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Generic Measures of Distributional Peakedness, Locational Differences and Modally Focused Kurtosis and Relative Variation Coefficients: Tools for Mode-based Analyses.

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Abstract.

More often than not in summarizing the level of and variation in outcomes, the Mean or Median are the focus, however, as the most frequently observed outcome, there are good reasons for using the Mode in that role. Its pertinence for that purpose hinges on the extent to which it differs from the other centrality statistics and its prominence or "Peakedness" in the distributional density profile. Peakedness is usually calibrated using Pearsons Kurtosis quotient however, that measure has been shown to be more about the fatness of a distribution's tails rather than its Peakedness. Here, generic probabilistically based measures of the extent to which location points differ and the extent to which distributions are peaked, suitable for any discrete or continuous potentially multidimensional data environment are proposed together with modally focused analogues of Pearsons Relative Variation and Kurtosis measures, appropriate for distributions which are not symmetric unimodal. Applications in five very different distributional environments demonstrate the usefulness and general applicability of these new measures.

Key Words. Skewness, Kurtosis, Peakedness.

C1; D0; E0; G0; I0.

1. Introduction.

More often than not the mean is used as a summary statistic to represent a population's outcomes. In fixed effects models and treatment-control experiments differences in conditional means in terms of average outcomes are compared. In implicit or explicit representative agent models, the average outcome is assumed to represent the average agent. In evaluating both absolute and relative variation, the mean is used as the focal point. However, as the most commonly observed outcome, there are substantive theoretical and empirical arguments for using the mode or conditional mode as the focal point of analysis (Anderson 2026). Its most frequently observed outcome status renders it at least as representative as average or median central location measures and, in many cases, it is much more representative and can offer a very different perspective in a wide variety of non-symmetric unimodal situations. In treatment effect models differences in the most likely outcomes of the treated and control groups may be more pertinent than differences in their respective expected outcomes. In models concerned with wellbeing and inequality, since relative frequency is a measure of the extent of commonality at a point in a distribution, the mode provides a useful alternative measure of centrality from which to assess the extent of diversity in a population (Anderson and Yalonetzky 2025, Anderson 2026, 2026a). Indeed, for Utilitarian wellbeing analysts following the Benthamite dictate of "the greatest good for the greatest number" (Bentham 1781), the mode, as the point of greatest commonality or equity, should be the starting point of any inequality-based wellbeing analysis.

Clearly the symmetric unimodality of standard normality assumptions regularly invoked in applied work renders the mean, median and mode identical so that employing a modal focus only matters for analysis when it is substantively different from the other centrality parameters and is prominent in the distributions profile. Measures of the extent to which this is true, generic to all data paradigms, would be helpful in analysis. Answering the question of how different the mode is from another location measure in both ordered categorical and cardinally measurable environments is most simply resolved by taking a probabilistic approach and examining the likelihood that an outcome could emerge

between them. The modes prominence hinges upon how much the Probability Density Function (PDF) value at the mode exceeds that of other points in the PDF, which is a property of the functions "Peakedness". In essence it is about how "large" the value of PDF is at its maximum relative to PDF values at other points in the distributions range which captures how distinctive the modal point is. As such, Peakedness can be construed as a measure of the relative certainty or surety with which the modal outcome will be realized or as a measure of the lack of diversity or unequalness in the population.

Peakedness has often been confounded with Kurtosis. Westfall (2014) attributes the confusion to Pearson (1905) who categorized the degree of "flat-toppedness" of a distribution relative to the standard normal curve in terms of a kurtosis measure K. Given a continuous random variable X defined on the real line with typical value x, probability density function $f(x)$ and mean μ, K is given by:

$$K = \frac{\int (x-\mu)^4 f(x)dx}{(\int (x-\mu)^2 f(x)dx)^2}$$

As the ratio of the 4th moment about the mean to the squared 2nd moment about the mean, K became the conventional measure of Kurtosis. Pearson clearly had symmetric unimodality in mind, noting that $K = 3$ for a Standard Normal variate, he declared that distributions whose K was less than 3 were platykurtic, distributions whose K was more than 3 were leptokurtic and distributions whose K was equal to 3 were mesokurtic. Westfall (2014) exemplifies[1] the Peakedness as Kurtosis problem by citing three distributions with identical K values of 2.4, but very different Peakedness topologies, effectively demonstrating that a given Kurtosis value was neither a necessary nor a sufficient condition for the extent of Peakedness and appeared to have little to do with it. In effect K measures the relative "fatness" of a PDF's tails oriented around the location of the mean. Pearson was effectively categorizing distributions as thin tailed (less than 3), fat tailed (greater than 3) or Normal Tailed (equal to 3).

[1] Figures 2 and 3 in Westfall (2014) illustrate Triangular, Devils Tower and Slip Dress distributions with identical Kurtosis statistics of 2.4.

Pearson (1896) also probably had symmetric unimodality in mind when he proposed the coefficient of variation measure COV as a measure of relative variation where he specified COV as:

$$COV = \frac{\sqrt{\int (x-\mu)^2 f(x)dx}}{\mu}$$

COV has been used extensively in comparing relative variation in outcomes[2] and its inverse, the Sharpe ratio, has been employed in various forms (Sharpe 1994, Sortino 1994) to order risk modulated excess portfolio returns where the standard deviation from the mean has a measure of risk interpretation and μ is the average excess (above the safe rate) portfolio return.

While COV and K make sense as Relative Variation and Kurtosis measures when considering symmetric unimodal distributions, it is not clear that they capture the same phenomena in the context of asymmetric or multimodal distributions. Furthermore, the mean isn't always the location of a distributions peak, in skewed distributions the mode is, and in this situation COV and K statistics are measuring moments about a location point which is not the location of the distributions peak. Of course, one could contemplate Relative Variation and Kurtosis measures based upon moments about a mode θ viz:

$$COV = \frac{\sqrt{\int (x-\theta)^2 f(x)dx}}{\theta} \;\; \& \;\; K(\theta) = \frac{\int (x-\theta)^4 f(x)dx}{(\int (x-\theta)^2 f(x)dx)^2}$$

Note that when $f(x)$ is symmetric unimodal these become the more conventionally defined statistics. However, the fact remains that $K(\theta)$ is still a measure of the relative fatness of the tails, in this case oriented around the location of the mode, and it is still not necessarily a measure of the Peakedness of a distribution.

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[2] Pearson proposed COV in response to Galtons' practice of using the 13 to 12 male-female size ratio to obtain comparable male equivalents in his work on Natural Inheritance (Galton 1894) and used the mean focused variation measure standardized by the mean to address the comparative variation of organ sizes (usually skull and bone dimensions) across race and gender

So how should "Peakedness" be construed? Following Proschan (1965), let X and Y be real random variables with respective probability density functions $f_X(x)$ and $f_Y(y)$ defined on the same real line and let a and b be constants with $\partial f_X(a)/\partial a = 0, \partial^2 f_X(a)/\partial a^2 < 0$ and $\partial f_Y(b)/\partial b = 0, \partial^2 f_Y(b)/\partial b^2 < 0$ [3], then if:

$$P\{|X - a| \geq t\} \leq P\{|Y - b| \geq t\} \; \forall \; t \geq 0$$

Or alternatively [1]

$$P\{|X - a| < t\} > P\{|Y - b| < t\} \; \forall \; t > 0$$

X is said to be more peaked about a than is Y about b (when $a = b$, X is said to be more peaked than Y). In words, if X has more probability mass in a defined region surrounding a peak point than does Y in its correspondingly defined region, then X is said to be more peaked than Y around their respective points.

In the following, in order to validate the use of the mode, simple generic modally focused locational difference and Peakedness measures applicable in any discrete or continuous potentially multidimensional distributional setting will be developed in the spirit of [1] together with modally based analogues of Pearsons Coefficient of Variation and Kurtosis measures. To demonstrate their general applicability and practicality, these measures will be exemplified in five very different ordinal and cardinal empirical settings of health outcomes, portfolio returns, life expectancy analysis, income distribution analysis and national fertility rate time profiles.

2. Generic Probabilistic Distance Based Locational Difference and Peakedness Measures.

In measuring Kurtosis, Pearson's focus was on tail size relationships relative to those of a standard normal distribution which is symmetric unimodal with equality between mean and modal (and indeed median) locations. When working with distributions that are not symmetric unimodal, centrality locations can be different and their difference a matter of

[3] Proschan (1065) implicitly assumes rather than explicitly states this. X and Y should be defined on the same real line to ensure the commonality of the meaning and value of t and hence the region space around a and b where a and b are respective locational peaks.

consequence, so a generic measure of distance between centrality parameters appropriate for both cardinally measurable and ordered categorical data paradigms will be useful. Note the measure will be developed in terms of the mean and modal difference but it can be employed to measure the distance between any two points in the range of the distribution.

Generic Locational Distance Measures

In a continuous cardinal paradigm, let the random variable X with particular value x where $-\infty < W < x < Y < \infty$ have a smooth continuous probability density function $f(x)$ with a cumulative distribution function $F(x) = \int_W^x f(z)dz$, a mean $\mu = \int_W^Y xf(x)dx$ and a mode θ where $f(\theta) > f(x)\ \forall\ x \neq \theta$. When $f(x)$ is unimodal symmetric $\mu = \theta$ with $F(\mu) = F(\theta) = 0.5$, when $f(x)$ is unimodal left skewed $\mu < \theta$ with $F(\mu) < F(\theta)$ and when $f(x)$ is unimodal right skewed $\mu > \theta$ with $F(\mu) > F(\theta)$. The probabilistic distance between the mean and the mode is given by $PD(\mu, \theta)$ where:

$$PD(\mu, \theta) = F(\mu) - F(\theta) \qquad [2]$$

$PD(\mu, \theta)$ is such that $-1 < PD(\mu, \theta) < 1$, when $PD(\mu, \theta) = 0$ mean and mode are equal, when $PD(\mu, \theta) < 0, \theta >\ \mu$ and when $PD(\mu, \theta) > 0, \theta <\ \mu$. Its magnitude $|PD(\mu, \theta)|$ provides a measure of the probabilistic distance between μ and θ whose significance, given a random sample of size n drawn from $f(x)$, is straightforward to examine. Following Rao (2009), on the null hypothesis of $|PD(\mu, \theta)| = 0, \sqrt{n}\big(\widehat{|PD|} - |PD|\big) \sim_a N(0, \frac{|PD|(1-|PD|)}{n})$.

The measure is readily extended to ordered categorical paradigms. Suppose random variable X has I alternative outcomes ordered in terms of the index $i = 1, .., I$ which has a Probability Density Distribution $p_i\ for\ i = 1, .., I$ where $p_i \geq 0$ and $\sum_{i=1}^{I} p_i = 1$ with a cumulative density function $c_i, i = 1, .., I$ where $c_i = \sum_{j=1}^{i} p_j\ \ i = 1, .., I$ and a Survival Function $s_i = 1 - c_i\ i = 1, .., I$. Let $i = mo$ be the designated modal category and let $i = me$ be the designated mean category then supposing $me > mo$, intuitively $PD(\mu, \theta) = c_{me} - c_{mo} = \sum_{j=mo+1}^{me} p_j$. However, when mean and modal categories are next to each

other this would result in $PD(\mu,\theta) = p_{me}$ and if the probabilistic distance was determined by the survival functions $PD(\mu,\theta) = p_{mo}$ to avoid this contradiction

$$PD(\mu,\theta) = \sum_{j=mo}^{me} p_j - 0.5(p_{me} + p_{mo}) \qquad [2a]$$

In the bivariate case with a second dimension of H outcomes $h = 1,..,H$ yielding a Probability Density Distribution $p_{i,h}\ for\ i = 1,..,I\ and\ \ h = 1,..,H$ where $p_{i,h} \geq 0$ and $\sum_{i=1}^{I}\sum_{h=1}^{H} p_{i,h} = 1$, for location points (a,b) and (c,d) where $a \leq c\ and\ b \leq d$, with strict inequality somewhere, $PD(\mu,\theta) = \sum_{i=a+1}^{c-1}\sum_{h=b+1}^{d-1} p_{i,h} + 0.5\left(p_{a,b} + p_{c,d}\right)$. Inference follows Rao (2009) as before.

Generic Modal Peakedness Measures.

Suppose the random variable X has I alternative outcomes[4] indexed $i = 1,..,I$ which have a Probability Density Distribution $p_i\ for\ i = 1,..,I$ where $p_i \geq 0$ and $\sum_{i=1}^{I} p_i = 1$. $RL_{i,j} = \frac{p_i}{p_j}$, for $i,..,j\ \in [1,..,I]$ is the relative likelihood of outcome i as opposed to outcome j being realized. Denote the Modal or "Most Likely" outcome M where M is such that $p_M = \max_{i=1,..,I} p_i$ and define the i'th outcomes modally based relative likelihood o_i for $i = 1,..,I$ as:

$$o_i = {}^{p_i}/_{p_M}$$

o_i is the outcome i versus modal outcome likelihood ratio reporting the relative chance that outcome i occurs as opposed to the modal outcome occurring. Note $0 \leq o_i \leq 1\ \forall\ i = 1,.,I$ with $o_i = 0$ when outcome i does not occur and $o_i = 1$ when outcomes are uniformly distributed. Consider $A = \sum_{i=1}^{I}(1 - o_i)$ and note that $0\ \leq A \leq I - 1$ where $A = I - 1$ when $o_i = 1$ for some i and 0 elsewhere and $A = 0$ when outcomes are uniformly distributed $(o_i = 1\ \forall\ i)$. Then MP will be an index of the Modal Peakedness of the distribution in terms of the extent of density concentration at the mode where:

$$MP = \frac{1}{I-1}\sum_{i=1}^{I}(1 - o_i) \qquad [3]$$

[4] None of the following is materially altered when X is multidimensional or an ordered categorical variate.

Note $0 \leq MP \leq 1$ where $MP = 0$ when outcomes are uniformly distributed and no modal point exists and $MP = 1$ when $o_i = 1$ for some i i.e. there is complete concentration at the mode. In essence when $MP = 0$ there is no peak in the distribution and when $MP = 1$ the distribution is all peak.

MP measures the extent of commonality or equality in the distribution, rendering $1 - MP$ a measure of diversity or inequality when there is no order to the outcomes[5] (thinking of choice of alternative car colour and upholstery pairing options, it is a measure of how diverse the demand for those pairing options is). MP can also be construed as a measure of the certainty with which the Modal outcome is realized, when it is 1, no other outcome can be realized (everyone chooses the same colour and upholstery pairing), when it is 0 all colour-upholstery outcome pairs have the same chance $(1/I)$ of being realized and the modal most popular outcome pairing cannot be determined.

Note that MP is readily defined in multivariate environments and has a continuous variable analogue. For PDF $f(x)$ defined on $-\infty < x < \infty$ where M is such that $f(M) = \max_x f(x)$, let $o(x) = f(x)/f(M)$ and consider MP where:

$$MP = \int_{-\infty}^{\infty} f(x)\,(1 - o(x))dx \qquad [4]$$

Note, $0 \leq \text{MP} \leq 1$ with $MP = 0$ when outcomes are uniformly distributed (i.e. $o(x) = 1\ \forall\ x$) and $MP = 1$ when $o(x) = 0\ \forall\ x \neq M$ i.e. there is complete concentration at the mode. In effect MP is a measure of the surety with which the modal value will be realized.

Generic Modally Focused Coefficient of Variation and Kurtosis Measures.

Modally focused theoretical analogues of COV and K in the continuous paradigm have already been outlined above. Suppose $x_i\ i = 1, .., n$ is a random sample of size n drawn from the population and m is determined to be the modal value of X, then their empirical counterparts would be:

$$COV = \frac{\sqrt{\sum_{i=1}^{n}(x_i - m)^2}}{m} \quad \& \quad K = \frac{\sum_{i=1}^{n}(x_i - m)^4}{(\sum_{i=1}^{n}(x_i - m)^2)^2}$$

[5] Inequality measurement changes when the categories are ordered since distance from the point of most commonality matters Anderson and Yalonetzky 2025.

Suppose the discrete random variable X defined above is an Ordered Categorical variate with I outcomes x_i indexed $i = 1,..,I$ with respective probabilities p_i and cumulative density $C_h = \sum_{i=1}^{h} p_i \;\; h = 1,..,I$. In the past, to facilitate analysis, artificial scaling was attributed to ordinal categories (Cantril 1965) but this has met with criticism (Bond and Lang 2019) because scale dependency of estimators can lead to ambiguity and equivocation (equally valid alternative scaling can lead to contradictory results and conclusions) so here resort is made to probabilistic distance techniques (Mendelson 1985). Define $d_{h,j}$ the probabilistic distance between x_h and x_j for $h < j, \;\; h, j \in 1{:}\, I$ as:

$$d_{h,j} = \sum_{i=h}^{j} p_i - 0.5(p_h + p_j)$$

And let x_M be the modal outcome where $M \in 1,..,I$ and $p_M \geq p_i \forall i \in 1,..,I$ then coefficient of variation and Kurtosis analogues can be written as:

$$COV = \frac{\sum_{i=1}^{M} d_{i,M}^2 + \sum_{i=M}^{I} d_{M,i}^2}{C_h} \quad \& \quad K = \frac{\sum_{i=1}^{M} d_{i,M}^4 + \sum_{i=M}^{I} d_{M,i}^4}{\left(\sum_{i=1}^{M} d_{i,M}^2 + \sum_{i=M}^{I} d_{M,i}^2\right)^2}$$

3. Five Diverse Empirical Applications.

Initially, to illustrate the notion that the Modal Peakedness Statistic is capturing something beyond Kurtosis, [3] was calculated based upon random samples of 800 drawn from the identically kurtotic Triangular, Devils Tower and Slip Dress distributions examined in Westfall (2014). The respective Modal Peakedness Statistics (Standard Errors) of 0.3349 (0.0051), 0.1744 (0.0056) and 0.3664 (0.0110) for the three distributions are all mutually significantly different from each other and clearly identify differences in their respective Peakedness not captured by the conventional Kurtosis measure which, by construction, is common to all. For comparison purposes, based upon the fact that the modal value of the PDF of a Standard Normal (N(0,1)) random variable is $1/\sqrt{2\pi}$, a random sample of 10000 standard normal variates yielded an MP of 0.2936 with a standard error of 0.0028. So, the Triangular and Slip Dress distributions would be considered significantly more Peaked than N(0,1) whereas the Devils Tower would be considered significantly less peaked. The results also confirm the visual impression that the Slip Dress distribution is more peaked than Triangular distribution which is in turn more peaked than Devils Tower distribution.

Here five illustrative applications are reported, a multivariate ordered categorical analysis of self-reported health and age group distributions in the UK for the years 2010 and 2018, a study of portfolio returns distributions from the Fama and French Data base, an examination of Canadian Age At Death distributions for 2017 and 2022, a study of the progress of income distributions in the UK over the period 2010 to 2018, (all drawn from Anderson 2026) and a study of Fertility Rate trends of concern in Sustainable Population Studies (drawn Anderson 2026a).

3.1 Health and Age Analysis.

The location and prominence of the most frequently observed health outcomes is of intrinsic interest and obviously important for health policy purposes. Tables 1 and 2 report the details of the of the age group – health outcome joint probability density distributions and their corresponding marginal distributions.

Table 1. 2010 Age Group - Health Outcome Joint and Marginal Distributions Overall.

Age Group	Age Group Marginals	Dissatisfied				Satisfied		
		Completely	Mostly	Somewhat	Neither	Somewhat	Mostly	Completely
>65	0.1794	0.0095	0.0148	0.0271	0.0177	0.0253	0.0671	0.0178
55-64	0.1530	0.0081	0.0127	0.0227	0.0119	0.0200	0.0638	0.0137
45-54	0.1712	0.0094	0.0129	0.0263	0.0135	0.0227	0.0699	0.0165
35-44	0.1931	0.0077	0.0128	0.0267	0.0154	0.0293	0.0793	0.0219
25-34	0.1554	0.0043	0.0087	0.0199	0.0115	0.0238	0.0654	0.0217
<25	0.1478	0.0036	0.0064	0.0159	0.0133	0.0251	0.0584	0.0251
Health PDF		0.0427	0.0685	0.1387	0.0834	0.1462	0.4040	0.1167

Modal Peakedness for the Joint Distribution 0.7169 (Mostly Satisfied 35-44 year olds), Modal Peakedness for Health 0.7540 (Mostly Satisfied), Modal Peakedness for AGE 0.1644 (35-44 year-olds)

Table 2. 2018 Age Group - Health Outcome Joint and Marginal Distributions Overall.

Age Group	Age Group Marginals	Dissatisfied				Satisfied		
		Completely	Mostly	Somewhat	Neither	Somewhat	Mostly	Completely
>65	0.2271	0.0093	0.0175	0.0320	0.0232	0.0353	0.0912	0.0185
55-64	0.1699	0.0094	0.0161	0.0256	0.0161	0.0244	0.0657	0.0126
45-54	0.1876	0.0075	0.0168	0.0289	0.0185	0.0288	0.0723	0.0146
35-44	0.1609	0.0050	0.0117	0.0223	0.0155	0.0283	0.0637	0.0145
25-34	0.1250	0.0041	0.0076	0.0157	0.0123	0.0236	0.0474	0.0143
<25	0.1295	0.0034	0.0068	0.0138	0.0128	0.0224	0.0478	0.0226
Health PDF		0.0387	0.0766	0.1383	0.0985	0.1628	0.3881	0.0971

Modal Peakedness for the joint distribution 0.7570 (Mostly Satisfied 45-54 year olds), Modal Peakedness for Health 0.7372 (Mostly Satisfied), Modal Peakedness for age groups 0.3193 (45-54 year-olds)

While a simple application of a Cantril (1965) Scale would record average health and age group as neither dissatisfied nor satisfied, and 45-54 in both years, the modal points in the distributions were determined by the mostly satisfied 35–44-year-olds in 2010 and the mostly satisfied 45-54-year-olds in 2018. The probabilistic distance (standard error) between the mean and modal locations was 0.2301 (0.0188) in 2010 and 0.1196 (0.0145) in 2018 indicating significant differences but closer proximity in 2018. The likelihood ratios of observing most likely health-age as opposed to average health age outcomes were 5.8754 and 3.9078 in the respective years. Peakedness in the joint distribution increased (i.e. became more concentrated) over the period, albeit at a different, higher age group location reflective of an ageing population (the 2018 age group distribution first order stochastically dominates the 2010 age group distribution). Effectively the most likely health-age outcome was almost six times more likely to be observed than the average health-age outcome in 2010 and nearly four times more likely in 2018 raising questions as to what should be considered the representative outcome.

3.2 Excess Returns Distributions.

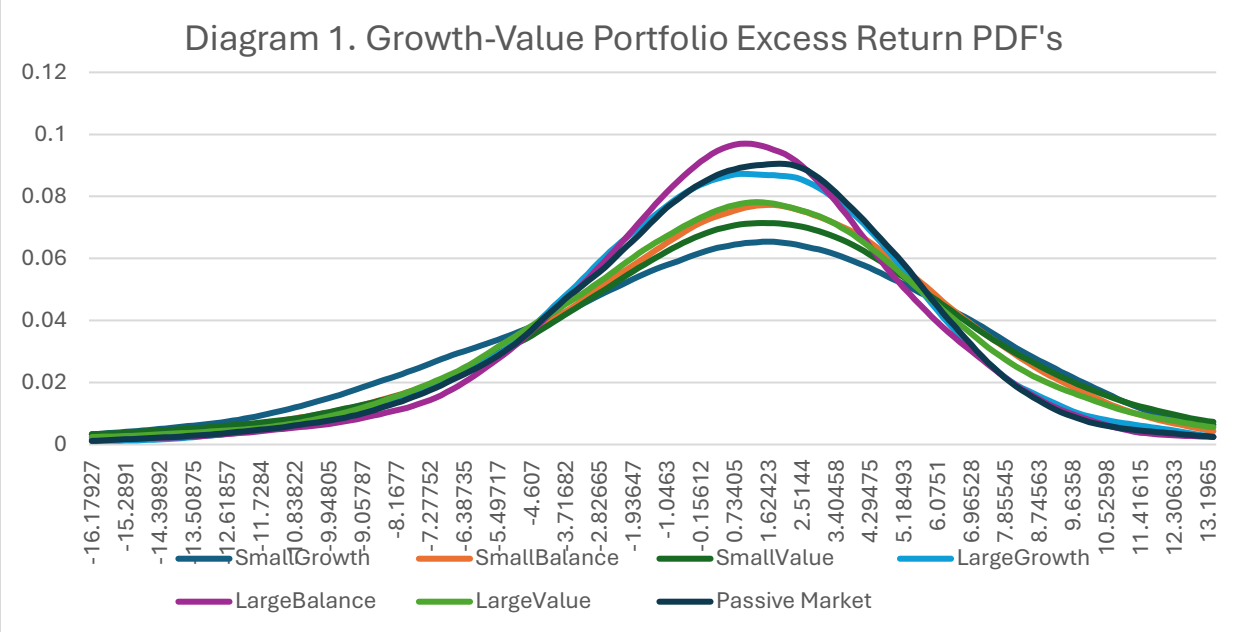
In the finance literature, whereas expected returns have been the focus of interest in portfolio choice analysis, recently there has been an emergence of interest in most likely outcomes in the guise of probability dominance (Blavatskyy 2006, Decidue, Levy and Levy 2020). Using Fama and French (1992,1996) data resources Table 3 reports the results of the Growth – value Portfolio Excess Returns Distributions analysis, distributions which are pictured in Diagram 1.

The modal excess returns are significantly different from the mean excess returns with respective probabilistic distance measures (standard errors) -0.0811 (0.0122), -0.0710 (0.0115), -0.0560 (0.0103), -0.0395 (0.0087), -0.0442 (0.0092), -0.0482 (0.0096) and -0.1222 (0.0146) indicating significant left skewness.

With regard to the Peakedness versus Kurtosis issue, except for one minor change in ranking, the ordering is unchanged whether the index is Modal Peakedness, Mean focused Kurtosis or Modally focused Kurtosis. This is not surprising given how close the

Table 3.

	Mean	Sharp	Rank	Mode	Sharp	Rank	MP	Rank	Mean K	Rank	Mode K	Rank
Small Growth	0.4660	0.0797	7	1.6242	0.2725	3	0.5133	7	2.7032	7	2.9389	7
Small Balance	0.8442	0.1599	1	1.6242	0.3043	2	0.5831	5	3.0811	6	3.2802	5
Small Value	0.8648	0.1559	2	1.4759	0.2645	4	0.5577	6	3.0465	5	3.1966	6
Large Growth	0.6163	0.1313	6	1.0308	0.2187	7	0.6235	3	3.2895	3	3.3907	3
Large Balance	0.6025	0.1338	4	1.0308	0.2278	6	0.6643	1	3.6335	1	3.7634	1
Large Value	0.7481	0.1439	3	1.3275	0.2537	5	0.5896	4	3.1351	4	3.2616	4
Passive Market	0.6122	0.1319	5	1.9209	0.3985	1	0.6377	2	3.4208	2	3.7395	2



distributions appear to be to symmetric unimodality, however, note that they are around twice as peaked as a standard normal distribution (0.2914) whilst being similarly Kurtotic (3 was Pearsons Standard Normal comparator). While the Peakedness ranking is completely aligned with the Modal Focused Kurtosis ranking it is slightly misaligned with the Mean focused Kurtosis Ranking suggesting the modal point is having an impact on the mean oriented distributional tails. The most strident differences emerge when analysis is pursued in terms of the modally oriented Sharpe Ratio following a most likely outcome approach as opposed to a mean based Sharpe Ratio following an expectations-based approach. Sharpe Ratios are employed to order portfolios and the orderings under the alternative approaches change substantially.

3.3 Life Expectancy and Age at Death Distributions.

Pay As You Go Government and privately sponsored pension plans are facing a fiscal challenge[6] brought about by an aging population (with subscribers diminishing in numbers relative to beneficiaries) and the fact that, given heavily skewed age at death distributions, the average age at death (i.e. life expectancy) upon which the plan is based is exceeded by the most likely age at death causing the Most Likely Pension Payout Period to exceed the planned Average Pension Payout Period. Table 4 and Diagram 2 reports the details of Age At Death distributions in Canada in 2017 and 2022 where it can be seen that the Average Payout Period is exceeded by the Most Likely Payout Period by at least 10 years.

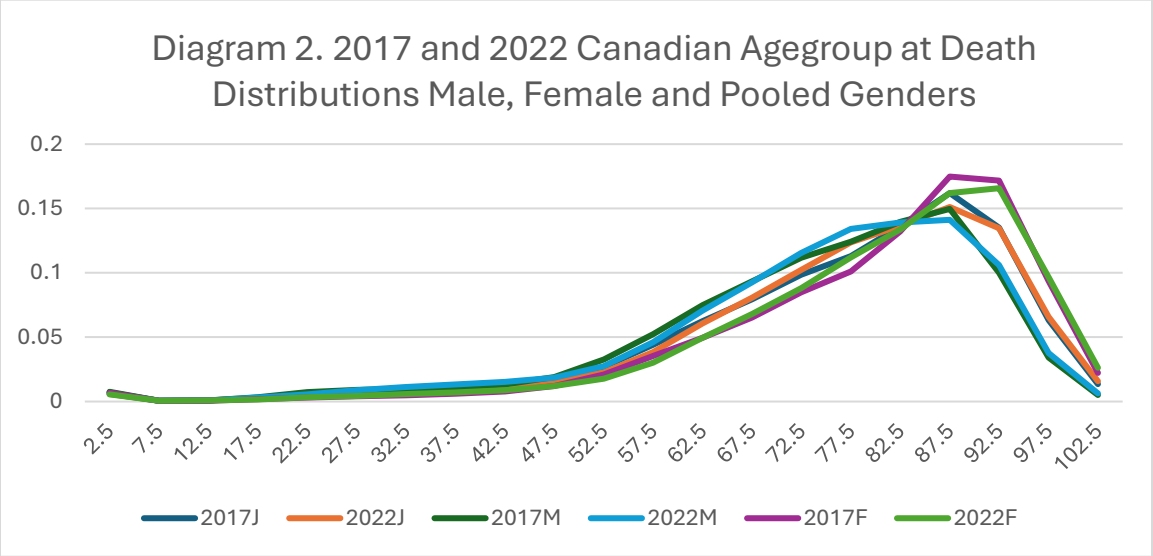


Table 4.

	Mean	Mean Location	Modal Location	MP	Peak Rank	Mean K Kurtosis	Kurtosis Rank	Mode K Kurtosis	Rank
2017 Joint	76.7187	77.5000	87.5000	0.7060	3	5.8808	3	6.1191	2
2022 joint	76.9092	77.5000	87.5000	0.6850	4	5.7336	4	6.0072	3
2017 Male	73.8759	72.5000	87.5000	0.6818	5	5.4102	5	5.1811	5
2022 Male	74.3193	72.5000	87.5000	0.6627	6	5.2897	6	5.1539	6
2017 Female	79.6801	77.5000	87.5000	0.7276	1	6.8998	1	7.5408	1
2022 Female	79.7264	77.5000	92.5000	0.7127	2	6.6816	2	5.9498	4

The modal life expectancies are significantly different from the mean life expectancies with respective probabilistic distance measures (standard errors) of -0.4110 (0.0220), -0.4113

[6] See for example Heer, Polito, Wickens 2020, Foltin 2018, Inagaki 2021, Elder and Wagner 2015

(0.0220), -0.4136 (0.0220), -0.4144 (0.0220), -0.3071 (0.0206) and -0.4615 (0.0223) indicating left skewness. Note that mean and modal focused Kurtosis measures generate very different rankings from each other with the modally focused measure offering a greater relative variation in values than the mean focused measure. The Modal Peakedness measure offers the least relative variation and generates the same ranking as the mean focused kurtosis measure (again it is of the order of twice the magnitude of an N(0,1) peakedness). This is not so surprising when it is noted that in skewed distributions, when tails are defined by the mean, the modal peak will appear in the tail with greater peakedness engendering fatter tails in a monotonic increasing relationship. The difference between the Modally focused Kurtosis measure and the Modal Peakedness measure reinforces the Westfall (2014) contention that Kurtosis is not about Peakedness.

3.4 Income Distributions.

Income distributions are notoriously right skewed which makes for differences between mean and modal outcomes which can make a difference when a centrality parameter is used to represent the typical agent in an economy. A classic example would be growth models where average incomes are used to represent the progress (or lack thereof) of incomes in the economy.

In this example of monthly income distributions in the UK drawn from Anderson (2025) reported in Table 5 and Diagram 3, the modal incomes are significantly different from the mean incomes with respective probabilistic distance measures (standard errors) of 0.2855 (0.0202), 0.2883 (0.0203), 0.2925 (0.0203) and 0.2688 (0.0198) indicating significant right skewness. There is no discord between the peakedness and the mean and modally based kurtosis measures in terms of the ordering of the distributions and, while the mean kurtosis measure is a bit larger than the modal measure in three of four cases, their relationship to the corresponding N(0,1) parameters is similar being at least twice as large. Of more concern here, when choosing a representative value of income level, is that for females, modal incomes are between 35% and 46% more likely to be observed than average incomes and for males the figure would be in the region of 28% more likely. This has an impact on how income growth is measured with Mean Annualized Income Growth Rates for

Females and Males respectively being 0.0313 and 0.0331 whereas the corresponding Modal Growth Rates would be 0.0329 and 0.0417.

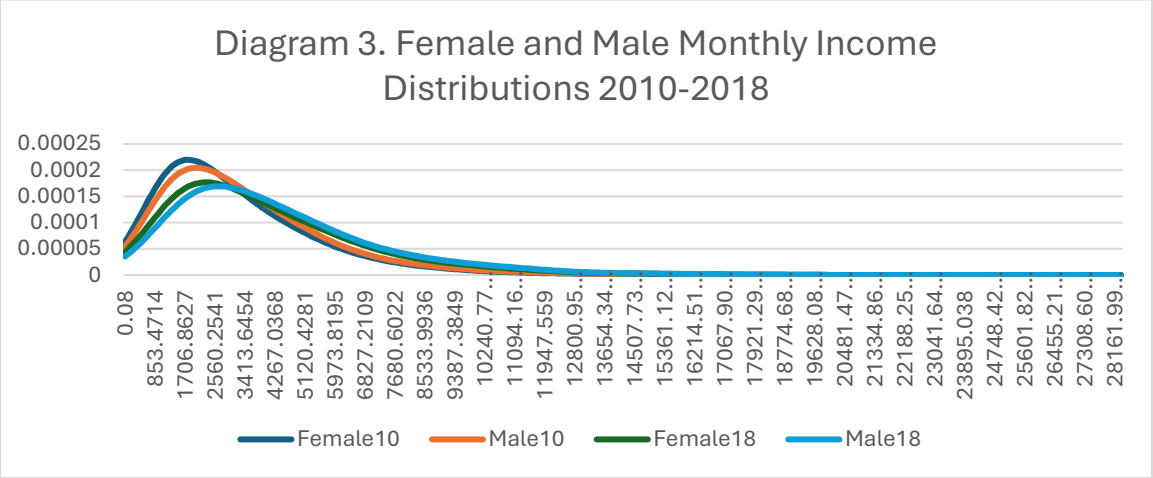


Table 5.

	Mean Location	Modal Location	Peakedness	Mode/ Mean odds	Peake Rank	Mean Kurtosis	Kurtosis Rank	Modal Kurtosis	Rank
2010 Females	3310.3224	1801.6840	0.8474	1.3487	1	9.7891	1	8.9365	1
2018 Males	3529.5550	1991.3265	0.8352	1.2980	2	8.7751	2	8.2494	2
2010 Females	4138.2952	2275.7903	0.8084	1.4560	3	7.2875	3	7.0416	3
2018 Males	4406.9920	2655.0753	0.7990	1.2708	4	6.4925	4	6.6805	4

It is important to note that modal growth rates represent the income growth of a greater proportion of the population than the growth rates of the average member of the population do and, because of the right skewness of the distribution, they represent growth in the incomes of a poorer group in society.

3.5 Fertility and Sustainability.

Evidence from the sustainability literature (Daily et.al 1994, Dasgupta 2019, Lianos and Pseiridis 2015, Pimentel et. al. 1994, Tucker 2019) suggests the current world population of approximately 8.1 billion is not sustainable and needs to be at most half of what it is[7]. Given current infant mortality rates, a stable population which routinely replicates itself requires a fertility rate (the number of live births per adult female) of 2.1, more than this results in population growth, less results in population shrinkage. Thus, the main route for

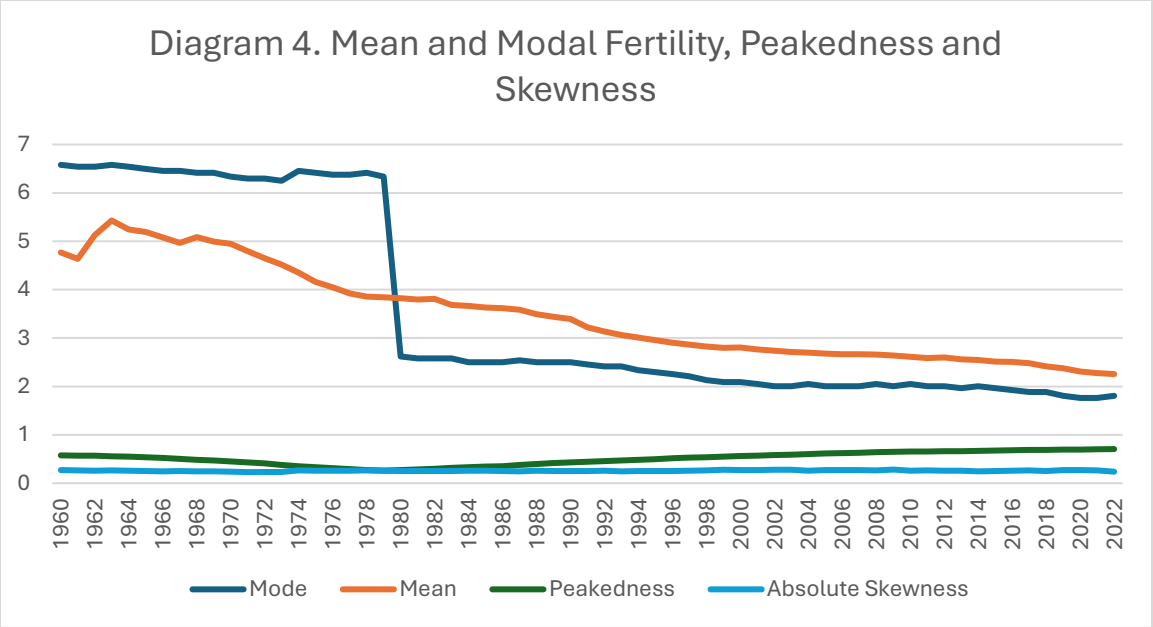
[7] United Nations (2024) projects the world's population to grow to over 10 billion by the end of the century.

transiting to a sustainable population level requires maintenance of a fertility rate less than 2.1 for several generations, followed by a return to a fertility rate of 2.1 thereafter.

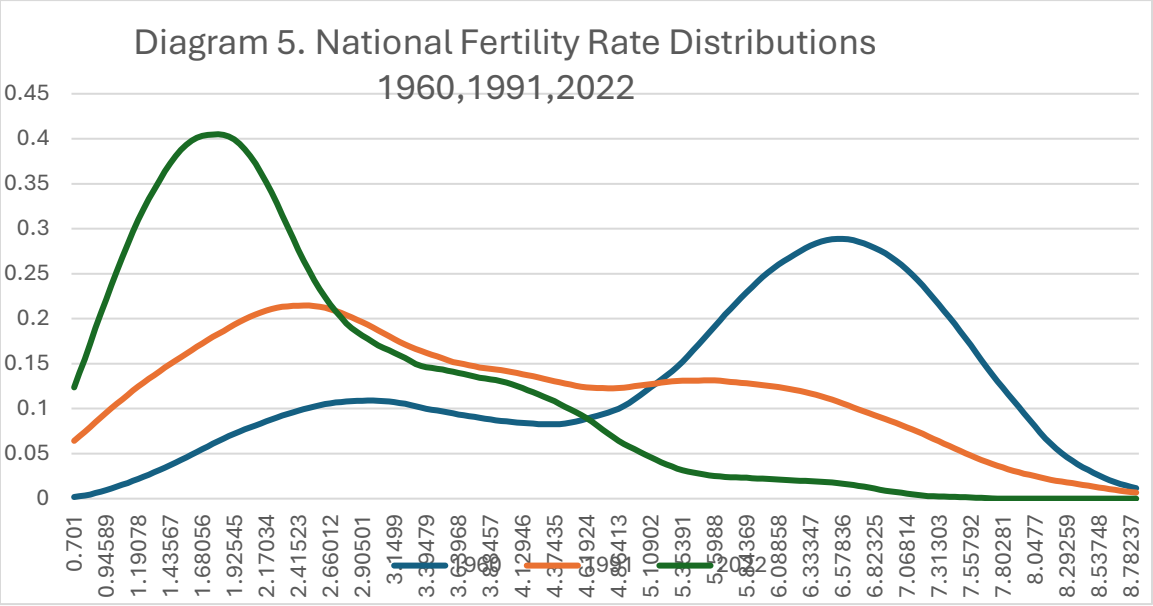
Using data drawn from the United Nations (2024), Anderson (2026) employed a representative agent framework (in this case the agent was a nation) to examine Fertility Rate trends from 1960 to 2022 for 204 nations. Two nation types were considered, the average nation, determined by the average national fertility rate and the most likely nation determined by the modal national value[8] in the distribution across nations in a given year. Clearly the distinction between the two approaches hinges on the extent to which the mean and the mode differ and the extent to which the mode is prominent which is related to the extent to which the distribution peaks at the mode.

The salient statistics are recorded in Table A1 in the appendix, the details of which are more readily seen diagrammatically. Diagram 4 reports the time paths of the Average and Most Likely representative nations fertility rates together with the Peakedness and Absolute Skewness statistic. Absolute Skewness values between 0.22 and 0.29 with standard errors in the region of 0.03 clearly indicate significant differences between the mean and the mode with maximal peakedness at the beginning of the period, when the distribution was left skewed, and at the end of the period when it was right skewed. While Average and Most Likely representative nations both exhibited downward trends over time, their profiles were very different with the most likely representation exhibiting the steepest descent. There does appear to be a sea change in 1978-1980 which coincided with the instigation of Chinas one child policy.

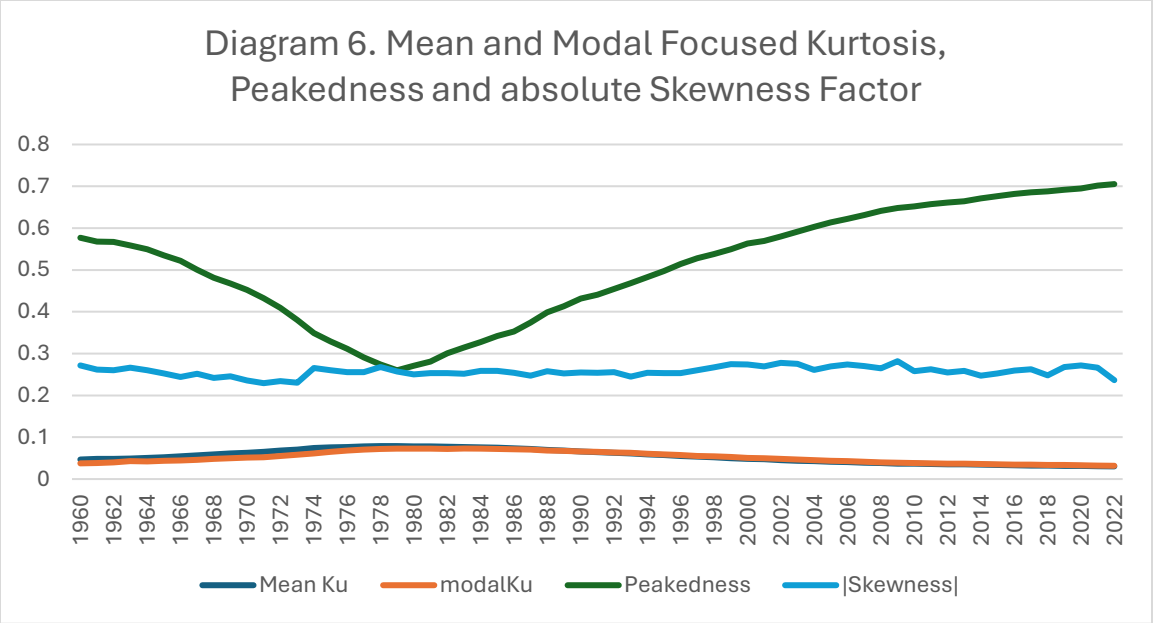
[8] Modes were calculated using an Epanechnikov kernel estimate (Silverman 1986) of the probability density function over nations in a given year and finding the value which maximizes the pdf.



To illustrate the transition over the period, fertility distributions for 1960, 1991 and 2022 are pictured in Diagram 5 where it may be seen that distributions have switched from left skewed to right skewed over the period.



For comparison purposes the Peakedness statistic is pictured with its precursors, the mean and modally focused kurtosis statistics where it can be seen to be doing a better job of picking up peakedness.



4. Conclusions.

Generic Probabilistic Locational Distance and Modal Peakedness measures quantifying the extent of locational differences and the degree of commonality within a distribution have been proposed. The measures are applicable in any discrete or continuous univariate or multivariate distributional environment. In addition, a modally focused version of the Kurtosis measure most commonly used for continuous variates has been proposed appropriate for distributions that are not symmetric unimodal.

The instruments can be used in a variety of situations where use of the mode as a summary statistic needs validation. They have been exemplified in a multivariate ordinal data-based health and age group framework as well as in continuously measured portfolio returns, age at death, income and fertility rate distributional analyses. The results of applications confirm the Westfall (2014) assertion that Kurtosis is not about Peakedness and highlight the pertinence of the mode as opposed to other centrality measures as a representative outcome in asymmetric situations.

Appendix.

	mode	mean	P(<mode)	P(<mean)	left skew	std err	Peakness	ModalKu	MeanKu
1960	6.57836	4.76644	0.66864	0.39730	0.27134	0.03113	0.57682	0.04689	0.03751
1961	6.53755	4.63864	0.66173	0.39967	0.26206	0.03079	0.56785	0.04814	0.03866
1962	6.53755	5.12322	0.65825	0.39789	0.26036	0.03072	0.56700	0.04814	0.03966
1963	6.57836	5.42957	0.66996	0.40337	0.26658	0.03096	0.55851	0.04936	0.04301
1964	6.53755	5.24129	0.66683	0.40650	0.26033	0.03072	0.54949	0.05049	0.04243
1965	6.49673	5.19293	0.66436	0.41201	0.25235	0.03041	0.53512	0.05240	0.04368
1966	6.45592	5.07954	0.66248	0.41802	0.24446	0.03009	0.52169	0.05434	0.04455
1967	6.45592	4.96753	0.67393	0.42216	0.25177	0.03039	0.50041	0.05691	0.04606
1968	6.41510	5.08757	0.67155	0.42959	0.24196	0.02998	0.48123	0.05914	0.04860
1969	6.41510	4.98976	0.68058	0.43492	0.24566	0.03014	0.46789	0.06098	0.04952
1970	6.33347	4.94897	0.67233	0.43641	0.23592	0.02973	0.45215	0.06290	0.05106
1971	6.29266	4.79324	0.67358	0.44429	0.22930	0.02943	0.43237	0.06500	0.05247
1972	6.29266	4.64778	0.68315	0.44895	0.23420	0.02965	0.40936	0.06779	0.05488
1973	6.25184	4.51436	0.68542	0.45536	0.23006	0.02947	0.38028	0.07066	0.05800
1974	6.45592	4.34885	0.73331	0.46736	0.26595	0.03093	0.34902	0.07391	0.06149
1975	6.41510	4.16026	0.73304	0.47289	0.26015	0.03072	0.32871	0.07574	0.06533
1976	6.37429	4.04799	0.73211	0.47612	0.25599	0.03056	0.31114	0.07675	0.06781
1977	6.37429	3.92050	0.73916	0.48333	0.25583	0.03055	0.29062	0.07803	0.07074
1978	6.41510	3.85566	0.75271	0.48501	0.26770	0.03100	0.27370	0.07889	0.07230
1979	6.33347	3.84228	0.74598	0.48857	0.25741	0.03061	0.26006	0.07883	0.07269
1980	2.61930	3.81920	0.24023	0.49046	-0.25023	0.03033	0.27117	0.07826	0.07263
1981	2.57849	3.79451	0.24185	0.49490	-0.25305	0.03044	0.28116	0.07810	0.07297
1982	2.57849	3.80939	0.24648	0.49954	-0.25305	0.03044	0.30039	0.07720	0.07193
1983	2.57849	3.68660	0.25201	0.50402	-0.25201	0.03040	0.31426	0.07671	0.07302
1984	2.49686	3.66400	0.24549	0.50436	-0.25887	0.03067	0.32757	0.07578	0.07276
1985	2.49686	3.62873	0.25157	0.51040	-0.25883	0.03067	0.34179	0.07481	0.07222
1986	2.49686	3.61489	0.25616	0.51020	-0.25403	0.03048	0.35308	0.07388	0.07148
1987	2.53767	3.58303	0.26970	0.51720	-0.24750	0.03022	0.37423	0.07237	0.07014
1988	2.49686	3.49248	0.26754	0.52550	-0.25796	0.03063	0.39835	0.06968	0.06853
1989	2.49686	3.44106	0.27506	0.52756	-0.25250	0.03042	0.41353	0.06814	0.06741
1990	2.49686	3.39491	0.28076	0.53534	-0.25458	0.03050	0.43179	0.06597	0.06559
1991	2.45604	3.22096	0.28178	0.53603	-0.25425	0.03049	0.44102	0.06466	0.06535
1992	2.41523	3.13437	0.28217	0.53776	-0.25559	0.03054	0.45488	0.06278	0.06391
1993	2.41523	3.06491	0.29507	0.54031	-0.24524	0.03012	0.46820	0.06125	0.06249
1994	2.33360	3.00993	0.28826	0.54240	-0.25414	0.03048	0.48288	0.05912	0.06079
1995	2.29278	2.95464	0.29042	0.54397	-0.25355	0.03046	0.49747	0.05736	0.05921
1996	2.25197	2.90614	0.29320	0.54665	-0.25345	0.03046	0.51416	0.05520	0.05719
1997	2.21116	2.86319	0.29429	0.55472	-0.26043	0.03073	0.52769	0.05329	0.05540
1998	2.12953	2.82711	0.28755	0.55462	-0.26707	0.03098	0.53796	0.05176	0.05421
1999	2.08871	2.79721	0.28587	0.56084	-0.27498	0.03126	0.54911	0.05001	0.05256
2000	2.08871	2.80225	0.29472	0.56852	-0.27380	0.03122	0.56353	0.04796	0.05041
2001	2.04790	2.76467	0.29478	0.56436	-0.26958	0.03107	0.56897	0.04721	0.04978
2002	2.00708	2.73636	0.29244	0.57031	-0.27787	0.03136	0.58031	0.04556	0.04821
2003	2.00708	2.71107	0.30024	0.57565	-0.27541	0.03128	0.59173	0.04402	0.04654
2004	2.04790	2.69961	0.31661	0.57744	-0.26083	0.03074	0.60279	0.04284	0.04508
2005	2.00708	2.67867	0.31022	0.57990	-0.26969	0.03107	0.61332	0.04141	0.04372
2006	2.00708	2.66786	0.31438	0.58828	-0.27391	0.03122	0.62234	0.04031	0.04254
2007	2.00708	2.66282	0.31660	0.58701	-0.27041	0.03110	0.63118	0.03914	0.04129
2008	2.04790	2.65780	0.32831	0.59327	-0.26496	0.03090	0.64126	0.03795	0.03985
2009	2.00708	2.64020	0.31985	0.60168	-0.28183	0.03150	0.64804	0.03692	0.03891
2010	2.04790	2.61239	0.34197	0.60014	-0.25817	0.03064	0.65214	0.03658	0.03833
2011	2.00708	2.58766	0.33734	0.59951	-0.26218	0.03079	0.65720	0.03593	0.03776
2012	2.00708	2.59663	0.33967	0.59485	-0.25518	0.03052	0.66141	0.03532	0.03714
2013	1.96627	2.55726	0.33324	0.59184	-0.25861	0.03066	0.66382	0.03505	0.03694
2014	2.00708	2.54333	0.35220	0.59905	-0.24686	0.03019	0.67094	0.03426	0.03592
2015	1.96627	2.51018	0.34491	0.59762	-0.25271	0.03043	0.67634	0.03351	0.03521
2016	1.92545	2.50872	0.33901	0.59830	-0.25928	0.03068	0.68207	0.03266	0.03447
2017	1.88464	2.48248	0.33572	0.59797	-0.26225	0.03080	0.68526	0.03228	0.03416
2018	1.88464	2.41470	0.34666	0.59484	-0.24818	0.03024	0.68794	0.03215	0.03387
2019	1.80301	2.37256	0.32520	0.59316	-0.26796	0.03101	0.69136	0.03154	0.03342
2020	1.76219	2.30506	0.31934	0.59127	-0.27193	0.03115	0.69509	0.03108	0.03291
2021	1.76219	2.27309	0.32058	0.58669	-0.26611	0.03094	0.70143	0.03040	0.03210
2022	1.80301	2.25586	0.34837	0.58462	-0.23625	0.02974	0.70525	0.03027	0.03176

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