The operator $\hat{\Psi}$ is a contraction mapping in the sup-norm. The sup-norm Lipschitz constant of Ψ is $\sup_{\boldsymbol{\lambda}} \max_j \sum_k |\frac{\partial \Psi_j}{\partial \lambda_k}|$. Since $\frac{\partial \hat{\Psi}_j}{\partial \lambda_j} = 0$ and $\frac{\partial \hat{\Psi}_j}{\partial \lambda_k} = \frac{\partial \hat{\Gamma}_j/\partial \lambda_k}{|\partial \hat{\Gamma}_j/\partial \lambda_j|}$ for $k \neq j$, we have $\rho = \sup_{\boldsymbol{\lambda}} \max_j \frac{\sum_{k\neq j} \partial \hat{\Gamma}_j/\partial \lambda_k}{|\partial \hat{\Gamma}_j/\partial \lambda_j|} < 1$ by diagonal dominance. Combined with a similar non-smooth argument established in Lemma 3, $\hat{\Psi}$ is a contraction, and fixed-point iteration converges geometrically from any initial point.

Third, the outer loop on $\mathbf{H}$ also converges under a generalized Newton iteration. The convergence of the outer loop is hard to verify because it is neither a contraction mapping nor order-monotone (so results like Tarski's fixed-point theorem, as used in Garrido (2024), do not apply). Moreover, F is not necessarily monotone in the variational inequality sense [9]. By Theorem 2, the Jacobian J_F is a Z-matrix with positive weighted column sums $J_F^\top \mathbf{L} \gg 0$, which guarantees uniqueness. But a positive semidefinite symmetric part of the Jacobian is not implied by the structure of J_F. Cross-market effects can be asymmetric ($\frac{\partial S_m}{\partial H_k} \neq \frac{\partial S_k}{\partial H_m}$) when binding capacity constraints create unbalanced spillovers, causing the symmetric part to have some negative eigenvalues even when all M-matrix conditions hold. However, damped Newton iteration with line search remains valid when the Jacobian is nonsingular. In addition, semismooth Newton theory extends these guarantees when all generalized Jacobians are non-singular (Qi and Sun, 1993). In practice, non-smoothness requires no special handling in simulations because iterates almost never land exactly on regime boundaries. At each iteration, the inner loop determines the binding/slack partition, and the Jacobian is computed using the current regime's formulas. A backtracking line search ensures descent even near regime boundaries. Alternatively, finite-difference Jacobian approximations provide a black-box approach that automatically accommodates the piecewise structure. Natural candidates for starting points are the bounds $\bar{H}_m = 1 + \sum_j \exp(\delta_{jm})$ and $\underline{H}_m = 1$.

More broadly, this aggregator-based approach offers a general strategy for simplifying equilibrium computation in differentiated products markets. Even in settings where the specific contraction or monotonicity properties of our model do not hold, reformulating the problem in terms of market-level sufficient statistics can dramatically reduce dimensionality. This key insight applies

[9] An operator F is VI-monotone if: $\langle F(\mathbf{x}) - F(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \ge 0 \quad \forall \mathbf{x}, \mathbf{y}$ where $\langle \cdot, \cdot \rangle$ denotes the inner product, which is equivalent to the symmetric part of the Jacobian being positive semi-definite: $\frac{1}{2}(J_F + J_F^\top) \succeq 0$.

whenever the demand system admits a suitable aggregator representation. This framework extends naturally to other demand specifications such as nested logit, mixed logit, and CES, though the specific convergence guarantees will depend on the structure of each model.[10] Nevertheless, the dimensional reduction remains valuable even when theoretical convergence guarantees are unavailable, as it substantially simplifies numerical exploration and reduces computational burden.

[10] We leave the formal analysis of convergence properties under alternative demand specifications for future research.

Algorithm 1 Nash Equilibrium via Nested fixed-point

Require: Initial guess $\mathbf{H}^{(0)}$ in $\mathbb{R}_{++}^{|\mathscr{M}|}$; tolerance $\varepsilon > 0$
Ensure: Equilibrium aggregator $\mathbf{H}^*$; equilibrium prices p^*

1: $n \leftarrow 0$ ▷ *Outer Loop on* $\mathbf{H}$ *starts*
2: **repeat**
3: Initialize $\lambda^{(0)} = \mathbf{0}$; set $k \leftarrow 0$ ▷ *Middle Loop on* λ *Starts*
4: **repeat**
5: Solve for the unique $\nu^{(k)}$ where each component $\nu_m^{f,(k)}$ satisfies: ▷ *Inner Loop on* ν

$$\nu_m^{f,(k)} = \frac{1}{\alpha} + \sum_{j' \in \mathscr{J}^f} \nu_m^{f,(k)} s_{j'm}\big(\nu_m^{f,(k)}; \lambda^{(k)}, \mathbf{H}^{(n)}\big)$$

for each firm $f \in \mathscr{F}$ and market $m \in \mathscr{M}$.
6: **(Update λ)** For each product $j \in \mathscr{J}$, update λ_j via operator Ψ_j:

$$\lambda_j^{(k+1)} = \hat{\Psi}_j\big(\lambda^{(k)}\big) = \begin{cases} 0, & \text{if } \hat{\Gamma}_j\big(0; \nu^{(k)}, \lambda_{-j}^{(k)}, \mathbf{H}^{(n)}\big) \le \kappa_j \\ \hat{\Gamma}_j^{-1}\big(\kappa_j; \nu^{(k)}, \lambda_{-j}^{(k)}, \mathbf{H}^{(n)}\big), & \text{otherwise} \end{cases}$$

7: $k \leftarrow k+1$
8: **until** $\|\lambda^{(k+1)} - \lambda^{(k)}\| < \varepsilon$ ▷ *Middle Loop Ends*
9: Set $\hat{\lambda} \leftarrow \lambda^{(k)}$ and $\hat{\nu} \leftarrow \nu^{(k)}$
10: **(Recover Prices)** For all $j \in \mathscr{J}^f$ and $m \in \mathscr{M}$: $\hat{p}_{jm} \leftarrow c_{jm} + \hat{\nu}_m^f + \hat{\lambda}_j$ for $j \in \mathscr{J}^f$
11: **(Update H)** For each m: $F_m \leftarrow H_m^{(n)} - 1 - \sum_{j \in \mathscr{J}} \exp\big(\delta_{jm} - \alpha \hat{p}_{jm}\big)$; Solve d from $J_F(\mathbf{H}^{(n)})\, d = -F(\mathbf{H}^{(n)})$.
12: Update $\mathbf{H}^{(n+1)} \leftarrow \mathbf{H}^{(n)} + t \cdot d$ with line search.
13: **until** $\|\mathbf{H}^{(n+1)} - \mathbf{H}^{(n)}\| < \varepsilon$ ▷ *Outer Loop Ends*
14: **return** $\mathbf{H}^* \leftarrow \mathbf{H}^{(n)}$; $p_{jm}^* \leftarrow c_{jm} + \hat{\nu}_m^f(\mathbf{H}^*) + \hat{\lambda}_j(\mathbf{H}^*)$ for $j \in \mathscr{J}^f$

Notes: $\hat{\Gamma}_j(\lambda_j; \nu, \lambda_{-j}, \mathbf{H}) := \sum_{m \in \mathscr{M}} L_m s_{jm}(\lambda_j; \nu, \lambda_{-j}, \mathbf{H})$ denotes the aggregate quantity of product j; $\hat{\Gamma}_j^{-1}(\kappa_j; \cdot)$ returns the unique $\lambda_j > 0$ such that $\hat{\Gamma}_j = \kappa_j$; λ_{-j} denotes λ with the j-th component removed; J_F may be computed analytically or via finite differences $[J_F]_{mk} \approx [F_m(\mathbf{H} + \varepsilon e_k) - F_m(\mathbf{H})]/\varepsilon$; $t \in (0,1]$ is the step size from backtracking line search ensuring sufficient decrease. The Newton step in line 11 is locally quadratically convergent, so in practice we warm-start $\mathbf{H}^{(0)}$ by iterating $H_m \leftarrow 1 + \sum_j \exp(\delta_{jm} - \alpha\, p_{jm}(\mathbf{H}))$ for each m until $\|F(\mathbf{H})\|$ is small to lie inside the basin of Newton's method.

4 Applications

In this section, we study two standard comparative statics: changes in the quality of the outside option and changes in the quality of inside options. We then illustrate merger implications through a stylized example. With capacity constraints, mergers introduce mechanisms absent from the unconstrained benchmark, especially when the merged firm can reallocate capacity across products (possibly at a cost).

4.1 Comparative Statics

We first allow the outside option to deliver positive indirect utility rather than being normalized to zero and study its effects. The excess demand function becomes:

$$F_m(\mathbf{H};\mathbf{H}_0) = \underbrace{\left(1-\frac{H_{0m}}{H_m}\right)}_{\text{Required Inside Share}} - \underbrace{S_m(\mathbf{H})}_{\text{Actual Inside Share}} = 0 \tag{12}$$

where $\mathbf{H}_0$ is the outside option's contribution to the aggregator, equal to the exponential of its quality in our model.

An increase in $\mathbf{H}_0$ shifts the $\mathbf{F}$-curve downward. At the original equilibrium $\mathbf{H}^*$, the required inside share $1 - H_{0m}/H_m^*$ decreases because consumers now have a better outside option. The actual inside share $S_m(\mathbf{H}^*)$ is unchanged, creating excess supply ($F_m < 0$). To restore equilibrium, H_m^* must increase: a higher H_m raises the required inside share and lowers the actual share, both effects closing the gap.

Cross-market spillovers are weakly non-negative: they are positive when some capacities bind and zero when all products are slack. When capacity binds and H_k increases, market shares in k fall, so binding products reallocate sales toward other markets m to maintain their capacity constraints. This increases supply S_m, creating excess supply in market m, which pushes H_m^* upward to restore equilibrium. When all products are slack, markets decouple and firms face no capacity trade-offs, so changes in H_k have no effect on H_m^* for $m \neq k$.

For firm profits, the rise in H_m^* reflects a more competitive environment. The envelope theorem shows firms cannot avoid this obstacle: re-optimizing prices cannot offset the direct effect of increased competition (since $s_{jm} \propto 1/H_m$). Profits fall because exogenously increased competition erodes market power. Hence the effect on total welfare is ambiguous. These results are summarized in the following Lemma:

Lemma 4. *The equilibrium mapping $\mathbf{H}^* : \mathscr{H}_0 \to \mathscr{D}$ is continuous and monotone: $\mathbf{H}_0' \geq \mathbf{H}_0 \Rightarrow \mathbf{H}^{*\prime} \geq \mathbf{H}^*$ (componentwise). When $\mathbf{H}_0$ increases, consumer surplus is weakly increasing, firms' profits are weakly decreasing, and the effect on total welfare is ambiguous.*

Proof. See Online Appendix C. □

Now consider comparative statics with respect to inside product quality shocks. The equilibrium system now becomes:

$$F_m(\mathbf{H};\boldsymbol{\delta}) = \left(1 - H_m^{-1}\right) - S_m(\mathbf{H};\boldsymbol{\delta}) = 0 \quad \forall m \in \mathscr{M}$$

The following Lemma studies the effect of increasing δ_{jk} for product $j \in \mathscr{J}^f$ in market k.

Lemma 5. *Locally, a quality improvement δ_{jk} for a slack product raises H_m^* in all markets, benefiting consumers and hurting rivals, but the effect on the firm's own profit is ambiguous; a quality improvement for a binding product has ambiguous welfare effects.*

Proof. See Online Appendix C. □

When product j is capacity constrained, the effect of a quality improvement on δ_{jk} is ambiguous due to the interaction between direct quality benefits and indirect capacity pricing effects. When product j's quality in market k increases, the direct effect makes product j more attractive. Holding shadow costs λ fixed, this raises the market power markup ν_k^f, increasing firm f's supply in market k. If this were the only effect, the equilibrium response would be unambiguously positive. However, since j is binding, the quality improvement creates excess demand for product j's capacity, forcing its shadow cost λ_j to increase. This shadow cost represents the opportunity cost of capacity: when quality improves, each unit of capacity becomes more valuable. The increased shadow cost feeds back into firm f's pricing and higher λ_j reduces the market power markup in all markets. This negative indirect effect operates everywhere, including market k itself. The net effect is therefore ambiguous because the firm cannot freely expand sales without paying the opportunity cost of scarce capacity, which can offset the direct quality benefit.

When product j is slack, the firm can freely expand sales, so H_m^* rises unambiguously in all markets. Consumers benefit and rivals are hurt. However, own-firm profit remains ambiguous: the direct effect $L_k \nu_k^f s_{jk} > 0$ captures the benefit of higher quality, but the rise in H_m^* intensifies competition and reduces profits. Which effect dominates depends on the magnitude of the equilibrium response relative to the direct quality benefit.

These comparative statics are local; for large quality changes that trigger regime switches, the cumulative effects may differ. Note, however, that the ambiguity above is structural: it arises from opposing effects, not from crossing regime boundaries.

4.2 Merger Implications

With shared capacity constraints, mergers in our model generate new insights relative to the unconstrained benchmark. We highlight two additional channels. First, shadow costs λ create a new mechanism ("shadow costs relief"): when capacity is scarce, each product's λ_j acts as an endogenous marginal-cost shifter, and it can move in the opposite direction from the standard market power markup ν_m^f. Second, mergers can reallocate capacity across the merged portfolio, shifting capacity from low-value products to high-value ones. This is common in practice. Firms may have strong products but limited capacity, and a key motivation for merging is precisely to move capacity toward higher-value goods. Importantly, this reallocation channel overturns the standard implication from unconstrained MNL models, in which mergers harm consumers by raising prices and reducing consumer surplus unless they get additional synergies (e.g., Nocke and Schutz (2025)). This is because reallocating scarce capacity creates an additional allocative efficiency gain from capacity reallocation that exists only when capacity constraints bind.

4.2.1 Shadow Cost Relief

First, we consider the "shadow costs relief" mechanism. Recall that by Equation (5), each product's price can be viewed as the sum of three parts: marginal cost c_{jm}, a market power markup ν_m^f, and a capacity shadow cost λ_j. The term ν_m^f captures the usual Bertrand market-power effect: ownership and market size contribute to market power and therefore raise markups. Because capacity is shared across markets, λ_j summarizes the opportunity cost of selling an extra unit of product j in any market: expanding sales in one market tightens the aggregate capacity constraint and crowds out capacity available elsewhere.

After a merger, the merged firm internalizes substitution and competes less aggressively, so market power increases and the market-power component ν_m^f of prices rises. However, there is an offsetting mechanism that comes from capacity. A higher price reduces demand for product j, which lowers total capacity usage. That relaxation makes the pressure from the capacity constraint less severe, so the shadow cost falls and this decline mechanically offsets part of the merger-induced increase coming from ν_m^f. The overall price impact depends on which component moves more.

We show a concrete numerical example of this effect:

Example 1. *Consider a two-product logit model in a single market with price coefficient* $\alpha = 1$. *Products have mean utilities* $\delta = (2.0, 1.8)$, *marginal costs* $c = (0.5, 0.5)$, *and capacity limits* $\kappa = (0.15, 0.15)$. *We consider a pure ownership change. Initially, each product is produced by a separate firm, and both firms are capacity constrained. The two firms then merge, and the merged firm remains capacity constrained.*

Table 1: Pre/Post Merger Outcomes

	Pre-merger			Post-merger		
Product	Price	ν	λ	Price	ν	λ
1	3.54	1.18	1.86	3.54	1.43	1.61
2	3.34	1.18	1.66	3.34	1.43	1.41

Example 1 illustrates the offset mechanism. In this calibration, both products are capacity constrained both before and after the merger. As a result, equilibrium prices are pinned down by the binding capacity limit $\kappa = 0.15$ together with marginal cost $c = 0.5$ (symmetric across products), so prices do not change after the merger. Because both products remain binding, equilibrium quantities are fixed at capacity, and prices adjust only to clear those quantities.

However, the price decomposition does change. Post-merger, the market power markup increases by 0.25, reflecting the merged firm's stronger incentive to raise markups. But higher markups reduce demand and therefore relax the pressure from the binding capacity constraint. The shadow cost falls by exactly 0.25 for both products, leaving prices unchanged. This example shows why capacity constraints can materially dampen standard merger price effects through "shadow-cost relief," and why decomposing prices into a market power markup and a capacity shadow cost is useful for interpreting merger impacts in constrained industries.

4.2.2 Capacity Reallocation

We now consider merger impacts when the merged firm can reallocate capacity across products. We use a stylized merger example with three firms and three products, in which two firms merge. The purpose is not to provide a full characterization of merger effects under arbitrary market structures, but to isolate the key mechanisms introduced by capacity constraints, especially the role of capacity reallocation across products within the merged firm. These mechanisms are not specific to the particular numbers of firms or products; the example is chosen solely for transparency. We begin with a numerical example and a figure illustrating the mechanism, then give the formal proof.

Example 2. *Consider a three-product logit model in a single market (market size is normalized to one) with an outside option and price coefficient $\alpha = 1$. Products have mean utilities $\delta = (2.0, 1.8, 1.6)$, marginal costs $c = (0.5, 0.5, 0.5)$, and pre-merger capacity limits $\kappa^{pre} = (0.05, 0.5, 0.2)$. Thus, product 1 has the highest quality but the tightest capacity. Initially, each product is produced by a separate firm. We then consider a merger between firms 1 and 2. Post-merger, the merged firm is allowed to reallocate its combined capacity across products 1 and 2, subject to*

$\kappa_1^{post} + \kappa_2^{post} = 0.55$, *while firm 3's capacity remains fixed at* $\kappa_3^{post} = 0.2$. *The resulting welfare effects are shown in Figure 1.* [11]

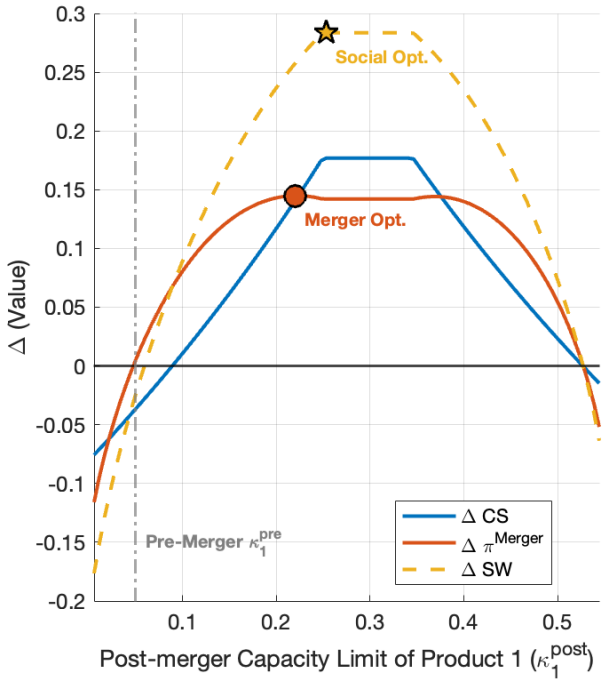

Figure 1: Welfare effects of post-merger capacity reallocation

Now consider the results in Figure 1. The x-axis is the post-merger capacity assigned to product 1, κ_1^{post}. Increasing κ_1^{post} mechanically means taking capacity away from product 2. The vertical dashed line marks the no-reallocation benchmark where $\kappa_1^{post} = \kappa_1^{pre}$ (i.e., product 1's capacity is kept at its pre-merger level).

At the dashed line, the blue curve shows $\Delta CS < 0$, which implies that the merger harms consumers when capacity is not reallocated toward product 1. Intuitively, ownership changes raise market power, but capacity stays "misallocated", with too little capacity on the high-value product. In this case, only the standard merger effect operates, so the merger generates the usual consumer-harm outcome.

As κ_1^{post} increases, ΔCS rises and becomes positive over a wide range. This reflects an additional allocative efficiency gain from capacity reallocation, that shifting capacity toward the higher-quality product improves the match between capacity and demand. Over that same interior region, the red curve $\Delta\pi_{\text{Merger}}$ and the dashed yellow curve ΔSW becomes strongly positive. Thus, in this example, reallocation can make consumers, the merged firm, and total welfare better off. However, at very low or very high κ_1^{post}, the curves fall and can turn negative. Too little reallocation leaves

[11] Figure B1 in Appendix B provides additional detail on how each outcome variable changes.

product 1 constrained, while too much reallocation creates severe scarcity for product 2 and hurts all agents.

Now we compare the merger-optimal and socially optimal allocations of capacity in Figure 1. The orange dot ("Merger Opt.") is where the merged firm chooses κ_1^{post} to maximize its own profit gain $\Delta\pi_{\text{Merger}}$; the star ("Social Opt.") is where total welfare is maximized. The figure shows the merged firm reallocates less capacity to product 1 than a planner would. The social planner's optimum lies on the "flat top" of the social welfare curve, which corresponds to the regime where both product 1 and product 2 are unconstrained. In that region, the shadow costs for both products are zero, so reallocation no longer affects feasibility. By contrast, the curvature around the merged firm's optimum shows that the firm faces a profit trade-off and therefore does not choose the fully unconstrained allocation. To explain this gap, we now derive the following proposition.

Proposition 1. *Consider the merger experiment in Example 2. The social optimal allocation is κ_1^S, which lies in the regime where products 1 and 2 are slack and product 3 is binding. The merger's profit-maximizing allocation κ_1^M lies in the regime where product 1 is binding, product 2 is slack, and product 3 is binding. Hence $\kappa_1^M \neq \kappa_1^S$.*

Proof sketch. The full proof is in Online Appendix C, and we summarize the main steps here. We consider two regimes: product 1 binding but product 2 slack (Regime I), and both products slack (Regime II). Product 3 is always binding under all κ_1.[12] The proof proceeds by analyzing comparative statics with respect to the merged firm's capacity κ_1 in Regime I. Expanding capacity reduces the shadow cost λ_1, which raises the firm's markup ν^M and the market aggregator H. Prices move in opposite directions, where the binding product's price falls while the slack product's price rises. The independent firm is harmed as higher H erodes its shadow cost of capacity λ_3. For the merged firm, the profit effect can be decomposed into a direct marginal gain (captured by λ_1 on the additional units) and a marginal market expansion loss (expanding output due to the increasing κ_1 raises market competitiveness H, which hurts the merger). In particular, we can show that:

$$\frac{d\pi^M}{d\kappa_1} = \underbrace{\lambda_1}_{\text{Direct gain}} + \underbrace{\frac{S^M}{1-S^M} \cdot \frac{d\pi^I}{d\kappa_1}}_{\text{Market expansion loss}},$$

where π^I is the profit of the independent firm. This trade-off is positive for small κ_1 but negative near the slack boundary, explaining why the merged firm optimally chooses a binding-capacity allocation. Consumer surplus rises throughout Regime I but is maximized in Regime II, where capacity constraints no longer bind. Then, we have $\kappa_1^M \neq \kappa_1^S$.

□

[12] Note that we can easily relax this by considering more regimes, which complicates the argument.

Now we can explain why the merged firm's preferred reallocation κ_1^M differs from the social optimum using Proposition 1. Consider the range $[\underline{\kappa}_1, \bar{\kappa}_1]$. In this region, product 1 remains binding and becomes slack exactly at $\bar{\kappa}_1$, while product 2 is slack throughout.[13] The merged firm's profit can therefore be written as:

$$\pi^M = \sum_{j\in\{1,2\}} (p_j - c_j)s_j = \left(\nu^M + \lambda_1\right) \cdot \kappa_1 + \nu^M s_2 \tag{13}$$

where ν^M is the merged firm's market power markup and λ_1 is the shadow cost of capacity for product 1.

As κ_1 increases, two forces move in opposite directions. First, λ_1 decreases because reallocating capacity toward product 1 relaxes the pressure from binding constraint on the high-quality product 1, thereby bringing down the shadow cost. Second, the merged firm becomes more influential in the market and the equilibrium adjusts in a way that changes market power markups and quantities: ν^M increases but s_2 falls due to within-firm cannibalization. Therefore, the optimal point for merger is when the marginal gain from quantity κ_1 and market power markup ν^M exactly equals the marginal loss from λ_1 and cannibalization s_2.

It is also worth noting that our results extend to more realistic settings in which capacity reallocation is not frictionless but requires resources (e.g., adjustment costs in money, time, or operational disruption). Modeling these frictions explicitly would require a richer specification that is less convenient for illustration. Conceptually, however, such frictions simply steepen the effective marginal cost of shifting capacity, which our framework can still accommodate. The main message therefore remains unchanged.

Formally, these effects are summarized by the marginal-profit decomposition in Proposition 1. On the one hand, increasing κ_1 generates a direct gain because it relaxes the binding constraint on the high-quality product. This benefit is captured by the shadow cost λ_1, which equals the marginal profit from expanding capacity by one unit when product 1 is binding. On the other hand, expanding κ_1 also toughens the competitive environment, generating a loss captured by the market-expansion term in Proposition 1. This term is proportional to $d\pi^I/d\kappa_1$: reallocating capacity toward product 1 steals business from the rival, so the rival's equilibrium profit falls. Moreover, the merger's loss is larger when the merged firm's market share S^M is higher, because more of its sales are exposed to the intensified competitive environment.

Finally, the planner prefers to fully eliminate scarcity at the "flat top," where both products become slack. The merged firm, however, has an incentive to preserve some scarcity: keeping the high-quality product constrained sustains pricing power and avoids a price war. As shown above,

[13] Our goal is to show that these effects arise in a relevant region, not that they hold generically across all environments.

raising κ_1 intensifies competition, which reduces the merged firm's profits and leads it to stop short of the fully unconstrained allocation. This creates a role for merger policy when capacity reallocation is feasible, because remedies or commitments that encourage (or require) efficient post-merger reallocation can move outcomes closer to the social optimum.

5 Extensions

We now consider several extensions of the baseline model. First, we allow for convex cost functions. Second, we consider a finer geographic structure. Third, we discuss how our results may extend to alternative demand systems.

5.1 Convex Cost Functions

In this case, firm f's profit maximization problem is

$$\Pi_f = \sum_{j \in \mathscr{J}^f} \left[\sum_{m \in \mathscr{M}} L_m p_{jm} s_{jm}(p_m) - TC_j \left(\sum_{m \in \mathscr{M}} L_m s_{jm}(p_m) \right) \right],$$

where $TC_j(\cdot)$ denotes the convex total cost function of product j. Denote total quantity across all markets by $\Gamma_j = \sum_{m \in \mathscr{M}} L_m s_{jm}$. We assume the associated marginal cost $c_j(\Gamma_j) := TC_j'(\Gamma_j)$ is strictly increasing and diverges to infinity as $\Gamma_j \to \kappa_j$, so κ_j acts as an asymptotic capacity limit.

Hence, the first-order condition with respect to prices yields[14]

$$p_{jm} = c_j(\Gamma_j) + \nu_m^f,$$

where the market power markup ν_m^f is given by

$$\nu_m^f := \frac{1}{\alpha} + \sum_{k \in \mathscr{J}^f} \big(p_{km} - c_k(\Gamma_k)\big) s_{km}.$$

However, notice that the competition happens in price space, not quantity space. Since the cost function links prices and quantities through Γ_j, we can directly measure how much marginal cost rises above its baseline level due to capacity pressure, defined as *incremental marginal cost*:[15]

$$\lambda_j := c_j\left(\Gamma_j\right) - c_j(0)$$

[14] This readily extends to settings with market-specific marginal costs of the form $c_{jm} = c_j(\Gamma_j) + \tau_{jm}$, where τ_{jm} denotes a market-specific transportation cost.

[15] In the benchmark model, λ_j is literally the shadow cost of the capacity constraint.

Then the first-order condition admits an analogous decomposition:

$$p_{jm} = \underbrace{c_j(0)}_{\text{baseline cost}} + \underbrace{\lambda_j}_{\text{incremental marginal cost}} + \underbrace{\nu_m^f}_{\text{market power markup}}$$

This is a smooth approximation to the hard capacity-constraint model: there are no binding/slack regimes and no kinks induced by complementary slackness. For any fixed rivals' strategy summarized by H^{-f}, equilibrium λ^f solves the smooth system:

$$G_j(\lambda^f; H^{-f}) := c_j^{-1}\left(\lambda_j + c_j(0)\right) - \hat{\Gamma}_j(\lambda^f; H^{-f}) = 0.$$

Since c_j is strictly increasing, there is a one-to-one mapping $\Gamma_j = c_j^{-1}\left(\lambda_j + c_j(0)\right)$. Moreover, $\hat{\Gamma}_j(\lambda^f; H^{-f}) = \sum_m L_m \hat{s}_{jm}(\lambda^f; H_m^{-f})$, where shares are evaluated at prices $p_{km} = c_k(0) + \lambda_k + \nu_m^f$.

Unlike the benchmark setting, the convex cost formulation makes the entire equilibrium system $G(\lambda) = 0$ smooth. As a result, we can proceed by first establishing that equilibrium costs cannot blow up, which allows us to work on a well-behaved domain. On this domain, we compute the Jacobian of the full system and apply the Gale–Nikaidō theorem to the resulting smooth mapping. The same idea applies to specifications where marginal cost diverges as output grows large arbitrarily. Online Appendix D provides the detailed proof. In contrast, the benchmark setting yields a non-smooth mapping due to complementary slackness, which precludes direct application of Gale–Nikaidō and motivates our use of Pourciau (1983) for non-smooth mappings.

Theorem 3. *If the total cost function $TC_j : [0, \kappa_j) \to \mathbb{R}_+$ is $\mathscr{C}^2$ with strictly increasing marginal cost $c_j := TC_j'$, and $\lim_{\Gamma_j \to \kappa_j} c_j(\Gamma_j) = \infty$, then a unique Nash equilibrium exists in which all equilibrium quantities are interior, i.e., $\Gamma_j^* < \kappa_j$.*

Proof. See Online Appendix D. □

To the best of our knowledge, this paper is the first to establish existence and uniqueness of Nash equilibrium under a convex cost specification in which each product's cost depends on its aggregate output across all markets. Our framework provides an alternative to the *hockey-stick* cost specifications commonly used in empirical work (e.g., Ryan (2012), Miller and Osborne (2014), Miller et al. (2025)). Relative to the benchmark setting, both the convex cost specification studied here and hockey-stick cost functions can be challenging to estimate in some applications, as they introduce additional curvature parameters that often require calibration (see, for example, Miller and Osborne (2014)). When marginal cost is approximately constant over the relevant output range and the capacity limit is known (or can be credibly calibrated), a researcher can estimate demand together with constant marginal cost. The researcher can then compute equilibrium prices or counterfactual outcomes using our algorithm, without identifying the additional curvature parameters

implied by a hockey-stick cost function (see, for example, Brandt et al. (2026)). Naturally, this simplification is less compelling in settings where marginal costs vary substantially with output.

5.2 Regional Capacity Constraints

Our baseline model assumes that each product's capacity is pooled across all markets, captured by a single scalar shadow cost λ_j. This is a natural benchmark when production is centralized and distribution is flexible, but in many industries capacity is shaped by finer geographic structure. Formally, suppose markets are partitioned into disjoint regions indexed by r, with $\mathscr{M}_r \subset \mathscr{M}$ the set of markets in region r and $r(m)$ the region of market m. Each product j now faces a separate capacity constraint in each region:

$$\sum_{m \in \mathscr{M}_r} L_m s_{jm} \leq \kappa_{j,r}, \quad \forall r. \tag{14}$$

The price decomposition generalizes to

$$p_{jm} = c_{jm} + \lambda_{j,r(m)} + \nu_m^f, \tag{15}$$

where $\lambda_{j,r}$ is the region-specific shadow cost, replacing the single pooled λ_j of the baseline.

All of our main results extend under the substitution $\lambda_j \mapsto \lambda_{j,r(m)}$. The key observation is that each market's pricing FOC still involves exactly one shadow cost per product—specifically, the shadow cost of the region containing that market—so the within-market diagonal dominance condition underlying KKT sufficiency and equilibrium uniqueness is preserved. Cross-market spillovers through binding capacity now operate within regions rather than across all markets, yielding a sparser spillover structure without changing the underlying economic mechanisms.

This regional extension broadens the applied scope of the framework. Natural settings include multinational firms whose capacity is partitioned by regulatory jurisdiction, shipping firms whose fleets are partitioned by trade lane, chain retailers whose inventory is partitioned by distribution center service area, and regulated utilities facing state-level constraints.

5.3 Beyond MNL

Next, we briefly discuss extensions beyond the multinomial logit. Our arguments suggest analogous conclusions under alternative demand systems, but the analysis becomes substantially more involved, so we do not provide formal proofs. Nocke and Schutz (2018) show that pricing games with Nested MNL (and Nested CES) can be rewritten as auxiliary games with standard MNL (and standard CES) by assigning each firm a transformed "type" that absorbs the nesting structure

(see their online appendix). They establish equilibrium uniqueness for the standard MNL and CES games, and for the nested versions under the additional restriction that each firm's products lie in a single nest. Under this mapping, extending our results from standard MNL to Nested MNL is essentially a relabeling of firm-level parameters, since the strategic structure of the pricing game is unchanged.[16] However, the extension is still technically more involved because one must translate our capacity constraint KKT system into the transformed-type representation and verify that the regularity conditions used in the baseline proof continue to hold under the transformed parameters.

The CES/Nested CES case is more delicate in our multimarket capacity setting. Unlike MNL-type demand, CES-type demand can make it optimal to set some prices at infinity, effectively withdrawing products from certain markets, which can break the cross-market linkage created by a binding capacity constraint. Still, conditional on a given vector of shadow costs, the within-market pricing subproblem remains uniquely solvable by applying Nocke and Schutz's uniqueness results. The main additional step is to prove that the induced mapping from shadow costs to implied capacity utilization admits a fixed-point. We expect the baseline logic to carry over, but establishing it formally for CES/Nested CES requires further work.

Finally, extending our analysis to mixed logit (random coefficients) is substantially more challenging, and the existing theory in this direction is quite limited, especially once we allow multiproduct firms and capacity constraints. Mixed logit effectively averages over a continuum of heterogeneous consumer types, which weakens the clean market-level aggregation structure that we exploit under standard logit. As a result, the mapping from prices to shares and markups no longer admits a simple closed form, and the associated multiproduct pricing Jacobian becomes harder to characterize and control, particularly when capacity constraints introduce additional cross-market linkages through shadow costs. Aksoy-Pierson et al. (2013) provide sufficient conditions for uniqueness in mixed-logit pricing games (Theorems 4.4 and 4.5), but their results apply only to single-product firms. Extending their approach to multiproduct settings with binding capacity constraints would require new arguments to handle the richer within-firm substitution structure and the constraint-induced interactions. We leave this extension for future work.

6 Conclusion

Capacity constraints fundamentally alter oligopoly pricing by creating cross-market linkages: changes in demand, supply, or policy in one market can affect firms' incentives and equilibrium outcomes in others. By combining an aggregative-game formulation with fixed-point arguments

[16] The analysis can also be extended to demand systems that arise from discrete–continuous choices—i.e., consumers choose a product and a quantity, as in Nocke and Schutz (2018).

and tools from nonsmooth analysis, we establish existence and uniqueness of equilibrium under multinomial logit demand.

This foundation delivers tractable comparative statics and new merger insights. First, shadow-cost relief partially offsets the market-power effect, so post-merger prices rise by less than in the unconstrained benchmark. Second, with binding capacity, a merger can increase consumer surplus by reallocating capacity toward higher-value products. However, the merged firm reallocates capacity to maximize its own profit, which need not coincide with the socially optimal allocation. The policy takeaway is that merger review should explicitly account for capacity-driven reallocation incentives and, when needed, consider remedies that steer reallocation toward the social objective. We also show how the framework can incorporate finer geographic structure, allowing merger effects and capacity reallocations to vary across local markets. Then, we establish existence and uniqueness of Nash equilibrium under a smooth convex cost specification. Finally, we discuss how the framework might extend to other demand systems, including mixed logit, while leaving a formal proof for future work.

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A Univalence Lemma

Lemma A.1 (Lipschitz Univalence Lemma)**.** *Let $\mathscr{X} \subset \mathbb{R}^n$ be a compact domain, and let $\mathbf{h} = (h_1, \ldots, h_n)$ where each $h_j : \mathscr{X} \to \mathbb{R}$ is $\mathscr{C}^1$. Define the operator $\Psi : \mathscr{X} \to \mathbb{R}^n$ component-wise by $\Psi_j(x) = \max\big(0, h_j(x)\big)$. Then Ψ is globally Lipschitz on $\mathscr{X}$.*

Moreover, for any Lipschitz $F(x)$, let $\partial F([a,b])$ denote the Clarke generalized Jacobian over the segment $[a,b]$, and $\partial F(x)$ the Clarke generalized Jacobian at x. Univalence holds if every $\tilde{J} \in \partial F([a,b])$ is non-singular. (For fixed-point problems where $F(x) := x - \Psi(x)$, we have $\tilde{J}_F = I - \tilde{J}_\Psi$, so non-singularity requires $\det\big(I - \tilde{J}_\Psi\big) \neq 0$.). The following are sufficient conditions ensuring all such $\tilde{J}$ are non-singular:

1. *All Jacobians at differentiable points, and all limit Jacobians at non-differentiable points, are strictly diagonally dominant Z-matrices.*

2. *All Jacobians $\mathbf{J}_F(x)$ (at differentiable points and limits at non-differentiable points) are Z-matrices, and there exists $L \gg 0$ such that $\mathbf{J}_F(x)^\top L \gg 0$ for all such Jacobians.*

Proof. Since h_j is $\mathscr{C}^1$ on the compact domain $\mathscr{X}$, its Lipschitz constant $L_j := \sup_{x \in \mathscr{X}} \big\|\nabla h_j(x)\big\|_\infty$ is finite. Since $\max(0, \cdot) : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz, the composition $\Psi_j(x) = \max(0, h_j(x))$ is L_j-Lipschitz. For any $x, x' \in \mathscr{X}$, by definition of the supremum norm:

$$\big\|\Psi(x) - \Psi(x')\big\|_\infty = \max_j \big|\Psi_j(x) - \Psi_j(x')\big| \leq \underbrace{\max_j L_j}_{:=L} \cdot \big\|x - x'\big\|_\infty$$

Hence Ψ is globally Lipschitz on $\mathscr{X}$.

Now consider the unique solution problem for any Lipschitz F, (i.e., whether x^* satisfying $F(x^*) = 0$ is unique), note this includes fixed-points of Lipschitz Ψ since we can let $F(x) := x - \Psi(x)$ and x^* is also the fixed-point of Ψ. Let $\operatorname{dom} F'$ denote the set of all $x \in \mathscr{X}$ where the Jacobian $\mathbf{J}_F$ exists. The Clarke generalized Jacobian at $x_0 \in \mathscr{X}$ is:

$$\partial F(x_0) = \bigcap_{\delta > 0} \Delta_\delta(F, x_0)$$

where

$$\Delta_\delta(F, x_0) = \overline{\text{co}}\left\{J_F(x) : x \in B_\delta(x_0) \cap \operatorname{dom} F'\right\}$$

and $B_\delta(x_0) = \{z \in \mathbb{R}^n : |z - x_0| < \delta\}$ is the open ball of radius δ centered at x_0, and $\overline{\text{co}}$ denotes the closed convex hull. Intuitively, $\partial F(x_0)$ collects all "limiting Jacobians" near x_0 and captures the local behavior by looking at Jacobians from nearby differentiable points and taking their convex

hull.[17] And

$$\partial F([a,b]) = \overline{\text{co}} \bigcup_{x\in[a,b]} \partial F(x)$$

This takes another convex hull over all the $\partial F(x)$ sets along the entire path. So we need to consider mixtures of all Jacobians from interior points of different regimes and limiting Jacobians from regime boundaries.

Suppose x^* and x^{**} are two zeros. By the Mean Value Theorem (Pourciau (1983), Theorem 2.1), there exists $\tilde{J} \in \partial F\left([x^*, x^{**}]\right)$ such that:

$$F\left(x^{**}\right) - F\left(x^*\right) = \tilde{J}(x^{**} - x^*)$$

Since $F\left(x^*\right) = F\left(x^{**}\right) = 0$:

$$0 = \tilde{J}(x^{**} - x^*)$$

Then $x^{**} = x^*$ if $\tilde{J}$ is non-singular (i.e., $\det\left(\tilde{J}\right) \neq 0$).[18] Therefore, uniqueness holds if every $\tilde{J} \in \partial F([x^*, x^{**}])$ is non-singular.

We now characterize the structure of all limit Jacobians for $\Psi_j = \max(0, h_j)$ with $h_j \in \mathscr{C}^1$. Every limit Jacobian at any point $x \in \mathscr{X}$ has the form $\mathbf{J}_\Psi(x) = D \cdot \mathbf{J}_\mathbf{h}(x)$, where $D = \text{diag}(d_1, \ldots, d_n)$ with $d_j \in \{0, 1\}$. Specifically, $d_j = 1$ if approaching from a region where $h_j > 0$, $d_j = 0$ if approaching from a region where $h_j < 0$, $d_j \in \{0, 1\}$ (either choice is achievable) if $h_j(x) = 0$. Thus, at each point x, there are at most 2^n possible limit Jacobians, corresponding to the 2^n diagonal matrices with entries in $\{0, 1\}$.

Define the set of all limit Jacobians along the segment $[a, b]$:

$$\mathscr{L} = \{D \cdot \mathbf{J}_\mathbf{h}(x) : x \in [a,b],\, D = \text{diag}(d_1, \ldots, d_n),\, d_j \in \{0,1\},\, D \text{ achievable at } x\}$$

Let $\mathscr{D} = \{D = \text{diag}(d_1, \ldots, d_n) : d_j \in \{0, 1\}\}$ denote the set of all 2^n diagonal 0-1 matrices. For each $D \in \mathscr{D}$, define the set of points where D is achievable:

$$\mathscr{X}_D := \{x \in [a,b] : h_j(x) \geq 0 \text{ if } d_j = 1, \text{ and } h_j(x) \leq 0 \text{ if } d_j = 0\}$$

[17] For example, our operator $\Psi_j = \max(0, h_j)$, non-differentiability occurs at the regime boundary where $h_j = 0$. At such points, $\partial F(x_0)$ includes convex combinations of Jacobians with $\frac{\partial \Psi_j}{\partial \lambda_k} = 0$ if $h_j < 0$ and $\frac{\partial \Psi_j}{\partial \lambda_k} = \frac{\partial h_j}{\partial \lambda_k}$ if $h_j > 0$.

[18] Conversely, if $\tilde{J}$ is singular, then $x^{**} \neq x^*$ is possible but not guaranteed.

Since each h_j is continuous, both $\{x : h_j(x) \geq 0\}$ and $\{x : h_j(x) \leq 0\}$ are closed by the preimage theorem for continuous functions, and then:

$$\mathscr{X}_D = \bigcap_{j:d_j=1} \underbrace{\{x : h_j(x) \geq 0\}}_{\text{closed}} \cap \bigcap_{j:d_j=0} \underbrace{\{x : h_j(x) \leq 0\}}_{\text{closed}} \cap \underbrace{[a,b]}_{\text{closed}}.$$

This is an intersection of finitely many such closed sets, hence closed. Hence $\mathscr{X}_D$ is compact as a closed subset of the compact segment $[a,b]$. Note that the sets $\mathscr{X}_D$ are not disjoint: when $h_j(x) = 0$, the point x belongs to all $\mathscr{X}_D$ for which the remaining constraints are satisfied, regardless of whether $d_j = 0$ or $d_j = 1$. This correctly reflects that both limit Jacobians (with j-th row equal to $\nabla h_j(x)$ or 0) are achievable at such boundary points.

The map $x \mapsto D \cdot \mathbf{J_h}(x)$ is continuous because each $h_j \in \mathscr{C}^1$ implies that the entries of $\mathbf{J_h}(x)$, which are the partial derivatives $\frac{\partial h_j}{\partial x_k}$, vary continuously with x. Matrix multiplication by a fixed matrix is a linear map, hence continuous. Therefore, $\mathscr{L}_D := \{D \cdot \mathbf{J_h}(x) : x \in \mathscr{X}_D\}$ is compact as the continuous image of a compact set. Then, since $\mathscr{L} = \bigcup_{D \in \mathscr{D}} \mathscr{L}_D$ is a finite union of compact sets, $\mathscr{L}$ is compact.

We now verify the first sufficient condition. By assumption, every matrix in $\mathscr{L}$ is a strictly diagonally dominant Z-matrix. Define the diagonal dominance margin:

$$\gamma(J) = \min_j \left(|J_{jj}| - \sum_{k \neq j} |J_{jk}| \right)$$

The function γ is continuous on the space of matrices. Since $\mathscr{L}$ is compact and $\gamma(J) > 0$ for all $J \in \mathscr{L}$ by assumption, the infimum is attained and strictly positive:

$$\gamma^* := \min_{J \in \mathscr{L}} \gamma(J) > 0$$

Since $\mathscr{L} \subset \mathbb{R}^{n \times n}$ is compact (hence closed and bounded) and $\mathbb{R}^{n \times n}$ is finite-dimensional, $\mathrm{co}(\mathscr{L})$ is also compact and $\overline{\mathrm{co}}(\mathscr{L}) = \mathrm{co}(\mathscr{L})$, by Theorem 17.2 of Rockafellar (1970). Therefore, taking closed convex hull, $\partial F([a,b]) \subseteq \overline{\mathrm{co}}(\mathscr{L}) = \mathrm{co}(\mathscr{L})$.

For any $\tilde{J} \in \mathrm{co}(\mathscr{L})$, write $\tilde{J} = \sum_i \alpha_i J^{(i)}$ with $J^{(i)} \in \mathscr{L}$, $\alpha_i \geq 0$, and $\sum_i \alpha_i = 1$. The Z-matrix property (non-positive off-diagonal entries) is preserved under convex combinations: for $j \neq k$,

$$\tilde{J}_{jk} = \sum_i \alpha_i J^{(i)}_{jk} \leq 0$$

since each $J^{(i)}_{jk} \leq 0$ and $\alpha_i \geq 0$. For diagonal dominance, using the Z-matrix property (which implies $|J_{jk}| = -J_{jk}$ for $k \neq j$):

$$\tilde{J}_{jj} + \sum_{k \neq j} \tilde{J}_{jk} = \sum_i \alpha_i J^{(i)}_{jj} + \sum_{k \neq j} \sum_i \alpha_i J^{(i)}_{jk} = \sum_i \alpha_i \left(J^{(i)}_{jj} + \sum_{k \neq j} J^{(i)}_{jk} \right) \geq \sum_i \alpha_i \gamma^* = \gamma^* > 0$$

Thus every $\tilde{J} \in \partial F([a,b]) \subseteq \mathrm{co}(\mathscr{L})$ is a strictly diagonally dominant Z-matrix. By Theorem 6.2.3 of Berman and Plemmons (1994), $\tilde{J}$ satisfies condition I_{28}, so it is a non-singular M-matrix.

Alternatively, we consider the second sufficient condition. Suppose all Jacobians and limit Jacobians are Z-matrices satisfying $\mathbf{J}_F(x)^\top L \gg 0$ for some fixed $L \gg 0$. For any $\tilde{J} \in \partial F([a,b])$, writing $\tilde{J} = \sum_i \alpha_i J^{(i)}$ as above:

$$\tilde{J}^\top L = \sum_i \alpha_i (J^{(i)})^\top L \gg 0$$

since each $(J^{(i)})^\top L \gg 0$ and $\alpha_i \geq 0$ with $\sum_i \alpha_i = 1$. Thus $\tilde{J}^\top$ is a Z-matrix with $\tilde{J}^\top L \gg 0$, making it a non-singular M-matrix by Theorem 6.2.3 of Berman and Plemmons (1994) condition I_{27}. Hence $\tilde{J}$ is also non-singular.

□

Notice that, we cannot only use all $\det(\mathbf{J}_F) \neq 0$ as a sufficient condition since it does not preserve under $\overline{\mathrm{co}}$ operation. Pourciau (1983) provides two additional sufficient conditions, the first one is by Lemma 3.8: for all $x \in [a,b]$ and all $\mathbf{J}_F \in \partial F(x)$, the symmetric part $\mathbf{J}_{F,S} = \frac{1}{2}(\mathbf{J}_F + \mathbf{J}_F^\top)$ is a P-matrix. The second one is by his Theorem 3.7 for each $x \in [a,b]$ and each $\mathbf{J}_F \in \partial F(x)$ that $\det \mathbf{J}_F > 0$ and every principal minor of $\mathbf{J}_{F,S}$ is non-negative. These two conditions are weaker than diagonal dominance, and may be used in other problems, but they also must be verified directly on $\partial F(x)$ for all x, not just on the underlying Jacobians, making them harder to check in practice.

B Additional Figures

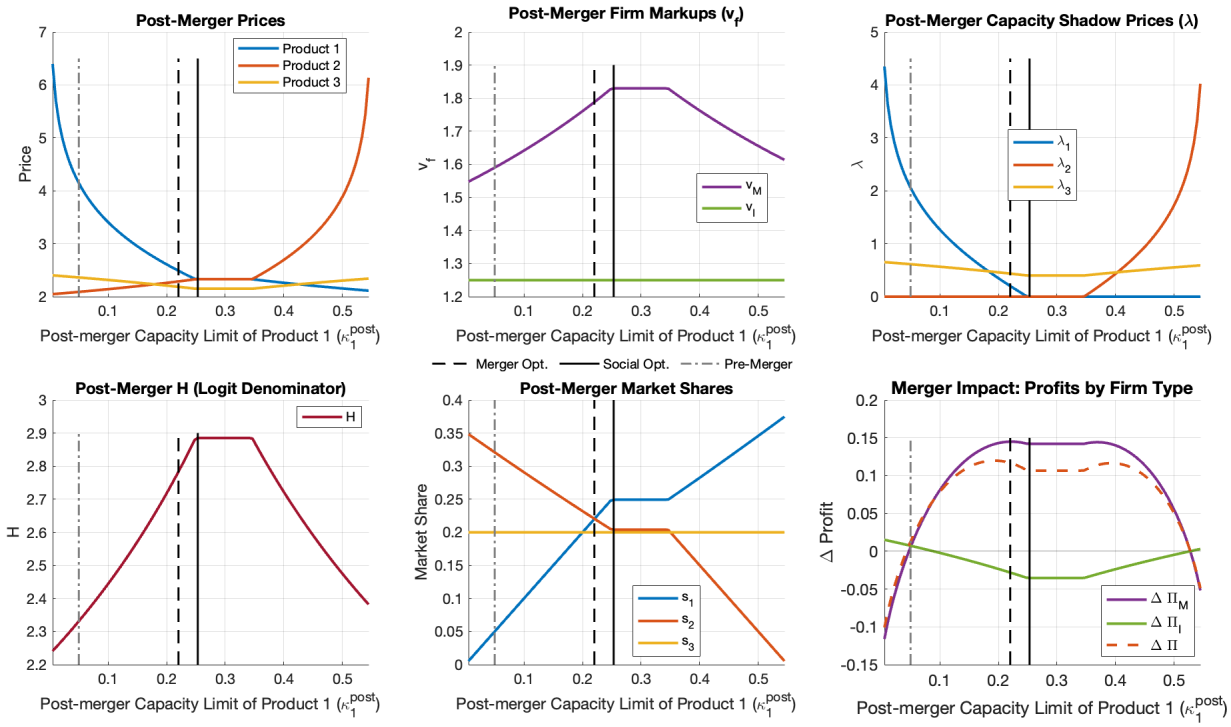

Figure B1: Effects of post-merger capacity reallocation

Notes. This figure reports additional equilibrium objects for the merger experiment in Example 2. The horizontal axis varies the post-merger capacity assigned to product 1, κ_1^{post}. The merged firm can reallocate its combined capacity across products 1 and 2 subject to $\kappa_1^{post} + \kappa_2^{post} = 0.55$ (so increasing κ_1^{post} mechanically reduces κ_2^{post}), while firm 3's capacity is fixed at $\kappa_3^{post} = 0.2$. The gray dash-dot vertical line marks the no-reallocation benchmark $\kappa_1^{post} = \kappa_1^{pre}$. The black dashed line marks the merged firm's profit-maximizing allocation ("Merger Opt.", κ_1^M) and the black solid line marks the welfare-maximizing allocation ("Social Opt.", κ_1^S). The panels display post-merger prices, firm markups ν_f, shadow costs λ_j, the logit denominator H, market shares s_j, and profit changes by firm type.

Online Appendix for "Multiproduct Multimarket Price Competition under Capacity Constraints"

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A KKT Sufficiency Proof

To prove Theorem 1 from the main paper, we first show that all equilibrium prices are above marginal cost (Lemma A.1). For each firm's profit maximization problem, the best response exists once we compactify the price space to include infinity (Lemma A.2). The necessity of the KKT conditions is established by Lemma A.3. To prove sufficiency, we begin by showing that setting prices to infinity is suboptimal (Lemma A.4), so it suffices to consider finite-price solutions. We then show that the solution to the KKT conditions with finite prices is unique; existence and uniqueness together imply sufficiency. Fixing all opponents' prices, we use the stationary conditions to express a firm's optimal prices as a function of its own marginal costs and a market power markup term, which arises from the IIA property. We then show this market power markup is a unique function of the firm's shadow costs using the Implicit Function theorem (Lemma 1 from the main paper). Therefore, we can express both the firm's prices and market shares as functions of its own shadow costs. Next we show that these functions are well-behaved and satisfy a diagonal dominance condition (Lemma A.5). Next, based on complementary slackness, primal feasibility, and dual feasibility, we construct a marginal cost operator. We prove this operator is bounded and exist by Brouwer (Lemma A.6), and show that the uniqueness of the fixed-point follows from a diagonal dominance property and nonsmooth techniques, using the Univalence Lemma in the Appendix A of the main paper and Lemma 2 of the main paper. Hence, there is a unique solution satisfying KKT conditions. Putting everything together, we conclude that for any fixed strategy

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profile of opponents, the KKT conditions are both necessary and sufficient for a firm's global best response (Theorem 1 from the main paper).

We first argue that, even after incorporating cannibalization, it is possible to identify a valid price range and show that the profit maximization problem for an arbitrary firm f admits a solution.

Lemma A.1 (Price Range)**.** *In any Nash Equilibrium p, for every firm $f \in \mathscr{F}$, there exists at least one product $j \in \mathscr{J}^f$ such that $p_{jm} < \infty$ in at least one market. Moreover, $p_{jm} \geq c_{jm}$ for all $j \in \mathscr{J}^f$ and $m \in \mathscr{M}$.*

Proof. First, observe that if firm f sets $p_{jm} = \infty$ for all $j \in \mathscr{J}^f$ and $m \in \mathscr{M}$, it earns zero profit. Naturally, the firm would have an incentive to deviate by setting $p_{jm} > c_{jm}$ for at least one product $j \in \mathscr{J}^f$ in order to earn strictly positive profit. We then need to verify that such a deviation is feasible. Without loss of generality, suppose the firm chooses to set a finite price for some product j. Since the capacity constraint is a finite, positive number, the firm can always choose a sufficiently high but finite price p_{jm} such that the resulting demand is small enough to satisfy the capacity constraint. Moreover, choosing $p_{jm} > c_{jm}$ ensures that the firm earns positive profit. Therefore, in any equilibrium, there must exist at least one product $j \in \mathscr{J}^f$ such that $p_{jm} < \infty$ in at least one market m.

Next, we show that $p_{jm} \geq c_{jm}$ for all j, for an arbitrary fixed $H_m^{-f} = 1 + \sum_{k \notin \mathscr{J}_f} \exp(\delta_{km} - \alpha p_{km})$. Suppose, for the sake of contradiction, that there exists a Nash equilibrium in which some product has $p_{jm} < c_{jm}$ in certain markets. Select an arbitrary market m where at least one product satisfies $p_{jm} < c_{jm}$. Fix the prices and quantities in all other markets $m' \neq m$. For each product j, its remaining available capacity for market m is given by $\kappa_j - \sum_{m' \in \mathscr{M} \setminus \{m\}} L_{m'} s_{jm'} > 0$, where $s_{jm'}$ are the fixed market shares in those other markets. We first show that the price vector in market m is strictly dominated by some feasible price vector such that $\bar{p}_{jm} \geq c_{jm}$ for all $j \in \mathscr{J}^f$. By sequentially applying this adjustment to each market, we can then construct a new price vector in which $\bar{p}_{jm} \geq c_{jm}$ holds for all $j \in \mathscr{J}^f$ and $m \in \mathscr{M}$.

We begin by establishing this result for a single market. Let $J_+^f := \{j : j \in \mathscr{J}^f, p_{jm} \geq c_{jm}\}$. We first raise the prices to ∞ for all $j \notin J_+^f$. Then, for each $j \in J_+^f$, there exists a price $\bar{p}_{jm} > c_{jm}$ such that the resulting market share matches that under the original price vector p:

$$\frac{\exp(\delta_{jm} - \alpha p_{jm})}{\sum_{j \in \mathscr{J}^f} \exp(\delta_{jm} - \alpha p_{jm}) + H_m^{-f}} = \frac{\exp(\delta_{jm} - \alpha \bar{p}_{jm})}{\sum_{j \in J_+^f} \exp(\delta_{jm} - \alpha \bar{p}_{jm}) + H_m^{-f}}.$$

Since p is assumed to be feasible, this new price vector remains feasible as well. We then show that this new pricing strategy yields strictly higher profit.

Notice that we can equivalently rewrite the equal-market-share condition as:

$$\exp\big(\delta_{jm}-\alpha\bar{p}_{jm}\big)=\exp\big(\delta_{jm}-\alpha p_{jm}\big)\cdot\frac{\sum_{j\in J_+^f}\exp\big(\delta_{jm}-\alpha\bar{p}_{jm}\big)+H_m^{-f}}{\sum_{j\in\mathscr{J}^f}\exp\big(\delta_{jm}-\alpha p_{jm}\big)+H_m^{-f}}:=\exp\big(\delta_{jm}-\alpha p_{jm}\big)\cdot\underbrace{\frac{\bar{H}_m}{H_m}}_{:=r}$$

Therefore, we have $\sum_{j\in J_+^f}\exp\big(\delta_{jm}-\alpha\bar{p}_{jm}\big)=r\cdot\sum_{j\in J_+^f}\exp\big(\delta_{jm}-\alpha p_{jm}\big)$. Next, we also have

$$r\cdot H_m=\sum_{j\in J_+^f}\exp\big(\delta_{jm}-\alpha\bar{p}_{jm}\big)+r\cdot\left(\sum_{j\notin J_+^f}\exp\big(\delta_{jm}-\alpha p_{jm}\big)+H_m^{-f}\right)$$

Then, we can combine these equations using $\bar{H}_m=r\cdot H_m$:

$$\sum_{j\in J_+^f}\exp\big(\delta_{jm}-\alpha\bar{p}_{jm}\big)+H_m^{-f}=\sum_{j\in J_+^f}\exp\big(\delta_{jm}-\alpha\bar{p}_{jm}\big)+r\cdot\left(\sum_{j\notin J_+^f}\exp\big(\delta_{jm}-\alpha p_{jm}\big)+H_m^{-f}\right)$$

which implies

$$r=\frac{H_m^{-f}}{\sum_{j\notin J_+^f}\exp\big(\delta_{jm}-\alpha p_{jm}\big)+H_m^{-f}}$$

Since $H_m^{-f}\geq 1$, we can conclude that $r\in(0,1)$. This gives us a direct formula for r that depends only on the original prices vector p and the fixed H_m^{-f}. Therefore, from this construction we ensure that the new price vector

$$\bar{p}_{jm}=\begin{cases}p_{jm}-\frac{1}{\alpha}\ln\left(\frac{H_m^{-f}}{\sum_{j\notin J_+^f}\exp\big(\delta_{jm}-\alpha p_{jm}\big)+H_m^{-f}}\right), & \text{if } j\in J_+^f\\ \infty, & \text{otherwise}\end{cases}$$

does not violate any of firm f's capacity constraints. Notice that $\bar{p}$ generates strictly higher profit than p in market m:

$$\begin{aligned}\pi_m^f\big((p_{jm})_{j\in\mathscr{J}^f},H_m^{-f}\big)&=\sum_{j\in\mathscr{J}^f}(p_{jm}-c_{jm})s_{jm}<\sum_{j\in J_+^f}(p_{jm}-c_{jm})s_{jm}<\sum_{j\in J_+^f}(\bar{p}_{jm}-c_{jm})\bar{s}_{jm}\\&=\pi_m^f\big((\bar{p}_{jm})_{j\in\mathscr{J}^f},H_m^{-f}\big)\end{aligned}$$

because $\bar{s}_{jm}=s_{jm}$ while $\bar{p}_{jm}>p_{jm}$ for all $j\in J_+^f$. Note that this holds for all markets, which completes the proof.

□

Since $(H_m^{-f})_{\forall m}$ is an arbitrary vector of fixed positive constants, we do not need to consider whether the prices of rival firms remain feasible or adjusts in response to firm f's strategy change from p^f to $\tilde{p}^f$. Verifying that all firms simultaneously satisfy optimality and feasibility conditions corresponds to establishing the existence of a Nash equilibrium, which requires analyzing mutual best responses, a broader task addressed later. With this price range result established, we can show that each firm f's profit maximization problem admits at least one solution for any given set of rival prices, after the compactification to infinite prices.

Lemma A.2 (Solution Existence for Firm f)**.** *For an arbitrary firm f, given any*

$$H^{-f} := \left(1 + \sum_{j \notin \mathscr{J}^f} \exp\left(\delta_{jm} - \alpha p_{jm}\right)\right)_{m \in \mathscr{M}},$$

the firm f's profit maximization problem has a solution.

Proof. Notice by Lemma A.1 we showed that any price $p_{jm} < c_{jm}$ is strictly dominated. Then we can write firm f's problem as

$$\max_{p^f \in \prod_{j,m}[c_{jm},\infty]} \pi^f(p^f, H^{-f}) = \sum_m L_m \sum_{j \in \mathscr{J}^f} \left(p_{jm} - c_{jm}\right) s_{jm} \quad \text{s.t.} \quad \sum_m L_m s_{jm} \leq \kappa_j \text{ for all } j \in \mathscr{J}^f.$$

We first show the objective function is continuous. Without loss of generality, consider a price vector p_m^f in market m such that $p_{jm} = \infty$ for $1 \leq j \leq K$ and $p_{jm} < \infty$ for $K+1 \leq j \leq |\mathscr{J}^f|$ where $1 \leq K \leq |\mathscr{J}^f|$. Notice that $H_m^{-f} > 0$, and for an arbitrary index j, recall:

$$\lim_{\tilde{p}_m^f \to p_m^f} s_{jm}(\tilde{p}_m^f, H_m^{-f}) = \begin{cases} 0, & \text{if } 1 \leq j \leq K \\ \frac{\exp\left(\delta_{jm} - \alpha p_{jm}\right)}{\sum_{j=K+1}^{|\mathscr{J}^f|} \exp\left(\delta_{jm} - \alpha p_{jm}\right) + H_m^{-f}}, & \text{otherwise} \end{cases} = s_{jm}(p_m^f, H_m^{-f})$$

Thus, allowing $p_{jm} = \infty$ is a continuous extension. In addition, notice that $\lim_{x \to \infty} x e^{-x} = 0$ by L'Hôpital's Rule, hence:

$$\lim_{\tilde{p}_m^f \to p_m^f} \pi_m^f(\tilde{p}_m^f, H_m^{-f}) = \frac{\sum_{j=K+1}^{|\mathscr{J}^f|} (p_{jm} - c_{jm}) \exp\left(\delta_{jm} - \alpha p_{jm}\right)}{\sum_{j=K+1}^{|\mathscr{J}^f|} \exp\left(\delta_{jm} - \alpha p_{jm}\right) + H_m^{-f}} = \pi_m^f(p_m^f, H_m^{-f})$$

and thus our extended objective function in each market is also continuous. The objective function π^f is a linear combination of π_m^f, which implies that $\pi^f(p^f, H^{-f})$ is continuous on $\prod_{j \in \mathscr{J}^f, m \in \mathscr{M}} \left[c_{jm}, \infty\right]$.

Next, we show the action space is compact. Notice that the ambient space, $\prod_{j,m} \left[c_{jm}, \infty\right]$, is compact by Tychonoff's theorem, as each $\left[c_{jm}, \infty\right]$ is compact in our extended real line topology.

Define the constrained action space to be $\mathscr{A}^f$, we prove $\mathscr{A}^f \subseteq \prod_{j,m} \left[c_{jm}, \infty\right]$ is closed. First, clearly $p^f = (\infty, \ldots, \infty) \in \mathscr{A}^f$ so it is non-empty. Let $(p^n)_{n=1}^{\infty}$ be a sequence in $\mathscr{A}^f$ that converges to some point $p^* \in \prod_{j,m} \left[c_{jm}, \infty\right]$. We need to show that $p^* \in \mathscr{A}^f$. By continuity of $s_{jm}(p_m^f, H_m^{-f})$ and the Limit Inequality Theorem, we have:

$$\kappa_j \geq \lim_{\forall m,\ p_m \to p_m^*} \sum_m L_m s_{jm}(p_m, H_m^{-f}) = \sum_m L_m s_{jm}(p_m^*, H_m^{-f})$$

Therefore, $p^* \in \mathscr{A}^f$ and $\mathscr{A}^f$ is closed. Since $\mathscr{A}^f$ is a closed subset of a compact space, it follows that $\mathscr{A}^f$ is compact.

Because the action space $\mathscr{A}^f$ is compact and $\pi^f(p^f, H^{-f})$ is continuous, the firm f's profit maximization problem must have a solution.

□

Recall that the KKT conditions are:

Definition 1 (KKT Conditions)**.** *The KKT conditions of the multiproduct firm's profit maximization problem in multiple markets hold at price vector $p \in \mathscr{P}$, if for each firm f with price vector p^f and for any fixed H^{-f}, there exists a Lagrange multiplier $\lambda_j \geq 0$ such that for every $j \in \mathscr{J}^f$ the following hold.*

1. ***Interior case*** *($p_{jm} < \infty$ for all m): the firm sets a finite price in every market.*

   (a) Stationarity*: at the optimum, the marginal profit from raising p_{jm} is exactly offset by the marginal shadow cost of the capacity it consumes,*

   $$\frac{\partial \mathscr{L}(p^f, H^{-f})}{\partial p_{jm}} = \frac{\partial \pi^f}{\partial p_{jm}} - L_m \sum_{j' \in \mathscr{J}^f} \lambda_{j'} \frac{\partial s_{j'm}}{\partial p_{jm}} = 0.$$

   (b) Primal feasibility*: total quantity sold cannot exceed capacity,*

   $$\sum_m L_m s_{jm} \leq \kappa_j \quad \text{for all } j \in \mathscr{J}^f.$$

   (c) Dual feasibility*: shadow costs are non-negative,*

   $$\lambda_j \geq 0 \quad \text{for all } j \in \mathscr{J}^f.$$

   (d) Complementary slackness*: capacity has positive shadow value only when it binds,*

   $$\lambda_j \left(\sum_m L_m s_{jm} - \kappa_j \right) = 0 \quad \text{for all } j \in \mathscr{J}^f.$$

2. ***Corner case*** *($p_{jm} = \infty$ for some m): the firm optimally withdraws product j from market m rather than serving it at any finite price. This is optimal only if no feasible deviation to a finite price would raise profits, i.e.,*

$$\pi^f(p^f, H^{-f}) \geq \pi^f\left(\left(\tilde{p}_{jm}, (p_{jm'})_{m' \in \mathscr{M}\setminus\{m\}}, (p_{km})_{k \in \mathscr{J}^f\setminus\{j\}, m \in \mathscr{M}}\right), H^{-f}\right)$$

for all finite $\tilde{p}_{jm}$ such that the resulting allocation still satisfies the capacity constraint

$$\sum_m L_m s_{jm}\left(\left(\tilde{p}_{jm}, (p_{jm'})_{m' \in \mathscr{M}\setminus\{m\}}, (p_{km})_{k \in \mathscr{J}^f\setminus\{j\}, m \in \mathscr{M}}\right), H^{-f}\right) \leq \kappa_j.$$

Then, we present the following lemma, which shows that the conditions in Definition 1 are necessary conditions for optimality.

Lemma A.3 (Necessity of KKT)**.** *Let a strategy profile $p \in \mathscr{P}$ represent a Nash equilibrium of the capacity-constrained pricing game, then for each firm f producing products $j \in \mathscr{J}^f$, there exists a Lagrange multiplier $\lambda_j \geq 0$ for each $j \in \mathscr{J}^f$ such that p satisfies KKT defined in Definition 1.*

Proof. Let $p^* \in \mathscr{P}$ be a Nash equilibrium profile. For each firm f, p^{*f} solves the constrained maximization problem:

$$\max_{p^f \in \mathscr{P}^f} \pi^f\left(p^f, H^{-f}\right) \quad \text{subject to} \quad \sum_m L_m s_{jm} \leq \kappa_j \quad \forall j \in \mathscr{J}^f$$

First, consider the case $p^*_{jm} < \infty$, notice that both the profit function and the capacity constraint (i.e., the Lagrangian) are continuously differentiable with respect to finite prices. Following the necessary part of KKT theorem,[1] there exists a Lagrange multiplier $\lambda_j \geq 0$ such that p^*_{jm} satisfies Condition 1 (a)-(d) in Definition 1.

For $p^*_{jm} = \infty$, standard differentiation does not apply. Instead, we must verify that no feasible deviation to a finite price improves the objective. Let $\tilde{p}_{jm}$ be any finite price such that the capacity constraint remains satisfied when firm f deviates from $p^*_{jm} = \infty$ to $\tilde{p}_{jm}$, keeping all other prices unchanged. Since p^* is a Nash equilibrium, deviating to $\tilde{p}_{jm}$ cannot improve the firm's profit, thus establishing condition 2 in Definition 1.

□

The next lemma establishes that in a multinomial logit (MNL) model, it is always suboptimal for a firm to set an infinite price for any of its products, given any fixed set of rival firms' prices.

[1] In fact, we can verify the constraints' qualification, which trivially holds here.

Lemma A.4 (Suboptimality of Infinite Pricing)**.** *Let a firm f, in multiple markets with MNL demand, offer a set of products $\mathcal{J}^f \subseteq \mathcal{J}$, while rival firms set prices $\mathbf{p}^{-f}$. Then, for any product $j \in \mathcal{J}^f$, setting price $p_{jm} = \infty$ is strictly suboptimal for the firm's profit maximization.*

Proof. First, we extend the idea of Garrido (2024), which states that infinite pricing is optimal only if the limit of first-order condtions approaches zero from above. In our case, we have the analogous stationary condition:

$$s_{jm}\left(\underbrace{1-\left(p_{jm}-c_{jm}-\lambda_j\right)\alpha+\sum_{k\in\mathcal{J}^f}\left(p_{km}-c_{km}-\lambda_k\right)\alpha s_{km}}_{\psi_{jm}}\right)=0$$

and we need $\lim_{p_{jm}\to\infty}\psi_{jm}\geq 0$. In our KKT setting the stationarity direction ψ_{jm} is not a marginal derivative but can be viewed as a direction that is only required to be non-negative at feasible finite prices. If capacity is already violated for all finite prices, nothing forces the firm to keep moving the price upward once $\psi_{jm}<0$.[2]

With the above argument, observe that for any finite λ_j, we have $\lim_{p_{jm}\to\infty}\psi_{jm}<0$, so setting $p_{jm}=\infty$ cannot be optimal unless $\lambda_j=\infty$. Now, suppose $\lambda_j=\infty$. Then, for any market m' such that $p_{jm'}<\infty$, the corresponding stationarity condition cannot hold: the market share $s_{jm'}$ remains strictly positive, while $\psi_{jm'}\to\infty$, violating the stationarity condition. Therefore, if $\lambda_j=\infty$, we must have $p_{jm}=\infty$ for all markets $m\in\mathcal{M}$.

To complete the proof, we now rule out this case as a Nash Equilibrium by showing that the pricing strategy is strictly dominated by a feasible finite price vector. Without loss of generality, we let $p_{jm}=\infty$ for $1\leq j\leq K$ in all markets. Note that $1\leq K\leq|\mathcal{J}^f|-1$, because we know that each firm will at least set a finite price by Lemma A.1. And for each $K+1\leq j\leq|\mathcal{J}^f|$, we have $c_{jm}<p_{jm}<\infty$. Now we consider an alternative price vector. Specifically, for each $1\leq j\leq K$ and $m\in\mathcal{M}$, denote $L:=\max_m L_m$ and consider a price $\tilde{p}_{jm}$ such that:

$$\tilde{p}_{jm}=\begin{cases}\frac{\delta_{jm}-\ln\left(\frac{\varepsilon}{K\cdot|\mathcal{M}|\cdot L}\right)}{\alpha}, & \text{if } 1\leq j\leq K\\ p_{jm}, & \text{otherwise}\end{cases}$$

[2]Economically, a negative direction would tell the firm to lower the price rather than increase it to ∞. In other words, we interpret the KKT stationarity condition as the "price-adjustment rule," a firm would never keep raising a price so long as the "escape direction" becomes negative.

We need this new price to be higher than c_{jm}; otherwise, it is strictly dominated. Denote

$$n_1 := K \cdot |\mathscr{M}| \cdot L \cdot \exp\left(\underbrace{\min\{(\delta_{jm} - \alpha c_{jm})_{1 \le j \le K, m \in \mathscr{M}}\}}_{\beta} \right),$$

notice that any $\varepsilon \in (0, n_1)$ can make $\tilde{p}_{jm} > c_{jm}$ for all j, m. Next, we fix the prices of the other firms, and denote $H_m := H_m^{-f} + \sum_{j \in \mathscr{J}^f} \exp\left(\delta_{jm} - \alpha p_{jm}\right)$ as the aggregator under the original price vector $p_m = (p_{jm})_{j \in \mathscr{J}^f}$.[3] Under the new price vector $\tilde{p}_m$, the profit change of firm f from products $1 \le j \le K$ in market m is:

$$L_m \sum_{1 \le j \le K} (\tilde{p}_{jm} - c_{jm}) \frac{\exp\left(\delta_{jm} - \alpha \tilde{p}_{jm}\right)}{1 + \sum_j \exp\left(\delta_{jm} - \alpha \tilde{p}_{jm}\right)} = L_m \sum_{1 \le j \le K} \frac{\delta_{jm} - \alpha c_{jm} - \ln\left(\frac{\varepsilon}{K \cdot |\mathscr{M}| \cdot L}\right)}{\alpha} \cdot \frac{\frac{\varepsilon}{|\mathscr{M}| \cdot L}}{K\left(H_m + \frac{\varepsilon}{|\mathscr{M}| \cdot L}\right)}$$

Due to the cannibalization effect, firm f selling previously unserved products under $\tilde{p}_m$ makes market shares of its other products decline. We then compute the profit loss for products $j \in \{K+1, \ldots, |\mathscr{J}^f|\}$ resulting from the price change from p_m to $\tilde{p}_m$:

$$L_m \sum_{K+1 \le j \le |\mathscr{J}^f|} (p_{jm} - c_{jm})[s_{jm}(\tilde{p}) - s_{jm}(p)] = -\frac{\frac{\varepsilon}{|\mathscr{M}| \cdot L}}{H_m + \frac{\varepsilon}{|\mathscr{M}| \cdot L}} L_m \underbrace{\sum_{K+1 \le j \le |\mathscr{J}^f|} (p_{jm} - c_{jm}) \cdot \frac{\exp\left(\delta_{jm} - \alpha p_{jm}\right)}{H_m}}_{:= \zeta_m^f}$$

Then, we have:

$$\begin{aligned}
\pi_m^f(\tilde{p}_m) - \pi_m^f(p_m) &= L_m \left[\sum_{1 \le j \le K} \frac{\delta_{jm} - \alpha c_{jm} - \ln\left(\frac{\varepsilon}{K \cdot |\mathscr{M}| \cdot L}\right)}{\alpha} \cdot \frac{\frac{\varepsilon}{|\mathscr{M}| \cdot L}}{K\left(H_m + \frac{\varepsilon}{|\mathscr{M}| \cdot L}\right)} - \frac{\frac{\varepsilon}{|\mathscr{M}| \cdot L}}{H_m + \frac{\varepsilon}{|\mathscr{M}| \cdot L}} \zeta_m^f \right] \\
&> L_m \frac{\frac{\varepsilon}{|\mathscr{M}| \cdot L}}{H_m + \frac{\varepsilon}{|\mathscr{M}| \cdot L}} \left[\frac{\beta - \ln\left(\frac{\varepsilon}{K \cdot |\mathscr{M}| \cdot L}\right)}{\alpha} - \zeta_m^f \right]
\end{aligned}$$

Hence, for $\pi_m^f(\tilde{p}_m) - \pi_m^f(p_m) > 0$, we need a small ε such that $\frac{\beta - \ln\left(\frac{\varepsilon}{K \cdot |\mathscr{M}| \cdot L}\right)}{\alpha} > \zeta_m^f$. Define $n_2 := \min_m \left\{ K \cdot |\mathscr{M}| \cdot L \cdot \exp\left(\beta - \alpha \zeta_m^f\right) \right\}$, we have $\pi_m^f(\tilde{p}_m) > \pi_m^f(p_m)$ for all markets if $\varepsilon \in (0, n_2)$. In other words, this deviation generates a strictly positive change in total profit for firm f across all markets.

[3] Throughout the analysis, we assume the existence of an outside option by default.

Finally, consider the capacity constraint. For products $K+1 \leq j \leq |\mathscr{J}^f|$, obviously their capacity constraints are not violated, because $s_{jm}(\tilde{p}_m) < s_{jm}(p_m)$. For products $1 \leq j \leq K$, we need to make ε small enough such that capacity constraints are not violated. Consider:

$$\sum_m L_m s_{jm}(\tilde{p}_m) = \sum_m L_m \frac{\frac{\varepsilon}{|\mathscr{M}|\cdot L}}{K\left(H_m + \frac{\varepsilon}{|\mathscr{M}|\cdot L}\right)} \leq \frac{\varepsilon \cdot |\mathscr{M}| \cdot L}{K(|\mathscr{M}| \cdot L \cdot H + \varepsilon)} \leq \kappa^f$$

where $H = \min_m H_m$ and $\kappa^f = \min_{j \in \mathscr{J}^f} \kappa_j$. Notice that we have all capacity constraints satisfied if the last inequality above holds. We should then discuss by cases. If $\kappa^f \cdot K < |\mathscr{M}| \cdot L$, then we need $\varepsilon \in (0, n_3)$ where $n_3 := \frac{\kappa^f \cdot K \cdot L \cdot |\mathscr{M}| \cdot H}{|\mathscr{M}| \cdot L - \kappa^f \cdot K}$, which establishes an upper bound for ε. If $\kappa^f \cdot K \geq |\mathscr{M}| \cdot L$, then this inequality trivially holds for all $\varepsilon \in \mathbb{R}_{++}$.

Thus, by selecting any small $\varepsilon \in (0, \min(\{n_1, n_2, n_3\} \cap \mathbb{R}_{++}))$, we can have a new price vector $\tilde{p} \in \prod_{j \in \mathscr{J}^f} [c_j, \infty)$ that is strictly better than p without violating any capacity constraints, which finishes the proof.

□

Lemma A.4 is important because it allows us to restrict attention to price vectors with only finite components. Any price vector containing an infinite price cannot be globally optimal for firm f. In other words, **maximum is interior**. Therefore, it suffices to show that condition 1 in Definition 1 admits a unique solution.

Also, recall that Lemma 1 proved in the main text:

Lemma (Lemma 1 from the main paper)**.** *For firm f producing products $j \in \mathscr{J}^f$, fix an arbitrary shadow cost vector $\lambda^f = (\lambda_j)_{j \in \mathscr{J}^f} \in \mathbb{R}_+^{|\mathscr{J}^f|}$ and arbitrary rivals' strategies, represented through H_m^{-f}, there exists a unique vector of prices $p_m^f = (p_{jm})_{j \in \mathscr{J}^f}$ that satisfies the KKT conditions in Definition 1. Moreover, the unique mapping $\nu_m^f(\lambda^f) : \mathbb{R}_+^{|\mathscr{J}^f|} \to \mathbb{R}_+$ is continuously differentiable (of class $\mathscr{C}^1$).*

Based on the price operator defined above, the market share operator for a given vector of shadow costs is:

$$s_{jm}(\lambda^f) = \frac{\exp\left(\delta_{jm} - \alpha(c_{jm} + \nu_m^f(\lambda^f) + \lambda_j)\right)}{H_m^{-f} + \sum_{j \in \mathscr{J}^f} \exp\left(\delta_{jm} - \alpha(c_{jm} + \nu_m^f(\lambda^f) + \lambda_j)\right)},$$

which is also $\mathscr{C}^1$ in all components of λ^f. We first establish some properties of these functions.

Lemma A.5. *The function $p_{jm}(\nu_m^f(\lambda^f), \lambda_j) \to \infty$ and $s_{jm}(\lambda^f) \to 0$ if and only if $\lambda_j \to \infty$, while for any other $\lambda^f \in \prod_{j=1}^{|\mathscr{J}^f|} [0, \infty]$, $p_{jm}(\nu^f(\lambda^f), \lambda_j)$ remains finite and $s_{jm}(\lambda^f) \in (0,1)$. Moreover,*

let J be the Jacobian matrix on market m with elements $J_{jk} = \frac{\partial s_{jm}}{\partial \lambda_k}$ *of market shares with respect to shadow cost multipliers. Then* $\frac{\partial s_{jm}}{\partial \lambda_j} < 0$, $\frac{\partial s_{jm}}{\partial \lambda_k} > 0$ *for* $k \in \mathscr{J}^f \backslash \{j\}$, *and* $\left|\frac{\partial s_{jm}}{\partial \lambda_j}\right| > \sum_{k \in \mathscr{J}^f \backslash \{j\}} \left|\frac{\partial s_{jm}}{\partial \lambda_k}\right|$.

Proof. We first show that $\nu_m^f(\lambda^f)$ is bounded below. Recall from the proof of Lemma 1 from the main paper, $\nu_m^f(\lambda^f)$ is the solution to the equation $\hat{F}(\nu_m^f, \lambda^f) = 0$. Notice that the limit of $\lim_{\lambda_j \to \infty} \exp\left(\delta_{jm} - \alpha\left(\nu^f + c_{jm} + \lambda_j\right)\right) = 0$ is well-defined. Consider any λ^{*f} has some components approaching infinity, without loss of generality assume that $\lambda_j^* = \infty$ for all $1 \leq j \leq N$ for some $1 \leq N \leq |\mathscr{J}^f|$. We have

$$\lim_{\lambda^f \to \lambda^{*f}} \sum_{j \in \mathscr{J}^f} \exp\left(\delta_{jm} - \alpha\left(\nu_m^f + c_{jm} + \lambda_j\right)\right) = \sum_{j=N+1}^{|\mathscr{J}^f|} \exp\left(\delta_{jm} - \alpha\left(\nu_m^f + c_{jm} + \lambda_j^*\right)\right) \in \mathbb{R}_+$$

Therefore,

$$0 = \hat{F}(\nu_m^f, \lambda^f) = H_m^{-f}\left(\nu_m^f - \frac{1}{\alpha}\right) - \frac{1}{\alpha} \sum_{j \in \mathscr{J}^f} \exp\left(\delta_{jm} - \alpha\left(\nu_m^f + c_{jm} + \lambda_j\right)\right) \leq H_m^{-f}\left(\nu_m^f - \frac{1}{\alpha}\right)$$

implying $\lim_{\lambda^f \to \lambda^{*f}} \nu_m^f(\lambda^f) \in [\frac{1}{\alpha}, \infty)$ for any $\lambda^{*f} \in [0,\infty]^{|\mathscr{J}^f|}$, where $\lim_{\lambda^f \to \lambda^{*f}} \nu_m^f(\lambda^f) = \frac{1}{\alpha}$ if and only if $\lambda_j^* = \infty$ for all $j \in \mathscr{J}^f$. By the price decomposition $p_{jm}(\nu_m^f, \lambda_j) = \nu_m^f + c_{jm} + \lambda_j$, $p_{jm}(\nu_m^f(\lambda^f), \lambda_j) \to \infty$ and $s_{jm}(\lambda^f) \to 0$ if and only if $\lambda_j \to \infty$.

Next, we consider for an arbitrary market, we suppress the index m for clarity. Note that H^{-f} is determined by rival firms, we treat it as fixed for now. First, we analyze how market shares change with respect to multipliers:

$$\frac{\partial s_j(\lambda^f)}{\partial \lambda_j} = -\alpha s_j\left(1 - s_j - s_j\left(1 - \sum_{k \in \mathscr{J}^f} s_k\right)\right) = -\alpha s_j\left((1 - s_j) \sum_{k \in \mathscr{J} \backslash \mathscr{J}^f} s_k + \sum_{k \in \mathscr{J}^f \backslash \{j\}} s_k\right) < 0$$

Similarly, for $j \neq k$,

$$\frac{\partial s_j(\lambda^f)}{\partial \lambda_k} = \alpha s_j s_k\left(1 + \left(1 - \sum_{l \in \mathscr{J}^f} s_l\right)\right) = \alpha s_j s_k\left(1 + \sum_{k \in \mathscr{J} \backslash \mathscr{J}^f} s_k\right) > 0$$

For the strict diagonal dominance to hold, we need to verify: $\left|\frac{\partial s_j}{\partial \lambda_j}\right| > \sum_{k\neq j}\left|\frac{\partial s_j}{\partial \lambda_k}\right|$. Substituting the expressions:

$$\begin{aligned}\left|\frac{\partial s_j}{\partial \lambda_j}\right| &= \alpha s_j \left((1-s_j)\sum_{k\in\mathscr{J}\setminus\mathscr{J}^f} s_k + \sum_{k\in\mathscr{J}^f\setminus\{j\}} s_k\right) \\ &> \alpha s_j \left(\sum_{k\in\mathscr{J}^f\setminus\{j\}} s_k \cdot \sum_{k\in\mathscr{J}\setminus\mathscr{J}^f} s_k + \sum_{k\in\mathscr{J}^f\setminus\{j\}} s_k\right) = \sum_{k\in\mathscr{J}^f\setminus\{j\}}\left|\frac{\partial s_j}{\partial \lambda_k}\right|\end{aligned}$$

This inequality demonstrates that the Jacobian matrix J with elements $J_{jk} = \frac{\partial s_j}{\partial \lambda_k}$ satisfies the strict diagonal dominance.

□

With these results, we can consider the shadow cost operator defined by

$$\Psi_j(\lambda^f) = \begin{cases} 0, & \text{if } \Gamma_j(0;\lambda^f_{-j}) \le \kappa_j \\ \Gamma_j^{-1}(\kappa_j;\lambda^f_{-j}), & \text{if } \Gamma_j(0;\lambda^f_{-j}) > \kappa_j \end{cases} \tag{1}$$

where $\Gamma_j(\lambda_j;\lambda^f_{-j}) := \sum_m L_m s_{jm}(\lambda^f)$. To prove the existence, we will show that the domain of λ is compact. The lower bound of zero follows directly from the construction. For the upper bound, since $H_m^{-f} > 0$ is fixed for all m, when we set all other products' multipliers to ∞, we are creating a scenario where all other products have a zero market share. This leaves the least competitive environment for product j, so the firm must select the highest λ_j to decrease the market share and ensure its capacity constraint is exactly binding.

Lemma A.6 (Solution Existence)**.** *Ψ is well-defined, and $\Psi(\lambda^f) \in \prod_{i\in\mathscr{J}^f}[0,\bar{\lambda}_i]$ for any $\lambda^f \in \prod_i[0,\bar{\lambda}_i]$. Moreover, the solution exists.*

Proof. We first establish that Ψ_j is well-defined. By Lemma A.5,

$$\lim_{\substack{\lambda_j\to\infty \\ \lambda^f_{-j}\to\lambda^{*f}_{-j}}} s_{jm}(\lambda_j;\lambda^f_{-j}) = 0, \quad \forall \lambda^{*f}_{-j} \in \prod_{i=1}^{|\mathscr{J}^f|-1}[0,\infty]$$

Since $\Gamma_j(\lambda_j;\lambda^f_{-j}) := \sum_m L_m s_{jm}(\lambda^f)$ and all $L_m > 0$, we have $\frac{\partial \Gamma_j}{\partial \lambda_j} < 0$, $\frac{\partial \Gamma_j}{\partial \lambda_k} > 0$, and $\lim_{\lambda_j\to\infty}\Gamma_j(\lambda_j;\lambda^f_{-j}) = 0$ also by Lemma A.5. We consider three cases: First, if $\Gamma_j(0;\lambda^f_{-j}) > \kappa_j$, because $\Gamma_j(\lambda_j;\lambda^f_{-j})$ is $\mathscr{C}^1$ and $\lim_{\lambda_j\to\infty}\Gamma_j(\lambda_j;\lambda^f_{-j}) = 0 < \kappa_j < \Gamma_j(0;\lambda^f_{-j})$, by the Intermediate Value Theorem, there must exist a value of λ_j where $\Gamma_j\left(\lambda_j;\lambda^f_{-j}\right) = \kappa_j > 0$. The strict monotonicity property $\frac{\partial \Gamma_j}{\partial \lambda_j} < 0$ further

ensures that the solution is finite and unique. Therefore, $\Gamma_j^{-1}(\kappa_j;\cdot)$ is well-defined under this case, and it is $\mathscr{C}^1$ by Implicit Function Theorem.

Second, if $\Gamma_j\left(0;\lambda_{-j}^f\right) < \kappa_j$, then $\Gamma_j\left(\lambda_j;\lambda_{-j}^f\right) < \kappa_j$ for all $\lambda_j \geq 0$ by monotonicity. By construction $\Psi_j\left(\lambda_j;\lambda_{-j}^f\right) = 0$, which is clearly continuous.

Third, we consider the boundary case. If $\Gamma_j\left(0;\lambda_{-j}^f\right) = \kappa_j$, the equation $\Gamma_j\left(\lambda_j;\lambda_{-j}^f\right) = \kappa_j$ is satisfied by $\lambda_j = 0$. Thus $\Psi_j\left(\lambda_j;\lambda_{-j}^f\right) = 0$, consistent with Case 2. Continuity across the boundary follows from joint continuity of Γ_j and uniqueness of solutions: $\Gamma_j^{-1}\left(\kappa_j;\lambda_{-j}^f\right)$ is continuous in λ_{-j}^f, and at the boundary $\Gamma_j^{-1}\left(\kappa_j;\lambda_{-j}^{f,*}\right) = 0$ since $\Gamma_j\left(0;\lambda_{-j}^{f,*}\right) = \kappa_j$. Therefore Ψ_j is continuous across the boundary.

Therefore, Ψ_j is well-defined. We extend Ψ_j continuously to $\prod_{i=1}^{\left|\mathscr{J}^f\right|}[0,\infty]$ by defining

$$\Gamma_j\left(\lambda_j^*;\lambda_{-j}^{*f}\right) := \lim_{\lambda^f \to \lambda^*} \Gamma_j\left(\lambda_j;\lambda_{-j}^f\right),$$

which exists by Lemma A.5. The above arguments apply after this extension since the limit preserves monotonicity and continuity properties. Therefore, $\Psi_j\left(\lambda_j;\lambda_{-j}^f\right)$ is well-defined and jointly continuous on $\prod_{i=1}^{\left|\mathscr{J}^f\right|-1}[0,\infty]$, and $\Psi\left(\lambda^f\right) := \left(\Psi_j\left(\lambda_j;\lambda_{-j}^f\right)\right)_{j\in\mathscr{J}^f}$ maps the compact set $\prod_{i=1}^{\left|\mathscr{J}^f\right|}[0,\infty]$ continuously to itself.

Next, we show that the $\Psi_j(\cdot;\lambda_{-j}^f)$ restricts to a continuous self-map on a compact convex set. Notice that $\lambda_j \in \mathscr{R}_+ \cup \{\infty\}$ so it is bounded from below by 0. For the upper bound, consider

$$\bar{\lambda}_j := \sup_{\lambda_{-j}^f \in \prod_{i=1}^{\left|\mathscr{J}^f\right|-1}[0,\infty]} \{\Psi_j(\lambda_j;\lambda_{-j}^f)\}$$

Notice that we do not need to take supreme over λ_j because by construction the output of $\Psi_j(\lambda_j;\lambda_{-j}^f)$ is irrelevant to the value of input λ_j. Let $\lambda_{-j}^f(n)$ be any monotonic increasing sequence element-wise where each component approaches infinity as $n \to \infty$. Because $\frac{\partial \Gamma_j}{\partial \lambda_k} > 0$, the sequence $\Gamma_j(0;\lambda_{-j}^f(n))$ is monotonically increasing in n. The application of limit operation on inequalities gives:

$$\Gamma_j(0;\lambda_{-j}^f(1)) < \cdots \leq \lim_{n\to\infty} \Gamma_j(0;\lambda_{-j}^f(n)) := \Gamma_j(0,\infty)$$

We then discuss by cases. First, if $\Gamma_j(0,\infty) \leq \kappa_j$, then $\Gamma_j(0;\lambda_{-j}^f) \leq \Gamma_j(0,\infty) \leq \kappa_j$ for any λ_{-j}^f. By the definition, we have $\Psi_j = 0$ for all inputs and $\bar{\lambda}_j = 0$ in this scenario.

Second, we now consider the case $\Gamma_j(0,\infty) > \kappa_j$. Then by Lemma A.5, we know $p_{km} = \infty$ for all $k \in \mathscr{J}^f \backslash \{j\}$ and $m \in \mathscr{M}$. By the same argument above, we know there is a finite

positive $\Gamma_j^{-1}(\kappa_j,\infty)$. We need to show that $\Gamma_j^{-1}(\kappa_j,\infty) \geq \Psi_j(\lambda^f)$ for any other λ^f. Notice that this is trivial if $\Gamma_j(0;\lambda_{-j}^f) \leq \kappa_j$ because $\Psi_j(\lambda^f) = 0$ in this case. That is, we only need to show $\Gamma_j^{-1}(\kappa_j,\infty) \geq \Gamma_j^{-1}(\kappa_j;\lambda_{-j}^f)$ for all λ_{-j}^f such that $\Gamma_j(0;\lambda_{-j}^f) > \kappa_j$. Again, by the argument above, $\Gamma_j^{-1}(\kappa_j;\lambda_{-j}^f)$ is well-defined for any such λ_{-j}^f. Suppose, for the sake of contradiction, we assume $\omega_1 := \Gamma_j^{-1}(\kappa_j;\lambda_{-j}^f) > \omega_\infty := \Gamma_j^{-1}(\kappa_j,\infty)$. Because $\frac{\partial \Gamma_j}{\partial \lambda_j} < 0$ and ω_1, ω_∞ finite, we have $\Gamma_j(\omega_\infty;\infty) > \Gamma_j(\omega_1;\infty)$. Next, let $\lambda_{-j}^f(1) = \lambda_{-j}^f$, then by the same sequence argument as above, we can show that

$$\Gamma_j(\omega_1;\lambda_{-j}^f) = \Gamma_j(\omega_1;\lambda_{-j}^f(1)) < \cdots \leq \lim_{n\to\infty} \Gamma_j(\omega_1;\lambda_{-j}^f(n)) = \Gamma_j(\omega_1,\infty)$$

Combine the inequalities, we get

$$\kappa_j = \Gamma_j(\omega_\infty;\infty) > \Gamma_j(\omega_1;\infty) \geq \Gamma_j(\omega_1;\lambda_{-j}^f) = \kappa_j$$

Hence $\kappa_j > \kappa_j$ gives the contradiction and therefore $\bar{\lambda}_j = \Gamma_j^{-1}(\kappa_j,\infty)$ under this case.

Putting it all together, we proved that $\bar{\lambda}_j$ is a well-defined, non-negative constant, such that $\Psi_j(\lambda^f) \in [0,\bar{\lambda}_j]$ for all $\lambda^f \in \prod_{j=1}^{|\mathscr{J}^f|}[0,\infty]$. Since this holds for all $j \in \mathscr{J}^f$, we can conclude that $\Psi(\lambda^f) \in \prod_{j=1}^{|\mathscr{J}^f|}[0,\bar{\lambda}_j]$ for all $\lambda^f \in \prod_{j=1}^{|\mathscr{J}^f|}[0,\infty]$. Therefore, we know a fixed-point λ^{f*} exists by Brouwer.

□

To prove the uniqueness of the mapping:

$$F(\lambda^f) = \lambda^f - \Psi(\lambda^f) = 0,$$

notice that at the boundary where $\Gamma_j\left(0;\lambda_{-j}^f\right) = \kappa_j$, the function Ψ_j transitions between a smooth function $\Gamma_j^{-1}\left(\kappa_j;\lambda_{-j}^f\right)$ and a constant zero is continuous but not differentiable, because the derivative from one side is zero but the other side is non-zero by the Implicit Function Theorem. Existing methods often require the mapping to be differentiable everywhere (such as Kellogg (1976) used in Konovalov and Sándor (2010)), cannot be applied here. We solve this problem by applying Clarke generalized Jacobian and Pourciau (1983) extension of global univalence theorem to non-smooth mappings. Intuitively, the Clarke generalized Jacobian collects limiting Jacobians near non-differentiable points; if these generalized derivatives preserve the sign structure needed for univalence, global uniqueness extends beyond smooth cases [4]. Formally, we show this in the Univalence Lemma in the Appendix A of the main paper.

[4] Alternatively, one can show that $\Psi(\lambda^f)$ is monotonic and then apply Tarski fixed-point theorem, but this does not directly give the uniqueness results, and one has to build a contradiction argument rely on the diagonally dominant property. However, this method is hard to implement when we solve the nested fixed-points of two layers of vectors later in Online Appendix B.

We now use the Univalence Lemma to show that the solution to $F(\lambda^f) = \lambda^f - \Psi(\lambda^f) = 0$ is unique.

Lemma (Lemma 2 from the main paper)**.** *The operator Ψ has a unique fixed-point in $\prod_{j\in\mathscr{J}^f}[0,\bar{\lambda}_j]$. Further, λ^* satisfies all KKT conditions if and only if λ^* is a fixed-point of Ψ.*

Proof. Notice that the value $\Psi_j\left(\lambda^f\right)$ is determined by solving $\Gamma_j\left(\Psi_j;\lambda^f_{-j}\right) = \kappa_j$ for Ψ_j, given λ^f_{-j}. And observe the definition of Ψ_j matches:

$$\max\left(0,h_j\right) = \begin{cases} h_j = \Gamma_j^{-1}\left(\kappa_j;\lambda^f_{-j}\right) & \text{if } \Gamma_j\left(0;\lambda^f_{-j}\right) > \kappa_j \\ 0 & \text{if } \Gamma_j\left(0;\lambda^f_{-j}\right) \leq \kappa_j \end{cases}$$

Crucially, Ψ_j does not depend on the input value, so $\frac{\partial\Psi_j}{\partial\lambda_j} = 0$ for all j. For $k \neq j$, when the constraint for product j is binding (i.e., $\Psi_j > 0$), we have $\Gamma_j\left(\Psi_j;\lambda^f_{-j}\right) = \kappa_j$. By the proof of LemmaA.6, we apply Implicit function theorem and can get $\frac{\partial\Psi_j}{\partial\lambda_k} = -\frac{\partial\Gamma_j/\partial\lambda_k}{\partial\Gamma_j/\partial\lambda_j} > 0$ for $k \neq j$. When the constraint is slack (interior of $\Psi_j = 0$), we have $\frac{\partial\Psi_j}{\partial\lambda_k} = 0$ for all k. Thus J_F is a Z-matrix with $(J_F)_{jj} = 1$ and $(J_F)_{jk} \leq 0$ for $k \neq j$. In addition, by Lemma A.5 we have

$$-\frac{\partial\Gamma_j}{\partial\lambda_j} > \sum_{k\neq j}\frac{\partial\Gamma_j}{\partial\lambda_k}$$

Dividing by $-\partial\Gamma_j/\partial\lambda_j > 0$, we get $1 > \sum_{k\neq j}\frac{\partial\Psi_j}{\partial\lambda_k}$. At non-differentiable points, the Clarke generalized Jacobian preserves the Z-matrix structure and the above diagonal dominant properties. Therefore, for any $\tilde{J} \in \partial F\left(\lambda^f\right)$:

$$\sum_{k\neq j}\left|(\tilde{J})_{jk}\right| \leq \sum_{k\neq j}\frac{\partial\Psi_j}{\partial\lambda_k} < 1 = \left|(\tilde{J})_{jj}\right|.$$

Every $\tilde{J} \in \partial F\left(\lambda^f\right)$ is a strictly diagonally dominant Z-matrix, and hence the existence of a fixed-point follows from LemmaA.6, and uniqueness follows from the Univalence Lemma in the Appendix A of the main paper.

Finally, we verify the equivalence of KKT solutions and fixed-points of Ψ. Notice that by the definition of Ψ and Lemma 1 from the main paper, any fixed-point of Ψ satisfies all KKT conditions. Conversely, let λ^* satisfy the KKT conditions. For each $j \in \mathscr{J}^f$, we show $\Psi_j(\lambda^{f*}) = \lambda^*_j$. By complementary slackness, we must have $\lambda^*_j\left(\sum_m L_m s_{jm}(\lambda^{f*}) - \kappa_j\right) = 0$. If $\lambda^*_j = 0$, then $\Gamma_j(0;\lambda^*_{-j}) = \sum_m L_m s_{jm}(\lambda^{f*}) \leq \kappa_j$ by primal feasibility. Thus $\Psi_j(\lambda^{f*}) = 0 = \lambda^*_j$ by definition of Ψ. Otherwise, if $\lambda^*_j > 0$, then $\Gamma_j(\lambda^*_j;\lambda^*_{-j}) = \sum_m L_m s_{jm}(\lambda^{f*}) = \kappa_j$ by complementary slackness. Since

$\frac{\partial \Gamma_j}{\partial \lambda_j} < 0$ and $\lambda_j^* > 0$, we have $\Gamma_j(0; \lambda_{-j}^*) > \Gamma_j(\lambda_j^*; \lambda_{-j}^*) = \kappa_j$. Therefore $\Psi_j(\lambda^*) = \Gamma_j^{-1}(\kappa_j; \lambda_{-j}^*)$. Since $\Gamma_j(\lambda_j^*; \lambda_{-j}^*) = \kappa_j$ and Γ_j is strictly monotone, $\lambda_j^* = \Gamma_j^{-1}(\kappa_j; \lambda_{-j}^*) = \Psi_j(\lambda^{f*})$.

□

Thus, we have shown that KKT conditions must be sufficient, not just necessary, so any profile satisfying them for all firms is a full Nash equilibrium rather than a merely local one. Uniqueness follows from the strict diagonal dominance of the market-share Jacobian $J_{jk} = \partial s_j / \partial \lambda_k$. The following theorem formally establishes that KKT conditions are both necessary and sufficient for characterizing the Nash equilibrium.

Therefore, the lemmas needed for the main-text proof of the following theorem are now in place, which completes the proof for Theorem 1 from the main paper.

Theorem (Theorem 1 from the main paper)**.** *The KKT conditions defined in Definition 1 are both necessary and sufficient for characterizing the solution of each firm's profit maximization problem. For any fixed rivals' aggregator vector $(H_m^{-f})_{m \in \mathscr{M}}$, there exists a unique solution satisfying these conditions, given by $p_{jm}^* = p_{jm}(\nu_m^f(\lambda^{*f}), \lambda_j^*)$ for all products of firm f (i.e. $j \in \mathscr{J}^f$).*

B Nash Equilibrium Proof

To prove Theorem 2 from the main paper, we show the existence of an equilibrium in which all firms' KKT conditions are satisfied simultaneously. We adopt an aggregator-based approach, where the aggregator is defined as the vector of total product attractiveness in each market. The proof follows a nested fixed-point structure. In the inner fixed-point problem, for any given aggregator vector, we show that there exists a unique vector of multipliers, which in turn determines a unique price vector for all products across all markets. In the outer fixed-point problem, we then show that there exists an aggregator vector that is consistent with the prices and resulting market shares.

We begin with the inner fixed-point problem, following an approach similar to the proof of Theorem 1 in the main paper. For any fixed aggregator vector, optimal prices are expressed as functions of firm-specific multipliers and a market power markup, which is uniquely and continuously determined by the multipliers (Lemma B.1). We then analyze the limits and derivatives, and establish diagonal dominance of its market share fitting-in functions (Lemma B.2). A fitting-in multiplier operator, constructed from complementary slackness, primal and dual feasibility, is shown to have a unique fixed-point (Lemma 3 from the main paper), completing the inner problem.

For the outer fixed-point problem, we treat the unique solution of the inner problem as an implicit function of the aggregator vector. However, since the multiplier operator is not differentiable, the standard Implicit Function Theorem does not apply. Instead, we rely on topological arguments. We construct a fixed-point operator from the aggregator definitions and show that it is continuous and bounded; by Lemma B.3 (via Brouwer's theorem) the operator has a fixed-point. Next, within each regime, the aggregate market share $S_m = \sum_{j \in \mathscr{J}} s_{jm}$ is differentiable, so we compute its derivatives with respect to own- and cross-market aggregators (Lemma B.4). We then define the excess demand system, and apply Pourciau (1983) to address non-differentiability at regime boundaries and complete the proof for Theorem 2 from the main paper.

We now show the detailed proof. We first characterize the price vector as a unique function of the aggregator H_m and the vector of multipliers.

Lemma B.1. *Fix an H_m, the partial derivative of the firm markup with respect to λ_j is $\frac{\partial \hat{v}_m^f}{\partial \lambda_j} = \frac{-\hat{s}_{jm}}{1-S_m^f+(S_m^f)^2} < 0$, where S_m^f is firm f's total market share in market m, and there exists a unique $\hat{v}_m^f$ that satisfies the stationary conditions for any given λ vector. Moreover, there exists a unique optimal price vector where each product price takes the form $\hat{p}_{jm}(\hat{v}_m^f(\lambda), \lambda_j) = c_{jm} + \hat{v}_m^f(\lambda) + \lambda_j$.*

Proof. From Lemma A.4, it suffices to consider the case where all prices are finite. For any firm f, product j and any fixed H_m, the stationary condition implies:

$$\hat{v}_m^f = \frac{1}{\alpha} + \sum_{k \in \mathscr{J}^f} \hat{v}_m^f \hat{s}_{km} = \frac{1}{\alpha} + \hat{v}_m^f S_m^f$$

where $S_m^f := \sum_{k \in \mathscr{J}^f} \hat{s}_{km}$ is firm f's total market share in market m. Substituting the definition of $\hat{s}_{jm}$, we can define the implicit function from the stationary condition:

$$G(\hat{v}_m^f, \lambda) = \frac{\hat{v}_m^f - \frac{1}{\alpha}}{\hat{v}_m^f} - \sum_{k \in \mathscr{J}^f} \frac{\exp\left(\delta_{km} - \alpha\left(c_{km} + \lambda_k + \hat{v}_m^f\right)\right)}{H_m}$$

Since G is composed of exponential functions and arithmetic operations, it is $\mathscr{C}^1$ on $\mathscr{R}_{++} \times \mathscr{R}_+^{|\mathscr{J}|}$. Notice that when $\hat{v}_m^f \notin \mathscr{R}_{++}$, we must have $S_m^f \geq 1$ since $\hat{v}_m^f = \frac{1}{\alpha(1-S_m^f)}$ in the equilibrium. However, such values of S_m^f cannot be part of a valid solution, and mathematically we can always find $\hat{v}_m^f > \frac{1}{\alpha}$. Therefore, it suffices to study the existence and uniqueness of a solution within the range $\mathscr{R}_{++}$. We then calculate the derivative:

$$\frac{\partial G}{\partial \hat{v}_m^f} = \frac{1}{\alpha(\hat{v}_m^f)^2} + \alpha \sum_{k \in \mathscr{J}^f} \frac{\exp\left(\delta_{km} - \alpha\left(c_{km} + \lambda_k + \hat{v}_m^f\right)\right)}{H_m} = \frac{1}{\alpha(\hat{v}_m^f)^2} + \alpha S_m^f > 0,$$

for any $\hat{v}_m^f \in \mathscr{R}_{++}$ and $H_m > 0$. Next, for any given λ, $\lim_{\hat{v}_m^f \to \infty} G > 0$, and $\lim_{\hat{v}_m^f \to \frac{1}{\alpha}} G < 0$. By the Intermediate Value Theorem and strict monotonicity, there exists a unique $\hat{v}_m^f \in (\frac{1}{\alpha}, \infty)$ such that $G(\hat{v}_m^f, \lambda) = 0$. The Implicit Function Theorem establishes a $\mathscr{C}^1$-diffeomorphism again, $\hat{v}_m^f$ is then a $\mathscr{C}^1$ function of λ. Notice that, holding $\hat{v}_m^f$ constant, we have $\frac{\partial \hat{s}_{km}}{\partial \lambda_j} = -\alpha \hat{s}_{jm}$ if $k = j$ and 0 otherwise. We then easily compute the following derivative:

$$\frac{\partial G}{\partial \lambda_k} = \frac{\exp\left(\delta_{km} - \alpha\left(c_{km} + \lambda_k + \hat{v}_m^f\right)\right)}{H_m} \cdot \alpha = \alpha \hat{s}_{km}.$$

Then by the Implicit Function Theorem,

$$\frac{\partial \hat{v}_m^f}{\partial \lambda_j} = -\frac{\partial G / \partial \lambda_j}{\partial G / \partial \hat{v}_m^f} = \frac{-\hat{s}_{jm}}{(1 - S_m^f)^2 + S_m^f}$$

where the last equality follows the fact that the partial derivatives are taking at $(\lambda, \hat{v}_m^f(\lambda))$ hence we can substitute $\hat{v}_m^f = \frac{1}{\alpha(1-S_m^f)}$.[5] Therefore, $\frac{\partial \hat{v}_m^f}{\partial \lambda_j} < 0$ given any $H_m > 0$.

□

Next, we analyze the limiting behavior of $\hat{p}_{jm}$ and $\hat{s}_{jm}$ as some multipliers approach infinity. This property allows us to extend the definitions of $\hat{p}_{jm}$ and $\hat{s}_{jm}$ continuously to include cases where certain multipliers take infinite values. We also derive own- and cross-effects of the multipliers on market shares. Furthermore, we show that the Jacobian of $\hat{s}_{jm}$ is diagonally dominant, which is useful for establishing the uniqueness of the multiplier vector $\{\lambda_j\}$ for any given H_m vector.

Lemma B.2. *$\hat{p}_{jm} \to \infty$ if and only if $\lambda_j \to \infty$, and consequently, $\hat{s}_{jm} \to 0$ if and only if $\lambda_j \to \infty$. For an arbitrary market m, fix a H_m, $\frac{\partial \hat{s}_{jm}}{\partial \lambda_j} < 0$, $\frac{\partial \hat{s}_{jm}}{\partial \lambda_k} > 0$ for any $k \in \mathscr{J}^f \backslash \{j\}$, and $\frac{\partial \hat{s}_{jm}}{\partial \lambda_k} = 0$ for any $k \notin \mathscr{J}^f$. In addition, it has the dominant diagonal property: $\left|\frac{\partial \hat{s}_{jm}}{\partial \lambda_j}\right| > \sum_{k \neq j} \frac{\partial \hat{s}_{jm}}{\partial \lambda_k}$ for all j.*

Proof. From Lemma B.1, we have shown that $\hat{v}_m^f(\lambda)$ is strictly decreasing with λ_j and continuous. Obviously, for any finite λ vector, we have proved that $\hat{v}_m^f(\lambda) \in (\frac{1}{\alpha}, \infty)$. Now we consider a limit case where some $\lambda_j \to \infty$. By monotonicity, $\lim_{\lambda \to \lambda^*} \hat{v}_m^f(\lambda)$ exist for any $\lambda^* \in [0,\infty]^{|\mathscr{J}|}$. In particular, since G is continuous on $\mathbb{R}_{++} \times [0,\infty]^{|\mathscr{J}|}$, taking the limit in equation $\lim_{\lambda \to \lambda^*} G(\hat{v}_m^f(\lambda), \lambda) = 0$ immediately implies $\hat{v}_m^f(\lambda^*) = \frac{1}{\alpha}$ if $\lambda^* = \infty$ for all components and $\hat{v}_m^f(\lambda^*) \in (\frac{1}{\alpha}, \infty)$ otherwise. Therefore, $\hat{v}_m^f(\lambda)$ is finite for any λ. By Lemma B.1, $\hat{p}_{jm}(\hat{v}_m^f(\lambda), \lambda_j) = c_{jm} + \hat{v}_m^f(\lambda) + \lambda_j$, hence $\hat{p}_{jm} \to \infty$ if and only if $\lambda_j \to \infty$, and consequently, $\hat{s}_{jm} \to 0$ if and only if $\lambda_j \to \infty$.

Next, also by Lemma B.1, we get the own-effect result:

$$\frac{\partial \hat{s}_{jm}}{\partial \lambda_j} = -\alpha \hat{s}_{jm} \left(1 + \frac{\partial \hat{v}_m^f}{\partial \lambda_j}\right) = -\alpha \hat{s}_{jm} \frac{(1-S_m^f)^2 + S_m^f - \hat{s}_{jm}}{(1-S_m^f)^2 + S_m^f}$$

Notice that $(1-S_m^f)^2 + S_m^f - \hat{s}_{jm} \geq (1-S_m^f)^2 + S_m^f - S_m^f = (1-S_m^f)^2 > 0$,[6] and the denominator is also positive, which implies $\frac{\partial \hat{s}_{jm}}{\partial \lambda_j} < 0$. For the cross-effect within the firm, that is, when $k \neq j$ but $k \in \mathscr{J}^f$:

$$\frac{\partial \hat{s}_{jm}}{\partial \lambda_k} = -\alpha \hat{s}_{jm} \frac{\partial \hat{v}_m^f}{\partial \lambda_k} = \frac{\alpha \hat{s}_{jm} \hat{s}_{km}}{1 - S_m^f + (S_m^f)^2} > 0$$

In addition, note that $\hat{v}_m^f = \frac{1}{\alpha(1-S_m^f)}$, and S_m^f depends only on the market shares of products in $\mathscr{J}^f$. Therefore, $\frac{\partial \hat{s}_{jm}}{\partial \lambda_k} = 0$ when $k \notin \mathscr{J}^f$.

[5] Note that the denominator is positive because the quadratic function $f(x) = (1-x)^2 + x \geq \frac{3}{4}$ for any $x \in \mathbb{R}$.

[6] Lemma B.1 excludes the case $S_m^f = 1$ because substituting this value into the definition of $\hat{v}_m^f$ forces the impossible identity $0 = \frac{1}{\alpha}$.

Finally, we show that the Jacobian matrix $\frac{\partial \hat{s}_m}{\partial \lambda}$ has the diagonal dominance property. Since $\frac{\partial \hat{s}_{jm}}{\partial \lambda_j} < 0$, $\frac{\partial \hat{s}_{jm}}{\partial \lambda_k} > 0$ for $k \in \mathscr{J}^f \backslash \{j\}$, and $\frac{\partial \hat{s}_{jm}}{\partial \lambda_k} = 0$ for $k \notin \mathscr{J}^f$, we need to show:

$$\left| \frac{\partial \hat{s}_{jm}}{\partial \lambda_j} \right| > \sum_{k \in \mathscr{J}^f, k \neq j} \frac{\partial \hat{s}_{jm}}{\partial \lambda_k}$$

Substituting:

$$\alpha \hat{s}_{jm} \frac{(1 - S_m^f)^2 + S_m^f - \hat{s}_{jm}}{(1 - S_m^f)^2 + S_m^f} > \sum_{k \in \mathscr{J}^f, k \neq j} \frac{\alpha \hat{s}_{jm} \hat{s}_{km}}{1 - S_m^f + (S_m^f)^2} = \alpha \hat{s}_{jm} \frac{S_m^f - \hat{s}_{jm}}{1 - S_m^f + (S_m^f)^2}$$

This reduces to:

$$1 - S_m^f + (S_m^f)^2 - \hat{s}_{jm} > S_m^f - \hat{s}_{jm},$$

which is equivalent to $(1 - S_m^f)^2 > 0$. Since this holds for all S_m^f, the dominant diagonal property holds for all f and m. [7]

□

Now we consider the best-response operator of shadow costs, for a given aggregator vector $\mathbf{H}$. We denote it as:

$$\hat{\Psi}_j(\lambda_j; \lambda_{-j}) = \begin{cases} 0, & \text{if } \hat{\Gamma}_j(0; \lambda_{-j}) \leq \kappa_j \\ \hat{\Gamma}_j^{-1}(\kappa_j; \lambda_{-j}), & \text{if } \hat{\Gamma}_j(0; \lambda_{-j}) > \kappa_j \end{cases} \tag{2}$$

where $\hat{\Gamma}_j(\lambda_j; \lambda_{-j}) := \sum_m L_m \hat{s}_{jm}(\lambda)$. We define the simultaneous best response operator as:

$$\hat{\Psi}(\lambda) := \left(\hat{\Psi}_j(\lambda)\right)_{\forall j}.$$

We then use these lemmas above to prove the existence and uniqueness of λ solution given any fixed $\mathbf{H}$, which is the following Lemma. Recall that the proof is given in the main text.

Lemma (Lemma 3 from the main paper)**.** *For any fixed* $\mathbf{H}$*, the operator* $\hat{\Psi}$ *is well-defined and maps* $\prod_{i \in \mathscr{J}^f} [0, \bar{\lambda}_i]$ *into itself. Moreover, the fixed-point of* $\hat{\Psi}$ *is unique.*

Next, consider

$$H_m = 1 + \sum_j \exp\left(\delta_{jm} - \alpha \hat{p}_{jm}(\mathbf{H})\right), \quad \forall m \in \mathscr{M}, \tag{3}$$

which defines the aggregator. It is important to note that, in all previous arguments, we consider only $H_m > 0$ to ensure positivity and avoid undefined terms at $H_m = 0$. In addition, H_m can even

[7] As a final remark, all of these strict inequalities hold only when $\hat{s}_{jm} \neq 0$. For the cross-partial derivatives, we require $\hat{s}_{km} \neq 0$ in addition. Otherwise, all the inequalities become equalities.

take values less than 1 , which still yields valid solutions for $\hat{p}_{jm}(\mathbf{H})$ and corresponding market shares $\hat{s}_{jm} \in (0,1)$ for each product j including the outside option. However, in such cases, the condition $\sum_j \hat{s}_{jm} = 1$ would no longer hold, as satisfying this condition requires $\mathbf{H} > 1$ by the fixed-point equation above. To establish the existence of a fixed-point aggregator vector $\mathbf{H}$, we apply Brouwer's fixed-point Theorem, which requires the function to be continuous and to map a non-empty, compact, convex set into itself. This means we must show that the fixed-point multipliers discussed in the previous section are continuous functions of $\mathbf{H}$.

We now show that the outer fixed-point problem has a solution.

Lemma B.3. *There exist* $\mathbf{H}^* \in \prod_{m\in\mathscr{M}}(1,\infty)$ *such that* $T(H^*) = H^*$ *where* $T(\mathbf{H}) = (T_m(\mathbf{H}))_{m\in\mathscr{M}}$ *and* $T_m(\mathbf{H}) = 1 + \sum_j \exp\big(\delta_{jm} - \alpha\hat{p}_{jm}(\mathbf{H})\big)$.

Proof. Consider the mapping

$$T_m(H) = 1 + \sum_j \exp\big(\delta_{jm} - \alpha\hat{p}_{jm}(H)\big).$$

We first show $T(H)$ is bounded. Since the exponential terms are strictly positive, it is natural to set the lower bound as $\underline{H}_m = 1$ for all $m \in \mathscr{M}$. By Lemma 3 from the main paper, consider an arbitrary $H \in \prod_{m\in\mathscr{M}}[1,\infty)$, it yields a unique well-defined vector $\hat{\lambda}(H) \in \prod_j [0,\bar{\lambda}_j]$.[8] Applying Lemma B.1, since $\hat{\lambda}(H)$ is finite, we obtain a unique finite value $\hat{v}_m^f(H) := \hat{v}_m^f(\lambda(H), H_m) > \frac{1}{\alpha}$ for each f and m, it then follows that $\hat{p}_{jm}(H)$ is finite and unique. Substituting into the definition of $T_m(H)$ gives:

$$T_m(H) = 1 + \sum_j \exp\big(\delta_{jm} - \alpha\hat{p}_{jm}(H)\big) > 1 = \underline{H}_m.$$

Since this inequality holds for every component m, it follows that $T(H) > \underline{H}$ for all $H \in \prod_{m\in\mathscr{M}}[1,\infty)$.

For the upper bound of $T(H)$, notice that we have proved $\hat{v}_m^f(H) \geq \frac{1}{\alpha}$ and $\hat{\lambda}_j(H) \geq 0$ for any strictly positive vector H. Therefore, $\hat{p}_{jm}(H) \geq 0$, and then we can let $\bar{H}_m := 1 + \sum_j \exp\big(\delta_{jm}\big)$, which is strictly larger than $\underline{H}_m = 1$. Then, for any $H \in \prod_{m\in\mathscr{M}}[1,\bar{H}_m]$, we have:

$$T_m(H) = 1 + \sum_j \exp\big(\delta_{jm} - \alpha\hat{p}_{jm}(H)\big) < \bar{H}_m,$$

which gives the upper bound. Let $\bar{H} := \max_{m\in\mathscr{M}} \bar{H}_m$ and $\underline{H} := 1$, it follows that if $H \in [\underline{H},\bar{H}]^{|\mathscr{M}|}$, then $T(H) \in [\underline{H},\bar{H}]^{|\mathscr{M}|}$. In other words, $[\underline{H},\bar{H}]^{|\mathscr{M}|}$ is compact (closed and bounded in $\mathbb{R}^{|\mathscr{M}|}$), convex (product of intervals), and invariant under T.

[8] Even though this upper bound on λ depends on H, this does not make this proof circular because we construct the bound on H only uses the lower bound of λ which is uniformly zero.

We next show that $T_m(\mathbf{H})$ is continuous. Consider the proof in Lemma B.1, by the Implicit Function Theorem applied to $G\left(\hat{v}_m^f, \lambda, H_m\right)$, the function $\hat{v}_m^f(\lambda, H_m)$ is $\mathscr{C}^1$, which implies $\hat{\Gamma}_j(\lambda, \mathbf{H})$ is $\mathscr{C}^1$ on $\mathbb{R}_+^{|\mathscr{J}|} \times \mathbb{R}_{++}^{|\mathscr{M}|}$. The bound $\bar{\lambda}_j(\mathbf{H}) := \hat{\Gamma}_j^{-1}(\kappa_j; \infty, \mathbf{H})$ is defined via the continuous extension of $\hat{\Gamma}_j$ to $\lambda_{-j} = \infty$ (Lemma 3 from the main paper). Since $\hat{\Gamma}_j(\lambda; \mathbf{H})$ is decreasing in each H_m (larger H_m implies smaller market shares for fixed λ), the bound $\bar{\lambda}_j(\mathbf{H})$ is decreasing in $\mathbf{H}$. Therefore, for $\mathbf{H} \in [\underline{H}, \bar{H}]^{|\mathscr{M}|}$:

$$\bar{\lambda}_j(\mathbf{H}) \leq \bar{\lambda}_j(\underline{H}) =: \bar{\lambda}_j < \infty$$

This provides a uniform bound.

We then show the correspondence $\Xi : \prod_m (0, \infty) \rightrightarrows \prod_j \left[0, \bar{\lambda}_j\right]$ defined by $\Xi(\mathbf{H}) = \{\lambda : \lambda = \hat{\Psi}(\lambda; \mathbf{H})\}$ induces a continuous function $\hat{\lambda}(\mathbf{H})$. Since $\hat{\Gamma}_j$ is continuous and strictly decreasing in λ_j, the operator $\hat{\Psi}(\lambda; \mathbf{H})$ is jointly continuous: on binding regions, $\hat{\Psi}_j = \hat{\Gamma}_j^{-1}\left(\kappa_j; \lambda_{-j}, \mathbf{H}\right)$ is continuous by the Implicit Function Theorem; on slack regions, $\hat{\Psi}_j = 0$; and both branches agree at the boundary. Define $F(\lambda, \mathbf{H}) := \lambda - \hat{\Psi}(\lambda; \mathbf{H})$. The graph $\{(\mathbf{H}, \lambda) : F(\lambda, \mathbf{H}) = 0\}$ is closed as the preimage of a closed set $\{0\}$ under a continuous map. Since $\hat{\lambda}(\mathbf{H})$ lies in the compact set $\prod_j \left[0, \bar{\lambda}_j\right]$ and is single-valued by uniqueness proved in Lemma 3 from the main paper, Ξ is upper hemicontinuous. A single-valued upper hemicontinuous correspondence is a continuous function.

Thus, we have shown that $\hat{v}_m^f(\hat{\lambda}(\mathbf{H}), H_m)$ and $\hat{\lambda}(\mathbf{H})$ are continuous. Hence, $\hat{p}_{jm}(\mathbf{H})$ is continuous, and thus $T_m(H)$ is also continuous. By the universal property of the product topology, $T(\mathbf{H})$ is continuous. By Brouwer's fixed-point theorem, T has a fixed-point on $[\underline{H}, \bar{H}]^{|\mathscr{M}|}$.

□

The standard closed graph theorem used to prove continuity requires a fixed compact codomain for $\hat{\lambda}(\mathbf{H})$, but the multiplier bounds constructed in Lemma 3 from the main paper depend on $\mathbf{H}$. However, the bounds on $\mathbf{H}$ are point-wise algebraic, not topological. In other words, building this bound does not require knowing what happens at nearby values of H (i.e. no continuity is needed). Therefore, we can construct an upper bound $\bar{\lambda}_j = \sup_{\mathbf{H} \in [\underline{H}, \bar{H}]^{|\mathscr{M}|}} \bar{\lambda}_j(\mathbf{H}) < \infty$ using the monotonicity established via the continuous extension. The bound $\bar{\lambda}_j(1)$ is a mathematical convenience for establishing uniform bounds, it needs not to correspond to an equilibrium of aggregators.

Next, the following lemma shows the partial effects of aggregators on firms' market share responses.

Lemma B.4. *For the aggregate market share $S_m = \sum_{j \in \mathscr{J}} s_{jm}$, $\frac{\partial S_m}{\partial H_m} \leq 0$, $\frac{\partial S_m}{\partial H_k} \geq 0$ for $m \neq k$, and $\frac{dS_m^{\mathscr{J}f}}{dH_k} \leq 0$ for all markets m for slack products. Equalities hold only if no product is binding.*

Proof. Pick an arbitrary firm $f \in \mathscr{F}$. The equilibrium markup satisfies $S_m^f = 1 - \frac{1}{\alpha \nu_m^f}$, Therefore:

$$\frac{dS_m^f}{dH_k} = \frac{1}{\alpha(\nu_m^f)^2} \frac{d\nu_m^f}{dH_k} \tag{4}$$

Consider the regime where all products in firm f are slack, all multipliers are fixed at $\lambda_j = 0$. Using the same implicit function from Lemma B.1, now treating H_m as variable (with λ fixed at zero in the all-slack regime), we compute

$$\frac{\partial F}{\partial H_m} = 1 - \frac{1}{\alpha \nu_m^f} = S_m^f > 0, \quad \frac{\partial F}{\partial \nu_m^f} = \frac{H_m}{\alpha \left(\nu_m^f\right)^2} + \alpha H_m S_m^f > 0$$

By the Implicit Function Theorem, $\frac{\partial \nu_m^f}{\partial H_m} = -\frac{\partial F / \partial H_m}{\partial F / \partial \nu_m^f} < 0$. For $k \neq m$: $\frac{\partial \nu_m^f}{\partial H_k} = 0$ because markets are decoupled in the all-slack regime. By equation (4), we show that $\frac{dS_m^f}{dH_m} < 0$ and $\frac{dS_m^f}{dH_k} = 0$ for $k \neq m$.

Now consider an arbitrary Binding Regime where some products in firm f are binding [9]. Let $\mathscr{B}^f \subset \mathscr{J}^f$ denote the set of binding products for firm f. The multipliers $\lambda_{\mathscr{B}^f} = (\lambda_i)_{i \in \mathscr{B}^f}$ are jointly defined by the capacity constraints:

$$\mathbf{G}(\lambda_{\mathscr{B}^f}; \mathbf{H}) := \left(\hat{\Gamma}_i(\lambda_{\mathscr{B}^f}; \mathbf{H}) - \kappa_i\right)_{i \in \mathscr{B}^f} = \mathbf{0}$$

where $\hat{\Gamma}_i(\lambda_{\mathscr{B}^f}; \mathbf{H}) = \sum_{m \in \mathscr{M}} L_m \hat{s}_{im}(\lambda_{\mathscr{B}^f}, \mathbf{H})$. Note that slack products within firm f have $\lambda_k = 0$ (fixed), so only binding multipliers appear as unknowns [10]. Since $\hat{s}_{im}$ is $\mathscr{C}^1$ in λ and $\mathbf{H}$, and the Jacobian

$$\mathbf{J} := \frac{\partial \mathbf{G}}{\partial \lambda_{\mathscr{B}^f}} = \left(\frac{\partial \hat{\Gamma}_i}{\partial \lambda_l}\right)_{i,l \in \mathscr{B}^f}$$

Notice that $-\mathbf{J}$ is a strictly diagonally dominant Z-matrix (Lemma B.2) and hence invertible, we can apply the Implicit Function Theorem:

$$\frac{\partial \lambda_{\mathscr{B}^f}}{\partial H_k} = -\mathbf{J}^{-1} \frac{\partial \mathbf{G}}{\partial H_k} = [-\mathbf{J}]^{-1} \frac{\partial \mathbf{G}}{\partial H_k}$$

By Theorem 6.2.3 of Berman and Plemmons (1994), $-\mathbf{J}$ satisfies condition I_{28}, so it is a non-singular M-matrix. In addition, a matrix with all non-zero entries corresponds to a complete directed graph, which is inherently strongly connected. Therefore, $-\mathbf{J}$ is irreducible by Graph

[9] We do not consider the non-differentiable created by the change of regimes, we leave that to the next Lemma. Now we just consider the effects within a regime.

[10] Notice that $\hat{s}_{km} = \hat{s}_{km}(\lambda_k = 0, \nu_m^f(\lambda_{\mathscr{B}^f}, \mathbf{H}), H_m)$ where $\lambda_k = 0$ fixed. Once we know $\lambda_{\mathscr{B}^f}(\mathbf{H})$ from the binding system, slack products' shares are automatically determined, and there is no equations to solve for the slack products.

Theory, and a non-singular irreducible M-matrix has property $[-\mathbf{J}]^{-1} \gg 0$ by Theorem 6.2.7 of Berman and Plemmons (1994). Also, holding λ fixed, we can use the same derivation as the slack case by treating λ as a fixed constant instead of zero, and derive:

$$\frac{\partial G_i}{\partial H_k} = L_k \frac{\partial \hat{s}_{ik}}{\partial H_k} = L_k \hat{s}_{ik} \left(-\alpha \left. \frac{dv_k^f}{dH_k} \right|_{\lambda \text{ fixed}} - \frac{1}{H_k} \right),$$

because H_m only affects another market indirectly through λ, recall that v_m^f is market-specific. We must determine the sign of the bracketed expression. When holding λ fixed, using an analogy to the all slack region, we have:

$$\left. \frac{\partial v_k^f}{\partial H_k} \right|_{\lambda \text{ fixed}} = -\frac{\partial F / \partial H_k}{\partial F / \partial v_k^f} = -\frac{S_k^f}{\frac{H_k}{\alpha \left(v_k^f \right)^2} + \alpha H_k S_k^f}.$$

We want to show that

$$-\alpha \left. \frac{\partial v_k^f}{\partial H_k} \right|_{\lambda \text{ fixed}} = \frac{\alpha S_k^f}{\frac{H_k}{\alpha \left(v_k^f \right)^2} + \alpha H_k S_k^f} < \frac{1}{H_k}$$

After cross-multiplying, this can be reduce to a trivially true inequality $0 < \frac{H_k}{\alpha \left(v_k^f \right)^2}$. Therefore:

$$\frac{\partial \lambda_{\mathscr{B}^f}}{\partial H_k} = (-\mathbf{J})^{-1} \frac{\partial \mathbf{G}}{\partial H_k} < \mathbf{0}.$$

From Lemma B.1:

$$\frac{\partial v_m^f}{\partial \lambda_i} = \frac{-\hat{s}_{im}}{1 - S_m^f + \left(S_m^f \right)^2} < 0 \quad \forall i \in \mathscr{B}^f$$

The total derivative is:

$$\frac{dv_m^f}{dH_k} = \underbrace{\left. \frac{\partial v_m^f}{\partial H_k} \right|_{\lambda \text{ fixed}}}_{=0} + \sum_{i \in \mathscr{B}^f} \frac{\partial v_m^f}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial H_k} = \sum_{i \in \mathscr{B}^f} \underbrace{\frac{\partial v_m^f}{\partial \lambda_i}}_{<0} \cdot \underbrace{\frac{\partial \lambda_i}{\partial H_k}}_{<0} > 0,$$

where $m \neq k$. Hence by equation (4), $\frac{dS_m^f}{dH_k} > 0$ for $m \neq k$.

Next, we decompose the products into $S_m^f = S_m^{\mathscr{B}f} + S_m^{\mathscr{S}f}$ under the same regime. For any slack product $j \in \mathscr{S}^f$ (with $\lambda_j = 0$ fixed), we have

$$\frac{ds_{jm}}{dH_k} = -\alpha s_{jm} \frac{dv_m^f}{dH_k} < 0,$$

and thus $\frac{dS_m^{\mathscr{S}f}}{dH_k} < 0$ for $m \neq k$ by summing over all $j \in \mathscr{S}^f$. Hence

$$\frac{dS_m^{\mathscr{B}f}}{dH_k} = \underbrace{\frac{dS_m^f}{dH_k}}_{>0} - \underbrace{\frac{dS_m^{\mathscr{S}f}}{dH_k}}_{<0} > 0.$$

But for binding products, the total weighted sales are fixed (i.e. $\sum_{m \in \mathscr{M}} L_m S_m^{\mathscr{B}f} = \sum_{j \in \mathscr{B}^f} \kappa_j$), which means:

$$\frac{dS_k^{\mathscr{B}f}}{dH_k} = -\frac{1}{L_k} \sum_{m \neq k} L_m \frac{dS_m^{\mathscr{B}f}}{dH_k} < 0.$$

Combining this with Equation (4) and $S_m^f = S_m^{\mathscr{B}f} + S_m^{\mathscr{S}f}$, we have:

$$\frac{1}{\alpha \left(v_k^f\right)^2} \frac{dv_k^f}{dH_k} = \frac{dS_k^{\mathscr{B}f}}{dH_k} - \alpha S_k^{\mathscr{S}f} \frac{dv_k^f}{dH_k} - \frac{S_k^{\mathscr{S}f}}{H_k}$$

which implies:

$$\frac{dv_k^f}{dH_k} \left(\frac{1}{\alpha \left(v_k^f\right)^2} + \alpha S_k^{\mathscr{S}f} \right) = \underbrace{\frac{dS_k^{\mathscr{B}f}}{dH_k}}_{<0} - \underbrace{\frac{S_k^{\mathscr{S}f}}{H_k}}_{>0} < 0$$

Therefore, $\frac{dv_k^f}{dH_k} < 0$ and hence $\frac{dS_k^f}{dH_k} = \frac{1}{\alpha \left(v_k^f\right)^2} \frac{dv_k^f}{dH_k} < 0$.

Then notice that, for slack products in market k, we can compute:

$$\frac{dS_k^{\mathscr{S}f}}{dH_k} = \sum_{j \in \mathscr{S}^f} s_{jk} \left(-\alpha \frac{dv_k^f}{dH_k} - \frac{1}{H_k} \right) = S_k^{\mathscr{S}f} \left(\underbrace{-\alpha \frac{dv_k^f}{dH_k}}_{\geq 0} - \underbrace{\frac{1}{H_k}}_{\geq 0} \right).$$

To determine the sign, notice that the total derivative is:

$$0 \geq \frac{d\nu_k^f}{dH_k} = \left.\frac{\partial \nu_k^f}{\partial H_k}\right|_\lambda + \underbrace{\sum_{i \in \mathscr{B}^f} \frac{\partial \nu_k^f}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial H_k}}_{\geq 0} \geq \left.\frac{\partial \nu_k^f}{\partial H_k}\right|_\lambda$$

Hence we have:

$$-\alpha \frac{d\nu_k^f}{dH_k} \leq -\alpha \left.\frac{\partial \nu_k^f}{\partial H_k}\right|_\lambda < \frac{1}{H_k}$$

Therefore,

$$\frac{dS_k^{\mathscr{S}^f}}{dH_k} = S_k^{\mathscr{S}^f} \left(-\alpha \frac{d\nu_k^f}{dH_k} - \frac{1}{H_k} \right) < 0.$$

Summing over all firms $f \in \mathscr{F}$ completes the proof.

□

Theorem (Theorem 2 from the main paper)**.** $\mathbf{H}^*$ *is unique.*

Proof. Recall we have $\mathbf{H} = 1 + \sum_{j \in \mathscr{J}} \left(\mathbf{H} \cdot s_j(\mathbf{H}) \right)$, which is

$$\underbrace{\sum_{j \in \mathscr{J}} s_j(\mathbf{H})}_{\text{Actual Total Inside Share}} = \underbrace{1 - \mathbf{H}^{-1}}_{\text{Required Inside Share}}.$$

We define the excess demand function for each market m as:

$$F_m(\mathbf{H}) = \left(1 - H_m^{-1}\right) - S_m(\mathbf{H})$$

where $S_m(\mathbf{H})$ is the total aggregated market share supply. The function $\mathbf{F}$ is not continuously differentiable everywhere due to the regime-switching of the capacity constraints (binding vs. slack). However, $\mathbf{F}$ is locally Lipschitz continuous: within each regime, $\mathbf{F}$ is C^1 by the Implicit Function Theorem (Lemmas B.1 and B.4), with derivatives bounded uniformly on the compact domain $[\underline{H}, \bar{H}]^{|\mathscr{M}|}$. We now establish global Lipschitz continuity. Note that we have finitely many regimes (i.e. $\mathscr{D} = \bigcup_{i=1}^N \mathscr{D}_i$ where $N \leq 2^{|\mathscr{J}|}$ and each $\mathscr{D}_i$ is the domain of a regime). For any $\mathbf{H}, \mathbf{H}'$, the line segment $[\mathbf{H}, \mathbf{H}']$ crosses finitely many regime boundaries [11]. Let $\mathbf{H} = z_0, z_1, \ldots, z_k = \mathbf{H}'$ denote the

[11] Since all functions involved are compositions of exponentials and arithmetic operations, and the Analytic Implicit Function Theorem preserves analyticity, $\hat{\Gamma}_j$ is piecewise real analytic. Zeros of analytic functions are isolated (unless it is identically zero, which is clearly not the case here), and the domain is compact, so the segment crosses each boundary finitely many times.

points where the segment $[\mathbf{H}, \mathbf{H}']$ crosses regime boundaries, ordered along the segment.[12] Since consecutive points z_i and z_{i+1} lie in the closure of a single regime, we can apply the MVT:

$$\|F(z_i + \varepsilon) - F(z_{i+1} - \varepsilon)\| \leq M\|(z_i + \varepsilon) - (z_{i+1} - \varepsilon)\|$$

where $M := \sup_{\mathbf{H} \in \Omega} \|J_F(\mathbf{H})\| < \infty$, with Ω denoting the set of points where J_F exists (regime interiors). Since there are finitely many regimes and derivatives are bounded within each regime on the compact domain, $M < \infty$. Taking $\varepsilon \to 0$ and using continuity of $\mathbf{F}$ (Lemma B.3):

$$\|F(z_i) - F(z_{i+1})\| \leq M\|z_i - z_{i+1}\|$$

By the triangle inequality:

$$\|F(\mathbf{H}) - F(\mathbf{H}')\| \leq \sum_{i=0}^{k-1} \|F(z_i) - F(z_{i+1})\| \leq M\|\mathbf{H} - \mathbf{H}'\|$$

Hence, $\mathbf{F}$ is globally Lipschitz continuous.

We next consider the properties of the standard Jacobian, and then extend them to the generalized Jacobian. Notice that the diagonal elements are:

$$J_{mm} = \frac{1}{H_m^2} - \frac{\partial S_m}{\partial H_m} \geq \frac{1}{H_m^2} > 0$$

where the inequality uses Lemma B.4. The off-diagonal elements are:

$$J_{mk} = -\frac{\partial S_m}{\partial H_k} \leq 0$$

by Lemma B.4, so $\mathbf{J}$ is a Z-matrix.

For the column sum condition, let $L = (L_1, \ldots, L_{|\mathscr{M}|})^\top \gg 0$:

$$L_k J_{kk} + \sum_{m \neq k} L_m J_{mk} = \frac{L_k}{H_k^2} - \sum_{m \in \mathscr{M}} L_m \frac{\partial S_m}{\partial H_k}$$

For binding products, $\sum_m L_m \frac{\partial s_{jm}}{\partial H_k} = 0$ since total capacity is fixed. For slack products, $\sum_m L_m \frac{\partial S_m^{\mathscr{S}^f}}{\partial H_k} \leq 0$ by Lemma B.4. Hence $\mathbf{J}^\top L \gg 0$. Notice that at non-differentiable points (regime boundaries), limit Jacobians are obtained by taking one-sided limits. Since these inequalities are non-strict in

[12] If not, the line lies in the interior of a regime, or lies along a smooth submanifold $\mathscr{M} := \bigcap_{j \in \mathscr{J}} \mathscr{B}_j$ where $\mathscr{B}_j := \{\mathbf{H} : \hat{\Gamma}_j(0; \lambda_{-j}(\mathbf{H}), \mathbf{H}) = \kappa_j\}$ is a regime boundary (a smooth hypersurface). Then $\mathbf{F}|_{\mathscr{M}}$ is $\mathscr{C}^1$ since the non-differentiability of $\mathbf{F}$ is only transverse to $\mathscr{M}$, and MVT applies directly.

the right direction (or strict with bounded margin), they are preserved under limits. Therefore, all limit Jacobians satisfy the condition (2) of the Univalence Lemma in the Appendix A of the main paper, thus $\mathbf{H}^*$ is unique.

□

C Comparative Statics and Merger Implications Proof

We allow the outside option to deliver positive indirect utility rather than being normalized to zero and consider its effects. We first show that the fixed-point solution $\mathbf{H}^*$ is a continuous function of $\mathbf{H}_0$, which represents the contribution of the outside option to the aggregator, equivalently, the exponential of the outside option's quality in our model.

Lemma (Lemma 4 from the main paper)**.** *The equilibrium mapping $\mathbf{H}^* : \mathscr{H}_0 \to \mathscr{D}$ is continuous and monotone: $\mathbf{H}_0' \geq \mathbf{H}_0 \Rightarrow \mathbf{H}^{*\prime} \geq \mathbf{H}^*$ (componentwise). When $\mathbf{H}_0$ increases, consumer surplus is weakly increasing, firms' profits are weakly decreasing, and the effect on total welfare is ambiguous.*

Proof. For any $\mathbf{H}_0 \in \mathscr{H}_0 := [1, \bar{H}_0]^{|\mathscr{M}|}$, the equilibrium $\mathbf{H}^*(\mathbf{H}_0)$ lies in a compact set $\mathscr{D} := [1, \bar{H}]^{|\mathscr{M}|}$. By Lemma B.3, $\mathbf{F}(\mathbf{H}; \mathbf{H}_0)$ is jointly continuous. Let $\mathbf{H}_0^n \to \mathbf{H}_0^0$. Since $\{\mathbf{H}^*(\mathbf{H}_0^n)\} \subset \mathscr{D}$ is compact, any convergent subsequence $\mathbf{H}^*(\mathbf{H}_0^{n_k}) \to \overline{\mathbf{H}}$ satisfies:

$$\mathbf{F}(\overline{\mathbf{H}}; \mathbf{H}_0^0) = \lim_{k \to \infty} \mathbf{F}(\mathbf{H}^*(\mathbf{H}_0^{n_k}); \mathbf{H}_0^{n_k}) = 0$$

By uniqueness (Theorem 2 from the main paper), $\overline{\mathbf{H}} = \mathbf{H}^*(\mathbf{H}_0^0)$. Hence $\mathbf{H}^*(\mathbf{H}_0^n) \to \mathbf{H}^*(\mathbf{H}_0^0)$.

Let $\mathbf{H}_0' \geq \mathbf{H}_0$ and denote $\mathbf{H}^* := \mathbf{H}^*(\mathbf{H}_0)$, $\mathbf{H}^{*\prime} := \mathbf{H}^*(\mathbf{H}_0')$. Then by definition of F_m:

$$F_m(\mathbf{H}^{*\prime}; \mathbf{H}_0) = \frac{H_{0m}' - H_{0m}}{H_m^{*\prime}} \geq 0$$

By the proof in Theorem 2 from the main paper, $\mathbf{F}(\cdot; \mathbf{H}_0)$ is Lipschitz in $\mathbf{H}$. By the Mean Value Theorem (Theorem 2.1, Pourciau (1983)), there exists $\overline{\mathbf{J}} \in \partial \mathbf{F}([\mathbf{H}^*, \mathbf{H}^{*\prime}])$ such that:

$$\mathbf{F}(\mathbf{H}^{*\prime}; \mathbf{H}_0) - \underbrace{\mathbf{F}(\mathbf{H}^*; \mathbf{H}_0)}_{=0} = \overline{\mathbf{J}}(\mathbf{H}^{*\prime} - \mathbf{H}^*)$$

By Theorem 2 from the main paper, the Jacobian is a Z-matrix with $\mathbf{J}^\top \mathbf{L} \gg 0$. Therefore, any Clarke Generalized Jacobian $\overline{\mathbf{J}}$ has this property which is preserved under convex hulls, as shown in the Univalence Lemma in the Appendix A of the main paper. Hence, $\overline{\mathbf{J}}$ is a nonsingular M-matrix with $\overline{\mathbf{J}}^{-1} \geq 0$ by Theorem 6.2.3 of Berman and Plemmons (1994). Hence:

$$\mathbf{H}^{*\prime} - \mathbf{H}^* = \overline{\mathbf{J}}^{-1} \mathbf{F}(\mathbf{H}^{*\prime}; \mathbf{H}_0) \geq \mathbf{0}$$

Finally, consumer surplus in market m is given by:

$$CS_m = \frac{L_m}{\alpha} \log\left(H_m^*\right)$$

Since $\log(\cdot)$ is strictly increasing and H_m^* is non-decreasing in $\mathbf{H}_0$, CS_m is non-decreasing in each H_{0k}.

In all-slack regimes, markets are decoupled: $J_{mk} = 0$ for $m \neq k$. In binding regimes, $J_{mk} < 0$ for $m \neq k$ (Lemma B.4). By the Fundamental Theorem of Calculus (FTC), we can show that if $J_{mk}(\gamma(t)) < 0$ on positive measure, then $\bar{\mathbf{J}}$ is a non-singular irreducible M-matrix with $\bar{\mathbf{J}}^{-1} > 0$ by Theorem 6.2.7 of Berman and Plemmons (1994). Then $H_m^{*\prime} > H_m^*$ if any $H_{0m}' > H_{0m}$.

By the envelope theorem, within each binding/slack regime:

$$\frac{\partial \Pi^{*f}}{\partial H_k} = \underbrace{\sum_{j,m} \frac{\partial \mathscr{L}^f}{\partial p_{jm}} \frac{\partial p_{jm}^*}{\partial H_k}}_{=0 \text{ (KKT stationarity)}} + \underbrace{\sum_j \frac{\partial \mathscr{L}^f}{\partial \lambda_j} \frac{\partial \lambda_j^*}{\partial H_k}}_{=0 \text{ (complementary slackness)}} + \frac{\partial \mathscr{L}^f}{\partial H_k} = -\frac{L_k \nu_k^f S_k^f}{H_k} < 0$$

Since $\Pi^{*f}(\mathbf{H})$ is Lipschitz, by Pourciau's MVT there exists $\bar{D} \in \partial \Pi^{*f}([\mathbf{H}, \mathbf{H}'])$ such that:

$$\Pi^{*f}(\mathbf{H}') - \Pi^{*f}(\mathbf{H}) = \bar{D} \cdot (\mathbf{H}' - \mathbf{H}),$$

where $\partial \Pi^{*f}([\mathbf{H}, \mathbf{H}'])$ is the Clarke generalized **gradient** over the segment $[\mathbf{H}, \mathbf{H}']$. Since $\bar{D}_k < 0$ for all k, and these inequalities hold at every limit gradients and are preserved in closed convex hulls[13], $\Pi^{*f}(\mathbf{H}') \leq \Pi^{*f}(\mathbf{H})$ whenever $\mathbf{H}' \geq \mathbf{H}$. Since $\mathbf{H}^*(\mathbf{H}_0)$ is non-decreasing in $\mathbf{H}_0$, we have $\Pi^{*f}(\mathbf{H}_0') \leq \Pi^{*f}(\mathbf{H}_0)$. The effect on total welfare is therefore ambiguous, as consumer surplus increases while firm profits decrease.

□

Next, we prove the lemma related to the comparative statics with respect to inside product quality changes.

Lemma (Lemma 5 from the main paper)**.** *Locally, a quality improvement δ_{jk} for a slack product raises H_m^* in all markets, benefiting consumers and hurting rivals, but the effect on the firm's own profit is ambiguous; a quality improvement for a binding product has ambiguous welfare effects.*

Proof. Continuity of $\mathbf{H}^*(\boldsymbol{\delta})$ follows by the same argument as for $\mathbf{H}_0$ in Lemma 4 from the main paper. Within each regime, $\mathbf{F}(\mathbf{H}; \boldsymbol{\delta})$ is $\mathscr{C}^1$; assuming this locally, from $\mathbf{F}(\mathbf{H}^*; \boldsymbol{\delta}) = \mathbf{0}$, the Implicit function theorem gives $\frac{d\mathbf{H}^*}{d\delta_{jk}} = \mathbf{J}^{-1}\mathbf{F}^S$ where $\mathbf{J} = \nabla_H \mathbf{F}$ and $F_m^S = \left.\frac{\partial S_m}{\partial \delta_{jk}}\right|_{\mathbf{H} \text{ fixed}}$. By Theorem 2 from

[13] The strict inequality holds uniformly and is preserved under convex combinations.

the main paper, $\mathbf{J}$ is a nonsingular M-matrix with $\mathbf{J}^{-1} \geq 0$ and $\mathbf{L}^T\mathbf{J} \gg \mathbf{0}^T$. Since firms $g \neq f$ are unaffected when $\mathbf{H}$ is fixed and $S_m^f = 1 - \frac{1}{\alpha \nu_m^f}$, we have:

$$F_m^S = \frac{1}{\alpha \left(\nu_m^f\right)^2} \frac{\partial \nu_m^f}{\partial \delta_{jk}}\bigg|_{\mathbf{H}}$$

We now consider the direct effect at fixed $\mathbf{H}$. When δ_{jk} changes, both ν_m^f and $\lambda_{\mathscr{B}^f}$ adjust. The total derivative is:

$$\frac{\partial \nu_m^f}{\partial \delta_{jk}}\bigg|_{\mathbf{H}} = \frac{\partial \nu_m^f}{\partial \delta_{jk}}\bigg|_{\lambda} + \sum_{i \in \mathscr{B}^f} \frac{\partial \nu_m^f}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial \delta_{jk}}$$

Further, at fixed $\mathbf{H}$ and λ, the market power markup ν_m^f is defined implicitly by:

$$F_m^{\nu}\left(\nu_m^f; \lambda, H_m, \delta\right) = H_m\left(1 - \frac{1}{\alpha \nu_m^f}\right) - \sum_{i \in \mathscr{J}^f} \exp\left(\delta_{im} - \alpha\left(c_{im} + \lambda_i + \nu_m^f\right)\right) = 0$$

For $m \neq k$, δ_{jk} does not appear, so $\frac{\partial \nu_m^f}{\partial \delta_{jk}}\Big|_{\lambda} = 0$. By the Implicit Function Theorem:

$$\frac{\partial \nu_k^f}{\partial \delta_{jk}}\bigg|_{\lambda} = \frac{s_{jk}}{\frac{1}{\alpha (\nu_k^f)^2} + \alpha S_k^f} > 0, \qquad \frac{\partial \nu_m^f}{\partial \delta_{jk}}\bigg|_{\lambda} = 0 \text{ for } m \neq k$$

First, we consider the case where j is slack. For binding $i \in \mathscr{B}^f$, the capacity constraint is $G_i = \sum_m L_m s_{im} - \kappa_i = 0$. Then at fixed λ:

$$\frac{\partial G_i}{\partial \delta_{jk}}\bigg|_{\lambda} = L_k \frac{\partial s_{ik}}{\partial \nu_k^f} \frac{\partial \nu_k^f}{\partial \delta_{jk}}\bigg|_{\lambda} = -\alpha L_k s_{ik} \frac{\partial \nu_k^f}{\partial \delta_{jk}}\bigg|_{\lambda} < 0$$

Higher quality raises ν_k^f, reducing s_{ik} and creating excess capacity. Since $\frac{\partial G_i}{\partial \lambda_i} < 0$ (Lemma B.2), restoring $G_i = 0$ requires $\frac{\partial \lambda_i}{\partial \delta_{jk}} < 0$. The total derivative:

$$\frac{\partial \nu_m^f}{\partial \delta_{jk}}\bigg|_{\mathbf{H}} = \underbrace{\frac{\partial \nu_m^f}{\partial \delta_{jk}}\bigg|_{\lambda}}_{\geq 0} + \sum_{i \in \mathscr{B}^f} \underbrace{\frac{\partial \nu_m^f}{\partial \lambda_i}}_{<0} \underbrace{\frac{\partial \lambda_i}{\partial \delta_{jk}}}_{<0} > 0$$

Hence $F_m^S > 0$ for all m. Since $\mathbf{J}^{-1} \geq 0$ with strictly positive diagonal (non-singular M-matrix), $\frac{dH_m^*}{d\delta_{jk}} = \sum_l \left[\mathbf{J}^{-1}\right]_{ml} F_l^S \geq [\mathbf{J}^{-1}]_{mm} F_m^S > 0$. Then the effect on consumer surplus is positive:

$$\frac{dCS_m}{d\delta_{jk}} = \frac{L_m}{\alpha H_m^*} \underbrace{\frac{dH_m^*}{d\delta_{jk}}}_{>0} > 0 \quad \forall m$$

Change in rivals' profit is negative in all markets by the envelope theorem (from $\mathbf{H}_0$ comparative statics):

$$\frac{d\Pi_m^g}{d\delta_{jk}} = -\frac{L_m v_m^g S_m^g}{H_m} \underbrace{\frac{dH_m^*}{d\delta_{jk}}}_{>0} < 0 \quad \forall m$$

But own firm's profit (hence social welfare) is ambiguous:

$$\frac{d\Pi^f}{d\delta_{jk}} = \underbrace{\left.\frac{\partial \Pi^f}{\partial \delta_{jk}}\right|_{\mathbf{H}}}_{>0} + \sum_m \underbrace{\frac{\partial \Pi^f}{\partial H_m}}_{<0} \underbrace{\frac{dH_m^*}{d\delta_{jk}}}_{>0}$$

Second, we consider the case where j is binding. At fixed λ:

$$\left.\frac{\partial G_j}{\partial \delta_{jk}}\right|_{\lambda} = L_k s_{jk} \left(1 - \alpha \left.\frac{\partial v_k^f}{\partial \delta_{jk}}\right|_{\lambda}\right) > 0,$$

where the inequality follows from $\alpha \left.\frac{\partial v_k^f}{\partial \delta_{jk}}\right|_{\lambda} < 1$ (since $s_{jk} \leq S_k^f$ and $\frac{1}{\alpha\left(v_k^f\right)^2} > 0$). By Lemma B.2, $\frac{\partial G_j}{\partial \lambda_j} < 0$. To restore $G_j = 0, \lambda_j$ must increase, so $\frac{\partial \lambda_j}{\partial \delta_{jk}} > 0$ due to the excess demand. This gives:

$$\left.\frac{\partial v_k^f}{\partial \delta_{jk}}\right|_{\mathbf{H}} = \underbrace{\left.\frac{\partial v_k^f}{\partial \delta_{jk}}\right|_{\lambda}}_{>0} + \underbrace{\frac{\partial v_k^f}{\partial \lambda_j} \frac{\partial \lambda_j}{\partial \delta_{jk}}}_{<0} + \sum_{i \in \mathscr{B}^f, i \neq j} \underbrace{\frac{\partial v_k^f}{\partial \lambda_i}}_{<0} \frac{\partial \lambda_i}{\partial \delta_{jk}}$$

Since $F_k^S = \frac{1}{\alpha (v_k^f)^2} \left.\frac{\partial v_k^f}{\partial \delta_{jk}}\right|_{\mathbf{H}}$ has ambiguous sign—the feedback through $\frac{\partial v_k^f}{\partial \lambda_j} < 0$ can offset the direct benefit—propagating this ambiguity through $\frac{d\mathbf{H}^*}{d\delta_{jk}} = \mathbf{J}^{-1}\mathbf{F}^S$ as in the slack case, all welfare effects (consumer surplus, rival profit, and own profit) inherit indeterminate sign.

□

Finally, we give the formal proof for the merger effects we discussed in the main text.

Proposition (Proposition 1 from the main paper)**.** *Consider the merger experiment in Example 2 of the main paper. The social optimal allocation is κ_1^S, which lies in the regime where products 1 and 2 are slack and product 3 is binding. The merger's profit-maximizing allocation κ_1^M lies in the regime where product 1 is binding, product 2 is slack, and product 3 is binding. Hence $\kappa_1^M \neq \kappa_1^S$.*

Proof. Throughout this proof, we use $S^M := \hat{s}_1 + \hat{s}_2$ for the merged firm's total market share and $S := S^M + \kappa_3$ for the total inside share. We write π_M for the merged firm's profit and π^I for the independent firm's profit.

Without loss of generality, we assume product 3 is binding for all values of κ_1 [14]. Then we consider two regimes [15]: product 1 binding but product 2 slack (Regime I), and both products slack (Regime II). Consider Regime I. Notice that, by the definition of H, we have $H = \frac{1}{1-\kappa_1-s_2-\kappa_3}$ in this regime, and κ_3 is fixed. Hence H is only a function of the quantities of product 1 and 2. Without loss of generality, let $c_1 = c_2 = c_3 = c$, and normalize market size to one.

For product 1, the best response satisfies $\hat{\Gamma}_1(\lambda_1; \lambda_2, \lambda_3) = \kappa_1$. Since $\hat{\Gamma}_1$ is strictly decreasing in λ_1 (by Lemma B.2: $\frac{\partial s_1}{\partial \lambda_1} < 0$), increasing κ_1 requires decreasing λ_1 to maintain the equality. Hence, $\frac{d\lambda_1}{d\kappa_1} = \frac{d\Gamma_1^{-1}(\kappa_1)}{d\kappa_1} < 0$.

Next, since $\lambda_2 = 0$ is constant and $\frac{d\lambda_1}{d\kappa_1} < 0$, by Lemma B.1:

$$\frac{d\hat{v}^M}{d\kappa_1} = \frac{\partial \hat{v}^M}{\partial \lambda_1}\frac{d\lambda_1}{d\kappa_1} = \underbrace{\frac{-s_1}{1 - S^M + (S^M)^2}}_{<0} \cdot \underbrace{\frac{d\lambda_1}{d\kappa_1}}_{<0} > 0$$

Moreover, from the binding constraint of product 1: $\exp(\delta_1 - \alpha\hat{p}_1) = \kappa_1 \cdot H$. Differentiating:

$$-\alpha\frac{d\hat{p}_1}{d\kappa_1} \cdot \exp(\delta_1 - \alpha\hat{p}_1) = H + \kappa_1\frac{dH}{d\kappa_1}$$

Since the RHS is positive (we will verify $\frac{dH}{d\kappa_1} > 0$ below), we have $\frac{d\hat{p}_1}{d\kappa_1} < 0$.

Furthermore, for product 2: $\frac{d\hat{p}_2}{d\kappa_1} = \frac{d\hat{v}^M}{d\kappa_1} > 0$.

Next, substitute $\lambda_2 = 0$ and taking logs to s_2, we have $\ln s_2 = \delta_2 - \alpha c - \alpha\hat{v}^M - \ln H$. Differentiating:

$$\frac{1}{s_2}\frac{ds_2}{d\kappa_1} = -\alpha\frac{d\hat{v}^M}{d\kappa_1} - \frac{1}{H}\frac{dH}{d\kappa_1} \tag{5}$$

[14] Note that we can easily relax this assumption by considering more regimes, which complicates the argument.

[15] Note that our goal is to show these effects are possible, rather than to prove that they hold generically. Accordingly, we do not need to claim these conclusions apply in every regime.

Now, $H = \frac{1}{1-S}$ implies $\ln H = -\ln(1-S)$, so we differentiate this and substituting back to (5) gives

$$\frac{ds_2}{d\kappa_1} = \frac{-\alpha \frac{d\hat{\nu}^M}{d\kappa_1} - H}{\frac{1+Hs_2}{s_2}} < 0$$

Next, notice that $\nu^I = \frac{1}{\alpha(1-\kappa_3)}$ is fixed. And from $\lambda_3 = \frac{\delta_3 - \ln(\kappa_3 H)}{\alpha} - c - \nu^I$:

$$\frac{d\lambda_3}{d\kappa_1} = -\frac{1}{\alpha H}\frac{dH}{d\kappa_1} < 0, \quad \frac{d\pi^I}{d\kappa_1} = \kappa_3 \frac{dp_3}{d\kappa_1} = \kappa_3 \frac{d\lambda_3}{d\kappa_1} < 0.$$

Next, we need to show $\frac{dH}{d\kappa_1} > 0$. Consider $\nu^M = \frac{1}{\alpha(1-\kappa_1-s_2)}$, and

$$\frac{d\nu^M}{d\kappa_1} = \frac{1}{\alpha}\frac{1}{(1-\kappa_1-s_2)^2}\left(1+\frac{ds_2}{d\kappa_1}\right) = \alpha\left(\nu^M\right)^2\left(1+\frac{ds_2}{d\kappa_1}\right)$$

Moreover, since $H = \frac{1}{1-\kappa_1-s_2-\kappa_3}$, we have:

$$\frac{dH}{d\kappa_1} = H^2\left(1+\frac{ds_2}{d\kappa_1}\right).$$

Substituting into (5),

$$\frac{1}{s_2}\frac{ds_2}{d\kappa_1} = -\left(\underbrace{(\alpha\nu^M)^2 + H}_{:=\Theta}\right)\left(1+\frac{ds_2}{d\kappa_1}\right),$$

which implies

$$\frac{ds_2}{d\kappa_1} = \frac{-\Theta s_2}{1+s_2\Theta} < 0.$$

Then

$$\frac{dS}{d\kappa_1} = 1 - \frac{\Theta s_2}{1+s_2\Theta} = \frac{1}{1+s_2\Theta} > 0, \quad \frac{dH}{d\kappa_1} = H^2 \cdot \frac{dS}{d\kappa_1} = \frac{H^2}{1+s_2\Theta} > 0, \quad \frac{d\nu^M}{d\kappa_1} = \frac{\alpha\left(\nu^M\right)^2}{1+s_2\Theta} > 0.$$

Next, since

$$\lambda_1 = \frac{\delta_1 - \ln(H) - \ln(\kappa_1)}{\alpha} - c_1 - \nu^M,$$

we have

$$\frac{d\lambda_1}{d\kappa_1} = -\frac{d\nu^M}{d\kappa_1} - \frac{1}{\alpha\kappa_1} - \frac{1}{\alpha H}\frac{dH}{d\kappa_1} = -\frac{1}{\alpha}\frac{\Theta}{1+s_2\Theta} - \frac{1}{\alpha\kappa_1}.$$

Now,

$$\frac{d\pi^M}{d\kappa_1} = \underbrace{\left(\nu^M + \lambda_1\right)}_{\text{margin on the new unit}} + \underbrace{\kappa_1\left(\frac{d\nu^M}{d\kappa_1} + \frac{d\lambda_1}{d\kappa_1}\right)}_{\text{change in margin on existing units}} + s_2\frac{d\nu^M}{d\kappa_1} + \nu^M\frac{ds_2}{d\kappa_1}$$

$$= \nu^M + \lambda_1 - \frac{1}{\alpha} - \frac{\kappa_1}{\alpha}\frac{H}{1+s_2\Theta} + \frac{-\nu^M s_2(\overbrace{\alpha\nu^M(\alpha\nu^M-1)+H}^{:=\beta})}{1+s_2\Theta}$$

$$= \lambda_1 + \frac{\nu^M - \frac{1}{\alpha} - \left(\frac{\kappa_1 H}{\alpha} + s_2\underbrace{(\nu^M\beta - \Theta(\nu^M - \frac{1}{\alpha}))}_{=H/\alpha}\right)}{1+s_2\Theta}$$

$$= \lambda_1 + \frac{\frac{1}{\alpha}\left(\alpha\nu^M - 1 - S^M H\right)}{1+s_2\Theta} = \lambda_1 - \frac{S^M}{1-S^M}\frac{\kappa_3 H}{\alpha\left(1+s_2\Theta\right)}$$

For the last equality, note that from $\nu^M = \frac{1}{\alpha(1-S^M)}$, we get $\alpha\nu^M - 1 = \frac{S^M}{1-S^M}$; and $H = \frac{1}{1-\kappa_3-S^M}$ and substitute. Clearly, $\lim_{\kappa_1\to 0}\frac{d\pi^M}{d\kappa_1} > 0$ as $\lim_{\kappa_1\to 0}\lambda_1 = \infty$; and $\left.\frac{d\pi^M}{d\kappa_1}\right|_{\lambda_1=0^+} < 0$, which explains why the merger picks a point outside of the flat zone.

Next, for the independent firm:

$$\pi^I = (p_3 - c_3)s_3 = \kappa_3(p_3 - c_3) = \kappa_3\left(\nu^I + \lambda_3\right)$$

where $\nu^I = \frac{1}{\alpha(1-s_3)} = \frac{1}{\alpha(1-\kappa_3)}$ is a constant since product 3 is always binding. From the binding constraint $s_3 = \kappa_3$, we can compute:

$$\lambda_3 = \frac{\delta_3 - \ln(\kappa_3 H)}{\alpha} - c_3 - \nu^I$$

and

$$\frac{d\lambda_3}{d\kappa_1} = -\frac{1}{\alpha H}\frac{dH}{d\kappa_1} = -\frac{H}{\alpha\left(1+s_2\Theta\right)} < 0$$

Hence

$$\frac{d\pi^I}{d\kappa_1} = -\frac{\kappa_3 H}{\alpha\left(1+s_2\Theta\right)} < 0$$

And we also have the identity:

$$\frac{d\pi^M}{d\kappa_1} = \underbrace{\lambda_1}_{\text{Direct gain}} + \underbrace{\frac{S^M}{1-S^M}\cdot\frac{d\pi^I}{d\kappa_1}}_{\text{Market expansion loss}}$$

Finally, for consumers, since $CS = \frac{1}{\alpha}\ln H$, we have $\frac{dCS}{d\kappa_1} = \frac{1}{\alpha H}\frac{dH}{d\kappa_1} > 0$ for all κ_1 in regime I (including the point where $\lambda = 0^+$). And $\left.\frac{dCS}{d\kappa_1}\right|_{\lambda_1=0^-} = 0$. So the optimal point for consumer is on Regime II.

□

Recall that λ_1 is the shadow cost of capacity for product 1, which represents the marginal profit from selling one more unit when capacity is binding. In other words, each additional unit of capacity directly earns profit λ_1, which is the "own benefit" from capacity expansion. However, when the merged firm expands capacity ($\kappa_1 \uparrow$), the market expands and becomes more competitive ($H \uparrow$), which erodes scarcity rents for all firms. The merged firm's loss from this market expansion is proportional to its market share: larger firms lose more because they have more sales exposed to price erosion. Therefore, the merged firm's incentive to expand capacity depends on whether the direct scarcity rent outweighs the market expansion loss. Larger merged firms face bigger market expansion losses, making them more inclined to restrict capacity — creating the asymmetry with the social planner.

The asymmetry from the proof arises from the fact that the independent firm is capacity-constrained. The merged firm strategically restricts its own capacity to keep H low, which erodes the competitor's scarcity rents. When the competitor (independent firm) is not capacity-constrained, this strategic channel disappears, and the merged firm's incentives align with the social planner's. When the independent firm is slack, its presence in the market creates a wedge between αv^M and H. The merged firm's actions still affect the competitive environment, but the independent firm can partially respond by adjusting quantity — reducing but not eliminating the strategic externality [16].

[16] This is difficult to prove theoretically because s_3 is now endogenous and creates interdependencies that require solving a larger system. But our numerical simulations confirm this.

D Extension Proof

We now prove the following theorem formally:

Theorem (Theorem 3 from the main paper)**.** *If the total cost function $TC_j : [0, \kappa_j) \to \mathbb{R}_+$ is $\mathscr{C}^2$ with strictly increasing marginal cost $c_j := TC_j'$, and $\lim_{\Gamma_j \to \kappa_j} c_j(\Gamma_j) = \infty$, then a unique Nash equilibrium exists in which all equilibrium quantities are interior, i.e., $\Gamma_j^* < \kappa_j$.*

Proof. Fix competitors' aggregator contribution:

$$H_m^{-f} = 1 + \sum_{k \notin \mathscr{J}^f} \exp(\delta_{km} - \alpha p_{km}) > 0$$

Firm f chooses prices $\left(p_{jm}\right)_{j \in \mathscr{J}^f, m \in \mathscr{M}}$. The share is:

$$s_{jm} = \frac{\exp\left(\delta_{jm} - \alpha p_{jm}\right)}{H_m^{-f} + \sum_{k \in \mathscr{J}^f} \exp(\delta_{km} - \alpha p_{km})}$$

So the firm's problem is:

$$\Pi_f = \sum_{j \in \mathscr{J}^f} \left[\sum_{m \in \mathscr{M}} L_m p_{jm} s_{jm} - TC_j\left(\Gamma_j\right) \right]$$

where the total quantity across all markets is $\Gamma_j = \sum_{m \in \mathscr{M}} L_m s_{jm}$. Recall that as in standard MNL model, we have $\frac{\partial s_{jm}}{\partial p_{jm}} = -\alpha s_{jm}\left(1 - s_{jm}\right)$ and $\frac{\partial s_{km}}{\partial p_{jm}} = \alpha s_{km} s_{jm}$ for $j \neq k$ within the same market m. And the cross-market derivative is $\frac{\partial s_{km'}}{\partial p_{jm}} = 0$, because the markets are linked through the cost functions in the equilibrium system, not through the individual FOC derivatives. Hence the FOC wrt prices p_{jm} is:

$$1 - \alpha\left(p_{jm} - c_j\left(\Gamma_j\right)\right) + \alpha \sum_{k \in \mathscr{J}^f} \left(p_{km} - c_k\left(\Gamma_k\right)\right) s_{km} = 0$$

This implies:

$$p_{jm} = c_j\left(\Gamma_j\right) + \nu_m^f$$

where the market power markup defined as:

$$\nu_m^f := \frac{1}{\alpha} + \sum_{k \in \mathscr{J}^f} \left(p_{km} - c_k\left(\Gamma_k\right)\right) s_{km}$$

First, we know the profit maximizing problem has a solution. Consider the price domain $P^f = \times_{j,m} \left[c_j(0), \infty\right]$, which is compact in the extended reals (one-point compactification) and π^f continuous, hence maximum exists by Weierstrass. For the upper bound, notice that FOC goes to $-\infty$ since the second term is bounded as market share goes to zero faster than prices go to infinity, hence $p_{jm} = \infty$ not optimal for any finite cost. If $c_j\left(\Gamma_j\right) = \infty$, then $\Gamma_j = \kappa_j$ so prices cannot be ∞, hence total profit is $-\infty$, and at any optimum the cost is finite. Now we show the lower boundary is not optimal. The same argument of Lemma A.1 applies with c_{jm} replaced by $c_j\left(\Gamma_j\right)$. Define $J_+^f := \left\{j : p_{jm} \geq c_j\left(\Gamma_j\right)\right\}$ and apply the same construction. For products in J_+^f, the construction raises prices to preserve their market shares in market m, so their total quantities and costs remain unchanged ($\bar{\Gamma}_k = \Gamma_k$). For withdrawn products $j \notin J_+^f$, their quantity decreases ($\bar{\Gamma}_j < \Gamma_j$), which reduces their cost ($c_j\left(\bar{\Gamma}_j\right) < c_j\left(\Gamma_j\right)$) and weakly improves profit in other markets. The profit improvement in market m follows as before. Putting it all together, the solution to firm's problem satisfies FOCs. We next show the sufficiency of the FOCs.

For the smooth non-linear cost method, the capacity constraint is embedded in the cost function via $c_j'\left(\Gamma_j\right) \to \infty$ as $\Gamma_j \to \kappa_j$. This unbounded derivative appears directly in the Jacobian, contaminating both diagonal and off-diagonal terms. In other words, the steep derivative $c_j'\left(\Gamma_j\right)$ near capacity amplifies small changes in Γ_j into large price changes. This amplification can overwhelm the logit's natural diagonal dominance.

However, notice that the competition happens in price space, not quantity space. Since cost function just maps prices and quantities, we can directly measure how much marginal cost has risen above baseline due to capacity pressure, defined as incremental marginal cost:

$$\lambda_j = c_j\left(\Gamma_j\right) - c_j(0)$$

In the kinked model, λ_j is literally the shadow cost of the capacity constraint. Then we have an analogy first order condition:

$$p_{jm} = \underbrace{c_j(0)}_{\text{baseline cost}} + \underbrace{\lambda_j}_{\text{incremental marginal cost}} + \underbrace{\nu_m^f}_{\text{market power markup}}$$

Notice that this is a simplification of our hard capacity constraint, because there is no binding and slack cases and no non-smoothness created by the kinks. The solution is simply a smooth equation for any given rivals' strategy H^{-f}:

$$G_j(\lambda^f; H^{-f}) := c_j^{-1}\left(\lambda_j + c_j(0)\right) - \hat{\Gamma}_j(\lambda^f; H^{-f}) = 0$$

since c_j is strictly increasing, there's a one-to-one mapping: $\Gamma_j = c_j^{-1}\left(\lambda_j + c_j(0)\right)$, and $\hat{\Gamma}_j(\lambda^f; H^{-f}) = \sum_m L_m \hat{s}_{jm}(\lambda^f; H_m^{-f})$ with shares evaluated at prices $p_{km} = c_k(0) + \lambda_k + \nu_m^f$.

Then notice that this is the same setup as our main proof, and by same argument as in Lemma 1 from the main paper and Lemma A.5, we can prove that $\frac{\partial s_{jm}}{\partial \lambda_k} = \alpha s_{jm} s_{km}\left(2 - S_m^f\right) > 0$ and $\frac{\partial s_{jm}}{\partial \lambda_j} = -\alpha s_{jm}\left[\left(1 - s_{jm}\right) - s_{jm}\left(1 - S_m^f\right)\right] < 0$, and the diagonal dominance condition can be simplifies to $(S_m^f - 1)^2 > 0$. Since there is an outside option, this always holds, and is completely irrelevant to the cost function.

Since $\hat{\Gamma}_j$ is the linear combination of s_{jm} across all m, the Jacobian:

$$(DG)_{jk} = \begin{cases} \dfrac{1}{c_j'\left(\Gamma_j\right)} - \dfrac{\partial \hat{\Gamma}_j}{\partial \lambda_j} > 0, & k = j, \\ -\dfrac{\partial \hat{\Gamma}_j}{\partial \lambda_k} < 0, & k \neq j. \end{cases}$$

And the diagonal dominance condition $\left|\frac{\partial \hat{\Gamma}_j}{\partial \lambda_j}\right| > \sum_{k \neq j} \frac{\partial \hat{\Gamma}_j}{\partial \lambda_k}$ holds.

Recall that we have proved the existence of a solution. It is worth noting that our specified non-linear cost function has self-correcting mechanism which guarantee that any solution is interior, since $G_j(\lambda^f) \to \kappa_j - 0 = \kappa_j > 0$ as $\lambda_j \to \infty$. Suppose two interior solutions λ^{f*} and λ^{f**} exist. Define the rectangular region:

$$R = \prod_{j \in \mathscr{J}^f} \left[\min\left(\lambda_j^*, \lambda_j^{**}\right), \max\left(\lambda_j^*, \lambda_j^{**}\right)\right]$$

Since diagonal dominance holds, by the same argument as before, the Jacobian is an non-singular M-matrix, and Gale-Nikaido gives uniqueness, hence contradiction. Hence, $\lambda^{f*} = \lambda^{f**}$.

Thus, we have proved that FOCs are the sufficient conditions for each firm's maximization problem. Unlike the kinked capacity constraint method, the smooth non-linear cost formulation makes the entire equilibrium system $G(\lambda) = 0$ smooth. Consequently, we can compute the global Jacobian DG across all products and all firms, verify diagonal dominance directly (which reduces to $S_m < 1$ from the outside option), and apply Gale-Nikaido to the entire system at once. In contrast, the kinked method has a non-smooth operator due to the complementary slackness conditions, which precludes the use of Gale-Nikaido and necessitates our extension to non-smooth operator using Pourciau (1983).

Next, we establish existence and uniqueness of Nash Equilibrium using a nested fixed-point argument. First, fix (λ, H_m), the markup ν_m^f is defined implicitly by:

$$F_m^\nu(\nu_m^f;\lambda,H_m) := H_m\left(1-\frac{1}{\alpha\nu_m^f}\right) - \sum_{k\in\mathscr{J}^f}\exp\Big(\delta_{km}-\alpha(c_k(0)+\lambda_k+\nu_m^f)\Big) = 0$$

We have:

$$\frac{\partial F_m^\nu}{\partial \nu_m^f} = \frac{H_m}{\alpha(\nu_m^f)^2} + \alpha H_m S_m^f > 0$$

As $\nu_m^f \to \frac{1}{\alpha}^+$: $F_m^\nu \to -H_m S_m^f < 0$. As $\nu_m^f \to \infty$: $F_m^\nu \to H_m > 0$. By IVT and strict monotonicity, there exists a unique $\hat{\nu}_m^f(\lambda, H_m) > \frac{1}{\alpha}$. By IFT, $\hat{\nu}_m^f$ is $\mathscr{C}^1$ in (λ, H_m), and

$$\frac{\partial \nu_m^f}{\partial \lambda_k} = -\frac{\partial F_m^\nu/\partial\lambda_k}{\partial F_m^\nu/\partial \nu_m^f} = -\frac{\alpha\hat{s}_{km}}{\frac{1}{\alpha\left(\nu_m^f\right)^2}+\alpha S_m^f} < 0$$

Second, fix $\mathbf{H}$, with $\nu_m^f = \hat{\nu}_m^f(\lambda, H_m)$ from Step 1, prices are:

$$p_{jm} = c_j(0) + \lambda_j + \hat{\nu}_m^{f(j)}(\lambda, H_m)$$

The equilibrium demand for product j is:

$$\hat{\Gamma}_j(\lambda;\mathbf{H}) := \sum_m L_m \hat{s}_{jm}(\lambda;H_m) = \sum_m L_m \frac{\exp\big(\delta_{jm}-\alpha p_{jm}\big)}{H_m}$$

The equilibrium condition from $c_j(\Gamma_j) = \lambda_j + c_j(0)$ is:

$$G_j(\lambda;\mathbf{H}) := c_j^{-1}(\lambda_j + c_j(0)) - \hat{\Gamma}_j(\lambda;\mathbf{H}) = 0 \quad \forall j \in \mathscr{J}$$

With fixed $\mathbf{H}$, products from different firms do not interact: $\frac{\partial \hat{s}_{jm}}{\partial \lambda_k} = 0$ for $k \notin \mathscr{J}^{f(j)}$. Hence the Jacobian $D_\lambda G$ is block diagonal by firm.

For products within firm f, define $A_m := \frac{1}{\alpha(\nu_m^f)^2} + \alpha S_m^f > 0$. Then:

$$\frac{\partial \hat{s}_{jm}}{\partial \lambda_j} = -\alpha\hat{s}_{jm}\frac{A_m - \alpha\hat{s}_{jm}}{A_m} < 0$$

$$\frac{\partial \hat{s}_{jm}}{\partial \lambda_k} = \frac{\alpha^2 \hat{s}_{jm}\hat{s}_{km}}{A_m} > 0 \quad \text{for } k \in \mathscr{J}^f \setminus \{j\}$$

The diagonal dominance condition requires

$$\alpha\hat{s}_{jm}\frac{A-\alpha\hat{s}_{jm}}{A} > \frac{\alpha^2\hat{s}_{jm}}{A}\left(S_m^f - \hat{s}_{jm}\right)$$

This reduces to $\frac{1}{\alpha(\nu_m^f)^2} > 0$, which always holds market-by-market. Since $\hat{\Gamma}_j = \sum_m L_m\hat{s}_{jm}$ is a positive linear combination, diagonal dominance holds for each firm's block. Since $c' > 0$, the Jacobian of G is:

$$(D_\lambda G)_{jk} = \begin{cases} \dfrac{1}{c_j'(c_j^{-1}(\lambda_j + c_j(0)))} - \dfrac{\partial\hat{\Gamma}_j}{\partial\lambda_j} > 0 & \text{if } k = j \\ -\dfrac{\partial\hat{\Gamma}_j}{\partial\lambda_k} < 0 & \text{if } k \in \mathscr{J}^{f(j)} \setminus \{j\} \\ 0 & \text{if } k \notin \mathscr{J}^{f(j)} \end{cases}$$

Define $T_j^\lambda(\lambda;\mathbf{H}) := c_j(\hat{\Gamma}_j(\lambda;\mathbf{H})) - c_j(0)$. As $\lambda_j \to \infty$, $\hat{\Gamma}_j \to 0$, so $T_j^\lambda \to 0$. Hence T^λ maps a compact set into itself. By Brouwer, a fixed-point exists. Suppose two solutions λ^* and λ^{**} exist. On the rectangular region $R = \prod_{j\in\mathscr{J}}[\min(\lambda_j^*, \lambda_j^{**}), \max(\lambda_j^*, \lambda_j^{**})]$, diagonal dominance implies $D_\lambda G$ is a non-singular M-matrix. By Gale-Nikaido, $\lambda^* = \lambda^{**}$. Thus, for any $\mathbf{H}$, there exists a unique $\lambda(\mathbf{H})$.

Finally, we consider the outer fixed-point problem in $\mathbf{H}$. Define $T : \prod_m [1, \bar{H}]^{|\mathscr{M}|} \to \mathbb{R}_{++}^{|\mathscr{M}|}$ by:

$$T_m(\mathbf{H}) := 1 + \sum_{j\in\mathscr{J}} \exp\left(\delta_{jm} - \alpha\hat{p}_{jm}(\mathbf{H})\right)$$

where $\hat{p}_{jm}(\mathbf{H}) = c_j(0) + \lambda_j(\mathbf{H}) + \hat{v}_m^{f(j)}(\lambda(\mathbf{H}), H_m) > 0$ and $1 < T_m(\mathbf{H}) \leq 1 + \sum_j \exp\left(\delta_{jm}\right) =: \bar{H}_m$. Since $D_\lambda G$ is non-singular, by IFT, $\lambda(\mathbf{H})$ is continuous. Hence T is continuous. By Brouwer, a fixed-point $\mathbf{H}^*$ exists.

Define the excess demand:

$$F_m(\mathbf{H}) := \left(1 - \frac{1}{H_m}\right) - S_m(\mathbf{H})$$

where $S_m(\mathbf{H}) = \sum_{j\in\mathscr{J}} s_{jm}(\lambda(\mathbf{H}), \mathbf{H})$. At a fixed-point, $\mathbf{H}^* = T(\mathbf{H}^*)$ implies $F(\mathbf{H}^*) = 0$. The Jacobian is:

$$J_{mk} := \frac{\partial F_m}{\partial H_k} = \frac{\mathbf{1}_{m=k}}{H_m^2} - \frac{\partial S_m}{\partial H_k}$$

By the Implicit Function Theorem applied to $G(\lambda;\mathbf{H})=0$:

$$\frac{\partial \lambda}{\partial H_k} = -\left(D_\lambda G\right)^{-1}\frac{\partial G}{\partial H_k}$$

For the partial derivative at fixed λ (so v_m^f is only a function of H_m not H_k for $m \neq k$):

$$\frac{\partial G_j}{\partial H_k} = -\left.\frac{\partial \hat{\Gamma}_j}{\partial H_k}\right|_\lambda = -L_k\left.\frac{\partial \hat{s}_{jk}}{\partial H_k}\right|_\lambda = -L\left(\alpha \hat{s}_{jk}\left.\frac{\partial v_k^f}{\partial H_k}\right|_\lambda - \frac{\hat{s}_{jk}}{H_k}\right)$$

By IFT, we have:

$$-\alpha\left.\frac{\partial v_k^f}{\partial H_k}\right|_\lambda = \frac{\alpha\left(1-\frac{1}{\alpha v_m^f}\right)}{\frac{H_k}{\alpha\left(v_k^f\right)^2}+\alpha H_k S_k^f} = \frac{\alpha S_k^f}{\frac{H_k}{\alpha\left(v_k^f\right)^2}+\alpha H_k S_k^f} < \frac{1}{H_k}$$

The last inequality always holds, because by cross-multiplying the inequility reduces to $0 < \frac{H_k}{\alpha\left(v_k^f\right)^2}$. Therefore,

$$\left.\frac{\partial \hat{s}_{jk}}{\partial H_k}\right|_\lambda = \hat{s}_{jk}\left(-\alpha\left.\frac{\partial v_k^f}{\partial H_k}\right|_\lambda - \frac{1}{H_k}\right) < 0$$

Hence $\frac{\partial G_j}{\partial H_k} > 0$. Again, by by Theorem 6.2.7 of Berman and Plemmons (1994), $D_\lambda G$ is a non-singular M-matrix, $\left(D_\lambda G\right)^{-1} \geq 0$ entrywise (with strict positivity within each firm's block). Therefore:

$$\frac{\partial \lambda_j}{\partial H_k} = -\sum_{l \in \mathscr{J}^{f(j)}}\left[\left(D_\lambda G\right)^{-1}\right]_{jl}\frac{\partial G_l}{\partial H_k} \leq 0$$

Since $\Gamma_j = c_j^{-1}\left(\lambda_j + c_j(0)\right)$ by definition, and c_j is strictly increasing:

$$\frac{\partial \Gamma_j}{\partial H_k} = \frac{1}{c_j'\left(c_j^{-1}\left(\lambda_j + c_j(0)\right)\right)}\frac{\partial \lambda_j}{\partial H_k} \leq 0$$

Next, notice that by the identity $S_m^f = 1 - \frac{1}{\alpha v_m^f}$:

$$\frac{\partial S_m^f}{\partial H_k} = \frac{1}{\alpha\left(v_m^f\right)^2}\frac{\partial v_m^f}{\partial H_k}$$

We consider own-market and cross-market effects separately. First, we prove the cross-market response result for $m' \neq m$. For $m' \neq m$, the markup in market m' has no direct dependence on H_m:

$$\frac{\partial \nu_{m'}^f}{\partial H_m} = \underbrace{\frac{\partial \nu_{m'}^f}{\partial H_m}\bigg|_{\lambda}}_{=0} + \sum_{l \in \mathscr{J}^f} \underbrace{\frac{\partial \nu_{m'}^f}{\partial \lambda_l}}_{<0} \cdot \underbrace{\frac{\partial \lambda_l}{\partial H_m}}_{\leq 0} \geq 0$$

And thus, $\frac{\partial S_{m'}^f}{\partial H_m} \geq 0$ for $m' \neq m$. Summing over firms: $\frac{\partial S_m}{\partial H_k} = \sum_f \frac{\partial S_m^f}{\partial H_k} \geq 0$, hence:

$$J_{mk} = -\frac{\partial S_m}{\partial H_k} \leq 0 \quad \text{for } m \neq k$$

Second, for the own-market, the total quantity response for firm f satisfies:

$$\sum_{m' \in \mathscr{M}} L_{m'} \frac{\partial S_{m'}^f}{\partial H_m} = \sum_{j \in \mathscr{J}^f} \frac{\partial \Gamma_j}{\partial H_m} < 0$$

Rearranging:

$$L_m \frac{\partial S_m^f}{\partial H_m} = \sum_{j \in \mathscr{J}^f} \frac{\partial \Gamma_j}{\partial H_m} - \sum_{m' \neq m} L_{m'} \underbrace{\frac{\partial S_{m'}^f}{\partial H_m}}_{\geq 0 (\text{ Case 1})} < 0$$

Therefore $\frac{\partial S_m^f}{\partial H_m} < 0$. Summing over firms:

$$J_{mm} = \frac{1}{H_m^2} - \sum_f \frac{\partial S_m^f}{\partial H_m} > \frac{1}{H_m^2} > 0$$

Thus, we proved J is a Z-matrix. Define the weight vector $L = \left(L_1, \ldots, L_{|\mathscr{M}|}\right)^\top \gg 0$. The weighted column sum is:

$$\sum_{m \in \mathscr{M}} L_m J_{mk} = \frac{L_k}{H_k^2} - \sum_{m \in \mathscr{M}} L_m \frac{\partial S_m}{\partial H_k} = \frac{L_k}{H_k^2} - \sum_{j \in \mathscr{J}} \frac{\partial \Gamma_j}{\partial H_k} \geq \frac{L_k}{H_k^2} > 0$$

By Theorem 6.2.3 of Berman and Plemmons (1994), J is a P-matrix.

Suppose two equilibria $\mathbf{H}^*$ and $\mathbf{H}^{**}$ exist with $F(\mathbf{H}^*) = F(\mathbf{H}^{**}) = 0$. Define the rectangular region:

$$R = \prod_{m \in \mathscr{M}} \left[\min\left(H_m^*, H_m^{**}\right), \max\left(H_m^*, H_m^{**}\right)\right]$$

Since J is a P-matrix on R, by the Gale-Nikaido univalence theorem, F is injective on R. Therefore $\mathbf{H}^* = \mathbf{H}^{**}$.

□

The special property of nonlinear marginal cost ($c_j \to \infty$ at finite κ_j) was only needed to ensure equilibrium is interior of some **hard capacity limit**. With general convex costs that go to infinity when $\Gamma \to \infty$, this is automatic since demand is bounded by the total market size:

$$T_j(\Gamma) = \sum_m L_m s_{jm} \leq \sum_m L_m := L < \infty$$

Therefore, Brouwer applies, and no equilibrium is on the boundary point ∞, and the uniqueness proof using Gale-Nikaido applies. The only sufficient condition is that $c_j\left(\Gamma_j\right)$ strictly monotone, so that we can apply the λ-transformation trick, and in our case it must strictly increasing to ∞. Otherwise, we do not have the uniqueness results, and multiple equilibria are economically real with dis-economies and economies of scale in different regimes.

Finally consider the case non-linear cost defined on $\mathbf{R}$ but with explicit capacity constraint $\Gamma_j \leq \kappa_j$, and this goes back to the benchmark kinked method but now with a non-linear marginal cost. As long as we maintain the additive separability of the cost function across products, the extension of our benchmark method is straightforward. [17] However, in practice it becomes very complicated due to the existence of cross-market effects in our more complicated linked multimarkets (shown below), one way to solve it could be using a similar trick to define $\lambda_j' = c_j\left(\Gamma_j\right) - c_j(0)$ as above, and together with a shadow cost λ_j as in benchmark method, so one will need an extra fixed-point loop added to the benchmark method. We are not aware of a formal, rigorous proof in the literature, and we do not provide one here.

[17] As Nocke and Schutz (2018) mentioned, relaxing the assumption of additive separability, thereby allowing for (dis-)economies of scope, would complicate the analysis as the marginal cost of a product would now depend on all of the firm's prices.

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