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Department of Economics

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Sequential entry into electoral competition when the possibility of ties is limited

By Martin J. Osborne

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Abstract

The members of a finite set of office-motivated politicians choose sequentially whether to become candidates in an electoral competition. Each candidate chooses a position from a set X that is a (possibly strict) subset of the set of all positions. I show that if X is a subset of a one-dimensional interval, a tie is possible only among candidates who choose the same position, and a candidate wins if her vote share exceeds $\frac{1}{2}$ and only if it is at least as large as any other candidate's vote share, then in every subgame perfect equilibrium the first and last politicians to move enter at one of the members of X closest to the median of the citizens' favorite positions and the remaining politicians do not enter. The assumption about ties is satisfied if the winner of the election is chosen from among the candidates with the highest vote shares by a mediator with strict preferences over positions or if the set X does not admit ties at distinct positions.

*Department of Economics, University of Toronto. I thank John Duggan and Jeffrey Rosenthal for helpful discussions.

1. Introduction

The members of a set of office-motivated politicians decide sequentially whether to stand as candidates in an election. Which ones choose to stand, and what positions do they take?

I analyze an extensive game with perfect information. The model departs from previous work in its handling of ties. In real elections, ties are rather unusual, and their possibility does not appear to much affect candidates' strategic reasoning. But in the standard analysis of multicandidate models of electoral competition with perfect information, ties play a key role, and the way in which they are handled can significantly affect the equilibria. A common assumption is that if each member of a set of candidates receives the same number of votes, each member of the set wins with the same probability. I explore the implications of assuming that such ties occur only if the candidates' positions are the same. That is, no tie is possible between candidates whose positions differ. One rationale for this assumption is that if candidates whose positions differ tie for the highest vote share, the winner is selected deterministically by a mediator with strict preferences over the set of possible positions. Another rationale is that the number of positions possible for candidates is finite and for no configuration of positions are any candidates whose positions differ tied for the highest vote share.

In the model, the set X from which each candidate chooses a position is a (possibly strict) subset of the set of all positions. I show (Proposition 1) that if the set of all positions is the real line, and (*i*) candidates whose positions are the same win with the same probability, (*ii*) any two candidates have positive probabilities of winning only if their positions are the same, (*iii*) a candidate whose vote share exceeds $\frac{1}{2}$ wins with certainty, (*iv*) a candidate wins only if her vote share is at least as high as every other candidate's vote share, and (*v*) the set X contains points close to the median of the citizens' favorite positions, then the game has a unique subgame perfect equilibrium, and in this equilibrium the first and last politicians to move choose the position in X closest to the median of the citizens' favorite positions and no other politician competes in the election.

2. Model

I analyze extensive games with perfect information in which each player in the set $N = \{1, \ldots, n\}$ in turn chooses either a *position*, in which case she becomes a *candidate* in an election, or not to compete, an action I denote ϕ. I assume that the set of possible positions, X, is a nonempty closed subset of $\mathbb{R}^q$ for some $q \geq 1$ and that the outcome of the election is given by a collection $(p_i)_{i \in N}$ of functions, where each function p_i associates with each action profile the probability that i wins the election.

Definition 1. An *election game* $\langle X, n, c, b, (p_i)_{i=1}^n \rangle$, where

- X is a nonempty closed subset of $\mathbb{R}^q$ for some integer $q \geq 1$ (the set of positions possible for candidates)
- $n \geq 2$ is an integer (the number of candidates)
- $c > 0$ (the cost of running) and $b > nc$ (the benefit of winning) are numbers
- for each $i \in \{1, \ldots, n\}$, $p_i : (X \cup \{\phi\})^n \to \mathbb{R}_+$ with $p_i(a) = 0$ if $a_i = \phi$ and $\sum_{\{i \in N : a_j \in X\}} p_j(a) = 1$ ($p_i(a)$ is the probability that candidate i wins when the action profile is a)

is an extensive game with perfect information with the following components.

Players $N = \{1, \ldots, n\}$.

Terminal histories (outcomes) $\{(a_1, \ldots, a_n) : a_i \in X \cup \{\phi\}$ for all $i \in N\}$.

Player function $P(h) = k + 1$ for every history h of length k, for all $k \in \{0, \ldots, n-1\}$.

Payoff function For each terminal history a, the payoff of each player $i \in N$ is

$$\begin{cases} p_i(a)b - c & \text{if } a_i \in X \\ 0 & \text{if } a_i = \phi. \end{cases}$$

I assume that $b > nc$ so that if all n players become candidates and win with the same probability, their common payoff is positive, and hence each of them prefers to be a candidate than to stay out of the competition.

I study subgame perfect equilibria of election games. I refer to a terminal history of the game (a sequence of actions, one for each player) as an *outcome*, and to a terminal history that is generated by a subgame perfect equilibrium as a *subgame perfect equilibrium outcome*.

The only restriction on the functions p_i imposed by the definition is that a player who does not enter the competition does not win. I now discuss additional restrictions. I assume first that the probabilities of winning depend only on the players' actions, not on their names.

Assumption 1 (Anonymity)**.** *For any permutation $\rho : N \to N$ and outcome a, define the outcome b by $b_i = a_{\rho(i)}$ for all $i \in N$. Then $p_i(b) = p_{\rho(i)}(a)$ for all $i \in N$.*

In particular, for any players i and j and any outcome a for which $a_i = a_j$ we have $p_i(a) = p_j(a)$.

My next assumption is more consequential: two players win with positive probability *only* if they choose the same action.

Assumption 2 (No dispersed ties)**.** *For any players i and j and any outcome a for which $p_i(a) > 0$ and $p_j(a) > 0$ we have $a_i = a_j$.*

This assumption distinguishes my model from previous analyses. In the last two subsections of the paper I discuss two models of the determination of the functions p_i that imply the assumption. In one case, ties are broken by an external agent with strict preferences over the set of positions. In the other case, the set X of possible positions simply does not admit ties: for no profile of positions do any candidates tie.

Assumptions 1 and 2 imply that every player's probability of winning is either 0 or $1/k$ for some $k \in \{1, \ldots, n\}$, and in particular takes one of finitely many values, ensuring that the game has a subgame perfect equilibrium. These assumptions imply further that in each subgame perfect equilibrium every candidate's position is the same and player n is one of the candidates.

Lemma 1. *Under Assumptions 1 and 2, every election game has a subgame perfect equilibrium, and for each subgame perfect equilibrium there is a position x such that every candidate's positions is x and player n is one of the candidates.*

Proof. For an outcome in which no player enters, player n can deviate to any position and win with probability 1, increasing her payoff from 0 to $b - c > 0$. Thus in any subgame perfect equilibrium in which no player $1, \dots, n-1$ is a candidate, player n is a candidate.

Now consider an outcome in which $k \geq 1$ of the players $1, \dots, n-1$ are candidates and player n is not. For this outcome to be generated by a subgame perfect equilibrium, each candidate must win with positive probability (because otherwise she can increase her payoff by exiting) and hence by Assumption 2 the positions of the candidates are the same. If player n deviates to become a candidate and chooses this common position, her probability of winning is at least $1/n$ by Assumption 1, so that her payoff increases from 0 to at least $b/n - c > 0$. □

To specify the additional restrictions on the functions p_i that I assume, I need to describe how the vote share of each candidate is determined.

I assume that in the background is a continuum of citizens, each of whom has a favorite point in $\mathbb{R}^q$. (These points may or may not be members of X.) I extend the model of an election game by specifying the distribution of these favorite points.

Definition 2. An *election game with a voter distribution* consists of an election game $\langle X, n, c, b, (p_i)_{i=1}^n \rangle$ and a nonatomic probability measure F on $\mathbb{R}^q$, where q is the dimension of X.

A position x chosen by one or more candidates attracts the votes of all citizens whose favorite positions are closer to x than to any other position chosen by a candidate. These votes are divided equally among the candidates whose positions are equal to x.

Precisely, let $\langle \langle X, n, c, b, (p_i)_{i=1}^n \rangle, F \rangle$ be an election game with a voter distribution, let a be a terminal history of the game, and let i be a player. Suppose that $a_i \in X$ (player i enters and chooses the position a_i) and denote by $S_i(a)$ the set of players who choose the position a_i:

$$S_i(a) = \{j \in N : a_j = a_i\}.$$

Then the *vote share* $v_i(a)$ of candidate i for the terminal history a is the fraction $1/|S_i(a)|$ of the measure, according to F, of the voters whose favorite positions are closer to a_i than to the position of any player whose position is different from a_i:

$$v_i(a) = \frac{F(\{x \in \mathbb{R}^q : \|x - a_i\| < \|x - a_j\| \text{ for all } j \in N \setminus S_i(a) \text{ with } a_j \in X\}}{|S_i(a)|}. \tag{1}$$

(Because F is nonatomic, the measure of voters whose favorite position is equidistant from a_i and the position of any player in $N \setminus S_i(a)$ is zero.) If $a_i = \phi$, then $v_i(a) = 0$.

I assume that a candidate who receives either more than half of the votes or exactly half of the votes and more than any other candidate is elected with probability one.

Assumption 3 (Majority rule)**.** *If (i)* $v_i(a) > \frac{1}{2}$ *or (ii)* $v_i(a) = \frac{1}{2}$ *and* $v_i(a) > v_j(a)$ *for all* $j \in N \setminus \{i\}$, *where* v_i *is given in (1), then* $p_i(a) = 1$.

3. One-dimensional positions

3.1 General analysis

I now restrict to election games in which the set of positions is one-dimensional, the support of the distribution of the voters' favorite positions is an interval, and this distribution has a unique median.

Definition 3. A *one-dimensional election game with a voter distribution* is an election game with a voter distribution $\langle\langle X, n, c, b, (p_i)_{i=1}^n\rangle, F\rangle$ for which $X \subseteq \mathbb{R}$, the support of F is an interval, and F has a unique median.

For any $x \in \mathbb{R}$, I denote simply by $F(x)$ the fraction $F((-\infty, x])$ of citizens with favorite positions at most x, and denote by m the median of F. Define $m^*(X)$ as follows. If $m \in X$, then $m^*(X) = m$. Otherwise, let $M(X)$ be the set of positions in X closest to m:

$$M(X) = \{x \in X : |x - m| \le |y - m| \text{ for all } y \in X\}. \tag{2}$$

This set is nonempty because X is closed, and it contains at most two positions. If it contains one position, $m^*(X)$ is that position. If it contains two positions, Assumptions 1 and 2 imply that only one of them has the property that a candidate who chooses that position has a positive probability of winning in an election in which each of the positions is chosen by a single player and no other player competes in the election (and this probability is 1); let $m^*(X)$ be that position. (Assumption 1 implies that the winning position in such an election does not depend on the identity of the candidates.)

Lemma 2. *Let $\langle\langle X, n, c, b, (p_i)_{i=1}^n\rangle, F\rangle$ be a one-dimensional election game with a voter distribution that satisfies Assumptions 1, 2, and 3. In every subgame perfect equilibrium of this game at least two players are candidates, the position of every candidate is $m^*(X)$, and player n is one of the candidates.*

Proof. By Lemma 1, every candidate's position is the same and player n is one of the candidates.

Step 1. *In every subgame perfect equilibrium at least two players are candidates.*

Proof. Suppose that player n is the only candidate. Consider a deviation by player $n-1$ to enter at $m^*(X)$. If, following this deviation, player n enters at a position different from $m^*(X)$ then her probability of winning is zero (by the definition of $m^*(X)$); if she enters at $m^*(X)$ then she and player $n-1$ each wins with probability $\frac{1}{2}$ by Assumption 1. Thus following player $n-1$'s deviation, the best action of player n is to enter at $m^*(X)$, in which case player $n-1$ wins with probability $\frac{1}{2}$, making her deviation profitable. $\lhd$

Step 2. *In every subgame perfect equilibrium every candidate's position is $m^*(X)$.*

Proof. Suppose that the candidates' common position is not $m^*(X)$ and that player n deviates to enter at $m^*(X)$. If $m \in X$, so that $m^*(X) = m$, or $M(X)$ (defined in (2)) contains a single position, n's vote share exceeds $\frac{1}{2}$, so that by Assumption 3 she wins the election with probability 1. If $M(X)$ contains two positions and two or more of the other players are candidates, n's vote share either exceeds $\frac{1}{2}$ (if the other candidates' common position is not in $M(X)$) or equals $\frac{1}{2}$ and exceeds the vote share of each of the other candidates (if the other candidates' common position is in $M(X)$). In both cases, player n wins the probability 1 by Assumption 3. Finally, if $M(X)$ contains two positions and exactly one of the other players is a candidate, player n's probability of winning is 1 by the definition of $m^*(X)$. Thus in every case player n's deviation raises her probability of winning. $\lhd$

3.2 Plurality rule

For any terminal history a, denote by $W(a)$ the set of candidates whose vote share is maximal:

$$W(a) = \{i \in N : a_i \in X \text{ and } v_i(a) \geq v_j(a) \text{ for all } j \neq i\}, \tag{3}$$

where v_i is given in (1).

The next assumption is a distinguishing feature of plurality rule: only candidates whose vote shares are maximal win with positive probability.

Assumption 4 (Win only if vote share maximal)**.** *If* $p_i(a) > 0$ *then* $i \in W(a)$.

My final assumption is that the set X of possible positions contains points close to the median of F. Specifically, X contains a point x less than m such that the midpoint of x and m exceeds $F^{-1}(\frac{1}{3})$ and also a point y greater than m such that the midpoint of m and y is less than $F^{-1}(\frac{2}{3})$. That is, X contains a point in $(F^{-1}(\frac{1}{3}) - (m - F^{-1}(\frac{1}{3})), m)$, or $(2F^{-1}(\frac{1}{3}) - m, m)$, and a point in $(m, 2F^{-1}(\frac{2}{3}) - m)$. This condition is illustrated in Figure 1 and stated precisely as follows.

Assumption 5 (Rich domain)**.** *The set X of positions contains a point in* $(2F^{-1}(\frac{1}{3}) - m, m)$ *and a point in* $(m, 2F^{-1}(\frac{2}{3}) - m)$.

My main result for election games with one-dimensional sets of positions is that under Assumptions 1–5 in any subgame perfect equilibrium only players 1 and n enter, and they do so at one of the positions in X closest to the median of the voters' favorite positions.

Proposition 1. *Let* $\langle\langle X, n, c, b, (p_i)_{i=1}^n\rangle, F\rangle$ *be a one-dimensional election game with a voter distribution. If this game satisfies Assumptions 1–5 then it has a unique subgame perfect equilibrium outcome, and in this outcome players* 1 *and* n *are candidates at* $m^*(X)$ *and players* $2, \ldots, n-1$ *do not compete in the election.*

Proof. Denote the game by Γ. By Lemma 2, in every subgame perfect equilibrium of Γ, player n and at least one other player are candidates and their common position is $m^*(X)$.

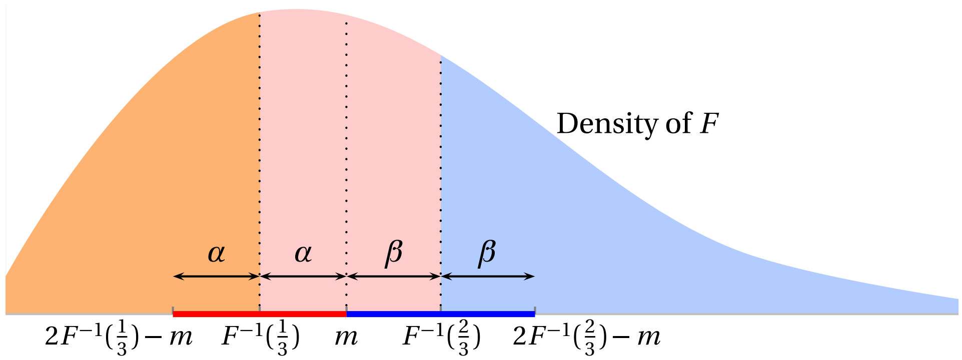

Figure 1: Assumption 5 requires that X contain a point in $(2F^{-1}(\frac{1}{3})-m, m)$ (the red interval) and a point in $(m, 2F^{-1}(\frac{2}{3})-m)$ (the blue interval).

Step 1. *In every subgame perfect equilibrium of* Γ*, player* n *and exactly one other player are candidates.*

Proof. Consider an equilibrium with two or more candidates in addition to player n. Suppose without loss of generality that $m^*(X) \leq m$ and that player n deviates to a position $y \in X \cap (m, 2F^{-1}(\frac{2}{3})-m)$. (Such a position exists by Assumption 5.) Then her vote share is

$$1 - F(\tfrac{1}{2}(m^*(X)+y)) > 1 - F(\tfrac{1}{2}(m + 2F^{-1}(\tfrac{2}{3}) - m)) = \tfrac{1}{3}.$$

At least two candidates still occupy the position $m^*(X)$, so that player n's vote share exceeds that of every other candidate, making her the unique member of $W(a)$, so that by Assumption 4 she wins with probability 1, obtaining the payoff $b-c$. Her previous payoff was either zero, if she was not a candidate, or at most $\frac{1}{3}b-c$, if she was, so that her deviation is profitable.

Thus in no subgame perfect equilibrium are there three or more candidates, so that in every equilibrium exactly one player in addition to n is a candidate. ◁

Step 2. *In every subgame perfect equilibrium of* Γ*, players 1 and* n*, and no other players, are candidates.*

Proof. Suppose that Γ has a subgame perfect equilibrium in which player 1 is not a candidate. Consider the subgame following player 1's deviation to become a candidate at

$m^*(X)$. In a subgame perfect equilibrium of this subgame, every player in $\{2, \ldots, n\}$ who becomes a candidate wins with positive probability (otherwise she can profitably deviate by exiting, obtaining the payoff 0 rather than $-c$). Thus by Assumption 2, the position of every such player is the same. If this position differs from $m^*(X)$, the candidates other than player 1 lose, by the definition of $m^*(X)$ and Assumption 3. Thus the common position of the candidates other than player 1 is the same as player 1's position, so that by Assumption 1, all candidates, including player 1, win with the same positive probability. Hence player 1's deviation to become a candidate is profitable. Thus Γ has no subgame perfect equilibrium in which player 1 is not a candidate. $\triangleleft$

3.2.1 Ties broken by a mediator

Suppose that for each action profile a the winner of the election is selected by a mediator from the set $W(a)$ of candidates i for whom the vote share $v_i(a)$ is highest. The mediator chooses the position she most prefers from those chosen by the candidates in this set and selects each of the candidates occupying that position with the same probability. Assume that $X = \mathbb{R}$ and the mediator's preference relation over X is strict: she is not indifferent between any two positions. (For example, for some strictly quasiconcave (single-peaked) function u on $\mathbb{R}$, she may choose the smallest position x among those for which $u(x)$ is largest.) The resulting functions p_i satisfy Assumptions 1–4, as assumed in Proposition 1.

3.2.2 Finite set of possible positions

Alternatively, suppose that the set X of positions is finite and has the property that for no action profile a is there a tie for the largest vote share $v_i(a)$ between candidates at different positions. If the winner is selected equi-probably from $W(a)$ (all of whose members must have the same position) then again the resulting functions p_i satisfy Assumptions 1–4. Intuition suggests that if for some positive integer k the set X is the result of k independent draws from a nonatomic distribution over $\mathbb{R}$, then as k increases without bound the probability that it has this property approaches 1.