University of Toronto
Department of Economics

VELUT ARBOR ÆVO

Working Paper 719

On Focused, Fit-For-Purpose Inequality Measurement.

By Gordon John Anderson

February 15, 2022

On Focussed, Fit-For-Purpose Inequality Measurement.

Gordon Anderson.

Economics Department

University of Toronto

February 2022

Abstract.

Many inequality indices are based on an aggregation of all individual outcome differences from some centrality measure or focus point, regardless of where that point is in the outcome spectrum. Often they are employed with normative intent, to highlight the lack of equality in outcomes, with equality viewed as a good thing wherever it occurs in the outcome spectrum. Yet, when outcomes are positively associated with wellbeing, while high inequality with respect to a high focus point, suggesting a preponderance of inferior outcomes, may be considered "bad", that same level of inequality measured with respect to a low focus point, suggesting a preponderance of superior outcomes, could be construed as relatively "good". In essence some inequality measures may be inappropriately focused. Here, a family of Focussed Inequality indices, together with their sampling distributions, is introduced. While the indices are formulated for multivariate unordered and ordered categorical data environments, they are readily extended to the continuous paradigm. They are exemplified in a study of the evolution of health and loneliness inequalities over the ageing process in China.

C13, C18, I14, I24, I31, I32

Key Words. Inequality Measurement, Inference, Focused Indices, Health Outcomes.

Introduction.

Since the work of Pigou (1920) and Dalton (1920) there has been normative interest in measuring Inequality. Popular inequality measures such as the Gini Coefficient (Gini 1921), or a Standardised Quantile Range, work with the totality of individual differences, the former measuring the average absolute difference and the latter measuring the maximum absolute difference between all possible agent pairs within some quantile range. They do so without reference to a point of centrality, other than for standardisation purposes, and can thus be considered Un-Focussed. On the other hand, measures such as the Coefficient of Variation or the Atkinson and Information Theoretic inequality measure families (Hendricks and Robey 1936, Theil 1967, Atkinson 1970, Maasoumi 1987) can be construed as Focussed, in that they work with relative distances from some centrality measure or Focus point, such as the mean or median. In these instances, it is the aggregated differences from the Focus Point that is of interest. However, exactly where the Focus Point is in the range of variation seems to be of no import, yet normative concerns with "the growing concentration of incomes within the hands of a small economic elite"[1] and phrases like "Equalizing opportunities", "Levelling the playing field" or "Levelling Up", frequently encountered in public discourse[2], suggest that inequality measures with a particular Focus Point may be appropriate[3] in some situations. With an underlying notion that all should be equal in some sense, this discourse frequently reflects a secondary imperative, articulating the need to recognize and rectify specific typologies of collective differences in terms of equality for all, at some "ideal" outcome level or distribution. Dependent upon the nature of the outcomes, the "ideal" outcome level or distribution may not be reflected in the current centrality measure, implying that commonly used inequality measures are not always Fit for Purpose in terms of having the appropriate Focus.

Levelling downwards, levelling upwards or simply levelling to some arbitrary centrality measure, are all ways of levelling playing fields. Each require distinctly different inequality metrics to be minimised, with the totality of differences from the lowest outcome being the object of concern in the first instance, the totality of differences from the highest outcome being the focus in the second and the totality of differences from the centrality measure being the Focus in the last instance. Furthermore, the nature of what is to be equal can affect the choice of equalizing metric so that, if equal chances or opportunities are the objective, simple equality of location measures will not suffice since equality of their respective location measures is not a sufficient statistic for equality of their respective chances between two groups. Questions then arise as to what, given a particular situation, is the appropriate inequality measure that is Fit for Purpose? These issues are relevant whether the objects of measurement are continuously measured, discretely measured, or ordered categorical variates and can be readily translated from one measurement paradigm to another. Here, since the measurement problem is more obscure, analysis will be conducted in terms of ordered and unordered categorical paradigms and extensions to other paradigms noted.

Measurement of Inequality in the context of ordered and unorderable multidimensional categorical data calls for measurement in the absence of cardinality. Given agents are considered equal when they exhibit identical outcomes, inequality measures are usually based upon an aggregation of cardinally

---

[1] Paul Krugman on the dust cover of Picketty (2014)

[2] The UK Government currently has a "Levelling Up" Secretary of State, Michael Gove.

[3] Perhaps the most influential diatribe on inequality in the 21st Century, Piketty (2014) eschews centrality focussed inequality measures and employs top quantiles as focus points.

measured agent outcome distances. In measuring a sense of variation around a given Focus Point, these measures can be construed as quantifying the extent to which there is a lack of clustering around the Point. Given agents are considered Polarized when they are clustered around distinct Points of Attraction, Polarization measures can be construed as quantifying some function of the distances between multiple focus points weighted by the respective propensities for clustering around them. It is these interpretations that motivate the following family of inequality and polarization measures for ordered and unordered categorical data.

Generally, even when it is ordered, categorical data does not afford the luxury of a cardinal distance measure, unless it is artificially endowed by attributing some arbitrary scale factor to the ordered categories (Likert 1936, Cantril 1965). Recently concerns have been raised regarding this practice (Shroder and Yitzhaki 2017, Bond and Lang 2019) since alternative equally valid scaling procedures can yield substantially different measured conclusions. In a Univariate Ordered Categorical Environment, Allison and Foster (2004), Cowell and Flachaire (2017) and Jenkins (2021) circumvented this issue by employing measures of probabilistic distance from a median or maximal category, the determination of which does not need artificial calibration. As to the question of which approach to take, the answer lies in the purpose of the measure. For example, if societal concern is that all should have the best of health then aggregate relative distance from the best health outcome (Cowell and Flachaire 2017) would be the appropriate measure. On the other hand, if concern is just that everyone should be the same without preference for level, then a median focus (Allison and Foster 2004) could be appropriate.

Whereas Allison and Foster (2004)), Cowell and Flachaire (2017) and Jenkins (2021) use the median or maximal category as a focus, here, the Maximum Modal category is proposed as a Focus for several reasons. Primarily, the maximum modal category is the category identified as most likely to command universal membership and hence equality. Its density value measures the extent to which the category is common to all so that 1 minus the density value becomes a very natural measure of the extent to which the population does not reside in a common category and is thus unequal. When the modal density is equal to 1 there is absolute equality in the collection, when equal to 1/N (its minimum possible value where N is the total number of categories) the distribution is discrete uniform and there is a sense of maximal inequality (Jenkins 2021) so the range of the inequality measure is [0, (N-1)/N]. Secondly, the median and maximal categories are not defined when data are not ordered and, when they are ordered and multidimensional (but not lexicographic), the median is ill defined and not necessarily unique, whereas, except in the unusual case of multiple maximal modes, the maximum modal category is invariably unique and well-defined and unifying in terms of the dimensions. With multiple modes, the median category need not coincide with the mutually most common category, indeed with a strongly bimodal or polarized distribution it may end up being a very unlikely category and its density value not a good measure of the degree of commonality. When well identified multiple modes exist, the possibility of a general multilateral polarization measure exists but granularity (the number of categories in each dimension) matters here.

To formulate inequality/polarization indices, the idea is to hypothesize a theoretically completely equal or totally polarized distribution based upon empirical modes identified in the data, and then measure the proximity or goodness of fit of the empirical distribution to the hypothesized distribution. In the case of inequality measurement, the closer is the empirical distribution to the hypothesized distribution the less inequality there is and, in the case of polarization, the closer is the empirical distribution to the hypothesized polarized distribution the more polarization there is. Measurement of proximity is based upon TR, Ginis' Transvariation statistic (Gini 1916) which is equal to 1-OV where OV is the overlap of two distributions (Anderson, Linton and Whang 2012). TR[4] is a measure on the unit interval of the extent of disparity between two probability density functions (PDF's) over their combined support, recording 0 when they are identical and 1 when they have mutually exclusive Supports. Here the Transvariation of interest is that between the PDF describing the empirical distribution of the population over the set of categories in question and a hypothetical PDF that would prevail under complete equality where the whole population resided in just one Focus category. If polarization is of concern, when the whole population resides in the M possible Poles of attraction the $TR = 1$ recording complete polarization in the population. As will be seen below, a similar exercise can be performed considering the proximity of the hypothesised and empirical cumulative densities.

In the context of inequality measurement, when they are uniquely defined, the Focus category could be the "average", "median", "highest" or "lowest" category, all of which have been employed in ordered categorical inequality measurement literature (see Jenkins 2021 for a discussion) though not in the context of a Transvariation measure. Here in the unordered case, the Focus will be the modal or category or categories since they are invariably uniquely defined and can be construed as the most likely categories within which all agents reside in a completely equal or polarized society. In the ordered case the focus could be any one of the lowest, highest or modal categories reflecting secondary societal imperatives for levelling down, leveling up or levelling to the most common societal norm. These correspond to clear, and well-defined focus points in multidimensional environments where joint average and median categories are not necessarily uniquely defined.

In the following, Section 1 develops the inequality and polarization measures for multidimensional ordered and unordered categorical data. Section 2 provides an illustrative application on self reported health and loneliness data from China, and Section 3 draws some conclusions.

Section 1.

Suppose categorical responses are recorded in K dimensions indexed $k = 1,..,K$, with I_k categories in the k'th dimension yielding $I = \prod_{k=1}^{K} I_k$ multidimensional categories in total. Consider $\underline{f}: f_i: i = 1,..,I$ to be a vectorized discrete PDF over I categories such that $f_i \geq 0 \; \forall \; i \; and \sum_{i=1}^{I} f_i = 1$. Since f_i is the probability that an agent falls into the $i'th$ category, when there is complete equality in the population, all agents will fall in the same category and the distribution would collapse to a unit probability for that category. Empirically, the most likely category for this to be the case is the most densely populated or modal category. Let i^* be the unique modal category where $\max_i f_i = f_{i^*}$ (note that $1 \geq f_{i^*} > 1/I$)[5] and let $\underline{f}^{eq}: f_i^{eq} = 0 \; \forall i \neq i^* \; and f_{i^*}^{eq} = 1$ be the Equally Distributed Modal density where everyone in the population experiences the same multi-dimensioned categorical outcome at the modal category. In a similar fashion it is also possible to quantify the extent of multilateral polarization, Suppose M < I polar categories are suspected at categories i*= m_1, m_2, .., m_M in this case let $\underline{f}^{eq}: f_i^{eq} = 0 \; \forall i \neq i^* \; and f_{i^*}^{eq} =$

[4] Note that $TR = 1 - OV$ where OV is a measure of the extent to which two PDF's overlap. Since OV has a well defined asymptotically normal distribution in any mixture of discrete, continuous, multidimensional environments (Anderson, Linton and Whang, 2012) so will TR have, rendering inference straightforward in all categorical environments.

[5] If $f_{i^*} = 1/I$ it would not be unique since all f_i would have the same value, in effect there would be societal indifference to design or policy type with each alternative commanding identical support.

$\theta_m \; for \; i^* = m_1, m_2, \; ., m_M \; where \; \sum_{m=1}^{M} \theta_m = 1, \theta_m > 0 \; \forall \; m$, if the Focus Points were equally balanced $\theta_m = \frac{1}{M} \forall M$. In this case ultimate polarity is defined by $\underline{f}^{eq}$.

Inequality and Polarization Indices, The Unordered Categorical Case.

Suppose subjects are offered I alternative colours for a car and asked to record their preferred colour. There is no ordering of categories, nor sense of distance between them, but a sense of the extent to which preferences are evenly or unevenly spread across the range of offered colours is clearly of interest which is a matter of commonality or equality of preferences. When all choose the same colour, preferences are concentrated, identically focussed on one category, when colour is a matter of indifference and colours randomly chosen, preferences will appear diverse with an equal number of subjects choosing each category.

Categorical Inequality, CI, is defined as proportionate to the Transvariation of the probability density function and its's corresponding Equally Distributed Modal density $\left(0.5 \sum_{i=1}^{I} |f_i - f_i^{eq}|\right)$ where the factor of proportionality is a scaling factor bringing CI into the full unit interval. Since $TR\left(\underline{f}, \underline{f}^{eq}\right)$, the transvariation of two distributions, is equal to $1 - OV\left(\underline{f}, \underline{f}^{eq}\right)$ where $OV\left(\underline{f}, \underline{f}^{eq}\right)$ is their Overlap[6] and the range of the transvariation in this case is $(1/_I, 1]$:

$$CI = \frac{I}{2(I-1)} \sum_{i=1}^{I} |f_i - f_i^{eq}| = \frac{I}{I-1}\left(1 - OV\left(\underline{f}, \underline{f}^{eq}\right)\right) = \frac{I}{I-1}\left(\sum_{i=1}^{I}\left(f_i - \min\left(f_i, f_i^{eq}\right)\right)\right) \quad [1]$$

Inference is straightforward since it is readily shown that, for independent random samples:

$$\sqrt{n}\left(\widehat{CI} - CI\right) \sim \sqrt{n}\left(\widehat{OV} - OV\right) \sim N\left(0, \left(\left(\frac{I}{I-1}\right)^2 \left(OV(1 - OV)\right)\right)\right).$$

Where $\widehat{CI}$ is the maximum likelihood estimate of CI. $\widehat{CI}$ will equal 0 when there is complete equality with all agents in the same category and it will equal 1 when membership of the modal category is at a minimum and spread of agents between the categories at its most diverse.

The Ordered Case.

When the categories are ordered, a sense of directional distance from the modal category is imparted so that categories at the extremes correspond to more serious deviations from the modal norm when it is not at the extreme. Hence some form of weighting scheme may be necessary where categories further from the modal category receive more weight than categories proximate to the modal category. Clearly CI works in this case since it is appropriate for any Categorical situation but it pays no attention to the ordering of categories. Suppose in the univariate case, i reflects the ordering of the categories so that category i is considered definitively "higher than" the next lower category $i-1$, $|i^{**} - i^*| - 1$ is the

[6] Both $OV\left(\underline{f}, \underline{f}^{eq}\right)$ and $TR\left(\underline{f}, \underline{f}^{eq}\right)$ have well defined sampling distributions where, given an independent random sample of size n $\sqrt{n}\left(\widehat{OV} - OV\right) \; and \; \sqrt{n}\left(\widehat{TR} - TR\right) \sim_{asy} N\left(0, \left(OV(1 - OV)\right)\right)$.

number of categories or categorical distance that has to be skipped in order to move from i^{**} to i^{*} and consider a weighting vector $\underline{w}$: $w_i\ i = 1,..,I$ where:

$$w_i^* = i^* - i + 1\ \forall\ i \le i^*\ and\ w_i^* =\ i - i^* + 1\ \forall\ i > i^*$$

and

$$w_i = \frac{w_i^*}{\sum_{i=1}^{I} w_i^*/I} \quad i = 1,..,I \qquad [2]$$

So that OCI, Ordered Categorical Inequality may be measured as:

$$OCI = \frac{I}{I-1}\sum_{i=1}^{I} w_i\left(f_i - \min\left(f_i, f_i^{eq}\right)\right)$$

Higher order indices are possible following Anderson and Leo 2021.

In truth, this is much like attaching an arbitrary Cantril scale to the data, endowing it with cardinal content which, in reality, it does not possess (Bond and Lang 2019, Schroder and Yitzhaki 2017) subjecting the measure to the scale dependency critique. Alternatively, following Allison and Foster (2004), Cowell and Flachaire (2017), the probability density function could be used. Indeed, if a "levelling upward" or a "levelling downward" imperative is in the background, a simpler approach would be to contemplate respective $OCILU$ and $OCILD$ indices where:

$$OCILU = \frac{I-1-\sum_{i=1}^{I}(1-F_i)}{(I-1)} = \frac{\left(\sum_{i=1}^{I} F_i\right)-1}{(I-1)}; \quad OCILD = \frac{I-\sum_{i=1}^{I} F_i}{(I-1)};$$

With $OCILU$, the focal point category is the highest category ($i^* = I$) and distance from it is measured by elements of the counter cumulative distribution. With $OCILD$, the focal point category is the lowest category ($i^* = 1$) and distance from it is measured by elements of the cumulative distribution.

Noting that $F_i = \sum_{j=1}^{i} f_j,\ i = 1,..,I$, inference with these measures is straightforward since, given a unit vector d of length I and C [7], an I dimensioned square cumulation matrix, $\underline{F}$, the vector of cumulative distribution values may be written as a function of $\underline{f}$ the corresponding vector of probability density values as $\underline{F} = C\underline{f}$ and $OCIU$ and $OCID$ respectively written as:

$$OCILU = \frac{d'C\underline{f}-1}{(I-1)}; \quad OCILD = \frac{I-d'C\underline{f}}{(I-1)} \qquad [3]$$

Note that $OCILU = 1 - OCILD$ i.e. one is the complement of the other. When all the population reside in category I, so $F_i = 0, i = 1,..,I-1$ and $F_I = 1$ so that $OCILU = 0$ recording complete equality and $OCILD = 1$ recording maximum inequality. When all the population reside in category 1, $F_i = 1 \forall\ i$ and $OCILU = 1$ recording maximum inequality and $OCILD\ = 0$ recording complete equality.

Following Rao (2009), since $\underline{\widehat{f_g}}$, the estimator of the vector of outcome probabilities $\underline{f_g}$, for group g is such that

[7] In the univariate case, C would be a square matrix with 1's in the lower triangle and zeros elsewhere in the multivariate case C would be a more complex but none-the-less known quare matrix of 1's and zeros.

$$\sqrt{n}\left(\underline{\widehat{f_g}} - \underline{f_g}\right) \sim N\left(\underline{0}, V_g\right)$$

where:

$$V_g = \begin{bmatrix} f_{1,g} & 0 & 0 & . & 0 \\ 0 & f_{2,g} & 0 & . & 0 \\ 0 & 0 & f_{3,g} & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & 0 & . & f_{5,g} \end{bmatrix} - \begin{bmatrix} f_{1,g} \\ f_{2,g} \\ . \\ . \\ f_{5,g} \end{bmatrix} \begin{bmatrix} f_{1,g} & f_{2,g} & . & . & f_{J,g} \end{bmatrix}$$

Noting that $\sum_{i=1}^{I} F_i = d'C\underline{f}$, it follows that for an independent sample of size n:

$$\sqrt{n}\left(\widehat{OCIU} - \mathrm{OICU}\right) = \sqrt{n}\frac{d'C}{(I-1)}\left(\underline{\widehat{f_g}} - f_g\right) \sim N\left(\underline{0}, \frac{d'C}{(I-1)} V_g \left(\frac{d'C}{(I-1)}\right)'\right) \quad [4]$$

$$\sqrt{n}\left(\widehat{OCID} - \mathrm{OCID}\right) = \sqrt{n}\frac{-d'C}{(I-1)}\left(\underline{\widehat{f_g}} - f_g\right) \sim N\left(\underline{0}, \frac{d'C}{(I-1)} V_g \left(\frac{d'C}{(I-1)}\right)'\right) \quad [5]$$

So that, for comparing the inequality indices of two independently sampled groups g' and g with a common sample size n for either $OICU$ or $OICD$ consider[8]:

$$\sqrt{n}\frac{d}{(I-1)}\left(\widehat{F_{g'}} - \widehat{F_g}\right) = \sqrt{n}\frac{d'C}{(I-1)}\left(\underline{\widehat{f_{g'}}} - \underline{\widehat{f_g}}\right) \sim N\left(\frac{d'C}{(I-1)}(\underline{f_{g'}} - \underline{f_g}), \frac{d'C}{(I-1)}(V_{g'} + V_g)\left(\frac{d'C}{(I-1)}\right)'\right)$$

When the modal category i^* is used as a focal point, distance from it can be calibrated using:

$$F^*_{i^*,i} = f_{i^*} \; and \;\; F^*_{i^*,i} = F^*_{i^*,i+1} + f_i \;\forall\, i < i^* \; and \; F^*_{i^*,i} = F^*_{i^*,i} + f_i \;\forall\, i > i^*$$

Then $OCIM$, the Modally focussed Ordered Categorical Inequality measure may be written as:

$$OCIM = \frac{I - d'C_{i^*}\underline{f}}{(I-1)} = \frac{I - d'F_{i^*}}{(I-1)}$$

Where for I = 6 and $i^* = 3$, C_{i^*} is of the form:

$$C_{i^*} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

When all agents are in the modal category, $OCIM = 0$ and when all agents are in the highest or lowest category $OCIM = 1$.

For inference purposes it is readily seen that:

[8] To examine the unambiguous increase/ decrease in the inequality measure, inference can be performed using the Stoline and Ury (1979) Maximum Modulus Distribution to explore the joint "non-negativity" of all the elements of the vector $C\left(\widehat{f_{g'}} - \widehat{f_g}\right)$.

$$\sqrt{n}\left(\widehat{OCIM} - \mathrm{OICM}\right) = \sqrt{n}\frac{d'c_{i^*}}{(I-1)}\left(\underline{\hat{f}_g} - f_g\right) \sim N\left(\underline{0}, \frac{d'c_{i^*}}{(I-1)} V_g \left(\frac{d'c_{i^*}}{(I-1)}\right)'\right) \qquad [6]$$

Formulating $\underline{\mathbf{f}}^{\mathbf{eq}}$.

To identify the potential modal category in the Modally focussed Inequality measures, seek $\max_i \hat{f}_i$. For Polarization measures, if the proposed M modal points are known and presumed to have equal attraction power, they should each be accorded 1/M, to relate them to the empirical likelihood, formulate the M dimensioned empirical polarization vector $\underline{f}^{MM}$with typical element $\hat{f}_i^{MM}$ $i = m_1, m_2, ., m_M$. Then let $\underline{f}^M$, the theoretical polarization focus vector be the vector of elements $\hat{f}_i^M = \frac{\hat{f}_i^{MM}}{\sum_{i=1}^{M} \hat{f}_i^{MM}}\, for\ i = m_1, m_2, \ ., m_M$ which will be located in the $\underline{f}^{eq}$ in the appropriate places with all other elements set to 0.

Multivariate Considerations.

In the bivariate case where both dimensions are ordered with $f_{i,j} \geq 0$: $i = 1,..,I, j = 1,..,J$ $\sum_{i=1}^{I}\sum_{j=1}^{J} f_{i,j} = 1$ with the ordering following the dimension indexing, cumulative and counter cumulative density functions are well defined with $F_{i,j} = \sum_{k=1}^{i}\sum_{l=1}^{j} f_{k,l}\ for\ i = 1,..,I, j = 1,..,J.$

In the modal case where $\max_{i,j} f_{i,j} = f_{i^*,j^*}$:

$$\text{Let } F_{i^*,j}^{**} = f_{i^*,j}\ j = 1,..,J\ and\ F_{i,j^*}^{**} = f_{i,j^*}\ i = 1,..,I$$

$$F_{i,j}^{**} = F_{i+1,j}^{**} + f_{i,j}\ \forall\, i < i^*\ and\ F_{i,j}^{**} = F_{i,j}^{**} + f_{i,j}\ \forall\, i > i^*, \forall j = 1,..,J$$

$$F_{i,j}^{*} = F_{i,j+1}^{*} + F_{i,j}^{**}\ \forall\, j < j^*\ and\ F_{i,j}^{*} = F_{i,j}^{*} + F_{i,j}^{**}\ \forall\, j > j^*, i = 1,..,I$$

Vectorizing the probability density function and arranging the cumulating matrix C accordingly facilitates bivariate versions of [4], [5] and [6] together with their distributions for inference purposes. Extensions to more than 2 dimensions follow a similar process.

Following Sen (1976), the axiomatic development of indices has been popular in inequality measurement. Here $CI, OCIM, OCIU\ and\ OCID$ indices can each be shown to satisfy axioms of continuity, scale independence and normalization and coherence. All the indices are continuous in the probability measure f_{i^*}, are scale independent by definition (any scale attributed to the categories does not appear in the formulae) and normalized, i.e. confined to the unit interval. In the case of coherence, the inequality measure should diminish when the probability of membership of the focus category increases so that ${dCI}/{df_{i^*}} < 0 |\ f_{i^*} < 1$. Since from [1] CI may be written as:

$$CI = \frac{I}{(I-1)}\sum_{\substack{i=1\\ i\neq i^*}}^{I} f_i = \frac{I}{(I-1)}(1 - f_{i^*})$$

$$dCI/df_{i^*} = -\frac{I}{(I-1)} < 0.$$

And from [3], since $OCILU = \frac{d'\underline{F}-1}{(I-1)}$ and $OCILU = 1 - OCILD$

$$\frac{dOCIU}{dF_i} = \frac{1}{I-1} > 0 \ and \ \frac{dOCID}{dF_i} = -\frac{1}{I-1} < 0 \ for \ i = 1,..,I-1$$

Confirming that increasing probabilistic distance from the focus point is inequality increasing in both inequality measures. Similarly:

$$\frac{dOCIM}{dF_{i^*,i}} = \frac{1}{I-1} > 0 \ \forall \ \mathrm{i} \neq i^* \ and \ \leq 0 \ for \ i = i^*$$

As for variables with cardinal measure, note that when the ordered categorical variables have cardinal measure, as for instance is the case when the real line is partitioned into a collection of mutually exclusive and exhaustive intervals, the analysis follows the same process and is readily extended to the cardinal paradigm.

Distributional Inequalities.

Sometimes the concern is the extent to which outcome distributions amongst a collection of groups differ. The Equality of Opportunity literature is a case in point where the policy objective is that the respective outcome distributions of a collection of circumstance groups be similar. Anderson et. al. (2021) proposed a family of distributional inequality measures, distributional analogues of the Gini coefficient and Coefficient of Variation, the latter of which can be construed as a focussed Distributional Inequality measure with its focus as the Average Distribution in the collection. In the present context, given G probability density functions $\underline{f_g}$ of G groups indexed $g = 1,..,G$ with respective population weights w_g, DCV, their Distributional Coefficient of Variation may be written as:

$$DCV = \frac{1}{(1-\sum_{g=1}^{G} w_g^2)} \sum_{g=1}^{G} \left(1 - OV_{gO}\right) \qquad [7]$$

Where OV_{gO} is the overlap of the g'th distribution and the focus or object distribution $\underline{f_O}$, which in this case is the average distribution $\underline{f_O} = \sum_{g=1}^{G} w_g \underline{f_g}$ so that $OV_{gO} = \sum_{i=1}^{I} min\left(f_{g,i}, f_{O,i}\right)$ where $f_{g,i}, f_{O,i}$ are respectively the i'th elements of the vectors $\underline{f_g}$ and $\underline{f_O}$.

Sometimes it is not just equality of distributions that are the order of the day, maybe the policy aspiration is for all groups to have the "Highest Ranked" or "Lowest Ranked" distribution with Levelling Up or Levelling Down the policy intent. Suppose that the distributions are ordered in some sense so that $\underline{f_g}$ stochastically dominates $\underline{f_{g'}}$ at some order when $g > g'$ for all $g \in 1,..,G$. Then $DCVU$, a Levelling Up sensitive index or $DCVD$, a Levelling Down sensitive distribution can be contemplated where $\underline{f_O} = \underline{f_G}$ in the former case and $\underline{f_O} = \underline{f_1}$ in the latter. The sampling distribution of DCV and hence $DCVU$ and $DCVD$ are developed in Anderson et al (2021).

An Example: Inequalities in Health and Loneliness Amongst the Aged in China.

The relationship between health and loneliness, and the ageing process is a major factor in the wellbeing of the elderly (Gerst-Emerson and Jayawardhana, 2015; Ong et al., 2016; World Health Organization, 2015) and, given its disproportionately ageing population (Zhang, 2017), issues surrounding health, loneliness and ageing are particularly pertinent in China. Health care provision for the older population, universally provided and a government responsibility until the advent of the economic reforms of the late 1970's, was in effect privately provided since that time until 2009 when China embarked upon major health care reform[9]. The intent was to provide all citizens with equal access to basic reasonable quality care and financial risk protection, in essence an Equal Opportunity in Health Care Provision Imperative with an aspiration to elevate health outcomes.

In a study of Health and Loneliness outcomes and the Aging process, Anderson et. al. (2021) analysed the health-loneliness wellbeing status of Chinese citizens, but it did not directly address the inequality measurement issue with respect to the health and loneliness experiences of individuals. Here, as an exemplar, the new Focussed inequality measures will be employed in studying the evolution of health-loneliness inequalities over the ageing process by comparing inequalities in the experiences of younger and older populations. The study employed survey data drawn from the China Health and Retirement Longitudinal Study 2013. Within each sampled household, respondents over 44 years of age, answered questions about their health status (poor, fair, good, very good, excellent) and personal sense of loneliness (not at all, a little, somewhat, quite a lot, very). Aside from age group and gender, respondents were identified by their partner status (partnered or single), and their urban – rural status. After eliminating incomplete records, the sample size was 13593. To examine the evolution of health and loneliness inequalities with the aging process, the sample was initially split into two population groups, under sixties and sixty and over. Tables 1 and 2 report the joint and marginal probability density values of ordered health and loneliness outcomes in the two populations under comparison.

Table 1. Joint Distributions for Under 60's, Over 60's and Overall.

		Probability Density Functions					Cumulative Distribution Functions.				
	Health	Loneliness					Loneliness				
		Very	Quite a lot	Somewhat	A little	Not at all	Very	Quite a lot	Somewhat	A little	Not at all
Under 60's	Poor	0.0071	0.0053	0.0054	0.0151	0.1013	0.0071	0.0123	0.0177	0.0329	0.1342
	Fair	0.0054	0.0064	0.0107	0.0294	0.3108	0.0125	0.0242	0.0402	0.0848	0.4970
	Good	0.0021	0.0034	0.0050	0.0243	0.2922	0.0146	0.0297	0.0507	0.1196	0.8240
	Very Good	0.0003	0.0005	0.0009	0.0059	0.1174	0.0149	0.0305	0.0524	0.1272	0.9490
	Excellent	0.0000	0.0002	0.0001	0.0021	0.0487	0.0149	0.0306	0.0527	0.1296	1.0000
Over 60's	Poor	0.0084	0.0079	0.0094	0.0200	0.1311	0.0084	0.0163	0.0257	0.0457	0.1769
	Fair	0.0075	0.0097	0.0133	0.0326	0.3021	0.0159	0.0335	0.0561	0.1088	0.5420
	Good	0.0020	0.0049	0.0069	0.0275	0.2763	0.0179	0.0404	0.0700	0.1501	0.8596
	Very Good	0.0006	0.0011	0.0013	0.0058	0.0928	0.0185	0.0421	0.0730	0.1589	0.9612
	Excellent	0.0001	0.0003	0.0006	0.0019	0.0359	0.0186	0.0425	0.0740	0.1618	1.0000
Overall	Poor	0.0077	0.0066	0.0074	0.0176	0.1165	0.0077	0.0144	0.0218	0.0394	0.1560
	Fair	0.0065	0.0081	0.0120	0.0310	0.3064	0.0142	0.0289	0.0483	0.0970	0.5199
	Good	0.0021	0.0042	0.0060	0.0260	0.2841	0.0163	0.0352	0.0605	0.1352	0.8422
	Very Good	0.0004	0.0008	0.0011	0.0058	0.1049	0.0167	0.0364	0.0629	0.1434	0.9552
	Excellent	0.0001	0.0002	0.0004	0.0020	0.0422	0.0168	0.0367	0.0636	0.1460	1.0000

[9] It is still the case that half the health care costs must be covered privately, with obvious consequences for the economic burden placed upon an ageing population.

Table 2. Marginal Distributions.

	Health					Loneliness				
	Poor	Fair	Good	V.Good	Excellent	Very	Quite a lot	Somewhat	A little	Not At All
<60's pdf	0.1342	0.3627	0.3270	0.1250	0.0511	0.0149	0.0158	0.0221	0.0769	0.8704
cdf	0.1342	0.4969	0.8240	0.9489	1.0000	0.0149	0.0306	0.0527	0.1296	1.0000
≥60's pdf	0.1769	0.3651	0.3177	0.1016	0.0388	0.0186	0.0239	0.0315	0.0878	0.8382
cdf	0.1769	0.5420	0.8597	0.9612	1.0000	0.0186	0.0425	0.0740	0.1618	1.0000
Overall pdf	0.1560	0.3640	0.3223	0.1130	0.0448	0.0168	0.0199	0.0269	0.0824	0.8540
cdf	0.1560	0.5199	0.8422	0.9552	1.0000	0.0168	0.0367	0.0636	0.1460	1.0000

Note that in the Joint and both Marginal Cumulative Densities, the Under 60's cdf is everywhere less than or equal to the corresponding 60 And Over cdf indicating the unambiguous superiority of the Health and Loneliness wellbeing of the former group over the latter group so that Health and Loneliness Wellbeing appears to be deteriorating with age. In both age groups in the Joint distribution, the modal category is that of Fair Health and No Loneliness, which is also reflected in the Marginal distributions. However, with respect to the health marginal, while modal and median outcomes are coincident in the 60 and Over group at the Fair outcome level, in the Under 60 group the modal health category is Fair, but the median health category is Good.

When the socially desirable state is that all should have the Best or Utopian outcome (Excellent Health and No Loneliness), the Maximally Focussed OCIU index is appropriate, whereas if the social imperative is simply that all should be equal, OCIM, the Modally Focussed index is appropriate. Table 3 reports the two indices for the respective age groups together with the Unordered Categorical Index CI.

Table 3. Joint Modally Focussed and Maximally Focussed Inequality Measures.

	Under 60's Joint			60 and Over Joint			Overall Joint		
	CI	OCIM	OCIU	CI	OCIM	OCIU	OCIM	OCIU	OCIM
Measure	0.7179	0.3983	0.1376	0.7270	0.4136	0.1549	0.7225	0.4061	0.1464
Standard Error	0.0059	0.0059	0.0042	0.0057	0.0060	0.0044	0.0058	0.0060	0.0043
Sample size	6662			6931			13593		

Note that the Modally Focussed inequality measures record a substantially greater level of inequality than does the Maximally oriented inequality measure suggesting that joint health and loneliness inequalities are not as bad as they would seem to be normatively speaking in that outcomes are more intensely concentrated around the Best Outcome (Excellent Health and No loneliness) than in the modal outcome (Fair Health and No Loneliness). As may be seen, while there is not a significant increase in the basic Unordered Modally Focussed index with the older age group (Difference 0.0091, Standard Error 0.0082, z 1.1093, P(Z>z) 0.1336) there is a significant increase when proximity to the Best or Utopian Outcome is considered (Difference 0.0173, Standard Error 0.0061, z 2.8442, P(Z>z) 0.0022) and with the Ordered Categorical Modally focussed index is used (Difference 0.0153, Standard Error 0.0084, z 1.8182 P(Z>z) 0.0345).

To further explore the age group inequality differences, univariate Health and Loneliness marginal distribution-based indices are reported in Tables 4 and 5. When Modal and Median categories are coincident OCIM is both a Median and a Modally Focussed index, when they are not, it is not. With respect to Health, since the median category differs across age groups both "Fair" focussed and "Good" focussed indices are reported for comparison purposes.

Table 4. Univariate Health Distribution Comparisons.

Health	Under 60's			60 and Over			Overall		
	OCIU	OCIM(F)	OCIM(G)	OCIU	OCIM(F)	OCIM(G)	OCIU	OCIM(F)	OCIM(G)
Measure	0.6010	0.4425	0.5511	0.6349	0.4507	0.5657	0.6183	0.4467	0.5585
Standard Error	0.0126	0.0184	0.0194	0.0122	0.0186	0.0191	0.0123	0.0183	0.0190

In this instance the Maximally Focussed Utopian indices record significantly larger inequality than do Modally focussed indices. The Maximally Focussed index records significantly greater Health Inequality in the 60 and Over Group than in the Under 60 group (Difference 0.0339, Standard Error 0.0175, z 1.9329, P(Z>z) 0.0266) whereas the Modally focused indices (OCIM(F)) do not (Difference 0.0082, Standard Error 0.0262, z 0.3134, P(Z>z) 0.3769). When Median Focussed indices are compared (i.e. comparing OCIM(F) for the 60 and Over group with OCIM(G) for the Under 60's), a significant reduction in inequality in the 60 and Over group as compared to the Under 60 group is revealed (60 and Over – Under 60 Difference -0.1004, Standard Error 0.0269, z -3.7357, P(Z>z) 0.9999) suggesting an improvement in the fortunes of the 60 and Over group in terms of health outcome inequality.

Turning to the distribution of loneliness outcomes, Maximal, Median and Modal categories are coincident so Table 5 simply reports the OICU indices (which are also the Median and Modal indices) where a marginally significant increase in inequality of loneliness outcomes is revealed (Difference 0.0172, Standard Error 0.0129, z 1.3345, P(Z>z) 0.0910).

Table 5. Univariate Loneliness Distribution Comparisons.

Loneliness	Under 60's OICU	60 and Over OICU	Overall OICU
Measure	0.0570	0.0742	0.0658
Standard Error	0.0086	0.0096	0.0091

Distributional Comparisons.

The intent of the 2009 reforms was to provide all citizens with equal access to basic reasonable quality care, in essence an Equal Opportunity in Health Care Imperative underlays the reforms. In this context it is the equality of outcome distributions of circumstance groups that signals equality of chances and hence equality of opportunity. Circumstance groupings will be defined by gender, partner type, urban/rural and age (young and old within the category) status forming 16 circumstance groups in each of the under and over 60's groupings. Table 6 reports the Average Distribution Focussed Distributional Coefficient of Variation and the Best[10] Distribution Focussed Distributional Coefficient of Variation. In this framework, Distributional Inequality and hence Inequality of Opportunity is much greater when Circumstance Class Outcome Distributions are compared to the "Utopian" distribution as opposed to the Average distribution.

[10] The Average Distribution is simply a population weighted average of the circumstance class distributions, the Best Distribution is defined as the Utopian distribution in the collection (Anderson et. al. 2020) which for a collection of K cumulative densities each with I ordered category values $F_{i,k}$ has a CDF $F_i^U = \min_{k=1,..,K} F_{i,k} \;\; i = 1,..,I.$

Table 6. Average and Utopian Distribution Focussed Distributional Coefficients of Variation.

	DCV Average Distribution Focussed		DCV Best Distribution Focussed	
	Under 60's	60 and Over	Under 60's	60 and Over
Measure	0.0403	0.0491	0.1499	0.1320
Standard Error	0.0025	0.0022	0.0025	0.0023

With respect to the Average Distribution Focus, a significant increase in distributional inequality and consequent loss of equality of opportunity is revealed for the older group as compared to the younger group (Difference 0.0088, Standard Error 0.0033, z 2.6425, P(Z>z) 0.0041). Whereas when a "Best" Distribution Focus is considered, there is a reduction in distributional inequality with a consequent increase in equality of opportunity (Difference -0.0179, Standard Error 0.0034, z -5.2693, P(Z>z) 1.000) for the Older grouping suggesting some policy success in equalizing health and loneliness outcome distributions across gender, partner type, urban/rural and age group determined circumstance groups.

Conclusions.

Inequality measurement usually proceeds under the normative presumption that aggregated differences from a centrality measure are generically "Bad", without regard to where that centrality measure is located in the spectrum of outcomes. However, while a given high level of variation around a location measure high in the bad-to-good outcome spectrum may be considered normatively bad, that same level of variation, around the same location measure, low in the outcome spectrum, could well be considered normatively good[11]. The problem is that most standard inequality measures lack Focus. Here, a family of Focussed Inequality Measures which account for where in the outcome spectrum inequalities reside, has been proposed, together with their sampling distributions to facilitate inference. They are applicable in a wide variety of multilateral and multidimensional circumstances and facilitate estimation and inference in more Focussed inequality comparison situations. While they have been articulated in the context of multidimensional ordered and unordered categorical data environments, they are readily extended to the continuous paradigm.

The relationship between health, loneliness and the ageing process is a major factor in the wellbeing of the elderly and, since the 2009 reforms in health care provision in China, disparities in the experiences of older populations in that country are of interest. Using data from the post health reform period, application of the measures was exemplified in a comparison of the joint health and loneliness inequality experiences of Under Sixty and Sixty and Over Age Groups in China. Inequality measures with a Modal and Best or Utopian outcome Focus suggest that joint health and loneliness outcome inequalities, in being more closely clustered around the best outcome than around the modal outcome, are not quite as bad as they would seem. However, when comparing older and younger age groups, the older age group records higher inequalities than does the younger group. Comparison at the respective health and loneliness marginal levels reveal higher inequality levels for the older age group. An Equality of Opportunity analysis of distributional similarities among groups defined by age, gender, partner and urban-rural status revealed diminishing similarities and thus lower Equality of Opportunity for the

[11] Interestingly enough the coefficient of variation would downgrade the same standard deviation – level of inequality at the high end more than it would at the lower end.

elderly when the “Average” distribution was the focus, but increasing similarities and thus more Equality of Opportunity for the elderly when the Best or Utopian distribution was the focus.

References.

Allison RA, Foster JE (2004). Measuring health inequality using qualitative data. Journal of Health Economics 23(3): 505-524.

Anderson G.J., Fu R. and Leo T.W. (2021) “Health, Loneliness and the Ageing Process in the Absence of Cardinal Measure: Rendering Intangibles Tangible” Forthcoming Journal of the Economics of Aging.

Anderson G., Linton O., Whang, Y.-J. (2012). “Nonparametric Estimation and Inference about the Overlap of Two Distributions”, Journal of Econometrics, 171(1), 1-23.

Anderson, G., Post, T., and Y-J Whang (2020) “Somewhere Between Utopia and Dystopia: Choosing From Multiple Incomparable Prospects” Journal of Business and Economic Statistics 38, 502-515

Anderson, G.J., Linton, O., Pittau, G. Whang Y-J. and R. Zelli (2021) “On Unit Free Assessment of The Extent of Multilateral Distributional Variation” The Econometrics Journal. 24, Issue 3, 502–518,

Apouey B (2007). Measuring health polarization with self-assessed health data. Health Economics 16(9): 875-894.

Apouey B, Silber J, Xu Y (2020). On inequality-sensitive and additive achievement measures based on ordinal data. Review of Income and Wealth 66(2): 267-286.

Atkinson, AB (1970) On the measurement of inequality. Journal of Economic Theory, 2 (3), pp. 244–263

Bond T.N. and K. Lang (2019) "The Sad Truth about Happiness Scales," Journal of Political Economy 127, no. 4: 1629-1640.

Cowell, F. A. and E. Flachaire, (2017) “Inequality with Ordinal Data,” Economica, 84, 290–321, 2017.

Hugh Dalton (1920) “The Measurement of the Inequality of Incomes” The Economic Journal

Vol. 30, No. 119 (Sep., 1920), pp. 348-361

Gerst-Emerson, K. and Jayawardhana, J. (2015). Loneliness as a Public Health Issue: The Impact of Loneliness on Health Care Utilization Among Older Adults. American Journal of Public Health, 105(5), 1013–1019.

Gini C (1916). “Il concetto di transvariazione e le sue prime applicazioni” Giornale degli Economisti e Rivista di Statistica.

Gini, C. (1921). Measurement of Inequality of Incomes. The Economic Journal 31 (121), 124-126.

Hendricks W. A. and K. W. Robey (1936) “The Sampling Distribution of the Coefficient of Variation” The Annals of Mathematical Statistics 7, 129-132

Stephen P. Jenkins 2021 “INEQUALITY COMPARISONS WITH ORDINAL DATA” Review of Income and Wealth Series 67, Number 3 547-563

Esfandiar Maasoumi (1986) “The Measurement and Decomposition of Multi-Dimensional Inequality” Econometrica, Vol. 54, No. 4, (Jul., 1986), pp. 991-997 Published by: The Econometric Society

MONTALVO, J., and M. REYNAL-QUEROL (2005) Ethnic polarization, potential conflict and civil wars, American Economic Review 95, 796–815.

Ong, A. D., Uchino, B. N., and Wethington, E. (2016). Loneliness and Health in Older Adults: A Mini-Review and Synthesis. Gerontology, 62(4), 443–449.

Picketty T. (2014) Capital In the Twenty-First Century. Harvard University Press.

IIÑAKI PERMANYER, CONCHITA D'AMBROSIO (2015) Measuring Social Polarization with Ordinal and Categorical Data Journal of Public Economic Theory 17, 311-327.

Pigou, A. C. (1920). The economics of welfare. London: Macmillan and Co.

Schroder, C. and Yitzhaki, S. (2017). Revisiting the evidence for cardinal treatment of ordinal variables. European Economic Review, 92:337-358.

Sen, A. K. (1976) “Poverty: An Ordinal Approach to Measurement” Econometrica 44, 219-231.

Theil H. (1967) Economics and Information Theory North Holland Amsterdam.

World Health Organization (2015). World Report on Ageing and Health. Technical Report.