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Should Expected or Most Likely Returns be the Focus in Investment Decisions? Introducing â•œMost Likelyâ•š Versions of Sharpe and Sortino Ratios

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Very Preliminary. Please do not Quote Yet.

Abstract.

For the last 60 years, Expected Utility Theory, Rational Expectations, and a tacit presumption of symmetry in outcome distributions have been the micro and macro foundations of decision-making paradigms which seek optimum risk tempered expected outcomes. The Sharpe ratio, in its use in evaluating portfolio performance and its focus on average returns, epitomizes the practice. When outcome distributions are symmetric unimodal, expected and most likely outcomes coincide, and choices can be construed as being made on the basis of either. However, when outcome distributions are asymmetric or multi-modal, expected outcomes are not the most likely and, in contradiction of rational expectations assumptions, expectations-based choice will engender systematic information laden surprises raising questions as to whether choice should be most likely or expected outcome based. Here, the impact of switching to a Most Likely view of the world is examined and "Most Likely" focused versions of the Sharpe and Sortino Ratios are introduced. Simple exercises performed on commonly used benchmark portfolio and stock returns data demonstrate that portfolio orderings change substantially when the focus is switched to most likely outcomes, all of which gives some pause for thought.

Keywords.

Portfolio choice, expected outcomes, most likely outcomes, Sharpe Ratio, Sortino Ratio

JEL classifications.

G11; C14; C18

Introduction.

Following von Neumann and Morgenstern (1944), Markowitz (1952, 1956, 1959), Muth (1961), Nerlove (1958) and Sharpe (1964), rational and adaptive expectations assumptions, expected utility theory, mean-variance analysis and a predisposition toward distributional symmetry have been the analytic underpinnings of choice under uncertainty models promoting the use of expected, i.e. average outcomes as the basis for choice. Such models have found expression in many micro and macro fields of economics, life-cycle income constrained consumption paths, human capital - career choice, capital asset pricing models and financial investment decisions of firms and portfolio selection to name but a few. Given historical information and a presumption that future patterns will be similar to past patterns, risk averse decision makers are assumed to choose between available alternatives on the basis of what their expected outcomes and associated uncertainties will be by comparing the historical risk adjusted average outcome of each alternative under consideration. The question raised here is why focus on expected outcomes if they are not the most likely outcomes?

For some intuition, suppose two gamblers, one a “Rational Expector”, the other a “Most Likely Hoper” are confronted with a potentially biased die with an unknown probability density function. They are allowed to observe and record repeated throws of the die before placing a bet on the outcome of the final throw. The Rational Expector would take the average value of the scores and base their bet on that value whereas the Most Likely Hoper would compute the relative frequency of the scores and bet on the most frequently observed value. If the true distribution was 1 (1/12) 2 (1/3) 3 (1/4) 4 (1/6) 5 (1/6) 6 (0) , the expected value- based bet would be 3 and be correct 25% of the time, the most likely based bet would be 2 and correct 1/3 of the time. An important issue to be addressed is when does the “expected” vs “most likely” distinction matter? It is readily shown that a gambler cannot do better than base the choice on the most likely outcome since when the outcome distribution is symmetric and unimodal the most likely outcome will be the expected outcome and when it is not, they are not. Even then, the distributional skewness may be so limited that the distinction is practically and empirically inconsequential, rendering the expected and the most likely good approximations of each other. Theoretically the distinction affects the way that models are formulated but practically it is of no consequence.

In the case of portfolio selection, the Sharpe Ratio[1] (Sharpe 1966, 1994) and its downside risk analogue the Sortino Ratio (Sortino and Price 1994) have been the workhorses, and each has an expected outcome focus. Much like its inverse, the Coefficient of Variation (Pearson 1896), the Sharpe Ratio is focused on the mean and a measure of variation around it, with the Expectations or "Averaging" Operator and its Variation[2] looming large in the analyses. The relevance of the Sharpe Ratio and the associated statistical inference in this context is based upon assumptions of normality or at least symmetric unimodality (Kan and Zhou 2007, Tu and Zhou 2011) even though many historically based returns distributions are demonstrably asymmetric with mounting evidence that such skewness has been priced in the market[3]. Indeed, in order to retain the basic form of the Sharpe Ratio, analysts have gone to great lengths to adjust the inference process to accommodate non-normality of returns (see Bailey and López de Prado 2012, Kourtis 2016, Ledoit and Wolf 2008 for example). All of which begs the question, why use the Sharpe Ratio with its mean and standard deviation focus in the first place? Unless an outcome distribution is symmetric and unimodal, what is expected to happen will not be the same as what is most likely to happen.

When investors base their judgements on expected outcomes in the context of skewed outcome distributions, contrary to the rational expectations hypothesis there will be systematic information laden surprises. Left skewness will result in investors being pleasantly surprised on average since they will receive better than expected returns relatively more frequently than worse than expected returns. On the other hand, for similar reasons, right skewness will result in investors being disappointed on average. Not that basing their judgements on Most Likely outcomes would result in the absence of systematic surprises, since only when the median outcome is the focus will it be the case that the relative frequency of pleasant and unpleasant surprises will be the same. However, when they are different, most likely

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[1] or its squared value (Kourtis 2016)

[2] Given $U(x)$ an individuals' Utility or Value Function is a monotonic increasing function of the random outcome variable x that they confront, where the possible outcomes are described by the probability density function $f(x)$, Expected Utility ($E\big(U(x)\big)$) is defined as $E\big(U(x)\big) = \int U(x)f(x)dx$ (effectively the Average Utility of x generated by the outcome distribution $f(x)$) and its standard deviation $SD\big(U(x)\big) = \sqrt{\int \Big(U(x) - E\big(U(x)\big)\Big)^2 f(x)dx}$. Letting $U(x) = x$ establishes the connection with the Sharpe Ratio.

[3] Using a variety of techniques to measure skewness, several papers have confirmed the basic prediction that more positively skewed stocks will have lower average returns (Boyer, Mitton, and Vorkink 2010; Bali, Cakici, and Whitelaw 2011; Conrad, Dittmar, and Ghysels 2013). In essence skewness is being priced in the market – positively skewed securities will be overpriced relative to the price it would command in a market with expected utility investors and would thus earn a lower average return (Barberis and Huang 2008).

based anticipated outcomes will always be realized more frequently than expectations based anticipated outcomes.

The simplicity of the averaging operator has clearly been instrumental in the formulation of these indices, it is much easier to compute historical averages of alternative outcome variables than it is to compute their historically most likely outcomes by identifying their respective modal locations[4]. Indeed, when distributions are symmetric unimodal, the variation muted most likely outcome would be the same as the variation muted expected outcome. But, given the possibility of computing modal outcomes and the variation around them (Bickel 2003) and using the same process of projecting the past onto the future, would it not be more rational to compare the variation muted most likely outcomes, rather than the variation muted average (and somewhat less likely) outcomes?

In the following Section 1 explores the notion that most likely outcomes could differ from expected outcomes and develops most likely outcome-based Sharpe and Sortino Ratios. Section 2 reports a comparison of the two alternative expectations based and most likely based Sharpe and Sortino measures for evaluating sets of portfolios drawn from the Fama and French data library and examines the daily stock returns of 441 stocks from the S&P500 for potential mean and modal differences. Conclusions are drawn in Section 3. Since these ideas are equally applicable to situations where outcome choices are ordered categorical in nature, the equivalent Most Likely Sharpe ratio for the ordinal outcome situation is developed in the appendix.

Section 1. Introducing the Most Likely Focused Sharpe and Sortino Ratios.

To fix ideas, for a continuously measured excess return variable x, with finite lower and upper bounds W and Y respectively so that $-\infty < W < x < Y < \infty$, denote the asset type t excess returns distribution $f_t(x)$ with a corresponding CDF: $P_t(X < x) = F_t(x) = \int_W^x f_t(z)dz$, Survival Function SF: $P_t(X \geq x) = S_t(x) = 1 - F_t(x)$, mean $\mu_t = E_{f_t(x)}(x) = \int_W^Y x f_t(x)dx$ and variance $\sigma_t^2 = E_{f_t(x)}((x-\mu_t)^2) = \int_W^Y (x-\mu_t)^2 f_t(x)dx$. Note that integrating the mean formula by parts reveals it to be the integral of the

[4] Finding the average value of a collection of numbers is a simple exercise when compared to estimating the mode via Half Sample techniques (Bickel and Frühwirth 2006) or locating the maximal value of the kernel estimate of a density function (Pagan and Ullah 1999).

survival function[5] so that $\mu_t = \int_W^Y S_t(x)dx$ which yields an alternative interpretation of the mean as the cumulated chances of higher outcomes than x over its range which will be useful when contemplating ordinal data environments.

Note that μ_t is very much the focus of the variance statistic and since:

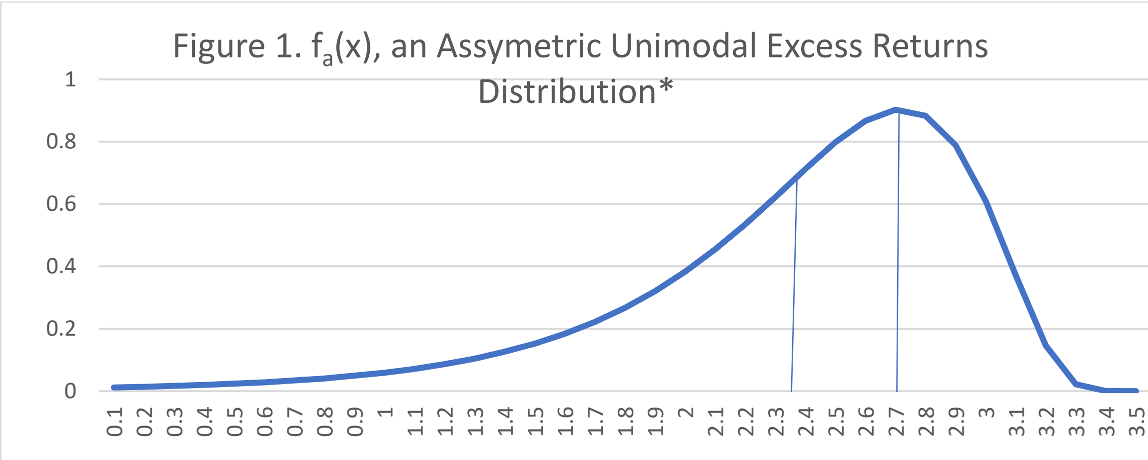
$$\frac{\partial \sigma_t^2}{\partial \mu_t} = -2\int_W^Y (x-\mu_t)\, f_t(x)dx = 0 \Rightarrow \int_W^Y (x)\, f_t(x)dx = \int_W^Y \mu_t\, f_t(x)dx = \mu_t\,, \text{ with } \frac{\partial^2 \sigma_t^2}{\partial \mu_t^2} = 2 > 0$$

it is the value of x that minimizes σ_t^2 which implies that a similar variation measure focused on any other value of x would invariably be at least as large. Furthermore, unless the distribution is symmetric unimodal, μ_t is not the most likely excess return to be observed, indeed in heavily bimodal distributions it can be a very unlikely outcome which gives pause for thought as to why one would want to focus any risk adjusted returns measure on such an unlikely outcome.

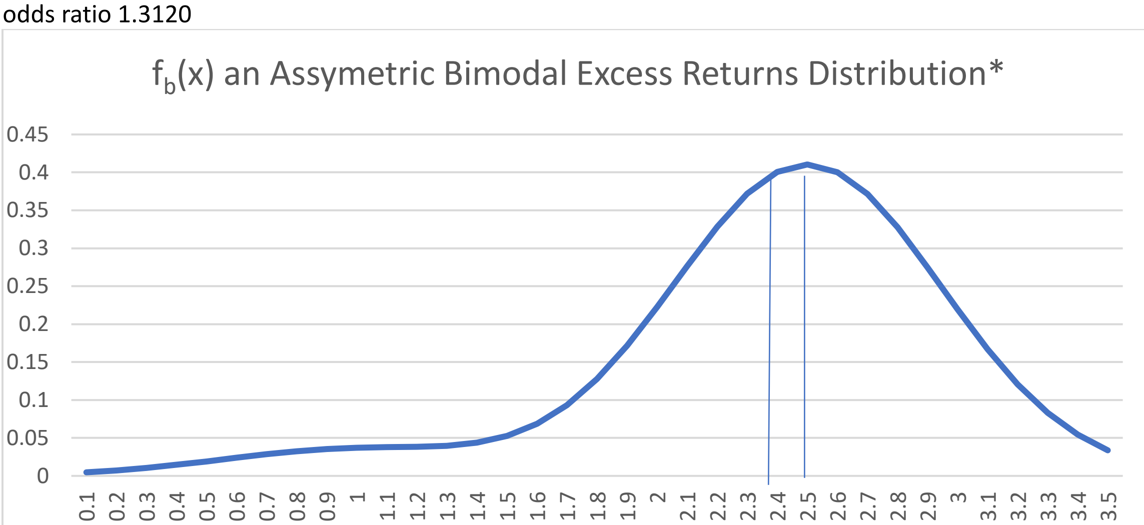
To illustrate these ideas, consider x to be the net return on an investment, where for simplicity the monotonic increasing Utility or Value function is $U(x) = x$ and consider $f_a(x)$ and $f_b(x)$, two synthetically contrived negatively skewed excess returns distributions designed to make the point, $f_a(x)$ is an asymmetric unimodal distribution and $f_b(x)$ is an asymmetric bimodal distribution, each are illustrated in figures 1 and 2. In these examples, Expected or Average Value is given by $E\big(U(x)\big) = \int U(x)f(x)dx = \bar{X}$, which is the average or expected value of x and the Most Likely Value is given by $\tilde{X}: f\big(\tilde{X}\big) = \max_{all\, x} f(x)$, which is the modal value of x. The Odds Ratio $f\big(\tilde{X}\big)/f(\bar{X}) \geq 1$ indicates how much better the chance is of observing $\tilde{X}$ rather than $\bar{X}$. Note that when $f(x)$ is unimodal and symmetric $\frac{f(\tilde{X})}{f(\bar{X})} = 1$, otherwise $\frac{f(\tilde{X})}{f(\bar{X})} > 1$. When $f(x)$ is left skewed $\tilde{X} > \bar{X}$ so the Average Return understates the Most Likely Return and when $f(x)$ is right skewed $\tilde{X} < \bar{X}$ so the Average Return overstates the Most Likely Return. Variation around the average return $\int (x-\bar{X})^2 f(x)dx$ (an estimate of the magnitude of risk associated with the average return) will always be lower than the corresponding variation around the most likely return $\int (x-\tilde{X})^2 f(x)dx$ since the mean is the value which minimizes the variation function.

[5] Given the integration by parts rule: $\int_W^Y u(x)v'(x)dx = [u(x)v(x)]_W^Y - \int_W^Y u'(x)v(x)dx$ letting $u(x) = x$ and $v'(x) = f(x)$, it can be seen that $\mu = \int_0^Y xf(x)dx = [xF(x)]_W^Y - \int_W^Y F(x)dx = Y - W - \int_W^Y F(x)dx = \int_W^Y (1-F(x))dx = \int_W^Y S(x)dx$.

Notice that, while Portfolio b yields a greater Expected Return than Portfolio a, its Most Likely return is lower, basically because the distributions have somewhat different topographies so the ordering of the portfolios could differ dependent on whether the ordering is based upon the most likely outcome or the average and expected outcome.



*This distribution was constructed from the log normal distribution of a random variable y where $\ln(y) \sim N(0,0.5)$ & $x = 3.5 - y$ Mean = 2.3669% f(2.3669%)= 0.6852: Mode = 2.7212% f(2.7212)= 0.8990 odds ratio 1.3120



*This distribution was constructed from a mixture of two normals with an 8% chance of N(1,0.2) and a 92% chance of N(2.5,0.2). Mean= 2.38% f(2.38%)=0.3948; Mode= 2.5%; f(2.5%)=0.4105, Odds Ratio 1.0398

Sharpe and Sortino ratios (Sharpe 1966, Sortino and Van Der Meer 1991)[6] epitomize characterizations of the return/risk relationship of interest in the portfolio choice problem. Implicitly assuming that asset

[6] Sortino Ratios use the magnitude of downside variation (basically the square root of the average squared deviations from the mean of all realizations below the mean) as a measure of risk whereas Sharpe Ratios use overall variation as a measure of risk.

returns distributions are stable over time, historical returns data are used to compute the average or expected excess return and its variation (measured in terms of the standard deviation which has the same unit of measurement as the mean) hence their ratio forms a unit free risk modulated expected rate of return for comparison purposes. When outcome distributions are symmetric and unimodal the expected outcome and the most likely outcome are coincident, but when distributions are asymmetric or multi-modal in nature, this is no longer true and a question arises as to whether the expected outcome or the most likely outcome should be the focus of the analysis. The Sharpe Ratio is the inverse of COV, Pearsons Coefficient of Variation (Pearson 1896) applied to Excess Returns. Formally, for a random variable x defined on the interval $a.b$ with PDF $f_t(x)$, the group t COV is given by:

$$COV_{f_t(x)}(x) = \frac{\sqrt{\sigma_t^2}}{\mu_t} = \frac{\sqrt{\int_a^b (x-\mu_t)^2 f_t(x)dx}}{\int_a^b S_t(x)dx} \qquad [1]$$

and SR, the Sharpe Ratio is given by:

$$SR = \left(COV_{f_t(x)}(x)\right)^{-1} \qquad [1a]$$

For the Sortino Ratio the numerator of [1] is replaced by $\sqrt{\int_a^{\mu_t} (x-\mu_t)^2\, f_t(x)dx}$ which interestingly defines a new "Downside Coefficient of Variation".

Practically, for a collection of N randomly sampled cardinally measurable values $x_i, i = 1,..N$, where the x_i's are in rank order the basic COV is given by[7]:

$$COV = \frac{\sqrt{\sum_{i=1}^{N}(x_i-\underline{x})^2/(N-1)}}{\underline{x}} = \frac{\hat{\sigma}}{\hat{\mu}}; where\ \hat{\mu} = \underline{x} = \sum_{i=1}^{N} x_i\,/N \qquad [2]$$

And the version of COV used for the Sortino Ration is given by:

$$COV = \frac{\sqrt{\sum_{i=1}^{N_\mu}(x_i-\underline{x})^2/(N_\mu)}}{\underline{x}} \qquad [2a]$$

As a measure of relative variation, COV can be seen to be the square root of the variance estimate, which is the average of the squared distances of the x_i's from the mean, divided by the mean. and division by it dilutes the standard deviation value rendering the statistic a unit

[7] When data are sampled from a set of K discrete cardinally measurable values x_k $k = 1,..,K$ where p_k is the proportion of the sample that took on the value x_k, [1] can be computed as:

$$COV = \frac{\sqrt{\sum_{k=1}^{K}(x_k-\underline{x})^2 p_k}}{\underline{x}} = \frac{\hat{\sigma}}{\hat{\mu}}; where\ \hat{\mu} = \underline{x} = \sum_{k=1}^{K} x_k\, p_k \qquad [2a]$$

free measure[8], the Sharpe Measure is simply the inverse of this. It should be noted that there is a problem with the Coefficient of Variation when the mean is close to 0 but this is not a problem for the Sharpe measure.

An investor may well prefer to base his judgement on what is most likely to arise in the future with regard to excess returns, so the question here is "From a likelihood perspective, is the mean a good measure of anticipated excess return and the standard deviation a good measure of risk and thus the Sharpe measure a good measure of the performance of an asset?". When the excess returns distribution is unimodal and symmetric, the mean is the most likely return and the standard deviation a good measure of the uncertainty or risk with which it can be viewed, but when the excess returns distribution is not unimodal and symmetric, the mean is not the most likely return, the mode is, and the standard deviation will always be an underestimate of the uncertainty or risk associated with the most likely return. Here a Sharpe measure based upon the "Most Likely Excess Returns" and a Most Likely Risk Measure which has a "Most Likely Excess Returns" focus is argued for. Letting θ_t be "the most likely" or "modal" excess return where θ_t is the value of $x \in [W, Y]$ that maximises $f_t(x)$. Basically θ_t is the value of x where:

$$\frac{\partial f_t(x)}{\partial x} = 0 \ ; \quad \frac{\partial^2 f_t(x)}{\partial x^2} < 0 \ \& \ f_t(\theta_t) \geq \ f_t(x) \ \forall \ x \neq \ \theta_t$$

Note that when $f_t(x)$ is symmetric unimodal $\theta_t = \mu_t$, when $f_t(x)$ is right skewed $\theta_t < \mu_t$ so that $\hat{\mu}$ overestimates or exaggerates the most likely excess return and when $f_t(x)$ is left skewed $\theta_t > \mu_t$, $\hat{\mu}$ underestimates or diminishes the most likely excess return. θ_t can be estimated using kernel estimates of $f_t(x)$ and finding the maximum value over the range of x (See Bickel 2003 for an alternative approach).

Then it is the spread around the most likely excess return that is the appropriate "most likely risk" measure viz $\sqrt{\sigma_t^2(\theta_t)}$ where[9]:

$$\sigma_t^2(\theta_t) = E_{f_t(x)}((x - \theta_t)^2) = \int_W^Y (x - \theta_t)^2 f_t(x) dx$$

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[8] Similar statistics can be contrived if other foci are of interest by making $\underline{x}$ the median or modal value of the collection, indeed dividing the standard deviation by any quantile value would render it a unit free measure relative to the designated quantile.

[9] The Most Likely Sortino ratio would use $\sigma_t^2(\theta_t) = \int_W^{\theta_t} (x - \theta_t)^2 f_t(x) dx$

So $SMLR$, the corresponding Sharpe "Most likely risk adjusted excess return ratio" would be:

$$SMLR = \frac{\theta_t}{\sqrt{\sigma_t^2(\theta_t)}}$$

Note that unless $f_t(x)$ is unimodal symmetric $SMLR \neq SR$ and $\sigma_t^2(\theta_t) > \sigma_t^2$ so that the level of most likely risk is generally understated by σ_t^2. Generally, when distributions are right skewed $SMLR < SR$ since $\theta_t < \mu_t$ and $\sigma_t^2(\theta_t) > \sigma_t^2$, when distributions are left skewed things are less clear since $\theta_t > \mu_t$ and $\sigma_t^2(\theta_t) > \sigma_t^2$. In terms of Rational Expectations, if investors base their judgements on expected returns, left skewness will result in them being systematically pleasantly surprised on average and right skewness will result in them being systematically disappointed on average, both of which contradict a rational expectations-based hypothesis of no systematic surprises. Basing their judgements on Most Likely outcomes would result in proportionately fewer pleasant surprises and proportionately more disappointments. If absence of systematic surprises is of great import then the coefficient should be based upon the median outcome but this would remove the "most likely outcome" basis for the statistic.

To a large degree the crux of the matter is whether there is a substantive difference between the mean and the mode. The most commonly used modally based statistic for unimodal skewness is $(\mu_t - \theta_t)/\sigma_t$ which is negative in the presence of left skewness, positive in the case of right skewness and zero in the absence of skewness but assessing significant magnitudes with this statistic is difficult without extensive simulations. A useful and simple alternative way of thinking about the difference is to examine the chance of an outcome between the mean and the mode being realized, in essence it is a measure of the probabilistic distance (Mendelson 1987) between the two. Given an independent random sample, $\hat{p}$ the relative frequency of outcomes between two given points in a random variables range[10], has a simple distribution, i.e. $\sqrt{n}(\hat{p} - p) \sim N(0, p(1-p))$, so the significance of the event is easy to compute.

Two Empirical Examples.

The Fama and French (1992) Data Base.

To examine what difference viewing portfolio selection from an average return view as opposed to a most likely return view would make, here the two benchmark portfolio data sets drawn from Kenneth French's data library that were analyzed in Anderson et. al. (2020) are explored. Each set consists of the monthly historical returns on six active portfolios of US common stocks designed with a particular choice

[10] Let the estimated cumulative distribution function of the random variable x be $\hat{F}(x)$ and the two points be a and b where $a \neq b$, then $\hat{p} = |\hat{F}(a) - \hat{F}(b)|$.

behavior in mind and choosing between these benchmark portfolios is considered without allowing for portfolio mixtures, since many active money managers specialize in security selection for a given market segment or investment style to exploit economies of scale and specialization. The first data set consists of 1082 monthly returns observations (up to 2016) on six active portfolios formed, and periodically re-balanced, on the basis of the market capitalization of equity ('size') and book-to-market equity ratio ('valuation'). The six portfolios are labeled as Small Growth (SG), Small Blend (SB), Small Value (SV), Large Growth (LG), Large Blend (LB) and Large Value (LV). The second benchmark set consists of 1076 monthly returns observations on six portfolios that are based on market capitalization and recent past return: Small Loser (SL), Small Neutral (SN), Small Winner (SW), Large Loser (LL), Large Neutral (LN) and Large Winner (LW). Past return is measured using a one-month lagged trailing window of 11 months, to avoid the short-term reversal effect (Jegadeesh (1990)) and exploit the intermediate-term momentum effect (Jegadeesh and Titman (1993)). They are of particular interest, because a wealth of empirical research, starting with Banz (1981) and Basu (1983), suggests that the low historical average returns to SG stocks and high average for SV stocks defy rational explanations based on investment risk.

Since unimodality, Symmetry and normality are of the essence for Sharpe Ratio comparisons, it is those aspects that are considered in the first instance. The excess returns data were N(0,1) standardized by subtracting their respective means and dividing by their respective standard deviations and the resultant data examined for Standard Normality and Symmetry using Pearson Goodness of Fit tests over the partition points {-2, -1.5, -1, -0.667, -0.333, 0, 0.333, 0.667, 1, 1.5, 2}[11]. The Test for Normality is of the form $\left(\sum \frac{(Obs-Exp)^2}{Exp}\right)$ which yields a Chi Squared test statistic with 9 degrees of freedom (0.5% critical value of 23.6). The test for symmetry is of the form $\left(\sum_{i=1}^{6} \frac{\left(Obs(i)-Obs(12-i+1)\right)^2}{Exp(i)}\right)$ which yields a Chi Squared test statistic with 3 degrees of freedom (0.5% critical value of 12.8).

The results of this exercise are reported in Table 1. Normality is strongly rejected for all portfolios, but it could still be the case that the expected value would be coincident with the modal point and the appropriate focus if distributions are symmetric unimodal. However, symmetry is also rejected at most usual confidence regions (up to and including 99%) for all but two portfolios (Large Loser and Small Value).

[11] Under a null of standard normality these partition points yield 12 cells with respective cell probabilities of {0.0227, 0.0441, 0.0918, 0.0937, 0.1172, 0.1304, 0.1304, 0.1172, 0.0937, 0.0918, 0.0441, 0.0227}.

Table 1 Normality and Symmetry Tests for Growth and Value Portfolios.

Growth/Value Portfolio	Chi-squared Normality Tests *Chi(9)*	P Value	Chi-squared Symmetry Tests *Chi(3)*	P Value
Passive market	80.43351	0.00000	36.33269	0.00000
Small Growth	81.22038	0.00000	31.67685	0.00000
Small Balanced	150.16763	0.00000	26.21068	0.00001
Small Value	207.71843	0.00000	10.87354	0.01243
Large Growth	61.06949	0.00000	25.74760	0.00001
Large Balanced	157.36789	0.00000	31.83706	0.00000
Large Value	166.74791	0.00000	26.28673	0.00001
Winner/Loser Portfolio	**Chi-squared Normality Tests *Chi(9)***	**P Value**	**Chi-squared Symmetry Tests *Chi(3)***	**P Value**
Passive market	82.50291	0.00000	35.56202	0.00000
Small Loser	204.34989	0.00000	12.78940	0.00511
Small Neutral	185.81969	0.00000	21.58750	0.00008
Small Winner	100.88078	0.00000	44.70748	0.00000
Large Loser	210.65747	0.00000	6.48721	0.09017
Large Neutral	121.13209	0.00000	32.82973	0.00000
Large Winner	62.75851	0.00000	51.33177	0.00000

Even if distributions are asymmetric, it may still be the case that the mode is sufficiently close to the mean so that the ordering of the portfolios is unaffected. Figures 1 and 2 facilitate visualization of the probability density functions of the various portfolios (each estimated using a univariate Epanichnikov (1969) kernel) and highlights the unimodality and left skewness of most of the distributions. Here modal values were conveniently determined by finding the point at which the kernel estimate of the pdf was maximized [12]. Whilst variation around modes and means is quite similar in each portfolio, differences between mean and modal returns is somewhat greater ranging from 1.4 percentage points for the LW portfolio down to 0.03 percentage points for the SL portfolio. The crux of the matter is whether the mean differs significantly different from the mode. Here the difference is examined on the basis of the probabilistic distance between the mean and the mode (Mendelson 1987) – basically the chance that outcomes will occur between the estimated mean and mode, in essence if there is a significant chance, they will be deemed to be probabilistically different[13].

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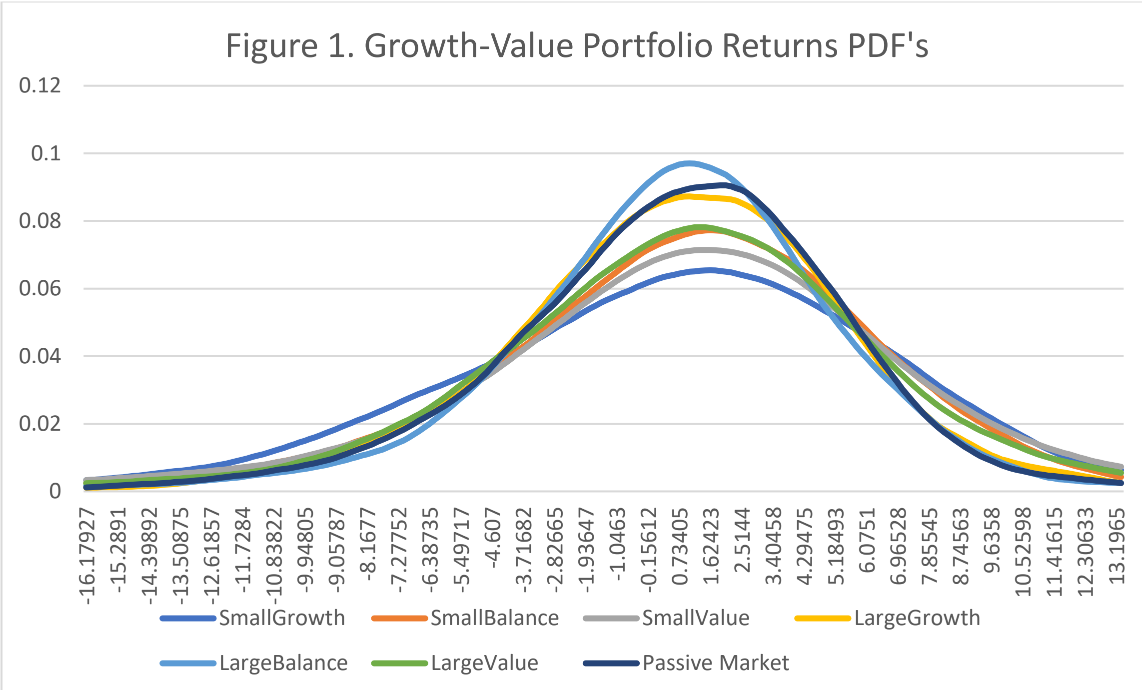
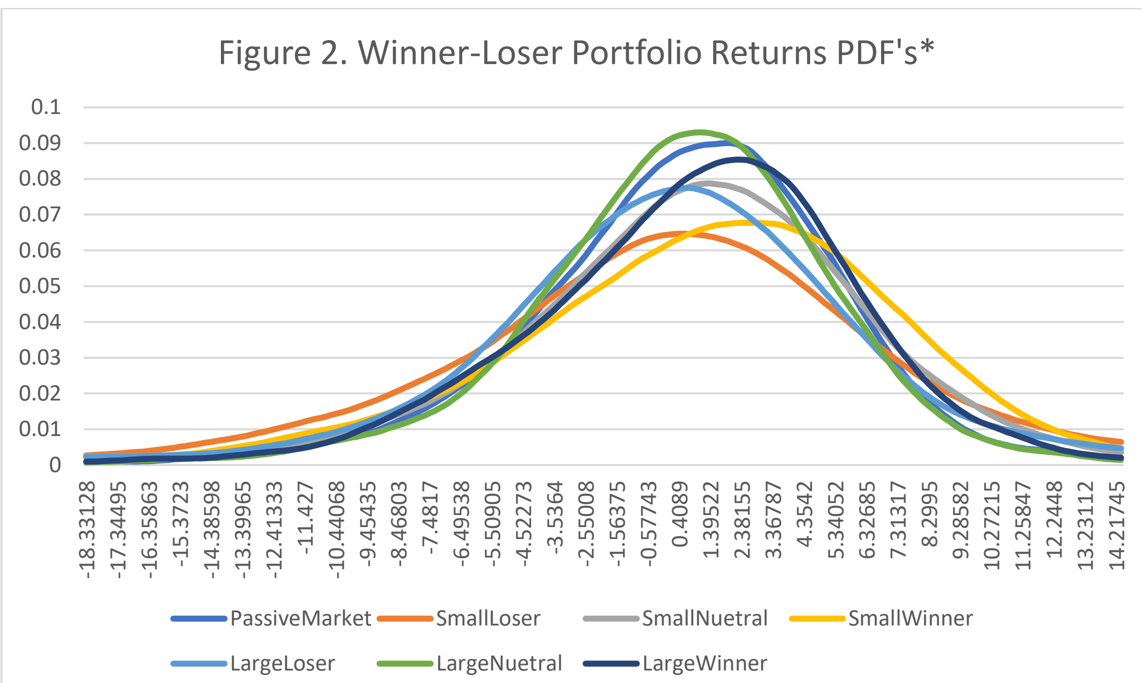
[12] Bickel (2003) provides a discussion of estimation of the mode but given the availability of kernel estimates this approach was deemed sufficient for present purposes.

[13] A simple test of a unimodal distributions' asymmetry is to consider (p_{dif}), the chance that two values which should be the same under symmetry (e.g. any pair of the mean, median or mode) are different. In the present case





Tables 2 reports the results together with the Skewness Factor, a measure of the extent and direction of skewness ((mean-mode)/Standard Deviation). As can be seen, for all but the Small Loser portfolio, the probabilistic distance between the mean and the mode is significantly greater than 0 and for the Small

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$p_{dif} = |F(\theta_t) - F(\mu_t)|$, which is the probabilistic distance between θ_t and μ_t, estimates of which can readily be shown to be $N(p_{dif}, \frac{p_{dif}(1-p_{dif})}{n})$ where n is the sample size.

Loser portfolio at is significantly greater for all confidence regions smaller than 95% and all but the Small Loser portfolio have negative skewness factors.

Table 2. Mean Mode differences.

Growth-Value	Probabilistic Distance	Standard Error	Z score	P(Distance>0)	Mean	Mode	Skewness Factor
Passmar	0.1323	0.0103	12.8366	1.0000	0.6503	1.9506	-0.24183
SmG	0.0665	0.0076	8.7825	1.0000	0.6874	1.6539	-0.12777
SmB	0.0536	0.0068	7.8285	1.0000	0.9805	1.6539	-0.09602
SmV	0.0185	0.0041	4.5140	1.0000	1.1854	1.4462	-0.03185
LaG	0.0277	0.0050	5.5548	1.0000	0.6271	0.9714	-0.06463
LaB	0.0518	0.0067	7.6848	1.0000	0.6876	1.0604	-0.06541
LaV	0.0434	0.0062	7.0096	1.0000	0.9084	1.3275	-0.05840
Winner-Loser	**Probabilistic Distance**	**Standard Error**	**Z score**	**P(Distance>0)**	**Mean**	**Mode**	**Skewness Factor**
Passmar	0.1310	0.0103	12.7382	1.0000	0.6466	1.8884	-0.23042
SmL	0.0028	0.0016	1.7345	0.9586	0.5417	0.5075	0.00372
SmN	0.0325	0.0054	6.0147	1.0000	0.9847	1.2966	-0.04300
SmW	0.0827	0.0084	9.8501	1.0000	1.3264	2.4802	-0.15877
LaL	0.0269	0.0049	5.4592	1.0000	0.3788	0.6390	-0.03427
LaN	0.0502	0.0067	7.5401	1.0000	0.6110	1.0336	-0.07588
LaW	0.1282	0.0102	12.5818	1.0000	0.9292	2.3158	-0.25064

The basic idea here is to use the modal return as the most likely return to be expected. Analogous to the use of the standard deviation as a measure of risk associated with the mean, the risk associated with the most likely return can be expressed as the square root of the variation around the mode. Table 3 reports the Mean and Modal values of the portfolio returns together with their respective standard deviations probability density function values, Sharpe and Sortino Ratios and respective ranks together with the Odds ratio. Note that with one exception (Small Loser) the modal value is always greater than the mean value so the distributions are generally left skewed. The standard deviation based upon the modal value is always greater than the standard deviation based upon the mean value which is as it should be since it is readily shown that the mean value minimizes the standard deviation function around it as the focus of that function.

With a sample standard error in the region of 0.2, the sample mean rarely sees the mode within its 95% confidence band and the odds ratio always records a greater likelihood of observing the modal value rather than the expected or mean value when they are different. Even if the Mode and the Mean are different, they may be sufficiently close together so that the ranking of the portfolios is unaffected. Tables 3 and 3a report the Most Likely or Modal focused Sharpe Ratio along with the Standard Expected Value or Mean focused Sharpe ratio together with their respective portfolio rankings.

Table 3. Growth/Value Sharpe Ratio Analysis

	Mode	σMode	f(mode)	Sharpe	rank	Mean	σMean	f(mean)	Sharpe	rank	odds
Passive Market	1.9506	5.5322	0.0905	0.3526		0.6503	5.3771	0.0885	0.1209		1.0229
Small Growth	1.6539	7.6256	0.0654	0.2169	2	0.6874	7.5640	0.0643	0.0909	6	1.0172
Small Balanced	1.6539	7.0453	0.0773	0.2348	1	0.9805	7.0131	0.0762	0.1398	2	1.0138
Small Value	1.4462	8.1923	0.0714	0.1765	6	1.1854	8.1881	0.0713	0.1448	1	1.0016
Large Growth	0.9714	5.3385	0.0873	0.1820	5	0.6271	5.3274	0.0867	0.1177	5	1.0061
Large Balanced	1.0604	5.7127	0.0970	0.1856	3	0.6876	5.7005	0.0965	0.1206	4	1.0058
Large Value	1.3275	7.1889	0.0782	0.1847	4	0.9084	7.1767	0.0776	0.1266	3	1.0077

Growth/Value Sortino Ratio Analysis

	Mode	σMode	f(mode)	Sharpe	rank	Mean	σMean	f(mean)	Sharpe	rank	odds
Passive Market	1.9506	5.8878	0.0905	0.3313		0.6503	5.6681	0.1147	0.0885		1.0229
Small Growth	1.6539	7.7257	0.0654	0.2141	2	0.6874	7.5058	0.0916	0.0643	6	1.0172
Small Balanced	1.6539	6.9977	0.0773	0.2363	1	0.9805	6.8889	0.1423	0.0762	4	1.0138
Small Value	1.4462	7.5387	0.0714	0.1918	4	1.1854	7.4961	0.1581	0.0713	5	1.0016
Large Growth	0.9714	5.7126	0.0873	0.1701	6	0.6271	5.6251	0.1115	0.0867	2	1.0061
Large Balanced	1.0604	5.6364	0.0970	0.1881	5	0.6876	5.6736	0.1212	0.0965	1	1.0058
Large Value	1.3275	6.8859	0.0782	0.1928	3	0.9084	6.8923	0.1318	0.0776	3	1.0077

Table 3a Winner/Loser Sharpe Ratio Analysis

	Mode	σMode	f(mode)	Sharpe	rank	Mean	σMean	f(mean)	Sharpe	rank	odds
Passive Market	1.8884	5.5306	0.0900	0.3414		0.6466	5.3893	0.0882	0.1200		1.0200
Small Loser	0.5075	9.1801	0.0647	0.0553	6	0.5417	9.1800	0.0647	0.0590	5	1.0000
Small Neutral	1.2966	7.2610	0.0787	0.1786	4	0.9847	7.2543	0.0784	0.1357	3	1.0044
Small Winner	2.4802	7.3580	0.0677	0.3371	2	1.3264	7.2669	0.0664	0.1825	1	1.0200
Large Loser	0.6390	7.5991	0.0775	0.0841	5	0.3788	7.5947	0.0773	0.0499	6	1.0018
Large Neutral	1.0336	5.5847	0.0930	0.1851	3	0.6110	5.5687	0.0926	0.1097	4	1.0046
Large Winner	2.3158	5.7035	0.0854	0.4060	1	0.9292	5.5322	0.0815	0.1680	2	1.0484

Winner/Loser Sortino Ratio Analysis

	Mode	σMode	f(mode)	Sharpe	rank	Mean	σMean	f(mean)	Sharpe	rank	odds
Passive Market	1.8884	5.8613	0.0900	0.3222		0.6466	5.6797	0.1138	0.0882		1.0200
Small Loser	0.5075	8.0370	0.0647	0.0631	6	0.5417	8.0390	0.0674	0.0647	6	1.0000
Small Neutral	1.2966	6.9146	0.0787	0.1875	3	0.9847	6.9247	0.1422	0.0784	3	1.0044
Small Winner	2.4802	7.7398	0.0677	0.3204	2	1.3264	7.5206	0.1764	0.0664	5	1.0200
Large Loser	0.6390	7.0274	0.0775	0.0909	5	0.3788	7.0370	0.0538	0.0773	4	1.0018
Large Neutral	1.0336	5.6027	0.0930	0.1845	4	0.6110	5.5872	0.1094	0.0926	1	1.0046
Large Winner	2.3158	6.3115	0.0854	0.3669	1	0.9292	6.0480	0.1536	0.0815	2	1.0484

Notice that, within the Growth / Value portfolio set, the orderings of the portfolios change substantially under the Most Likely Sharpe measure ordering as opposed to the Standard Expected Sharpe measure ordering but not so much under the Loser / Winner portfolio set. Indeed, in the latter, the “winner preferred to neutral preferred to loser” ordering is preserved under both Most Likely and Expected comparators the only change in the ordering is that Large is preferred to Small in corresponding Winner/Loser designations in the Most Likely paradigm whereas Small is preferred to Large in the Expected Return paradigm. Table 4 gives a sense of the magnitude of the differences in rankings (if outcomes were unimodal and symmetric and rankings were consistent these would all be 0.)

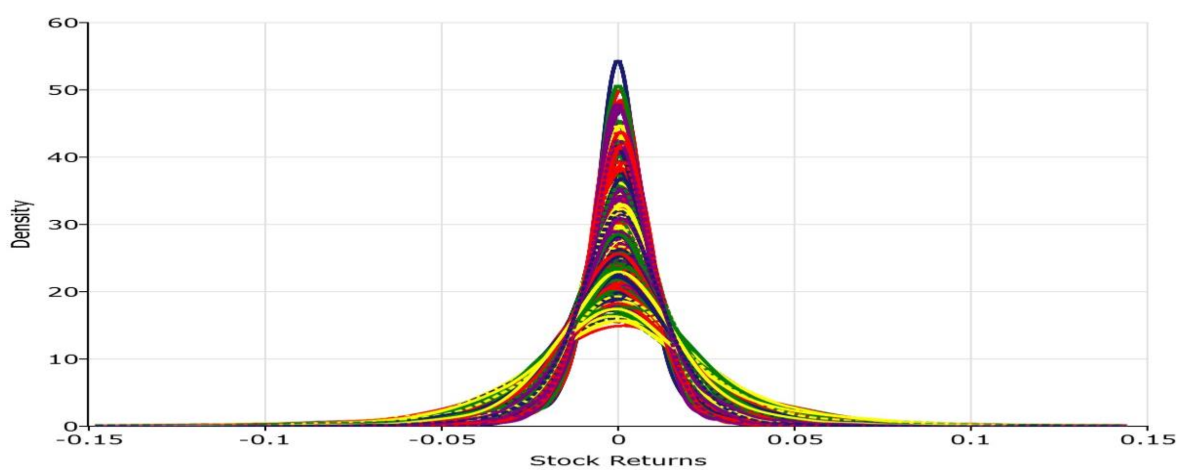
Table 4. Square Root of average squared difference of Ranks.

	Sharpe Mode v Mean	Sortino Mode v Mean		Sharpe v Sortino Mode	Sharpe v Sortino Mean
Growth/Value	2.7080	3.1091	Growth/Value	1.2910	0.5774
Winner/Loser	1.0000	1.8257	Winner/Loser	2.5166	2.2361

The Standard and Poor 500

It could be argued that the Fama-French portfolio returns data was constructed with a particular objective in mind i.e. studying portfolio returns patterns when the portfolios were designed to reflect particular market behaviour patterns with skewness properties effectively being self-selected. To check where similar results prevail more generally, 2732 daily returns on 441 SP500 stocks over the period January 2005 to November 2015 were studied for potential Expected vs Most Likely return differences. Figure 3 illustrates their respective Probability Density Functions which look reasonably normal with similar location parameters.

Figure 3.



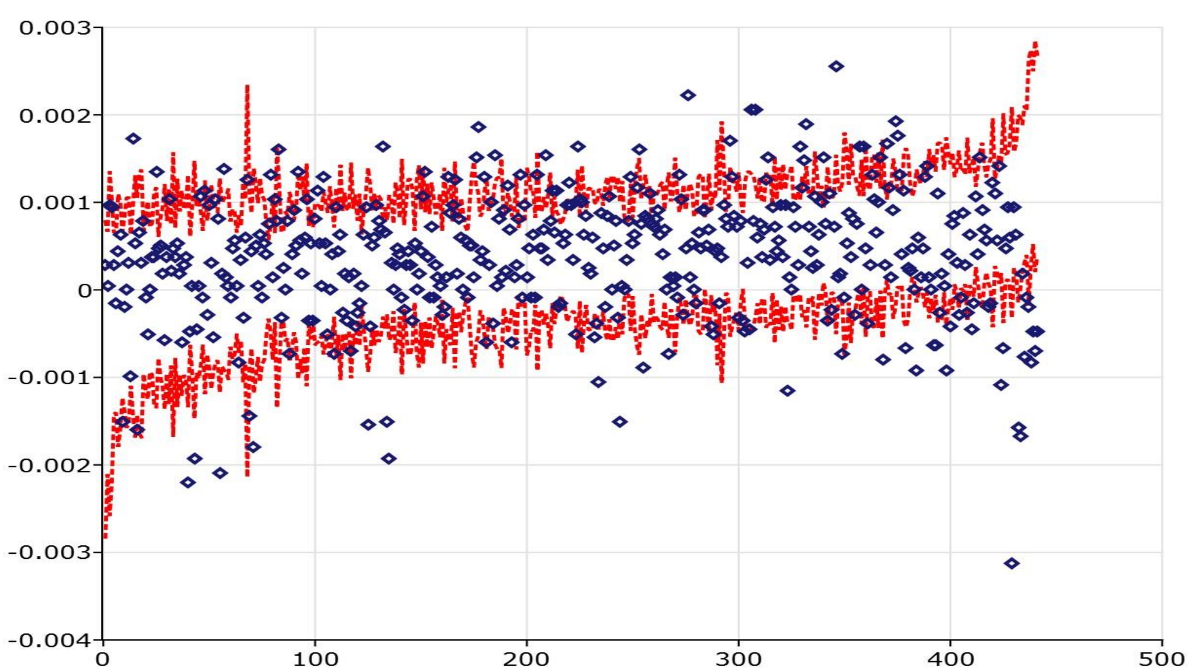
Their skewness was initially investigated in terms of their cumulative distribution function values at the mean (F_μ) and at the mode (F_θ). When the distribution is symmetric unimodal $F_\mu = F_\theta$, when left skewed $F_\mu < F_\theta$ and when right skewed $F_\mu > F_\theta$.

Table 5. F_μ , F_θ Cumulative Distribution Differences Proportion of rejections of $H_0: F_\mu = F_\theta$.

% $F_\mu \neq F_\theta$	% $F_\mu > F_\theta$	% Rejects 0.1% CV	% Rejects 0.5% CV	% Rejects 1.0% CV
96.8254	54.1950	87.0748	90.7029	91.8367

As Table 5 reports, a large portion of the stocks have statistically significant mean and modal difference values with right skewness being predominant. The average negative $\mu - \theta$ value was -0.0006713 with a standard error of 0.00004615 yielding a significantly below zero Z score of 14.5435 and the average positive $\mu - \theta$ value was 0.0005518 with a standard error of 0.00002574 yielding a significantly above zero Z score of 21.4411.

Figure 4. 95% confidence interval around the mean for the 441 stocks over 2005-2015 ordered from left to right by the level of the mean with modes identified as blue scatter plots.



Clearly there are many stocks for which the mode is outside the mean 95% confidence interval. To get a sense of the mean and modal juxtapositions, a simple regression over the 441 stocks of the mean -

mode difference on the median value (to see if there is a location effect) and the PDF value at the mode (to see if there is a concentration effect) was performed. Table 6 presents the results which indicate a clearly negative relationship between the mean modal difference and the location of the distribution, concentration around the mode seems to have little effect.

Even though means and medians are clearly different, what really matters is whether the differences are sufficient to affect the ordering of the stocks. The Spearman (1904) Rank Correlation statistic is 0.08813 with an asymptotic standard normal value of 1.8537995 which is significant at any confidence region smaller than 0.96812 indicating that the ordering is affected by whether choice is based upon Expected as opposed to Most Likely outcomes.

Table 6.

	Mean-mode	Median	f(mode)
Mean	0.0000304120	-0.00044238525	28.611449
Maximum	0.0020356613	0.00087602500	54.266055
Minimum	-0.0040077035	-0.0013302900	14.926926
Regression	Constant	Median	f(mode)
Coefficient	-0.0006345674	-1.3656496	0.00000213
Standard error	0.0001219894	0.0904584	0.00000409
Z score	-5.2018222	-15.096987	0.51980483
P(Z>|Z score|)	0.0000000987	0.0000000	0.30159981

Sigma 0.0000004117 R squared 0.34797067

Conclusions.

For the longest time, based upon notions of rational expectations, expected utility theory and a presumed symmetry of outcome distribution, investment decisions have been based upon risk adjusted expected outcomes with Sharp and Sortino ratios being the embodiment of this approach in portfolio selection. However, when the future returns to an investment are asymmetrically distributed, the expected return will not be the same as the most likely return and, contrary to the rational expectations hypothesis, investors who have based their decisions on expected outcomes will experience systematic surprises. The question then arises as to whether the investment decision should be based upon perceived expected outcomes or perceived most likely outcomes. Even when expected outcomes and most likely outcomes are demonstrably different, they may be sufficiently close together so that operating on the basis of the former is a good approximation to operating on the basis of the latter. Here, introducing the notion of a “Most Likely Focused” Sharpe ratio, these distinctions have been explored using two sets of portfolios, one formed on the basis of growth and value and the other formed on the basis of winners and losers, each drawn from the Fama and French data set. Portfolio returns were found to be decidedly asymmetric and usually left skewed. Ordering the portfolios on the basis of the most likely return made a difference in almost every case when compared to ordering on the basis of the Expected Return which gives some pause for thought when employing standard Sharpe Ratio methods. The Fama French portfolios were designed for a purpose which may have affected the skewness properties of their respective distributions. To counter this, daily returns on 441 stocks were examined and similar deviations from unimodal symmetry revealed in both left and right directions, furthermore, ordering the stocks on the basis of Expected Returns as opposed to Most Likely Returns was seen to change the ordering significantly. All in all this gives much pause for thought as to whether choice should be based on the Expected or Most Likely outcomes of alternatives.

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Appendix 1. A Most Likely Focused Coefficient of Variation and Sharpe Ratio for Ordered Categorical Data.

Sometimes investment returns may be ordinal in nature and ranked as very low, low, medium, high and very high for example. Anderson (2023) developed a Coefficient of Variation and corresponding Sharpe Ratio for such ordered categorical data environments. The problem in that paradigm is that ordinal data does not possess cardinal measure unless it is artificially endowed and there are problems with ambiguity and equivocation associated with such a practice (Bond and Lang 2019). However, by appealing to the notion of probabilistic distance (Mendelson 1987) it is possible to develop unambiguous and unequivocal measurement in ordinal paradigms. Suppose $\boldsymbol{K \geq 3}$ ordered categories indexed $\boldsymbol{k = 1,..,K}$ with higher $\boldsymbol{k}$ implying higher category. Endow the categories with a Probability Density Function $\boldsymbol{f}$ described by the probabilities $\boldsymbol{p_{fk}, k = 1,..,K}$, of being in the $\boldsymbol{k}$'th category under $\boldsymbol{f}$ where $\boldsymbol{p_{fk} \geq 0}$ and $\boldsymbol{\sum_{k=1}^{K} p_{fk} = 1}$. For $\boldsymbol{k = 1,..,K}$, the Cumulative Distribution Function $\boldsymbol{F}$ is given by $\boldsymbol{F_k = \sum_{i=1}^{k} p_{fi}}$ and the Survival Function $\boldsymbol{S}$ is given by $\boldsymbol{S_k = 1 - F_k}$. Analogous to the continuous paradigm formulation of the mean as the integral of the survival function over the range of $\boldsymbol{x}$, the sum of the Survival Function values over all categories could be considered as a “Mean Ordered Categorical” or $\boldsymbol{MOC}$ location measure[14] where:

[14]Note that, with a potential minimum value of 0 (when all probability mass is in the lowest category) and a maximum potential value of $K - 1$ (when all probability mass is in the highest category), MOC is not independent of K, the number of categories. While this is of no consequence when group outcomes are being compared across a common number of categories, it does matter when different groups have

$$MOC = \sum_{k=1}^{K} S_k$$

An ordered categorical measure of variation.

For a given outcome $k^* \in 1,..,K$ and outcomes $k = k^*+1,..,K$, define the Upper Cumulants of f with respect to k^* as $F_k^{U,k^*} = \sum_{i=k^*+1}^{k} p_{fi}$ (note for $k \leq k^*$,$F_k^{U,k^*} = 0$) and, for outcomes $k = 1,..,k^*-1$, define its Lower Cumulants as $F_k^{L,k^*} = \sum_{i=k}^{k^*-1} p_{fi}$ (note for $k \geq k^*$,$F_k^{L,k^*} = 0$). It may be seen that $\frac{F_k^{L,k^*}}{F_1^{L,k^*}}$ $k = 1,..,k^*-1$ is in effect the SF of the below k^* conditional PDF, whereas $\frac{F_k^{U,k^*}}{F_K^{U,k^*}}$ $k = k^*+1,..,K$ is the CDF of the above k^* conditional PDF. When $k > k^*$, F_k^{U,k^*} is the probability of an outcome between k^* and $k+1$ occurring which is monotonically non decreasing in k, when $k < k^*$, F_k^{L,k^*} is the probability of an outcome between k^* and $k-1$ occurring which is monotonically non-decreasing in k^*-k. Each record a sense of probabilistic distance of k from k^* in terms of the chance that an outcome will emerge between k and k^* which increases with $|k^*-k|$. Similarly defining G_k^{U,k^*}, G_k^{L,k^*}, the Upper and Lower Cumulants of g about k^*, then g constitutes an increasing spread of f with respect to outcome k^* when:

$$G_k^{L,k^*} \geq F_k^{L,k^*} \forall\, k = 1,.,k^*-1 \; and \; G_k^{U,k^*} \geq F_k^{U,k^*} \forall\, k = k^*+1,.,K \; with > somewhere. \quad [3]$$

The Mendelson (1987) condition [3] amounts to a first order stochastic dominance condition on the "downward looking" below k^* conditional distributions (i.e. imagine the category orderings below k^* were reversed) and the "upward looking" above k^* conditional distributions where f dominates g in each context. Intuitively, with respect to k^* inequality in g distribution is greater than inequality in f distribution with respect to k^* when the chance of below k^* outcomes and the chance of above k^* outcomes are both at least as great in g as they are in f with strictly greater than in at least one case.

These ideas can be employed to develop the concept of a modal preserving spread. Basically g constitutes a Modal Preserving Spread of f if [3] holds and k^*remains the modal outcome of g i.e. $p_{gk^*} = \max_k p_{gk}$. Given dispersion from the focus point k^* is maximized when k^*/K mass is allocated to the lowest outcome and $\frac{(K-k^*)}{K}$ is allocated to the highest outcome:

$$0 \leq IMPS(g,f) = \frac{\sum_{k=1}^{K}((G_k^{U,k^*}-F_k^{U,k^*})+(G_k^{L,k^*}-F_k^{L,k^*}))}{\left(\frac{k^*\sum_{i=1}^{k^*-1} i}{K}+\frac{(K-k^*)\sum_{i=k^*+1}^{K}(i-k^*)}{K}-\sum_{k=1}^{K}\left(F_k^{U,k^*}+F_k^{L,k^*}\right)\right)} \leq 1$$

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different numbers of categories. This can be resolved by dividing MOC by $K-1$ rendering it a number on the unit interval for all possible K.

provides an index measure on the unit interval of the extent of increased Modally Focused relative spread or inequality associated with a move from f to g. Suppose f^e was the distribution of a completely equal group with all elements experiencing outcome k^*, then $p_{fk^*} = 1$ and $p_{fk} = 0 \; \forall \; k \neq k^*$ so that $F_k^{U,k^*} = 0$ and $F_k^{L,k^*} = 0 \; \forall \; k$, then $IMPS(g, f^e)$ a measure of variation around an ordered categorical mode becomes:

$$IMPS(g, f^e) = \frac{\sum_{k=1}^{K}(G_k^{U,k^*} + G_k^{L,k^*})}{\left(\frac{(k^*-1)\sum_{i=1}^{k^*-1} i}{K} + \frac{(K-k^*)\sum_{i=k^*+1}^{K}(i-k^*)}{K}\right)} \qquad [4]$$

And the Ordered Categorical Coefficient of Variation becomes $IMPS(g, f^e)/MOC$ with its inverse becoming an Ordered Categorical Most Likely Sharpe Ratio.