University of Toronto
Department of Economics

Working Paper 715

Simultaneous Search and Adverse Selection

By Sarah Auster, Piero Gottardi and Ronald Wolthoff

January 04, 2022

Simultaneous Search and Adverse Selection*

Sarah Auster [‡] Piero Gottardi[§] Ronald Wolthoff [¶]

December 29, 2021

Abstract

We study the effect of diminishing search frictions in markets with adverse selection by presenting a model in which agents with private information can simultaneously contact multiple trading partners. We highlight a new trade-off: facilitating contacts reduces coordination frictions but also the ability to screen agents' types. We find that, when agents can contact sufficiently many trading partners, fully separating equilibria obtain only if adverse selection is sufficiently severe. When this condition fails, equilibria feature partial pooling and multiple equilibria co-exist. In the limit, as the number of contacts becomes large, some of the equilibria converge to the competitive outcomes of Akerlof (1970), including Pareto dominated ones; other pooling equilibria continue to feature frictional trade in the limit, where entry is inefficiently high. Our findings provide a basis to assess the effects of recent technological innovations which have made meetings easier.

*We thank Antonio Cabrales, Gabriel Carroll, Belen Jerez, Philipp Kircher, Thomas Mariotti, Stephan Lauermann and various seminar and conference participants for valuable comments. Sarah Auster gratefully acknowledges funding from the German Research Foundation (DFG) through Germany's Excellence Strategy—EXC 2126/1—390838866 and CRC TR 224 (Project B02). Ronald Wolthoff gratefully acknowledges financial support from the Social Sciences and Humanities Research Council.

‡Department of Economics, University of Bonn; auster.sarah@gmail.com

§Department of Economics, University of Essex; piero.gottardi@essex.ac.uk

¶Department of Economics, University of Toronto; ronald.p.wolthoff@gmail.com

1 Introduction

In this paper, we study an environment with two key ingredients, adverse selection and search frictions. Real-life markets that feature these ingredients are abundant and include labor markets, OTC markets, as well as insurance markets. In recent years, many of these markets have seen technological innovations giving rise to online platforms which made it easier for market participants to meet, thus lowering search frictions.[1] A natural question is how such innovations affect the strategies of traders and the resulting prices at which transactions occur and hence the properties of allocations obtained in those markets. An understanding of the welfare effects of lowering meeting barriers is important, also to guide possible regulatory interventions regarding the organization of trades in markets.

Our paper aims to provide a theoretical framework that allows to investigate the question of how facilitating contacts affects market outcomes in the presence of adverse selection. The main innovation is to embed a model where agents can contact multiple potential trading partners *simultaneously* into an otherwise standard framework of directed search with adverse selection. We demonstrate that this gives rise to a new trade-off: facilitating contacts between market participants not only means lowering search frictions, but also affects the ability to use information about the liquidity of markets in order to screen traders with private information. We show that the latter effect has significant implications for the properties of market outcomes. In contrast to search models that do not combine adverse selection and simultaneous search, equilibria in our setting may exhibit partial pooling and multiple equilibria may coexist. A striking result is that some of these equilibria continue to feature inefficient entry and thus frictional trade in the limit where agents can contact arbitrarily many other market participants and the exogenous search friction vanishes.

The analysis is cast in an environment as in Akerlof (1970), where sellers own an indivisible object and are privately informed about its quality. For expositional

[1] As discussed by Fermanian et al. (2016) and Riggs et al. (2020), recent technological innovations (as electronification) and regulatory changes (as the Dodd Frank act) had a very significant impact on the way many securities are traded in financial markets. These innovations, together with measures aiming to increase transparency in trades, generated a substantial increase in contacts among market participants in OTC markets, where corporate bonds and derivatives like swaps are mostly traded. In the new platforms which emerged customers can contact multiple dealers at the same time, both to have the quotes set by various dealers streamed to them (RFS) and to send a contemporaneous request for quote (RFQ) to a selected subset of dealers for a specific transaction.

purposes, we refer to a labor market situation throughout the paper: buyers are firms and sellers are workers, who have private information about their productivity and can accept at most one job.[2] We assume that productivity can be either low or high and that high-productivity workers have a higher outside option than those with low productivity. First, firms choose which wage to post, workers then send applications to $N \geq 1$ firms and, finally, firms make an offer to one of their applicants. A worker's strategy thus specifies an *application portfolio*, trading off higher wages against the lower associated probabilities of getting a job offer. The matching between firms and workers is complicated by the fact that workers may receive multiple offers and can choose which one to accept. Hence, a firm's offer may be rejected. We assume that, when this happens, the firm can keep making new offers until an applicant accepts or the firm exhausts the applicant pool, as in Kircher (2009).

We now describe our results in more detail. When workers can apply to only one firm—or, equivalently, when meetings are bilateral—there is always a unique search equilibrium which is separating. That is, different types of workers apply to different wages (see, e.g., Guerrieri et al., 2010). In contrast, we show that when workers send multiple applications, the existence of a fully separating search equilibrium is guaranteed only if adverse selection is severe, in the sense that the high types' outside option exceeds the productivity of low types (the so-called *lemons condition*). In such an equilibrium, the probability of being hired at a firm to which high types apply must be sufficiently low in order to ensure incentive compatibility for low types. The more applications low types have at their disposal, the tighter becomes this constraint. We then show that as the number of applications N each worker can send becomes large, the equilibrium probability that a high-type worker is hired by some firm converges to zero. In the limit, the allocation of the search equilibrium thus converges to the separating Walrasian equilibrium allocation found by Akerlof (1970). This result is notable, as it holds despite high types sending an infinite number of applications.

[2]The labor market is a natural application of our model. While application data is scarce, the available evidence indicates that the number of applications sent by workers has increased in recent decades (see e.g. Martinelli and Menzio, 2020), likely facilitated by the increased use of online job search since the beginning of this century, as documented by e.g. Faberman and Kudlyak (2016). Further, as discussed by Wolthoff (2018), various pieces of evidence highlight the importance of simultaneous search. First, data from online job boards shows that workers tend to send multiple applications within even the shortest time spans (a week or even a day). Second, surveys among employers indicate that the most common reason for a worker to reject a job offer is the simultaneous arrival of a more attractive offer.

If the lemons condition fails, some wages to which low types apply are also acceptable for high types. When N is sufficiently large, high types have so incentives to send some applications to those relatively low wages in order to hedge against the risk of remaining unmatched when applying to high wages. This precludes the existence of a fully separating equilibrium. We then show that, in this case, an equilibrium exists where low and high types send a subset of their applications to the same firms: there is thus a submarket where the two types of workers pool. For the low-type workers the wage in the pooling market is the highest to which they apply, while for the high types it is the lowest. Hence, low types applying in the pooling market are hoping for a 'lucky punch', whereas high types view jobs offered in this market as a fallback option in case their preferred applications fail to generate offers. The main driver for this result is the fact that, even though trade itself is exclusive—each worker can only be hired by one firm—the application process is not, since a firm is unable to observe the whole set of applications made by a worker. The opportunity cost for high types of sending an application to a low wage is small and the same is true for low types applying to a high wage. This limits the firm's ability to screen a worker on the basis of the liquidity of the market in which a worker is applying.

Finally, we show that, as the number of applications becomes large, the probability that workers are hired in the pooling market converges to one. There are, however, too many firms entering the market so that their hiring probability is bounded away from one in the limit. Since firms need to be compensated for their entry cost in equilibrium, workers pay for the excessive entry in the form of a wage below their average net productivity. Equilibrium trading is thus frictional, even in the limit as workers can apply to arbitrarily many firms. Other equilibria exist in this situation, both with a single mixed market and with multiple mixed markets. We show that in the latter case efficiency may obtain: there exists a sequence of equilibria where multiple pooling markets are active in equilibrium such that, in the limit as N grows to infinity, both types of workers are hired at a wage equal to the average net productivity. The limit allocation then corresponds to the pooling equilibrium allocation of Akerlof (1970).

To sum up, our limit results demonstrate that a convergence to the set of equilibria obtained in Walrasian markets á la Akerlof (1970) is possible but not necessary.[3] We

[3] As shown by Akerlof (1970), for parameter values such that the lemons condition holds and the share of high-type workers is sufficiently large, both a separating and a pooling Walrasian equilibrium

thus provide a new, search-theoretic foundation for Akerlof (1970). At the same time, we show that search frictions may persist in the limit as workers can contact arbitrarily many firms.

Our findings have also important implications with regard to the welfare consequences of facilitating contacts among traders and thus lowering search frictions. With adverse selection these consequences become ambiguous. While low types always gain, high types may end up losing, and total welfare may also decrease. The increased ability of traders to meet counterparts in the market limits the possibility of screening privately informed agents and may end up impairing their trading possibilities.

Related literature. Our paper contributes to various strands of literature. The first strand concerns models of simultaneous search, which dates back to Stigler (1961). His pioneering work was extended by Chade and Smith (2006) and embedded in an equilibrium setting by e.g. Albrecht et al. (2006), Galenianos and Kircher (2009), Kircher (2009), Wolthoff (2018) and Albrecht et al. (2020). Our work builds in particular on Kircher (2009), with respect to which we innovate by allowing for heterogeneity among searchers and introducing asymmetric information.

The second strand of literature consists of work on adverse selection in directed search environments, which includes Inderst and Müller (2002), Guerrieri et al. (2010) and Chang (2018). A robust prediction in this line of work is that for one-dimensional types the equilibrium must be separating. Our contribution is to show that this result hinges on the assumption that workers can meet at most one firm at a time. When instead workers can apply to multiple firms, the key innovation in our setup, equilibria with pooling markets may arise.

The effect of multiple applications in terms of the reduced ability of buyers to use the liquidity of the market in which they operate to screen sellers exhibits interesting similarities to that of non exclusivity in contracting. The consequences of the latter for the properties of equilibrium allocations in the environment considered by Akerlof (1970) have been examined by Attar et al. (2011). The relationship of our analysis with non exclusivity and this work will be discussed more in detail in the next sections.

Finally, some papers on frictional markets with adverse selection share important analogies in some aspects with our work, but ultimately focus on rather different

exist. We show that both these equilibrium outcomes can be obtained in the limit as the search friction valishes.

questions from ours. Lester et al. (2019) consider a market where sellers may meet either one or two buyers, but meetings are random. In their environment, the fact that sellers may meet multiple buyers affects the price at which they trade, but not their probability of trade. The main focus of their work is then on the effects of multiple meetings on buyers' market power. Kurlat (2016) and Board et al. (2020) also consider a labor market in which heterogeneous workers contact multiple firms, but the main emphasis is on the matching that arises when firms are heterogeneous in their ability to detect workers' types. Lauermann and Wolinsky (2016) and Kaya and Kim (2018) consider the effect of vanishing search frictions, but in a sequential random search environment with adverse selection and private noisy signals about the type of the informed party. Kim and Pease (2017) also study sequential search with adverse selection. In contrast to the previous papers, they allow the privately informed party to choose his search intensity and show that lower search costs may lead to worse equilibrium outcomes for the informed party. The result relies on the observability of the informed party's trading history, an important difference from our analysis in Section 5.2.

2 Environment

Agents. We consider a static labor market populated by a continuum of size one of workers and a large continuum of firms. Both types of agents are risk neutral. Workers supply and firms demand one unit of indivisible labor. All firms are identical but workers differ in their productivity, defining their type, which is private information. In particular, a fraction σ of workers have low productivity, while the remaining ones are of high productivity. We will index types by $i \in \{L, H\}$.

Market interaction. The market interaction between workers and firms proceeds in multiple subsequent stages. In the first stage, firms decide whether to become active or not. Active firms incur an entry cost $k > 0$ and subsequently choose and post the wage p that they will pay their potential hire. The support of the distribution of posted wages is denoted by $\mathcal{F}$.

Workers observe all posted wages before sending $N \in \{1, 2, \ldots\}$ job applications to firms in the second stage.[4] As standard in the directed search literature and motivated

[4] While we will generally focus on $N \geq 2$, we include $N = 1$ for completeness and to ease comparison with the existing literature.

by the idea that coordination among a continuum of agents in decentralized markets is unrealistic, we restrict workers to symmetric and anonymous strategies, which creates the search frictions we study. That is, for each application a worker selects a wage and then picks at random one of the firms posting such wage to whom he applies. A worker's application portfolio is thus a list of N wages. As we will show, whenever a worker has the opportunity to send an additional application, that application will be sent to a (weakly) higher wage than the previous ones. It is then convenient to order the applications sent in a weakly increasing order so a portfolio is described by $(p_1, \ldots, p_N) \in \mathcal{F}^N$, with $p_1 \leq \ldots \leq p_N$. Although the worker sends all N applications simultaneously, it will often be useful to refer to p_n, i.e. the n-th lowest application, as the worker's n-th application.

After the applications are sent, matches are formed. Following Kircher (2009), we model this in the spirit of deferred acceptance (Gale and Shapley, 1962). First, each firm with applicants randomly selects one of them and make him a job offer. Workers keep the best job offer they receive under consideration (without loss of generality, we take this to be the offer with the highest index) and reject all worse job offers. Firms whose job offers are rejected can then select a different applicant (as long as they still have one) and make a new job offer. After this, the process repeats until there are no more rejections. At that point, workers accept the job offer under consideration.[5]

Finally, after matches are formed, production takes place and payoffs are realized. In particular, a match between a firm and a worker of type i results in an output v_i, where $v_H \geq v_L$. The firm's payoff from the match is the difference between this output and the wage p that it pays. In contrast, the worker's payoff from the match is the difference between this wage and his outside option (or disutility from effort) c_i, where $c_H > c_L$. Unmatched workers and inactive firms receive a zero payoff.

Queues. Consider a *(sub)market* $p \in \mathcal{F}$, defined as the collection of all the firms posting this wage and of all the applications that they jointly receive. From the firms' perspective, each application has two unobservable but payoff-relevant characteristics: i) its position $n \in \{1, \ldots, N\}$ in the sender's application portfolio, which affects the firms' matching probability, and ii) the type $i \in \{L, H\}$ of its sender, which affects

[5] The same outcome can be motivated in other ways: i) as the stable matching on the network created by workers' applications, i.e. no firm remains unmatched while one of its applicants is hired at a lower wage, where ties are broken randomly; or ii) as the result of a process in which the market clears from the top, i.e. firms posting the highest wages make job offers first, followed by firms posting the next highest wages, etc.

the firms' payoff conditional on a match.

Define the *queue length* $\lambda_{n,i}(p) \in \mathbb{R}_+$ as the endogenous ratio of the number of applications with characteristics (n, i) to the number of firms in submarket p. As well-known in the literature, the number of applicants with characteristics (n, i) at a firm posting a wage p follows a Poisson distribution with mean equal to this queue length, independently of the number of applicants with other characteristics.[6]

Some of these applicants are irrelevant from the firm's point of view as they would turn down a potential job offer due to better offers from other firms. Denote the endogenous probability that an applicant with characteristics (n, i) would accept a job offer by $\xi_{n,i}(p) \in [0, 1]$. The number of *effective* applicants with characteristics (n, i) then follows a Poisson distribution with mean (or *effective queue length*) $\mu_{n,i}(p) = \xi_{n,i}(p)\lambda_{n,i}(p) \in \mathbb{R}_+$. For most of our analysis it will be convenient to aggregate these queues and define

$$\mu(p) \equiv \sum_n \sum_i \mu_{n,i}(p) \tag{1}$$

as the total effective queue length in market p, and

$$\gamma(p) \equiv \sum_n \mu_{n,L}(p)/\mu(p) \tag{2}$$

as the effective fraction of L-type workers.

Payoffs. Given $\mu(p)$ and $\gamma(p)$, we can construct the expected payoff $\pi(p)$ of a firm offering wage p. The firm incurs the entry cost and subsequently matches as long as it has at least one effective applicant, which occurs with probability $\eta(\mu(p)) \equiv 1 - e^{-\mu(p)}$. The hire will turn out to be an L-type worker with probability $\gamma(p)$ and an H-type worker with complementary probability. Therefore,

$$\pi(p) = \eta(\mu(p))\left(\gamma(p)v_L + (1 - \gamma(p))v_H - p\right) - k. \tag{3}$$

Active firms choose a posted wage p so as to maximize their profit $\pi(p)$. Free entry implies that in equilibrium these profits are zero.

Next, consider the expected payoff of a worker applying to $(p_1, \dots, p_N)$. The worker ends up earning a payoff $p_n - c_i$ if two conditions are satisfied. First, the ap-

[6] See Lester et al. (2015) and Cai et al. (2017) for a detailed discussion of this property, which they call *invariance*.

plication to p_n must result in a job offer, which happens with probability $\psi\left(\mu(p_n)\right) \equiv \eta\left(\mu(p_n)\right)/\mu\left(p_n\right)$. Second, none of the applications to higher wages $p_{n+1},\ldots,p_N$ must result in a job offer, which is the case with probability $\prod_{j=n+1}^{N}\left(1-\psi\left(\mu\left(p_j\right)\right)\right)$. The worker's expected payoff $u_{N,i}$ from sending N applications therefore equals

$$u_{N,i} = \max_{(p_1,\ldots,p_N)\in\mathcal{F}^N} \sum_{n=1}^{N} \prod_{j=n+1}^{N} \left(1-\psi\left(\mu(p_j)\right)\right) \psi\left(\mu_n(p)\right)\left(p_n - c_i\right).$$

As in Kircher (2009), this payoff can be rewritten in a recursive way, where

$$u_{n,i} = \max_{p\in\mathcal{F}} \psi\left(\mu(p)\right)\left(p-c_i\right) + \left(1-\psi\left(\mu(p)\right)\right) u_{n-1,i} \tag{4}$$

is the payoff of the first n applications, for all $n \in \{1,\ldots,N\}$, and $u_{0,i}=0$. Intuitively, the worker's n-th application results in a wage offer p with probability $\psi\left(\mu(p)\right)$; with complementary probability, the worker does not receive such an offer, but still has the chance that one of his applications to lower wages is successful, yielding a conditional payoff equal to $u_{n-1,i}$. Since $u_{n-1,i}$ is the expected payoff from sending $n-1$ applications to wages below p_n and trading at those wages occurs with probability less than 1, it follows from the above equation that $u_{n,i}$ is strictly increasing in n. Going forward, we will often refer to $c_i + u_{n-1,i}$ as the worker's *effective outside option* when sending his n-th application, and to $u_{n,i}$ as his *market utility* from sending n applications.

Beliefs. In order to decide whether to post a particular wage p, a firm needs to form beliefs about the applicant pool $(\mu(p), \gamma(p))$ that it will attract. If the wage is part of the equilibrium choices of firms, these beliefs are determined by the consistency conditions with firms' and workers' strategies, as described above. If instead the wage is not part of the equilibrium, we follow the standard assumption in the directed search literature that these beliefs are pinned down by the *market utility condition*, which aims to capture the consequences of deviations in our continuum economy in the spirit of subgame perfection.

To understand the market utility condition, consider an equilibrium wage $p \in \mathcal{F}$. For this wage, worker optimization implies that the effective queue length $\mu(p)$ must satisfy

$$u_{n,i} \geq \psi\left(\mu\left(p\right)\right)\left(p - c_i - u_{n-1,i}\right) + u_{n-1,i}, \tag{5}$$

with weak inequality for all (n, i) and with equality for at least one (n, i) if $\mu(p) > 0$. The market utility condition extends this idea to all p that are not part of an equilibrium. That is, a firm posting $p \notin \mathcal{F}$ expects an effective queue length $\mu(p)$ implying the smallest job offer probability that is needed to induce one of the workers' types to redirect one of their applications to p, indeed in the spirit of subgame perfection. This also pins down beliefs about the market composition: at this wage, the firm expects to attract applicants of a certain type only if (5) holds with equality for that type for some n. That is, for any $p \notin \mathcal{F}$, $\gamma(p)$ satisfies

$$\begin{cases} \gamma(p)\,\mu(p) = 0 & \text{if (5) holds with strict inequality for } i = L \text{ and all } n \\ (1-\gamma(p))\,\mu(p) = 0 & \text{if (5) holds with strict inequality for } i = H \text{ and all } n. \end{cases} \tag{6}$$

Equilibrium. We can then define an equilibrium as follows.[7]

Definition 1. *An equilibrium is a set of wages $\mathcal{F}$ posted by firms, effective queue lengths and compositions $(\mu(p), \gamma(p))$ for all p, and market utilities $u_{n,i}$ for all n and i, such that*

1. *Worker Optimization: a worker of type i sends his n-th application to wage $p \in \mathcal{F}$ only if (5) holds as equality.*

2. *Firm Optimization: $\pi(p) = 0$ for any $p \in \mathcal{F}$, and $\pi(p) \leq 0$ for any $p \notin \mathcal{F}$.*

3. *Consistency: for any $p \in \mathcal{F}$, $\mu(p)$ and $\gamma(p)$ are consistent with workers' and firms' strategies.*

4. *Out-of-Equilibrium Beliefs: for any $p \notin \mathcal{F}$, $\gamma(p)$ satisfies (6) and $\mu(p)$ satisfies (5) with weak inequality for any (n, i), and with equality for at least one (n, i) if $\mu(p) > 0$.*

3 Preliminaries

3.1 Indifference and Isoprofit Curves

Most of our analysis of workers' and firms' choices and hence of equilibria can be presented graphically by considering workers' indifference curves and firms' isoprofit

[7] To keep notation simpler in the main text, in the definition of equilibrium we state the consistency condition somewhat informally. We provide the full details in Appendix C.

curves. To facilitate this approach, we introduce these curves here and discuss some useful properties.

Isoprofit curves. As equation (3) shows, firms' profits depend not only on the price p and the effective queue length μ, but also on the queue composition γ. Hence, we need to specify the value of γ before being able to determine a firm's isoprofit curve as the set of all combinations of μ and p satisfying the free entry condition. The two extremes in which the firm respectively attracts only low (i.e. $\gamma = 1$) or high types ($\gamma = 0$) will prove to be particularly useful for our analysis. The isoprofit curves in those two cases is defined as follows:

$$\Pi_i \equiv \left\{ (\mu, p) \in \mathbb{R}^2 : \eta(\mu)(v_i - p) = k \right\}, \tag{7}$$

with $i \in \{L, H\}$.

Indifference curves. The indifference curve $I_{n,i}$ of a worker of type i sending his n-th application consists of all combinations of μ and p that solve (5) with equality.

$$I_{n,i} \equiv \left\{ (\mu, p) \in \mathbb{R}^2 : \psi(\mu)(p - c_i) + (1 - \psi(\mu)) u_{n-1,i} = u_{n,i} \right\}. \tag{8}$$

Differentiation of (5) reveals that the slope of a worker's indifference curve equals

$$\frac{d\mu}{dp} = -\frac{\psi(\mu)}{\psi'(\mu)} \frac{1}{p - c_i - u_{n-1,i}} > 0.$$

This expression highlights some helpful properties. In particular, the slope depends on the type of the worker (only) through the effective outside option $c_i + u_{n-1,i}$. As long as these effective outside options differ, the two types have different marginal rates of substitution between wage and matching probability for their n-th application, which creates scope for screening. For the first application, this is the case by assumption since $u_{0,L} = u_{0,H} = 0$ and $c_H > c_L$. For applications with higher indices ($n = 2, 3, \ldots$) however, the effective outside option is endogenous. It is easy to see that a worker's indifference curves becomes steeper as the index n of the application increases. Intuitively, as the effective outside option of a worker increases, he is willing to tolerate a larger increase in the effective queue length to obtain a higher wage. It is also clear that, for the same number of the application, the high type has steeper indifference curves, i.e. $c_L + u_{n-1,L} < c_H + u_{n-1,H}$ for all n. What is less obvious, however, is how $c_L + u_{n-1,L}$ compares to $c_H + u_{m-1,H}$ for $n > m$. This question will be at the center of our analysis in the following section.

3.2 Observable Types

It will be useful to first describe the equilibrium allocation that arises if worker types are observable to firms and hence incentive constraints are absent, as obtained by Kircher (2009).

Equilibrium allocation. Due to free entry, the equilibrium allocation can be determined for each type i worker in isolation. It is entirely pinned down by the free entry condition and the first-order condition of the firms' choice problem, taking into account beliefs as determined by market utility. Graphically, these beliefs are represented by the upper envelope of workers' indifference curves $I_{n,i}, n \in \{1, \ldots, N\}$ in the (p, μ) space. The effective queue lengths and wages for the N applications of each worker are then determined by the tangency points between the firms' isoprofit curve Π_i and this upper envelope, as illustrated in Figure 1. As shown by Kircher (2009), letting $p^*_{n,i}$ denote the wage to which a worker of type i sends his n-th application, one can combine these conditions to recursively characterize the equilibrium effective queue length $\mu^*_{n,i} \equiv \mu(p^*_{n,i})$ and the associated market utility $u^*_{n,i}$ for each application n and each type $i = L, H$. The procedure is as follows: set $u^*_{0,i} = 0$ and let $\{\mu^*_{n,i}, u^*_{n,i}\}_{i=1}^N$ be such that

$$k = \left(\eta\left(\mu^*_{n,i}\right) - \mu^*_{n,i}\eta'\left(\mu^*_{n,i}\right)\right)(v_i - c_i - u^*_{n-1,i}), \tag{9}$$

$$u^*_{n,i} = u^*_{n-1,i} + \eta'\left(\mu^*_{n,i}\right)(v_i - c_i - u^*_{n-1,i}). \tag{10}$$

Since the indifference curves become steeper as the index n of the application increases, the tangency point for this application moves up the firms' isoprofit curve to a higher wage and effective queue length.

The allocation implied by (9) and (10) will be different for workers of different types, since the firms' willingness to offer a wage p with queue length μ depends on the worker's type i, determining the firm's payoff v_i from hiring the worker. Similarly, workers of different types exhibit different preferences over portfolios of applications because their tradeoff between the wage and the probability of being hired depends on their outside option c_i. Hence, with heterogenous, observable types, the equilibrium features a separate submarket for each type and each application (at least generically).

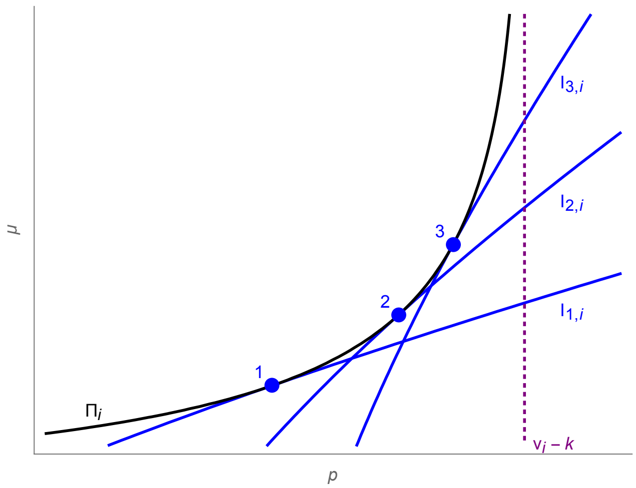

Figure 1: Equilibrium wages and effective queue lengths for the case where the type is observable and workers send three applications.

Vanishing search frictions. Kircher (2009) shows that, as the number of applications N goes to infinity, the equilibrium allocation tends to the Walrasian outcome in which all firms active in the market hire a worker with probability one and every worker finds a job. To see this, notice first that the difference $u_{N,i} - u_{N-1,i}$ converges to zero as $N \to +\infty$, since $u_{N,i}$ is strictly increasing in N and bounded above by the gains from trade, $v_i - c_i - k$. This property implies that the effective queue length $\mu^*_{N,i}$ tends to $+\infty$, as can be seen from (10).[8] Since $\lim_{n\to+\infty} \mu^*_{N,i} = +\infty$, every firm hires a worker, so the free-entry condition (9) requires that a worker's expected utility from his portfolio of applications, $u^*_{N,i}$, tends to $v_i - c_i - k$ as $N \to +\infty$. Because no firm would offer a wage greater than $v_i - k$, this implies that, in the limit, a worker is hired at a wage $v_i - k$ with probability one. It further means that the measure of firms posting wages that are bounded away from $v_i - k$ (or, equivalently, the probability of a firm attracting a finite effective queue length) tends to 0. In other words, all entering firms hire with probability one in the limit. The impact of the search friction thus disappears in the limit where each worker can submit infinitely many applications and the equilibrium allocation converges to the Walrasian outcome.

[8] Since $u_{N,i} - u_{N-1,i} \to 0$, (10) implies that $\eta'(\mu(p)) = e^{-\mu(p)} \to 0$ and hence $\mu(p) \to \infty$.

4 Equilibria with Adverse Selection

As seen in the previous section, allowing workers to apply simultaneously to multiple firms mitigates the search friction and therefore increases the trading probability of workers and firms when this friction is the only impediment to trade. We show next that the same result may not hold in environments with adverse selection because incentive constraints also limit trades. When workers submit multiple applications, the screening role of market liquidity is diminished, since workers can hedge against the possibility of not being hired in an illiquid market by sending some of their applications to more liquid markets. This second effect exhibits a close analogy to that of non-exclusivity in contracting, in which case the possibility of carrying out additional trades without being detected limits the possibility of screening agents via the design of contracts specifying different levels of trade. In markets with adverse selection, we thus face an interesting tradeoff: allowing workers to submit multiple applications reduces the search friction on the one hand, but restricts the possibility of screening workers on the other hand. In what follows, we will analyze how this trade-off shapes the properties of equilibrium allocations.

Incentive constraints. When types are unobservable, the allocation described in the previous section will often not be sustainable in equilibrium. The reason is that, due to the interdependence of values, L-type workers may find it profitable to send some applications to the submarkets designed for H-type workers. This point is illustrated in Figure 2, where we display the equilibrium allocation when both types are observable. Graphically, there are two relevant isoprofit curves for the firms, one for hiring the H-type and one for hiring the L-type. The H-isoprofit curve is shifted to the right with respect to the L-isoprofit curve, because, for each effective queue length μ, a firm is willing to pay a higher wage p for a worker of high productivity. In Figure 2, incentive compatibility is violated for L-type workers when they can send $N \geq 2$ applications. They can gain, for instance, by sending their second application to the market where H-type workers send their first.

Note that incentive constraints may be binding already in the case where workers send a single application. Multiple applications, however, tighten this constraint. Indeed, if v_H is strictly greater than v_L, incentive constraints necessarily bind whenever the number of applications that workers can send is sufficiently large. To see this, recall that the equilibrium with observable types converges to the Walrasian allocation when $N \to +\infty$. In this limit, both types of workers are hired with probability one, but the expected wage for the H-type, $v_H - k$, is strictly greater than that for the L-type, $v_L - k$. As a result L-type workers have strict incentives to send some of their applications to a market with a wage strictly above $v_L - k$. Hence the allocation found in Section 3.2 does not constitute an equilibrium when workers' productivity is only privately known by them.

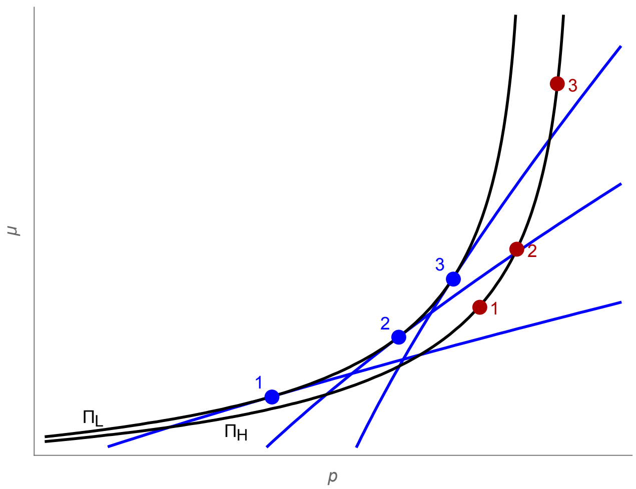

Figure 2: Equilibrium wages and effective queue lengths for the low type (blue) and the high type (red) when types are observable.

4.1 Market Segmentation

When workers can only send a single application, we know from Gale (1992) and Guerrieri et al. (2010) that complete market segmentation always obtains in equilibrium: L-type workers apply to a different market, with a lower price and a lower queue length, than the one to which high types apply, featuring a higher price and a higher queue length. We show first that this is no longer true when workers apply to several firms simultaneously: *with multiple applications, fully separating equilibria do not always exist.*

To characterize the conditions under which L- and H-type workers self-select into different markets, we introduce first the L-type's lower contour set in the allocation that is obtained when his type is observable, i.e. all the pairs (μ, p) that the L-type worker does not prefer to $(\mu^*_{n,L}, p^*_{n,L})$, characterized in Section 3.2, for all $n \in \mathbb{N}$:

$$U_L \equiv \left\{(\mu, p) \geq (\mu^*_{1,L}, p^*_{1,L}) : \forall n \in \mathbb{N},\ \psi(\mu)\left(p - c_L - u^*_{n-1,L}\right) \leq u^*_{n,L} - u^*_{n-1,L}\right\}.$$

In Figure 2, the set U_L is the area lying above (i.e., less preferred than) the indifference curves associated to all the applications chosen by type L (for $N \to \infty$). If the difference between the productivity of L- and H-type workers is sufficiently small, i.e. v_H is close to or equal to v_L, the set U_L has a non-empty intersection with the H-type isoprofit curve, Π_H. In this case, we can find a pair (μ, p) that yields zero profits with the H-type and does not attract L-type workers in that they prefer an L-type market over (μ, p) for all of their applications. As v_H increases, the set Π_H shifts down to the right in the figure, while U_L is unaffected, making the intersection of the two sets smaller until it vanishes at some point. Let $\bar{v}_H$ be the largest value of v_H such that $U_L \cap \Pi_H \neq \emptyset$. Figure 2 illustrates the case $v_H > \bar{v}_H$, where the two sets do not intersect. In this case, for any given N, the only incentive compatible pairs (μ, p) yielding zero profits with the H-type are the points in Π_H lying above the intersection with $I_{N,L}$.

Building on this, the following proposition establishes necessary and sufficient conditions for the existence of a (fully) separating equilibrium for any number of applications that workers can send.

Proposition 1. *If $v_L - k \leq c_H$, for all $N \geq 1$, there exists a separating equilibrium. If $v_H > \bar{v}_H$, for a separating equilibrium to exist for all $N \geq 1$, the condition $v_L - k \leq c_H$ is also necessary.*

The first part of Proposition 1 claims that a separating equilibrium exists, regardless of the number of applications that workers can send, under the condition $v_L - k \leq c_H$, that is, when the outside option of the H-type worker exceeds the productivity of the low type net of entry costs, referred to in the literature as the lemons condition (see e.g Daley and Green, 2012). As shown in the proof in the appendix, the incentive constraints of the high-productivity workers are slack in any separating equilibrium. Hence, the effective queue lengths and wages in markets for L-type workers are the same as in the unconstrained solution of Section 3.2, i.e. $\mu_{n,L} = \mu^*_{n,L}$ and $p_{n,L} = p^*_{n,L}$, for all $n = 1, ..., N$. When $v_L - k \leq c_H$, H-type workers send their applications to wages that are strictly higher than any wage to which the L-type workers apply, that is, $p_{j,H} > p^*_{n,L}$ for all n, j. As a consequence, the only incentive constraint that is potentially binding in equilibrium is the one associated to the low type's N-th application.

The wages and effective queue lengths in the H-type markets can then be constructed sequentially. If the unconstrained solution associated to the H-type's first application, $(\mu^*_{1,H}, p^*_{1,H})$ satisfies the L-type's incentive constraint associated to his N-th application, then the effective queue lengths and wages in all H-type markets are determined by the unconstrained solution. If, on the other hand, $(\mu^*_{1,H}, p^*_{1,H})$ violates the L-type's incentive compatibility constraint (and, as argued in the previous section, this always happens for N large enough), $(\mu_{1,H}, p_{1,H})$ is given by the smallest effective queue length and wage on the isoprofit curve Π_H such that incentive compatibility holds. Proceeding to the H-type's second application, we can determine the tangency between Π_H and the H-type worker's second indifference curve, i.e. the one corresponding to the effective outside option $c_H + u_{1,H}$. If this tangency point satisfies $\mu_{2,H} > \mu_{1,H}$, incentive compatibility is satisfied and the terms of trade in the markets for the H-type's remaining applications are determined in a similar way. Otherwise, incentive compatibility also binds for the H-type's second application, in which case the H-type workers send the first and second application to the same market. We repeat this procedure for the H-type's next application, until we reach the last application $n = N$. The feature that a worker may send multiple applications to the same submarket does not arise in the observable type case, as it is driven by the binding incentive constraints. We illustrate this property for the case of two applications in Figure 3.

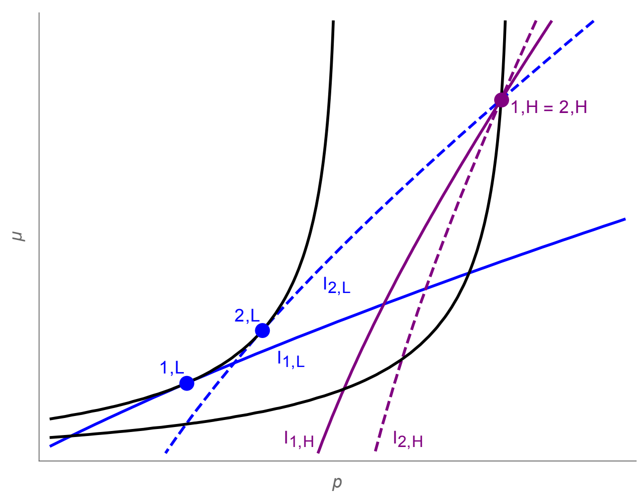

Figure 3: Equilibrium wages and effective queue lengths when workers send two applications and incentive constraints bind for both applications of the high type.

It is immediate to see that when the lemons condition does not hold and N is sufficiently large, the separating allocation described in the previous paragraphs no longer constitutes an equilibrium. If $v_L - k > c_H$, some of the wages to which low types apply are also acceptable for high types. Moreover, as N becomes sufficiently large, the low types' effective outside option associated with their N-th application, $c_L + u^*_{N-1,L}$, exceeds the high types' outside option associated with their first application, c_H. This follows from the property $u^*_{N,L} \to v_L - c_L - k$ as $N \to +\infty$, established in Section 3.2. The crossing of the two types' indifference curves associated to these applications is then reversed: the one of the first application of high types becomes flatter than the one of the N-th application of low types.[9] This implies that high types strictly prefer to send their first application to $p^*_{N,L}$ rather than to $p_{1,H}$, so that the candidate separating equilibrium under consideration is no longer an equilibrium. The condition $v_H > \bar{v}_H$ assures that no other kind of separating equilibrium—where H-type workers send their first application to a market with a wage smaller than $p^*_{N,L}$—exists either.

Remark 1. The restriction $v_H > \bar{v}_H$ requires that there is sufficient interdependence in the values of workers and firms. To understand the role of this assumption, consider the private value case ($v_L = v_H$), where the inequality $v_H > \bar{v}_H$ is clearly violated. Since in such case firms do not care which type of worker they hire, no type can gain by imitating the other type. Hence, incentive constraints are always slack and the allocation described in Section 3.2 for the observable type case continues to be the unique equilibrium when types are only privately observed. In this equilibrium, the intervals defined by the range of prices to which low and high types send their applications overlap: H-type workers send their first application to a firm posting a lower wage than the one offered by firms to which L-type workers send their last application. The same is true for $v_H - v_L$ positive but small: a separating equilibrium exists with overlapping price ranges, even when the number of applications N becomes large. If instead $v_H > \bar{v}_H$, overlapping price ranges cannot be sustained in equilibrium.

The lemons condition $v_L - k \le c_H$ also assures existence of a separating equilibrium in a Walrasian market environment à la Akerlof (1970), where there are no search frictions (typically with $k = 0$) and agents can trade at a single price, which they take as given. In such a market, the probability of trade, or the quantity traded, cannot be used to separate different types, except in the extreme case where some types

[9] As explained in Section 3.1, the slope of the indifference curve at (μ, p) is determined by a worker's effective outside option $c_i + u_{n-1,i}$. Hence, when $c_L + u_{n-1,L} > c_H$ the slope of the indifference curve for the first application of high types is flatter than that for the n-th application of low types.

choose not to trade at all in the market. For the economy we considered, with two levels of quality, it is well known that there are only two possible kinds of competitive equilibria with markets à la Akerlof (1970): one with a low price $p = v_L - k$ and only low types active in the market (i.e. a separating outcome), and one with a higher price $p = \sigma v_L + (1-\sigma)v_H - k$ and both types active in the market (i.e. a pooling outcome). The separating equilibrium exists if $c_H \geq v_L - k$, while the pooling equilibrium exists if $c_H \leq \sigma v_L + (1-\sigma)v_H - k$.

Vanishing search frictions. Proposition 1 shows that, when workers can send several applications, a separating equilibrium always exists if the same condition on the parameters of the economy holds under which a separating equilibrium exists in Walrasian markets à la Akerlof (1970). The equilibrium allocation is however different for the two market structures, since high types trade with positive probability in the search equilibrium. This difference is not surprising as the search friction still matters for any finite N. It is however of interest to examine the outcome when $N \to \infty$ and workers essentially face no constraints in their ability to contact firms. In that case, as we noticed for the observable type case, the search friction vanishes in the limit.

As we increase the number of applications that workers can send, the constraints imposed on the trading probability of H-type workers become tighter. Since both $\mu^*_{N,L}$ and $p^*_{N,L}$ increase with N, and the associated indifference curve becomes steeper, an increase in N pushes up the wage $p_{1,H}$ and the effective queue length $\mu_{1,H}$ in the market where high types apply. This is a clear evidence of the fact that market liquidity is less effective as a screening instrument when workers can send several applications: to separate themselves, high types must choose less and less liquid markets. Hence, as N increases, high types send more applications but also face increasingly congested markets. The following proposition shows that the latter effect outweighs the former and H-type workers are eventually driven out of the market.

Proposition 2. *Assume $v_L - k < c_H$. As $N \to +\infty$, the probability that an H-type worker is hired in a separating equilibrium tends to zero. The market utilities for L- and H-type workers take the following limits:*

$$\begin{aligned}
\lim_{N \to +\infty} u_{N,L} &= v_L - c_L - k, \\
\lim_{N \to +\infty} u_{N,H} &= 0.
\end{aligned}$$

To prove the result, we consider a candidate separating equilibrium that involves

H-type workers being hired with a strictly positive probability and construct a profitable deviation for low types. If low types follow the equilibrium strategy and send all their applications to the respective L-type markets, their probability of being hired tends to one and their wage to $v_L - k$. Suppose instead an L-type worker sends half of his applications to the first $N/2$ L-type markets and the remaining applications to the H-type market with the lowest effective queue length. Since N is arbitrarily large, the probability of being hired in one of the L-type markets is still arbitrarily close to one and the wage is arbitrarily close to $v_L - k$. We then show that sending half of the applications to the H-type market allows the L-type worker to be hired in that market with strictly positive probability. Since the wage in the H-type market is greater than c_H, which in turn is greater than $v_L - k$, the described portfolio of applications generates a strictly higher expected wage and thus constitutes a profitable deviation.

Summarizing, we conclude that, *as the search friction vanishes, the allocation in the separating search equilibrium converges to the one with Walrasian markets à la Akerlof (1970).*

4.2 Equilibria with Pooling Markets

We establish next another important implication of allowing workers to send multiple applications: *pooling markets may be active in equilibrium.* When this happens, there are typically multiple ways in which workers can pool some of their applications and hence multiple equilibria exist. This result is in stark contrast with the case in which workers can only send one application, where the equilibrium is unique and fully separating.

To properly explain our finding, it is useful to briefly review first the argument why equilibria with pooling cannot exist when workers can only send a single application. The reason is that in such a situation a profitable cream-skimming deviation always exists. To see this, consider a situation in which there is a pooling market $(\bar{\mu}, \bar{p})$ attracting both types, as illustrated by the green point in Figure 4. Since firms attract both types, the isoprofit curve associated with zero profits lies between Π_L and Π_H, as illustrated by the green curve in the figure. Due to the higher outside option, the indifference curve of the H-type passing through $(\bar{\mu}, \bar{p})$ is steeper than that of the L-type. This difference in marginal rates of substitution implies that high types are willing to tolerate longer effective queue lengths than low types in any market

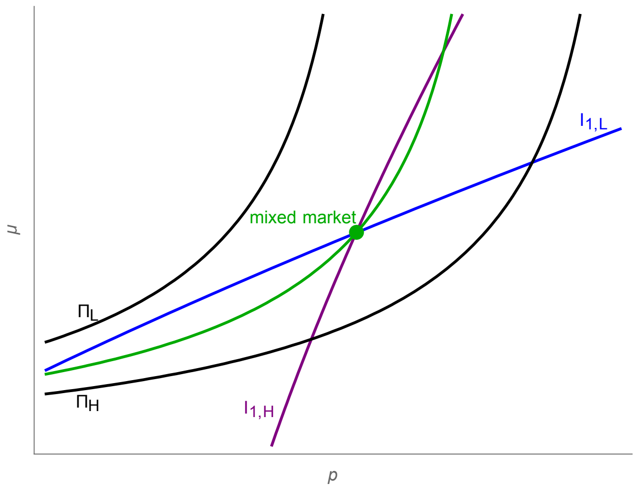

Figure 4: Cream-skimming deviation in the single-application case.

with a wage higher than $\bar{p}$. In other words, the H-type worker has more to gain by applying to a wage above $\bar{p}$ than an L-type worker. If a firm deviates and increases the wage above $\bar{p}$, it thus expects to attract only H-type workers. Hence, a marginal increase in the wage and the associated queue length leads to a discrete improvement in the composition of the applicant pool and thus constitutes a profitable deviation, effectively a cream-skimming deviation. As we will show, this argument is not always valid when workers send more than one application.

Example. Before stating our result formally, we illustrate it graphically for the same environment considered in Figure 4. Figure 5 describes an equilibrium where each worker sends two applications. There are three active markets: one with a low wage where each low-type worker sends his first application $(1, L)$, one with a high wage to which each high-type worker sends his second application $(2, H)$, and one with an intermediate wage where each low- (resp. high-) type worker sends his second (resp. first) application. We refer to the latter market as the *pooling market*, since both types send applications there. Low types apply to the pooling market hoping to receive an offer at the wage posted in that market, but insure themselves by sending also one application to a lower wage, where the chance of getting an offer is higher. In contrast, for high types the pooling market represents the fallback option in case their application to a firm offering a higher wage fails. As in Figure 4, let $\bar{p}$ and $\bar{\mu}$, respectively, denote the wage and the effective queue length in the pooling market. Note however that now the isoprofit curve (green curve) is different from the one in Figure 4: the effective composition in the pooling market is not equal to the population average but worse than that, because high types only agree to trade at the pooling wage $\bar{p}$ if they receive no offer in the high-wage market $2, H$.

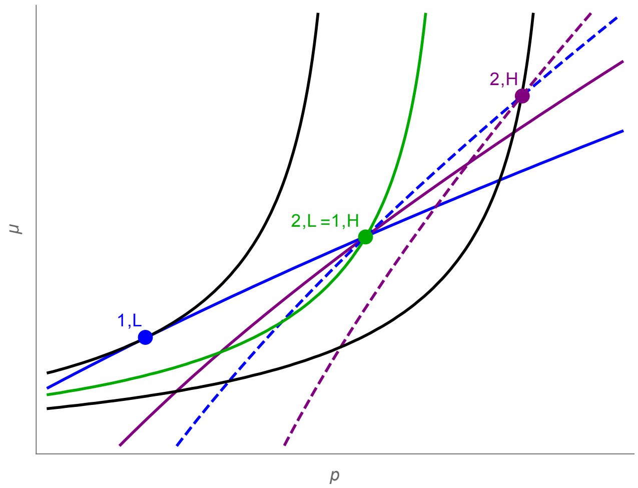

Figure 5: Equilibrium wages and effective queue lengths in an equilibrium with a pooling market with $N = 2$.

To be able to claim that the described allocation constitutes an equilibrium, we need to verify that firms have no incentives to deviate by offering a different wage. In particular, we must rule out the profitability of cream-skimming deviations like the ones we saw existed for the pooling allocation in Figure 4, when workers could only send one application. To assess the profitability of a deviation to a different wage, we must again determine which type of worker this wage is more likely to attract. In Figure 5, the L-type's indifference curve associated with his second application (the dashed blue curve) is steeper than the H-type's indifference curve associated with his first application (solid purple curve). As already noticed in Section 3.1, this happens when the effective outside option for this application of the L-type, $c_L + u^*_{1,L}$, is higher than the outside option for the H-type's first application, c_H. This reversal of the 'sorting condition' relative to Figure 4 implies that it is not the H-type who has most to gain from applying to wages slightly above $\bar{p}$ but rather the L-type with his second application. Hence, a firm contemplating to offer one of those wages expects to attract only L-type workers, which implies these cream skimming deviations are no longer profitable. For wages below $\bar{p}$, it is again the low type who has most to gain, this time by sending his first application. Hence, firms can only worsen the composition of the set of workers they attract by deviating to a wage slightly above or below $\bar{p}$, which means that no profitable cream-skimming deviation exists.

General result. We proceed now to formally establish conditions under which equilibria with pooling markets exist. As the previous discussion suggests, a key ingredient is the reversal of the sorting condition at any submarket where both low and high types send applications. This reversal cannot happen for the first application of L-type workers, since their outside option is $c_L < c_H$. Let us then define l as the highest value of n for which the sorting condition is still valid (if low types send their first n applications to separate markets with the same terms of trade $\mu^*_{j,L}, p^*_{j,L}$, $j = 1, .., n$ as in a separating equilibrium):

$$l \equiv \min\{n \in \mathbb{N} : u^*_{n,L} + c_L \geq c_H\}. \tag{11}$$

Such a value exists if and only if $v_L - k > c_H$, i.e. when the lemons condition is violated.[10] Figure 5 illustrates the case where $l = 1$, in which case the sorting condition is reversed for the second application of the low types.

Assuming that the lemons condition does not hold and $N > l$,[11] the construction in Figure 5 can then be generalized as follows. Low and high types send, respectively, their first and their last l applications to separate markets, while all remaining applications are sent to a single pooling market with wage $\bar{p}$. The terms of trade in the L-type markets are the same as in the separating equilibrium, while the terms of trade in the pooling market are such that the L-type is indifferent between sending his l-th application to the pooling market or to the respective L-type market. The wages and effective queue lengths of the H-type markets are determined by the L-type's incentive constraint associated to his N-th application, using a procedure analogous to the one described in Section 4.1.

To ensure that the allocation constructed in this way constitutes an equilibrium, we need to verify two properties. First, deviating to a price slightly below or above $\bar{p}$ does not allow a firm to improve the composition of the applicant pool received at $\bar{p}$, i.e. no profitable cream skimming deviation exists. The condition $u^*_{l-1,L} + c_L < c_H$, which follows from the definition of l, ensures that for wages below $\bar{p}$ it is the L-type who has most to gain, by redirecting his l-th application to such wages. Next,

[10] Recall that $u^*_{n,L} + c_L$ tends to $v_L - k$ as $n \to +\infty$.

[11] These are exactly the conditions under which the separating equilibrium we constructed in Section 4.1—featuring high types sending their applications to strictly higher wages than low types—fails to exist.

given that both types of workers send $N-l$ applications to the pooling market and, therefore, have the same chance of receiving an offer in that market, the condition $u^*_{l,L}+c_L \geq c_H$ implies $u^*_{N,L}+c_L \geq u_{N-l,H}+c_H$ (see the proof of Proposition 3). Hence, for wages just above $\bar{p}$ it is again the L-type, this time with his N-th application, who has most to gain from applying to these wages. Firms deviating to wages slightly below and above $\bar{p}$ thus expect to attract only low types.

The second property we need to verify is that firms do not find it profitable to attract L-type workers at any off-path wages. This property is satisfied as long as the L-type's indifference curves in the candidate equilibrium do not intersect the isoprofit curve Π_L. It is easy to see that it suffices to verify this property for the indifference curve associated to the N-th application, the last one sent to the pooling market. When N becomes large this indifference curve becomes vertical. To ensure the property holds for any N, the wage in the pooling market needs then to be higher than the vertical asymptote of Π_L and thus higher than $v_L - k$.

To find a sufficient condition guaranteeing this property, consider the pair (μ, p) determined by the intersection between the indifference curve $I_{l,L}$ and the isoprofit curve associated with zero profits when the fraction of low types in the market is the population value σ. This intersection is obtained as the solution to the following system of equations:[12]

$$
\begin{aligned}
u^*_{l,L} - u^*_{l-1,L} &= \psi\left(\mu\right)\left(p - \underline{c} - u^*_{l-1,L}\right), && (12)\\
\eta\left(\mu\right)\left(\sigma v_L + (1-\sigma) v_H - p\right) &= k. && (13)
\end{aligned}
$$

It is immediate to verify that the solution for p of this system is increasing in v_H and there exists a value $\hat{v}_H$ at which the solution equals $p = v_L - k$. This implies that when $v_H > \hat{v}_H$ there can be no intersection between the L-type's indifference curve $I_{N,L}$ passing through the point (p, μ), determined by (12-13), and the isoprofit curve Π_L, no matter how large is N.[13] The same property holds in equilibrium as long as the terms of trade in the pooling market are sufficiently close to the solution of (12-13), which, as argued below, is true whenever N is sufficiently large.

[12] As one can see from Figure 5, two intersections/solutions exist; the relevant one is that with the highest price.

[13] In the case where $v_H < \hat{v}_H$ and N is sufficiently large, there might be an intersection at a wage between $\bar{p}$ and the lowest wage in the H-type markets.

Proposition 3. *Assume $c_H < v_L - k$ and $v_H > \hat{v}_H$. Then, if N is sufficiently large, there exists an equilibrium where the low and the high types send, respectively, their last and first $N - l$ applications to the same market.*

The fact that workers can send more than one application limits the ability of employers to screen workers via distinct wage offers. This feature generates different outcomes for different values of the parameter space. We saw in Proposition 1 that when $c_H \geq v_L - k$, high and low types still trade in separate markets in equilibrium, but—compared to the case where a single application is sent—high types trade with a lower probability, which vanishes as $N \to \infty$. When instead $c_H < v_L - k$, Proposition 3 shows that separation breaks down in equilibrium, as high and low types send at least some of their applications to the same market.[14]

Remark 2. We should point out that, besides the equilibrium we illustrated in Figure 5 and whose existence we established more generally in Proposition 3, other equilibria exist where low types send less than l applications to separate markets and a larger number of applications than high types to a pooling market. In these equilibria, the composition of applicants in the pooling market is strictly worse than in the equilibrium we constructed (the pooling market lies on an isoprofit curve strictly to the left of the green curve in Figure 5), while the effective queue length and the wage are lower. In contrast, there are no equilibria where low types send more than l applications to separate markets and where the effective queue length and wage are higher in the pooling market than in the equilibrium described in Proposition 3. In this sense, the equilibrium we constructed constitutes an important benchmark, also for the analysis which follows. Note also that there cannot exist a fully pooling equilibrium in which both types send all their applications to the same market. In that case, the same cream-skimming deviation argument as in the one-application case applies.

Vanishing search frictions. When $c_H < v_L - k$, the Walrasian equilibrium à la Akerlof (1970) is unique and features all types trading at a single pooling price. Under the same condition, Proposition 3 shows that some of the trades of low and high types occur at the same price. We will now show that *as $N \to \infty$, equilibria with pooling*

[14]Both results require that v_H is sufficiently high, or that the economy considered is sufficiently different from the private-value case where $v_H = v_L$.

markets may not converge to the ones of Akerlof (1970) and exhibit inefficiency due to excessive entry.

To this end, it is important to point out one feature of the equilibrium constructed in the proof of Proposition 3: the switching point at which the low type starts applying to the pooling market does not depend on the total number of applications N. Hence, when N increases, the number of applications sent to the H- and L-type markets remains unchanged, equal to l. All additional applications go to the pooling market. What changes with N is the effective composition of applications in the pooling market. Even though both types send the same number of applications to that market, the high type sometimes gets an offer better than $\bar{p}$ and in that case would reject a wage offer equal to $\bar{p}$. Thus, as already mentioned, the effective composition is worse than the population average σ. However, as $N \to +\infty$ the probability that a high type trades in any of the H-type markets converges to zero, by the same argument as the one used for the separating equilibrium. As a consequence, the effective composition in the pooling market converges to the population average and the associated terms of trade (μ, p) converge to the solution of (12-13). The effective queue length in the pooling market thus remains finite in the limit (see Figure 6).

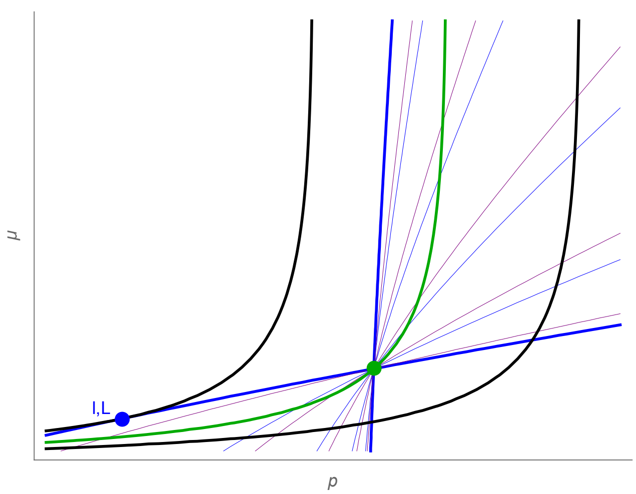

Figure 6: Equilibrium wages and effective queue lengths in an equilibrium with one pooling market.

The latter property has significant implications for the properties of the allocation obtained in the limit. Since the number of applications low and high types send to the pooling market tends to infinity as $N \to \infty$, a finite value of μ implies that the probability that any type ends up receiving an offer in the pooling market converges to one. In contrast, the probability that a firm hires a worker in the pooling market is bounded away from one and the wage is bounded away from $\sigma v_L + (1-\sigma)v_H - k$. Hence, there is excessive entry in the limit and, thus, a failure of convergence to the allocation obtained in the pooling equilibrium in Akerlof (1970). It also entails an efficiency loss relative to the latter outcome. This is an important result, as it shows that, in the presence of adverse selection, the inefficiency of the search equilibrium may not vanish in the limit when workers can send infinitely many applications to firms.

The first part of the next proposition establishes this result formally. As already noticed in Remark 2, uniqueness of the search equilibrium fails with multiple applications and we may have several equilibria with one pooling market. The second part of the proposition shows that the inefficiency result extends to all such equilibria.

Proposition 4. *Assume $c_H < v_L - k$. Then, as $N \to +\infty$,*

(i) at the equilibrium characterized in Proposition 3, the workers' probability of being hired in the pooling market converges to one and their market utility satisfies

$$\lim_{N\to+\infty} (\sigma u_{N,L} + (1-\sigma)u_{N,H}) < \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k, \tag{14}$$

hence there is excessive entry in the limit;

(ii) if $v_H > \bar{v}_H$, (14) also holds at any other equilibrium with a single pooling market.

The claim is established by contradiction. If (14) is violated and holds as an equality, this means that in the limit (a) all gains from trade in the market are exploited and (b) there is no excessive entry. We then show that (a) implies that the probability that workers are hired in the pooling market converges to 1, while from (b) it follows that the queue length in the pooling market tends to infinity. But this requires that low types send an arbitrarily large number of applications to separate markets which, under the stated conditions, contradicts incentive compatibility: high types would want to deviate and send some applications to L-type markets.

We show next that the multiplicity of equilibria extends beyond the situation discussed in Remark 2 and that this has important implications for the properties of equilibrium allocations in the limit as search frictions vanish. In particular, under the same parameter conditions as in Proposition 3, we can find a sequence of equilibria with two pooling markets where, as N tends to infinity, the probability that both types are hired converges to one and excessive entry vanishes. The key idea is the following: the first pooling market takes care of the incentives of high types to hedge by sending some applications to lower wages, which we saw in Proposition 4 leads to excessive entry when there is a single pooling market. But now a second pooling market is also active in equilibrium and the wage in this market increases with the number of applications, so that the queue length tends to infinity in the limit. Thus almost all applications are sent to the second pooling market and excessive entry vanishes, as we establish in the next proposition.[15] The construction is illustrated in Figure 7.

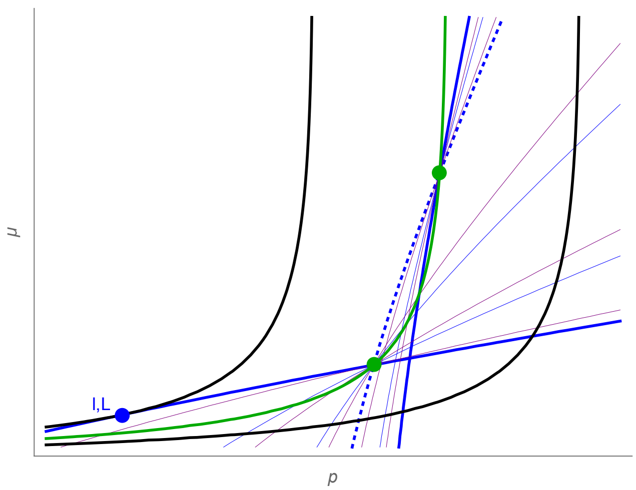

Figure 7: Equilibrium wages and effective queue lengths in an equilibrium with two pooling markets.

Proposition 5. *Assume $c_H < v_L - k$ and $v_H > \hat{v}_H$. For each $\varepsilon > 0$ arbitrarily close to zero, we can find some N_ε such that, for all $N > N_\varepsilon$, there exists an equilibrium with two pooling markets and*

$$\sigma u_{N,L} + (1-\sigma)u_{N,H} \geq \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k - \varepsilon. \qquad (15)$$

The consequences for the properties of equilibrium outcomes of allowing multiple applications in the presence of search frictions have interesting analogies to those

[15] Since the main steps of the proof are very similar to the ones of Proposition 3, we relegate the proofs of Proposition 5 and Proposition 6 in the following subsection to the Online Appendix.

of the non-exclusivity in contracting without such frictions. The latter also limits, though in different ways, the ability of firms to screen workers. Our environment features exclusivity in contracting, as each worker can accept only one offer, but not in applications as the worker can apply to many firms. When firms compete with non-exclusive contract offers, Attar et al. (2011) find that pooling obtains in equilibrium under the same no lemons condition as Proposition 3. Moreover, the equilibrium allocation is unique and coincides with the efficient pooling Walrasian equilibrium of Akerlof (1970). As demonstrated, our findings are different and suggest that non-exclusivity at the application stage has distinct implications from non-exclusivity in contracting.[16]

4.3 Co-Existence of Separating and Pooling Equilibria

In the previous section, we considered situations where the lemons condition is violated and established the existence of equilibria with pooling markets. We now show that such equilibria also exist when $c_H > v_L - k$, provided the proportion of H-type workers is sufficiently high, i.e. $c_H < \sigma v_L + (1-\sigma) v_H - k$. Under these two conditions, we know from Akerlof (1970) that when agents trade in Walrasian markets there are two equilibria, one separating and one pooling, the latter being Pareto dominating. We have seen in Propositions 1 and 2 that in this parameter region a fully separating equilibrium exists for all N and converges to the separating equilibrium of Akerlof (1970) as $N \to \infty$. We show in what follows that the pooling equilibrium can also be obtained in the limit.

To this end, we focus our attention on the case where the number of applications workers can send is large and establish a similar result to Proposition 5: a search equilibrium exists where the allocation is approximately the same as in the efficient pooling equilibrium of Akerlof (1970), i.e. both firms and workers are matched with probability one. The equilibrium differs however in some important aspects from the one characterized in Proposition 5. In particular, it features a single pooling market, where low types send a strictly greater number of applications than high types. The effective queue length in the pooling market approaches infinity in the limit and so does the number of applications sent by both types to that market.

To gain some understanding, we should recall that when the lemons condition is

[16] As discussed in the next section, an important role is also played by the fact that in our set-up firms can hire at most one worker.

violated, high types are willing to be hired at some of the wages in L-type markets. To ensure that the incentive constraints of high types are satisfied, separate markets for low types can then only exist for sufficiently low wages. In particular, as argued in Remark 2, they can send no more than l (defined in (11)) applications to separate markets. In contrast, when the lemons condition holds, high types are not willing to be hired in any of the L-type markets, hence their incentive constraints do not impose restrictions for the existence of these markets. We can thus have an arbitrarily large number of L-type markets (equivalently, $l = \infty$). As a consequence, we can set the number of applications sent by low types to separate markets at a sufficiently high level so that the queue length in the pooling market is arbitrarily large.

Proposition 6. *Let $c_H \in (v_L - k, \sigma v_L + (1-\sigma)v_H - k)$. For each $\varepsilon > 0$ arbitrarily close to zero, we can find some N_ε such that, for all $N > N_\varepsilon$, an equilibrium with a single pooling market exists and*

$$\sigma u_{N,L} + (1-\sigma)u_{N,H} \geq \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k - \varepsilon. \tag{16}$$

In our environment, as we showed in Section 4.2, multiple equilibria exist whenever we have pooling markets in equilibrium. The multiplicity persists in the limit, as the search friction vanishes. Proposition 6 shows that the *multiplicity of equilibria extends to the coexistence of equilibria with pooling markets and fully separating equilibria.* This stands in contrast with the findings of Kircher (2009) for the observable type case, where search equilibria are always unique. What is more striking, it also stands in contrast with the results obtained for the same parameter configuration of Proposition 6 when firms compete strategically in contract offers without search frictions. Both Attar et al. (2011), with general contracts under non-exclusivity, and e.g. Mas-Colell et al. (1995), with exclusive contracting when trade quantities are restricted to $\{0, 1\}$, find a unique equilibrium outcome, given by the Pareto dominant pooling allocation.

A key role behind this difference in equilibrium outcomes is played by the fact that in our environment firms are interested in hiring at most one worker, i.e. firms compete for workers but face an effective capacity constraint that limits the impact of their deviations on the market allocation. What ultimately matters is the presence of *some* capacity constraint, but not that the capacity is one. Without any capacity constraint, firms would find it profitable to deviate from the separating equilibrium by posting higher wages to attract all workers. In contrast, with a capacity constraint, firms would only attract the workers who are most keen to apply to higher wages and these are the low types (as illustrated in Figure 3). Our analysis therefore suggests that the presence of capacity constraints, as well as the decentralized nature of markets, have important consequences in markets with adverse selection.

Remark 3. Competitive search models with adverse selection and a single opportunity to contact a potential trading partner, such as Guerrieri et al. (2010), feature the stark property that the equilibrium outcome does not depend on the type distribution. Thus, the presence of low types severely distorts the equilibrium allocation for high types, even as the fraction of low types in the population vanishes. The discontinuity in the allocation at the point where this fraction is zero is sometimes viewed as unappealing, particularly when $c_H > v_L - k$ (see, for example, Lester et al., 2019). The same criticism applies to our model if we consider the separating equilibrium that exists in this case. However, as Proposition 6 shows, if σ is small, other equilibria with partial pooling exist and the efficient outcome can be approached in the limit as $N \to +\infty$. Hence, if we focus on the most efficient equilibrium, we can say that, comparing to the single-application benchmark, the discontinuity becomes smaller when workers can send multiple applications and disappears when $N \to +\infty$.

5 Discussion

5.1 Welfare

When workers' types are publicly observable, the only effect of allowing them to submit multiple applications is to alleviate the search friction. Hence, the welfare implications of increasing the number N of applications workers can submit are unambiguous: the welfare of all workers increases with N. In contrast, when the productivity of a worker is only privately observed by him, increasing N not only mitigates the search friction but also affects, as discussed earlier, the set of allocations that are incentive compatible. The welfare consequences of allowing multiple applications are thus no longer unambiguous.

When the lemons condition holds, that is under the parameter conditions of Proposition 1, and the fraction of low productivity workers in the population is not too low ($c_H > \sigma v_L + (1-\sigma)v_H - k$), L-type workers gain when they can send a large number of applications, but H-type workers lose. It is interesting to point out that the welfare of high types may be non-monotone in the number of applications: while it may be increasing in N for N small, it must eventually decrease since their probability of trade converges to zero as search frictions vanish ($N \to \infty$). Furthermore, provided the firms' entry cost is sufficiently small, the overall, ex-ante welfare of workers is strictly higher at the equilibrium with a single application than at the one with vanishing search frictions. The reason for these findings, rather distinct from the case where types are observable, is that the constraints imposed by incentive compatibility on admissible trades becomes stronger with multiple applications.

In contrast, when the lemons condition does not hold, the equilibria with one or more pooling market we constructed in the proofs of Propositions 3 and 5 Pareto-dominate the equilibrium with a single application. In this case, both L- and H-type workers gain when they can send a large number of applications compared to the case where they can only send a single application. In such a situation, not only the search friction is mitigated, but there is a dimension in which the diminished effectiveness of market liquidity as a screening device expands admissible trades, allowing pooling markets to be sustained in equilibrium. Moreover in this case, as we saw, multiple Pareto ranked equilibria exist.

The above considerations show that the welfare consequences of allowing workers to submit multiple applications can go in opposite directions for the different types and that it may not be possible to reach unambiguous conclusions.

5.2 Endogenizing Applications

In our model, we exogenously fixed the number of applications L- and H-type workers can send and assumed this number is the same for both types. The benefits from sending additional applications are however generally different for the two types of workers. If workers could choose how many applications to send facing a fixed, equal cost z per application, H- and L-type workers may thus make different choices. In what follows, we extend the analysis to this case. We show that the total number of applications sent by L-type workers is in fact higher than for H-type workers. The implication of this is that high types send fewer applications to separate markets and may not trade in such markets even away from the limit. Despite this difference, we show that the main properties of equilibrium allocations remain valid when the number of applications sent by each type is endogenously determined.

Let N_i denote the total number of applications a worker of type $i = L, H$ chooses to send in equilibrium. Given N_H, N_L, the definition of an equilibrium is analogous to the one in Definition 1. In addition, to assess the optimality of N_i, recall that for all $n \in \mathbb{N}$, the benefit for a worker of type i from sending one additional application to an optimally chosen market, after having sent $n-1$ of them, is equal to

$$u_{n,i} - u_{n-1,i} = \max_{p \in \mathcal{F}} \psi\left(\mu(p)\right)(p - c_i - u_{n-1,i}).$$

For N_i to be optimal, we need that for all $n \leq N_i$, the benefit $u_{n,i} - u_{n-1,i}$ exceeds the application cost z, while it is lower than z for all $n > N_i$. The fact that $u_{n-1,i}$ is increasing in n directly implies that the utility gain $u_{n,i} - u_{n-1,i}$ is decreasing in n. Hence, the total number of applications a worker of type i sends in equilibrium, N_i, is uniquely pinned down by the following condition:

$$N_i = \max\{n \in \mathbb{N} : u_{n,i} - u_{n-1,i} \geq z\}$$

To examine the consequences for the properties of equilibrium allocations, assume first that the lemons condition holds, $c_H \geq v_L - k$, and consider the separating equilibrium characterized in Proposition 1. As we saw, the markets for L-type workers coincide with the unconstrained solution described in Section 3.2. Hence, the total number of applications low types send is given by the largest number N_L that satisfies $u^*_{N_L,L} - u^*_{N_L-1,L} \geq z$. This condition ensures that L-type workers do not wish to send an additional application to a separate market (for which the utility gain is $u^*_{N_L+1,L} - u^*_{N_L,L} < z$). We also need that they have no incentives to send an additional application to the lowest wage to which high types apply.[17] Letting (μ_H, p_H) describe this market, we must have

$$\psi(\mu_H)(p_H - c_L - u_{N_L,L}) \leq z \tag{17}$$

for $u_{N_L,L} = u^*_{N_L,L}$. Since $c_H \geq v_L - k > u^*_{N_L,L} + c_L$, inequality (17) implies $\psi(\mu_H)(p_H - c_H) < z$. Hence, in equilibrium, incentive constraints limit the gains high types can achieve by trading in the market so much that they will prefer not to participate at all. Hence, with endogenous applications, a separating equilibrium exists under the conditions of Proposition 1 and features N_L application of low types and 0 applications

[17] This condition is different than the L-type incentive constraint relative to his last application, $u^*_{N_L,L} - u^*_{N_L-1,L} \geq \psi\left(\mu_H\right)(p_H - c_L - u^*_{N_L-1,L})$, which only guarantees that the L-type has no incentives to *divert* his last application to wage p_H.

of high types.

Turning then to the case $c_H < v_L - k$, consider the equilibrium with one pooling market described in Proposition 3. As we show in the proof of this proposition, in the equilibrium allocation we constructed, market utilities satisfy the condition

$$u_{\ell+n-1,L} + c_L < u_{n,H} + c_H \leq u_{\ell+n,L} + c_L \text{ for all } n \geq 0. \tag{18}$$

Letting $(\bar{\mu}, \bar{p})$ denote again the terms of trade in the pooling market, the total number N_L of applications that the low type sends must then be the largest number satisfying

$$\psi(\bar{\mu})(\bar{p} - c_L - u_{N_L-1,L}) \geq z. \tag{19}$$

Using (18), this implies $\psi(\bar{\mu})(\bar{p} - c_H - u_{N_L-\ell-1,H}) \geq z$, which means that, when low types are willing to send $N_L - \ell$ applications to the pooling market together with ℓ applications to separate markets, high types are also happy to send $N_L - \ell$ applications to the pooling market.

Suppose now that an H-type market exists with terms of trade (μ_H, p_H). For the considered allocation to be an equilibrium with endogenous applications, it must be that low types do not want to send any additional application to this market (inequality (17) is satisfied) nor to redirect any of their N_L applications to that market (ensured by the incentive constraints already imposed in the construction used in the proof of Proposition 3). By the second inequality in (18) we have $u_{N_L-\ell,H} + c_H \leq u_{N_L,L} + c_L$, so it is possible that $\psi(\mu_H)(p_H - u_{N_L-\ell,H} - c_H) \geq z$ and (17) are both satisfied. If that is the case, high types find it profitable to send one application to market (μ_H, p_H) while low types do not. However, due to the first inequality in (18), we also have $u_{N_L-\ell+1,H} + c_H > u_{N_L,L} + c_L$, so that sending a second application to market (μ_H, p_H) is never profitable. Hence, under the conditions stated in Proposition 3, there exists an equilibrium with a pooling market whenever z satisfies (19) for some $N_L > \ell$, and high types send at most one application to a separate market.

6 Conclusion

We study a market in which firms post wages to attract applications from workers with private information about their productivity. We demonstrate how increasing contacts in such a market not only decreases search frictions but also reduces firms' screening ability. The subtle interaction between these forces creates a rich set of outcomes. In particular, we find that—in contrast to a situation where each worker can only send a single application—the existence of a fully separating equilibrium is only guaranteed if adverse selection is sufficiently severe. When this condition is not satisfied, the equilibrium features pooling markets, giving rise to equilibrium multiplicity. Fully separating allocations and allocations with pooling may also co-exist in equilibrium, as in Akerlof (1970).

We analyze the properties of these equilibria as the number of applications grows large. While the allocation in the separating search equilibrium converges to the one with Walrasian markets à la Akerlof (1970), the same is not true for all equilibria with pooling markets: some of them exhibit frictional trade and thus inefficiency in the limit due to excessive entry. Finally, we show that, with adverse selection, the welfare consequences of facilitating contacts among market participants are ambiguous.

Appendix A Proofs

In what follows, we will denote by

$$I_i(u_{n-1,i}, u_{n,i}) = \{(\mu, p) : \psi(\mu)(p - c_i - u_{n-1,i}) = u_{n,i} - u_{n-1,i}\}$$

type $i = L, H$'s indifference curve associated to utility levels $u_{n-1,i}, u_{n,i}$ and by

$$\Pi_\gamma = \{(\mu, p) : \eta(\mu)(\gamma v_L + (1-\gamma)v_H - p) = k\}$$

the firms' isoprofit curve when the fraction of L-types is γ.

A.1 Proof of Proposition 1

Suppose a candidate separating equilibrium exists. In such an equilibrium, the market utilities $u_{n,L}$ and effective queue lengths $\mu_{n,L}$ for the L-type workers' applications are given by the unconstrained solution, unless at least one of the H-type's incentive constraint is binding. Towards a contradiction, suppose that the H-type's incentive compatibility is binding for some $n \leq N$. There exists then a market (m, L) such that the H-type is indifferent between sending his n-th application to (n, H) or sending it to (m, L). Since the L-type must weakly prefer to send his m-th application to market (m, L), single crossing implies that for wages $p_{m,L} + \varepsilon$ with $\varepsilon > 0$, we have

$\gamma(p)\mu(p) = 0$ as long as ε is sufficiently small (see the market utility condition). Hence, for wages slightly above $p_{m,L}$, firms believe to attract only the high type. Since $\mu(p)$ is continuous in p, offering a wage slightly higher than $p_{m,L}$ constitutes a profitable deviation, as the quality composition improves discretely. Hence, in a separating equilibrium, the property $u_{n,L} = u^*_{n,L}$ and $\mu_{n,L} = \mu^*_{n,L}$ holds for all $n = 1, 2, \ldots, N$. Notice that the associated wages in each of these markets, $p^*_{n,L}$, are strictly smaller than $v_L - k$.

Next, we consider the markets for H-type workers in a candidate separating equilibrium. When $v_L - k \leq c_H$, wages in the L-type markets are below the outside option of H-type workers, hence $p_{n,H} > p^*_{m,L}$ for all $n \leq N$ and $m \leq M$. Single-crossing of the L-type's indifference curves then implies that the only incentive constraint potentially binding is the one associated to the L-type's N-th application. The same property is satisified if $v_L - k \leq c_H$ is violated but $v_H > \bar{v}_H$ holds. In the latter case, there is a unique intersection between the upper envelope of the low type's indifference curves and Π_H. This intersection is with $I_L(u^*_{N-1,L}, u^*_{N,L})$, so again we get that the only incentive constraint potentially binding is the one associated to the L-type's N-th application. Given that either $v_L - k \leq c_H$ or $v_H > \bar{v}_H$ holds, incentive compatibility thus requires:

$$u^*_{N,L} \geq \psi\left(\mu_{n,H}\right)\left(p_{n,H} - c_L\right) + \left(1 - \psi\left(\mu_{n,H}\right)\right) u^*_{N-1,L}. \tag{20}$$

Let $(\underline{\mu}_H, \underline{p}_H)$ be the (unique) values of $(\mu_{n,H}, p_{n,H}) > (\mu^*_{N,L}, p^*_{N,L})$ satisfying (20) as an equality and $(\mu_{n,H}, p_{n,H}) \in \Pi_H$.

Suppose first $\mu^*_{n,H} \geq \underline{\mu}_H$ for all $n \geq 1$. In this case incentive constraints are not binding. We set for all n, $\mu_{n,H} = \mu^*_{n,H}$ and $u_{n,H} = u^*_{n,H}$. Notice that the associated wages satisfy $p^*_{1,L} < p^*_{2,L} < \ldots < p^*_{N,L} < p^*_{1,H} < p^*_{2,H} < \ldots < p^*_{N,H}$. For each p, we set

$$\mu(p) = \max\{\mu : \psi(\mu)(p - c_i - u_{n-1,i}) \leq u^*_{n,i} - u^*_{n-1,i} \text{ for some } i \in \{L, H\},\, n \leq N\}$$

and $\gamma(p) = 0$ for all p such that the previous max is attained at $i = H$ and $\gamma(p) = 0$ otherwise. It can be easily verified that this specification of the functions μ, γ satisfies the market utility condition and that, given μ, γ, firms have no profitable deviations.

If $\mu^*_{1,H} < \underline{\mu}_H$, we follow a recursive procedure to find the effective queue lengths and market utilities in the H-type markets. We start by setting $\mu_{1,H} = \underline{\mu}_H$ and $u_{1,H} = \psi(\mu_H)(\underline{p}_H - c_H)$. Given $u_{1,H}$, we calculate the unconstrained solution of $\mu_{2,H}$.

Setting $n = 2$, the solution is determined by

$$(1 - e^{-\mu_{n,H}} - \mu_{n,H} e^{-\mu_{n,H}})(v_H - c_H - u_{n-1,H}) = k \tag{21}$$

If the value of $\mu_{2,H}$ solving this condition is weakly greater than $\underline{\mu}_H$, it is the effective queue length in market $(2, H)$. The associated market utility is

$$u_{n,H} = e^{-\mu_{n,H}}(v_H - c_H) + (1 - e^{-\mu_{n,H}})u_{n-1,H} \tag{22}$$

The queue lengths and market utilities of the remaining markets ($N > 2$) are then determined by the same set of conditions.

If instead $\mu_{2,H}$ solving (21) for $n = 2$ is strictly smaller than $\underline{\mu}_H$, we set $\mu_{2,H} = \underline{\mu}_H$. The market utility $u_{2,H}$ is then determined by (22). We repeat the procedure for all $n > 2$. Having fixed market utilities in this way, the functions μ, γ can be specified as follows. For all $p < \underline{p}_H$ we set $\gamma(p) = 1$ and for all $p \geq \underline{p}_H$ we set $\gamma(p) = 0$. For wages $p < \underline{p}_H$, the queue length $\mu(p)$ is then determined as the upper envelope of the indifference curves $I_L(u^*_{n-1,L}, u^*_{n,L}), n = 1, ..., N$; for wages $p \geq \underline{p}_H$ it is determined as the upper envelope of the indifference curves $I_H(u_{n-1,H}, u_{n,H}), n = 1, ..., N$ with $\{u_{n,H}\}_{n=1}^N$ specified by the recursive procedure.

Existence: The equilibrium exists if and only if the H-type has no incentives to deviate and send a set of his applications to L-type markets. If the condition $c_H \geq v_L - k$ is satisfied, the wages in the L-type markets are strictly below the H-type's outside option, hence such deviation cannot be profitable. Therefore, if $c_H \geq v_L - k$, the separating equilibrium exists for all $N \geq 1$. What remains to be shown is that if $c_H < v_L - k$ and $v_H > \bar{v}_H$, there is an $\bar{N} > 1$ such that the separating equilibrium does not exists whenever $N \geq \bar{N}$. Letting $\psi_{n,i} \equiv \psi(\mu_{n,i})$ denote the probability of receiving an offer in market i, n, incentive compatibility generally requires that for any $(n, i) \neq (n', i')$ with $u_{n,i} + c_i \leq u_{n',i'} + c_{i'}$, we have $\psi_{n,i} \geq \psi_{n',i'}$, which follows from standard arguments. Notice then that as $N \to +\infty$, the L-type's outside option associated to his last application, $u^*_{N-1,L}$, converges to $v_L - c_L - k$, as proven by Kircher (2009). Hence, given $c_H < v_L - k$, we can find an N sufficiently large such that there is an $n < N$ with $u^*_{n,L} + c_L > c_H$. Incentive compatibility for the H-type then requires $\psi_{1,H} \geq \psi_{n,L}$ or equivalently $\mu_H \leq \mu^*_{n,L}$. However, as we have argued above, whenever $v_H > \bar{v}_H$ holds, incentive compatibility for the L-type requires that $\mu_{1,H}$ is weakly greater than $\underline{\mu}_{1,H}$, which is strictly greater than $\mu^*_{n,L}$ for all $n \leq N$. Hence, no separating equilibrium exists.

□

A.2 Proof of Proposition 2

A straightforward extension of Proposition 6 in Kircher (2009) shows that $\lim_{N\to+\infty} u_{N,L} = v_L - c_L - k$. We now want to prove that the probability with which the H-type is hired in equilibrium tends to zero. Since wages are bounded above by the firms' valuation (net of entry cost), this directly implies $\lim_{N\to+\infty} u_{N,H} = 0$. Letting $(\mu_{1,H}(N), p_{1,H}(N))$ describe the terms of trade in market $(1,H)$ when the number of available applications is N, we can define the probability of being hired when sending $\tilde{N}$ applications to market $(1,H)$:

$$\alpha(\tilde{N}, N) := 1 - \left(1 - \left(1 - e^{-\mu_{1,H}(N)}\right)/\mu_{1,H}(N)\right)^{\tilde{N}}.$$

Since $\mu_{1,H} \leq \mu_{n,H}$ for all n, $\alpha(N,N)$ is an upper bound for the equilibrium probability with which the H-type is hired when sending N applications.

Now suppose each worker has available $2n+j$ applications where $n \in \mathbb{N}$ and $j \in \{0,1\}$. If the L-type sends no applications to any of the H-type markets, his payoff is $u^*_{2n+i,L} < v_L - c_L - k$. If instead he sends $n+j$ applications to the L-markets with the lowest $n+j$ wages and n applications to market $(1,H)$, his payoff is

$$\alpha(n, 2n+j)(p_{1,H}(2n+j) - c_L) + (1 - \alpha(n, 2n+j))u^*_{n+j,L}. \tag{23}$$

In equilibrium, (23) must be smaller than $v_L - c_L - k$. Since $\lim_{n\to+\infty} u^*_{n+j,L} = v_L - c_L - k$ and $p_{1,H}(2n+j) - c_L > c_H - c_L > v_L - c_L - k$, this requires $\lim_{n\to+\infty} \alpha(n, 2n+j) = 0, j = 0, 1$.

Finally, we want to show that $\alpha(n, 2n+j) \to 0$ implies $\alpha(2n+j, 2n+j) \to 0$. To this end, notice that the function $\alpha(\cdot, 2n+j) : \mathbb{R} \to [0,1]$ is strictly increasing and strictly concave with $\alpha(0, 2n+j) = 0$. Hence,

$$\alpha(n, 2n+j) > \frac{n}{2n+j}\alpha(2n+j, 2n+j).$$

Given $\lim_{N\to+\infty} n/(2n+j) = 1/2$, this inequality and the property $\lim_{n\to+\infty} \alpha(n, 2n+j) = 0$ imply $\lim_{n\to+\infty} \alpha(2n+j, 2n+j)/2 = 0$. Hence, $\lim_{N\to+\infty} \alpha(N,N) = 0$. As we stated above, $\alpha(N,N)$ is an upper bound for the equilibrium probability with which

H-type workers are hired. In the limit this type is hired with probability zero and $\lim_{N\to+\infty} u_{N,H} = 0$.

$\square$

A.3 Proof of Proposition 3

Candidate equilibrium. We begin the proof by constructing a candidate equilibrium where L-types send the last m applications to the pooling market, while H-types send the first m' applications to that market and, for now, we allow m to differ from m'. The first $N-m$ applications of the L-type are sent to separate markets, which are the same as in equilibrium with observable types (or the separating equilibrium). Hence, the effective queue lengths and market utilities in these markets are $\mu_{n,L} = \mu^*_{n,L}$ and $u_{n,L} = u^*_{n,L}$, for all $n \leq N-m$.

We will first determine the effective queue lengths and wages in the pooling market and the H-type markets, taking as given the number of applications the two types send to the pooling market, m and m', and the composition in that market, given by the effective fraction $\bar{\gamma}$ of L-type workers. Let $\bar{\mu}$ and $\bar{p}$ be, respectively, the effective queue length and the wage in the pooling market. We set their values to be such that the L-type is indifferent between sending the $N-m$-th application to market $(N-m, L)$ and sending it to the pooling market. The terms of trade in the pooling market $(\bar{\mu}, \bar{p})$ must then satisfy

$$(\bar{\mu}, \bar{p}) \in \left(\Pi_{\bar{\gamma}} \cap I_L(u^*_{N-m-1,L}, u^*_{N-m,L})\right). \tag{24}$$

It is easy to verify that this condition has a unique solution on the domain $(\bar{\mu}, \bar{p}) > (\mu^*_{N-m,L}, p^*_{N-m,L})$. Let us denote such value with $\bar{\mu}(\bar{\gamma}), \bar{p}(\bar{\gamma})$ and set $\bar{\mu} = \bar{\mu}(\bar{\gamma}), \bar{p} = \bar{p}(\bar{\gamma})$.

To find the utility gains L- and H-types attain by trading in the pooling market, it is useful to define the probability of receiving an offer in a market with effective queue length μ when sending $n \geq 1$ applications to that market:

$$\beta(n, \mu) := 1 - (1 - \psi(\mu))^n \tag{25}$$

The market utility of H-type workers associated to their first m' applications is then $u_{n,H}(\bar{\gamma}) = \beta(n; \bar{\mu}(\bar{\gamma}))(\bar{p}(\bar{\gamma}) - c_H), n = 1, ..., m'$, while the market utility of L-type workers associated to their last m applications is $u_{N-m+n,L}(\bar{\gamma}) = \beta(n; \bar{\mu}(\bar{\gamma}))(\bar{p}(\bar{\gamma}) - c_L) + (1 - \beta(n, \bar{\mu}(\bar{\gamma})))u^*_{N-m,L}, n = 1, ..., m$.

To determine the separating markets to which H-types send their $(m+1)$-th and subsequent applications, let $(\mu_H(\bar{\gamma}), p_H(\bar{\gamma}))$ be the unique solution of

$$(\mu_H, p_H) \in (\Pi_H \cap I_L(u_{N-1,L}(\bar{\gamma}), u_{N,L}(\bar{\gamma}))$$

satisfying $(\mu_H, p_H) > (\bar{\mu}, \bar{p})$. Note that $p_H(\bar{\gamma})$ is the lowest wage to which only H-types are willing to apply. We then need to compare the utility they attain by sending applications to $p_H(\bar{\gamma})$ and to higher wages, at which incentive constraints no longer bind. When H-types send $n \geq 1$ applications to market $p_H(\bar{\gamma})$, they attain a utility level

$$u_{m'+n,H} = \beta(n; \mu_H(\bar{\gamma}))(p_H(\bar{\gamma}) - c_H) + (1 - \beta(n, \mu_H(\bar{\gamma})))u_{m',H}. \tag{26}$$

If the solution for μ of

$$(1 - e^{-\mu} - \mu e^{-\mu})(v_H - c_H - u_{m'+n-1,H}) = k \tag{27}$$

is greater than $\mu_H(\bar{\gamma})$, this means that the unconstrained solution for the $(m'+n)$-th application (starting from reservation utility $u_{m'+n-1,H}$) is feasible and hence preferred to market $p_H(\bar{\gamma})$. Let $\bar{n}$ be the lowest value of n for which this happens, that is, at which the L-type incentive constraint no longer binds. In equilibrium H-types will then send $\bar{n}-1 \geq 0$ applications to wage $p_H(\bar{\gamma})$. For all $n \geq \bar{n}$, we set $\mu_{m'+n,H}(\bar{\gamma})$ equal to the unconstrained solution, solving (27) for a level of the market utility $u_{m'+n,H}$ determined by (10), starting from the value $u_{m'+\bar{n}-1,H}$ pinned down by (26). Set then $\mu_{m'+n,H}(\bar{\gamma})$ equal to $\mu_H(\bar{\gamma})$ for $n = 1, .., \bar{n}-1$ and to the unconstrained solution, solving (27), for $n = \bar{n}, ..N - m'$.

Using these values we can derive the value of the probability with which an H-type worker is not hired in one of the H-type markets as a function of the effective composition $\bar{\gamma}$ in the pooling market:

$$\tau_H(\bar{\gamma}; m') = \prod_{n=1}^{N-m'} (1 - \psi(\mu_{m'+n,H})). \tag{28}$$

For any given $m, m' \geq 1$, the effective composition $\bar{\gamma}$ in the pooling market is determined by:

$$\bar{\gamma} = \frac{\sigma m}{\sigma m + \tau_H(\bar{\gamma}; m')(1-\sigma)m'} \tag{29}$$

To see that (29) has a solution, for any m, m', notice that both the left-hand side and the right-hand-side are continuous in $\bar{\gamma}$ on $(0,1)$.[18] Since $\tau_H(\bar{\gamma})$ belongs to $(0,1)$, the value of the right-hand side belongs to the interval $(\frac{\sigma m}{\sigma m+(1-\sigma)m'}, 1)$. As $\bar{\gamma} \to 0$ the left-hand side is then strictly smaller than the right-hand side, which is always greater than $\frac{\sigma m}{\sigma m+(1-\sigma)m'}$. In contrast, as $\bar{\gamma} \to 1$, the left-hand side is strictly greater than the right-hand side, since for any given N, m, m', $\lim_{\bar{\gamma}\to 1} \tau_H(\bar{\gamma}; m') > 0$. Hence, a solution of (29) always exists, constituting a candidate equilibrium for any $m, m' \geq 1$.

No profitable deviations: We focus our attention in what follows on a candidate equilibrium with $m = m' = N - l$ and l determined as in (11). By the assumption $c_H < v_L - k$ stated in Proposition 3, such a value of l exists whenever N is large enough (recall $\lim_{n\to+\infty} u^*_{n,L} + c_L = v_L - k$). We show that for $m = m' = N - l$, the candidate equilibrium constructed above is indeed an equilibrium for N sufficiently large, as no agent has a profitable deviation. The following lemma pins down the off-path beliefs regarding the composition in the candidate equilibrium. The proof is in the Online Appendix.

Lemma 7. *Consider the candidate equilibrium constructed in Section A.3 with $m = m' = N - l$. For all $p \in [0, \bar{p})$ and $p \in (\bar{p}, p_H)$, we have $\gamma(p) = 1$.*

According to Lemma 7, firms believe to attract the L-type when posting a wage below $\bar{p}$. Single crossing and L-type's indifference between sending the l-th application to $p^*_{l,L}$ and $\bar{p}$ imply that for any $p < \bar{p}$, the pair $(p, \mu(p))$ belongs to the upper envelope of the indifference curves of the L-type's first l applications. This property and $\gamma(p) = 1$ imply that there is no $p < \bar{p}$ such that $\eta(\mu(p))(v_L - p) > k$.

For wages p belonging to $(\bar{p}, p_H)$ the queue length $\mu(p)$ is such that

$$(\mu(p), p) \in I_L(u_{N-1,L}, u_{N,L}). \tag{30}$$

We need to show that any such pair $(p, \mu(p))$ yields a weakly negative profit for firms: $\eta(\mu(p))(v_L - p) \leq k$. Since $u_{n,L}$ is increasing in n and bounded from above, the difference $u_{N,L} - u_{N-1,L}$ converges to zero as $n \to +\infty$. Given that $\bar{\mu} = \mu(\bar{p})$ is finite (it lies on the indifference curve $I_L(u^*_{l-1,L}, u^*_{l,L})$), condition $(\mu(\bar{p}), \bar{p}) \in I_L(u_{N-1,L}, u_{N,L})$ implies that $\bar{p}$ tends to $c_L + u_{N-1,L}$ as $N \to +\infty$. Hence, for any $p > \bar{p} > c_L + u_{N-1,L}$

[18] It is immediate to verify that the map $\mu_H(\bar{\gamma})$, defined above, is continuous in $\bar{\gamma}$, while for $n \geq \bar{n}$ the map $\mu_{m'+n,H}(\bar{\gamma})$ is in fact independent of $\bar{\gamma}$.

the belief $\mu(p)$ determined by (30) tends to $+\infty$ as $N \to +\infty$. To guarantee that $\eta(\mu(p))(v_L - p) \leq k$ is satisfied for all $p \in (\bar{p}, p_H)$ as N becomes large, we thus need the wage in the pooling market to satisfy $\bar{p} \geq v_L - k$.

We show next that $\bar{p} \geq v_L - k$ is satisfied if $v_H > \hat{v}_H$. The fact that for $p > \bar{p}$, $\mu(p) \to +\infty$ as $N \to +\infty$ implies that the probability for H-type workers to be hired in an H-type market, τ_H, tends to zero as $N \to +\infty$ and, hence, that the effective composition $\bar{\gamma}$ tends to the population average σ (see (29)). Hence, as $N \to +\infty$, $\bar{p}$ tends to the wage lying at the intersection between the indifference curve $I_{l,L}(u^*_{l-1,L}, u^*_{l,L})$ and the isoprofit curve Π_σ. The threshold $\hat{v}_H$ is defined as the value of v_H such that the wage at this intersection is exactly $v_L - k$. By the assumption $v_H > \hat{v}_H$, the limit of $\bar{p}$ is then a number strictly greater than $v_L - k$. Hence, for N large, condition $\eta(\mu(p))(v_L - p) \leq k$ is satisfied for all $p \in (\bar{p}, p_H)$, ruling out a deviation to a wage in the interval $(\bar{p}, p_H)$.

Finally, standard arguments imply that firms do not want to deviate to wages $p > p_H$ (where $\gamma(p) = 0$), as such a deviation would constitute a move away from the unconstrained solution of the problem of attracting H-types, with reservation utility $u_{N-l,H}$. □

A.4 Proof of Proposition 4

Part (i) of the claim in the proposition was established in the main text, we thus focus here on the proof of claim (ii). Consider an arbitrary equilibrium with a single pooling market, that is, with a single wage level at which both H- and L-type workers send some applications. Let $\bar{p}$ denote the wage and $\bar{\mu}$ denote the effective queue length in that market. Towards a contradiction, suppose the equilibrium allocation satisfies

$$\lim_{N \to +\infty} \sigma u_{N,L} + (1-\sigma) u_{N,H} = \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k. \tag{31}$$

Under this condition workers extract all the surplus. This means that in the limit there is no welfare loss: all workers are thus hired with probability one and all firms hire with probability one. By an analogous argument to the one used in the proof of Proposition 2, we can exclude the possibility that high types are hired at strictly higher wages than low types with a probability that is positive in the limit. It thus follows that, as $N \to +\infty$ the probability of trades taking place outside the pooling market tends to zero. In order for firms to hire with probability one, the effective queue length in the pooling market $\bar{\mu}$ must then tend to $+\infty$ as $N \to +\infty$.

Let $\tilde{n}+1$ indicate the first application which L-types send to the pooling market. We allow $\tilde{n}$ to be equal to 0, in which case the first application of L-types is sent to the pooling market. When $\tilde{n} \geq 1$ the terms of trade in the separate markets where only L-types send applications, indexed by $n \leq \tilde{n}$, are determined as in the equilibrium where types are observable (see the argument in the proof of Proposition 1). In equilibrium L-types must then prefer to send their $(\tilde{n}+1)$-th application to the pooling market rather than to the L-type market where they send their $(\tilde{n}+1)$-th application in the equilibrium with observable types (if this condition is violated, posting wage $p^*_{\tilde{n}+1,L}$ constitutes a profitable deviation for firms). Hence, $\tilde{n}$ must be such that[19]

$$\psi(\bar{\mu})(\bar{p} - c_L - u^*_{\tilde{n},L}) + u^*_{\tilde{n},L} \geq \psi(\mu^*_{\tilde{n}+1,L})(p^*_{\tilde{n}+1,L} - c_L - u^*_{\tilde{n},L}) + u^*_{\tilde{n},L}.$$

As argued above, for (31) to hold, $\bar{\mu}$ must tend to $+\infty$ as $N \to +\infty$. Notice further that $\bar{p}$ is bounded above by the size of the gains from trade, i.e. $\sigma v_L + (1-\sigma)v_H - k$. It then follows that the left-hand side of the above inequality converges to $u^*_{\tilde{n},L}$ as $N \to +\infty$. In order for the inequality to hold, given $p^*_{\tilde{n},L} - c_L - u^*_{\tilde{n},L} > 0$, the effective queue length $\mu^*_{\tilde{n},L}$ must then also diverge to $+\infty$ as $N \to +\infty$. Hence, the index $\tilde{n}$ must tend to $+\infty$ as $N \to +\infty$: L-types send an infinite number of applications to L-type markets, where only such types apply.

Next, we can show that in any equilibrium with a single pooling market H-types send their first application to the pooling market. By the assumption $v_H > \bar{v}_H$, there is a unique intersection between the upper envelope of the L-types' indifference curves associated to the first $\tilde{n}$ applications and Π_H. This intersection is with indifference curve $I_L(u_{\tilde{n}-1,L}, u_{\tilde{n}})$. Since the wage at this intersection is strictly greater than $\bar{p}$ (Π_H lies to the right of $\Pi_{\bar{\gamma}}$ in the (p, μ) space), there cannot be a market with a wage $p < \bar{p}$ to which only H-types apply and firms make non-negative profits. High types must therefore send their first application to the pooling market, as claimed.

For the allocation to be incentive compatible and ensure H-types do not want to deviate and apply to any L-type market, since $\bar{\mu} > \mu^*_{\tilde{n},L}$,[20] the H-types' outside option associated to their first application must be greater than the L-types' outside option

[19] When $\tilde{n} = 0$, $u^*_{0,L} = 0$.

[20] The effective queue length is increasing in the index of the low types' applications.

associated to their $\tilde{n}$-th application, that is: $c_H \geq c_L + u^*_{\tilde{n}-1,L}$. Since $\tilde{n}$ tends to $+\infty$ as $N \to +\infty$, the market utility $u^*_{\tilde{n}-1,L}$ tends to $v_L - c_L - k$. Having assumed $c_H < v_L - k$, the term $c_L + u^*_{\tilde{n}-1,L}$ thus tends to a limit strictly greater than c_H as $N \to +\infty$. The above inequality is violated in the limit, which yields the contradiction.

□

References

Akerlof, G. A. (1970). The market for "lemons": Quality uncertainty and the market mechanism. *Quarterly Journal of Economics*, 84(3):488–500.

Albrecht, J. W., Cai, X., Gautier, P. A., and Vroman, S. B. (2020). Multiple applications, competing mechanisms, andmarket power. *Journal of Economic Theory*, 190:105121.

Albrecht, J. W., Gautier, P. A., and Vroman, S. B. (2006). Equilibrium directed search with multiple applications. *Review of Economic Studies*, 73(4):869–891.

Attar, A., Mariotti, T., and Salanie, F. (2011). Nonexclusive competition in the market for lemons. *Econometrica*, 79(6):1869–1918.

Board, S., Meyer-ter Vehn, M., and Sadzik, T. (2020). Recruiting talent. mimeo.

Cai, X., Gautier, P., and Wolthoff, R. (2017). Search frictions, competing mechanisms and optimal market segmentation. *Journal of Economic Theory*, 169:453–473.

Chade, H. and Smith, L. (2006). Simultaneous search. *Econometrica*, 74(5):1293–1307.

Chang, B. (2018). Adverse selection and liquidity distortion. *Review of Economics Studies*, 85(1):275–306.

Daley, B. and Green, B. (2012). Waiting for news in the market for lemons. *Econometrica*, 80(4):1433–1504.

Faberman, J. R. and Kudlyak, M. (2016). What does online job search tell us about the labor market? *Federal Reserve Bank of Chicago Economic Perspectives*, 40(2016-1):1–15.

Fermanian, J.-D., Guéant, O., and Pu, J. (2016). The behavior of dealers and clients on the european corporate bond market: the case of multi-dealer-to-client platforms. *Market microstructure and liquidity*, 2(03n04):1750004.

Gale, D. (1992). A walrasian theory of markets with adverse selection. *The Review of Economic Studies*, 59(2):229–255.

Gale, D. and Shapley, L. (1962). College admissions and the stability of marriage. *American Mathematical Monthly*, 69:9–15.

Galenianos, M. and Kircher, P. (2009). Directed search with multiple job applications. *Journal of Economic Theory*, 114:445–471.

Guerrieri, V., Shimer, R., and Wright, R. (2010). Adverse selection in competitive search equilibrium. *Econometrica*, 78(6):1823–1862.

Inderst, R. and Müller, H. M. (2002). Competitive search markets for durable goods. *Economic Theory*, 19:599–622.

Kaya, A. and Kim, K. (2018). Trading dynamics with private buyer signals in the market for lemons. *The Review of Economic Studies*, 85(4):2318–2352.

Kim, K. and Pease, M. (2017). Costly search with adverse selection: solicitation curse versus acceleration blessing. *The RAND Journal of Economics*, 48(2):526–553.

Kircher, P. (2009). Efficiency of simultaneous search. *Journal of Political Economy*, 117:861–913.

Kurlat, P. (2016). Asset markets with heterogeneous information. *Econometrica*, 84(1):33–85.

Lauermann, S. and Wolinsky, A. (2016). Search with adverse selection. *Econometrica*, 84(1):243–315.

Lester, B., Shourideh, A., Venkateswaran, V., and Zetlin-Jones, A. (2019). Screening and adverse selectionin frictional markets. *Journal of Political Economy*, 127(1):338–377.

Lester, B., Visschers, L., and Wolthoff, R. (2015). Meeting technologies and optimal trading mechanisms in competitive search markets. *Journal of Economic Theory*, 155:1–15.

Martinelli, P. and Menzio, G. (2020). Declining search frictions, unemployment, and growth. *Journal of Political Economy*, 128(12):4387–4437.

Mas-Colell, A., Whinston, M. D., Green, J. R., et al. (1995). *Microeconomic theory*, volume 1. Oxford university press New York.

Riggs, L., Onur, E., Reiffen, D., and Zhu, H. (2020). Swap trading after dodd-frank: Evidence from index cds. *Journal of Financial Economics*, 137(3):857–886.

Stigler, G. J. (1961). The economics of information. *Journal of Political Economy*, 69(3):213–225.

Wolthoff, R. P. (2018). Applications and interviews: Firms' recruiting decisions in a frictional labor market. *Review of Economic Studies*, 85(2):1314–1351.

— Online Appendix —

Appendix B Additional Proofs

B.1 Proof of Lemma 7

We show first that for wages $p < \bar{p}$ the market utility condition and (6) imply that firms' beliefs are $\gamma(p) = 1$. We begin by establishing the property for all $p \in (p^*_{l,L}, \bar{p})$. This is achieved by showing that, under the assumptions made, the following condition holds, for all $n = 1, .., N - l$:

$$\psi(\mu)(p - c_L) + (1 - \psi(\mu))\, u^*_{l-1,L} \;=\; u^*_{l,L}, \tag{32}$$

$$\psi(\mu)(p - c_H) + (1 - \psi(\mu))\, u_{n-1,H} \;<\; u_{n,H}. \tag{33}$$

Solving the first equation for $\psi(\mu)$ and substituting into the second inequality yields

$$\frac{u^*_{l,L} - u^*_{l-1,L}}{p - c_L - u^*_{l-1,L}} < \frac{u_{n,H} - u_{n-1,H}}{p - c_H - u_{n-1,H}}. \tag{34}$$

Recalling that, in the candidate equilibrium under consideration, $u^*_{l,L} = \psi(\bar{\mu})(\bar{p} - c_L) + (1 - \psi(\bar{\mu}))u^*_{l-1,L}$ and $u_{n,H} = \psi(\bar{\mu})(\bar{p} - c_H) + (1 - \psi(\bar{\mu}))u_{n-1,H}$, we have:

$$\frac{u^*_{l,L} - u^*_{l-1,L}}{\bar{p} - c_L - u^*_{l-1,L}} = \frac{u_{n,H} - u_{n-1,H}}{\bar{p} - c_H - u_{n-1,H}}. \tag{35}$$

Using this condition to substitute for $(u_{n,H} - u_{n-1,H}) / \left(u^*_{l,L} - u^*_{l-1,L}\right)$ in the above inequality and simplifying terms, we obtain:

$$p\left(c_H + u_{n-1,H} - c_L - u^*_{l-1,L}\right) < \bar{p}\left(c_H + u_{n-1,H} - c_L - u^*_{l-1,L}\right). \tag{36}$$

Finally, notice that for all $n \leq l - 1$, we have $u^*_{n,L} + c_L < c_H$ by definition of l and hence

$$u^*_{l-1,L} + c_L < u_{n-1,H} + c_H, \text{ for all } n = 1, ..N. \tag{37}$$

Hence inequality (36) reduces to $p < \bar{p}$, which establishes the claim. A similar argument applies to wages weakly below $p^*_{l,L}$—for this case, it is in fact the same as for the separating equilibrium.

Next, we consider wages in the interval $(\bar{p}, p_H)$. We will show that for all $p \in (\bar{p}, p_H)$, $\gamma(p) = 1$ again holds. By definition of l we have $u^*_{l,L} + c_L \geq c_H$. Hence for all $n = 0, 1, 2, ..., N - l - 1$, in the candidate equilibrium under consideration the following holds:

$$\begin{aligned} u_{l+n,L} + c_L &= \beta(n;\bar{\mu})\bar{p} + (1 - \beta(n;\bar{\mu}))(u^*_{l,L} + c_L) \\ &\geq \beta(n;\bar{\mu})\bar{p} + (1 - \beta(n;\bar{\mu}))c_H \\ &= u_{n,H} + c_H. \end{aligned} \tag{38}$$

This means that the reservation utility for the n-th application sent to the pooling market is greater for low than for high types, for all $n = 0, .., N - l - 1$. In particular, we have

$$u_{N-1,L} + c_L \geq u_{N-l-1,H} + c_H. \tag{39}$$

We also want to show that the reservation utility for the N-th application sent by low types to the pooling market is smaller than the one for the first application sent by high types to a H-type market, that is:

$$u_{N-1,L} + c_L < u_{N-l,H} + c_H. \tag{40}$$

Recalling that $u_{l+n,L} + c_L = \beta(n;\bar{\mu})\bar{p} + (1 - \beta(n;\bar{\mu}))(u^*_{l,L} + c_L)$, using the property $\beta(n;\cdot) = \beta(n-1;\cdot) + (1 - \beta(n-1;\cdot))\beta(1;\cdot)$ and the fact that $u^*_{l,L} = \beta(1;\bar{\mu})(\bar{p} - c_L) + (1 - \beta(1;\bar{\mu}))u^*_{l-1,L}$, when $n = N - l - 1$, we obtain

$$\begin{aligned} u_{N-1,L} &= \beta(N - l - 1;\bar{\mu})(\bar{p} - c_L) + (1 - \beta(N - l - 1;\bar{\mu}))u^*_{l,L} \\ &= \beta(N - l;\bar{\mu})(\bar{p} - c_L) + (1 - \beta(N - l;\bar{\mu}))u^*_{l-1,L}. \end{aligned} \tag{41}$$

This implies

$$\begin{aligned} u_{N-1,L} + c_L &= \beta(N - l;\bar{\mu})\bar{p} + (1 - \beta(N - l;\bar{\mu}))(u^*_{l-1,L} + c_L) \\ &< \beta(N - l;\bar{\mu})\bar{p} + (1 - \beta(N - l;\bar{\mu}))c_H \\ &= u_{N-l,H} + c_H, \end{aligned} \tag{42}$$

where the inequality in the second line follows from $u^*_{l-1,L} + c_L < c_H$, which as we already pointed out, holds by definition of l. This then establishes (40).

Having shown (39) and (40), we want to prove that the following conditions hold for all $p \in (\bar{p}, p_H)$ and $n = 1, ..., N$:

$$\begin{aligned} \psi(\mu)(p - c_L) + (1 - \psi(\mu))\, u_{N-1,L} &= u_{N,L} && (43) \\ \psi(\mu)(p - c_H) + (1 - \psi(\mu))\, u_{n-1,H} &\leq u_{n,H} && (44) \end{aligned}$$

Again solving for $\psi(\mu)$ the first equation and substituting into the second one yields

$$\frac{u_{N,L} - u_{N-1,L}}{(p - c_L - u_{N-1,L})} \leq \frac{u_{n,H} - u_{n-1,H}}{(p - c_H - u_{n-1,H})} \tag{45}$$

For $n \leq N - l$ the above inequality holds as an equality at $(\bar{\mu}, \bar{p})$. Following the same argument as above, we can use this equality to substitute for $(u_{n,H} - u_{n-1,H}) / (u_{N,L} - u_{N-1,L})$ and rewrite (45) as an inequality similar to (36):

$$p\left(c_H + u_{n-1,H} - c_L - u^*_{N-1,L}\right) \leq \bar{p}\left(c_H + u_{n-1,H} - c_L - u^*_{N-1,L}\right) \tag{46}$$

Due to condition (39), the terms in the brackets are negative for all $n \leq N - l$, so (45) holds. Hence, (43,44) is satisfied for $p \in (\bar{p}, p_H)$ and $n \leq N - l$.

Next, consider the applications that are sent by high types to the market with wage p_H: $n = N - l + 1, ..., N - l + \bar{n} - 1$. Using the property that for $n = N - l + 1, ..., N - l + \bar{n} - 1$ condition (45) holds as an equality at p_H (since in the candidate equilibrium we are considering, high types send those applications to p_H), we can again rewrite (45) as follows:

$$p\left(c_H + u_{n-1,H} - c_L - u_{N-1,L}\right) \leq p_H\left(c_H + u_{n-1,H} - c_L - u_{N-1,L}\right). \tag{47}$$

Under condition (40), we have $u_{N-1,L} + c_L < u_{n-1,H} + c_H$, so the inequality holds for all $p \in (\bar{p}, p_H)$.

For applications $n \geq N - l + \bar{n}$, the L-type incentive constraint is slack (by definition of $\bar{n}$) and the terms of trades for these applications are given by the unconstrained solution, described in (27). This implies that to attract applications from high types for which their reservation utility is given by $u_{N-l+\bar{n}-1}$, firms cannot make positive profits. Since for all $p \in (\bar{p}, p_H)$ and μ satisfying (43) firms would make positive profits if they could attract applications only from high types, i.e. $(1 - e^{-\mu})(v_H - p) > k$, it follows that (44) is satisfied for all $n \geq N - l + \bar{n}$.

$\square$

B.2 Proof of Proposition 5

Candidate equilibrium. The logic of the argument is very close to that used to prove Proposition 3. Consider the candidate equilibrium we constructed in the proof of that proposition A.3 with $m = m' = N - l$; that is, with the last $N - l$ applications and the first $N - l$ applications sent respectively by low and high types to the pooling market. Let us reassign the same fraction of these applications both for low and high types to a second pooling market, with a higher wage and effective queue length.

Let $\hat{n} > l$ indicate the application after which the low type switches from the first pooling market to the second one. The low types' application strategy consists thus in sending the first l applications to L-type markets, where only low types are present, the next $\hat{n} - l$ applications to pooling market 1 and the last $N - \hat{n}$ applications to pooling market 2. The high types' application strategy consists in sending the first $\hat{n} - l$ to pooling market 1, the next $N - \hat{n}$ applications to pooling market 2, and the last l applications to H-type markets. We show next that the effective composition in the two pooling markets, resulting from this reassignment, is the same. Let us denote it by $\bar{\gamma}$, while $(\bar{\mu}_1, \bar{p}_1)$ denote the terms of trade in the first pooling market and $(\bar{\mu}_2, \bar{p}_2)$ those in the second pooling market.

Proceeding similarly to the proof of Proposition 3, we also indicate with $\tau_{2,H}$ the probability that a high type receives no wage offer strictly above $\bar{p}_2$. In pooling market 2 low types send $N - \hat{n}$ effective applications (since all offers received are accepted) , while high types only send $\tau_{2,H}(N - \hat{n})$ effective applications. The effective composition in this market is thus given by the following expression, analogous to (29):

$$\frac{\sigma(N - \hat{n})}{\sigma(N - \hat{n}) + (1 - \sigma)\tau_{2,H}(N - \hat{n})} = \frac{\sigma}{\sigma + (1 - \sigma)\tau_{2,H}}$$

Let $\beta(N - \hat{n}; \bar{\mu}_2)$ denote again the probability for any of the two types of receiving an offer in pooling market 2, with effective queue length $\bar{\mu}_2$, when sending $n \geq 1$ applications to that market. It thus follows that the effective composition in pooling market 1 is:

$$\frac{\sigma\,(\hat{n} - l)\,(1 - \beta(N - \hat{n}; \bar{\mu}_2))}{\sigma(\hat{n} - l)(1 - \beta(N - \hat{n}; \bar{\mu}_2)) + (1 - \sigma)(\hat{n} - l)\tau_{2,H}(1 - \beta(N - \hat{n}; \bar{\mu}_2))} = \frac{\sigma}{\sigma + (1 - \sigma)\tau_{2,H}},$$

the same as the effective compositions in pooling market 1.

The terms of trade in pooling market 1 are determined by the same condition (24) pinning down the terms of trade in the single pooling market in Section A.3. In pooling market 2 they are then determined as the unique solution satisfying $\bar{p}_2 > \bar{p}_1$ of the analogous condition:

$$(\bar{\mu}_2, \bar{p}_2) \in (\Pi_{\bar{\gamma}} \cap I_L(u_{\hat{n}-1,L}, u_{\hat{n},L})) \,. \tag{48}$$

with $u_{\hat{n},L}$ obtained analogously to $u_{N,L}$ in Section A.3. It is easy to see[21] that such a solution exists whenever $\hat{n}$ is sufficiently large. The terms of trade in the high quality markets are determined by the same procedure as in Section A.3,[22] starting from the utility attained by high types from their applications to pooling markets 1 and 2

$$u_{N-l,H} = \beta(N-\hat{n}, \bar{\mu}_2)(\bar{p}_2 - c_H) + (1 - \beta(N-\hat{n}, \bar{\mu}_2)) \underbrace{\beta(\hat{n}-l; \bar{\mu}_1)(\bar{p}_1 - c_L)}_{=u_{\hat{n}-l,H}},$$

and the wage $p_{2,H}$ lying at the intersection of the low types' indifference curve associated to their last application to pooling market 2 and the H-isoprofit curve.

Having found the effective queue lengths in the H-type markets, the high types' probability of being hired in one of these markets $\tau_{2,H}$ can be determined as a function of $\bar{\gamma}$ in the same way as in (28). Proceeding as in Section A.3 allows us then to prove that a fixed point for $\bar{\gamma}$ exists. This fixed point depends on the switching point $\hat{n}$, as do the other equilibrium variables (except for the terms of trade in the low quality markets). In what follows we make this dependence explicit by writing the variables as functions of $\hat{n}$.

It will be useful to establish some limit properties of these variables. First, since $u_{\hat{n},L}$ is strictly increasing in $\hat{n}$ and bounded above by the gains from trade $\sigma v_H + (1-\sigma)v_L - k$, the difference $u_{\hat{n},L} - u_{\hat{n}-1,L}$ converges to zero as $\hat{n} \to +\infty$. Given this property and $\bar{p}_2(\hat{n}) > \bar{p}_1(\hat{n}) > c_L + u_{\hat{n}-1,L}$, condition (48) implies $\lim_{\hat{n}\to+\infty} \bar{\mu}_2(\hat{n}) = +\infty$. The fact that the effective queue length in the second pooling market tends $+\infty$ implies that also the effective queue lengths in the high type markets tend to $+\infty$.[23]

[21] A solution of (48) is always given by $\bar{\mu}_1, \bar{p}_1$. Note that the isoprofit curve of pooling market 1 is convex while the indifference curve of the $\hat{n}$-th application of the low types (sent to pooling market 1) is concave. Hence if the latter is steeper than the first one at $\bar{\mu}_1, \bar{p}_1$, a property satisfied for $\hat{n}$ sufficiently high, a second solution exists and features $\bar{p}_2 > \bar{p}_1$.

[22] In particular, see equations (27), (26) and the text immediately below them.

[23] Recall that the effective queue length increases in the index of the application—in this case the

Noticing that the number of applications that high types send to these markets is l and thus independent $\hat{n}$, it follows that the probability with which high types receive an offer in one of the high type markets tends to zero as $\hat{n} \to +\infty$. Hence, $\lim_{\hat{n}\to+\infty} \tau_{2,H}(\hat{n}) = 1$. Due to this property, the effective composition $\bar{\gamma}(\hat{n})$, as determined by (29) with $m' = m = \hat{n}$, tends to σ as $\hat{n} \to +\infty$.

No profitable deviations. Next, we need to show that there are no profitable deviations. For wages $p < \bar{p}_1(\hat{n})$ and $p > p_{2,H}(\hat{n})$ the proof in Section A.3 directly applies. Considering wages $p \in (\bar{p}_1(\hat{n}), p_{2,H}(\hat{n}))$, we want to show that for any p in this interval, $\gamma(p) = 1$ holds except at $p = \bar{p}_2(\hat{n})$. For wages in the interval $(\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n}))$ we can again apply the proof in Section A.3, conditions (39) and (40), simply replacing N with $\hat{n}$. Thereby, we obtain $u_{\hat{n}-1,L}(\hat{n}) + c_L \geq u_{\hat{n}-l-1,H}(\hat{n}) + c_H$ and $u_{\hat{n}-1,L}(\hat{n}) + c_L < u_{\hat{n}-l,H}(\hat{n}) + c_H$, thus proving $\gamma(p) = 1$ for all $p \in (\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n}))$.

Next, consider the interval $(\bar{p}_2(\hat{n}), p_{2,H}(\hat{n}))$. To show $\gamma(p) = 1$ for wages in this interval, we must prove that analogous inequalities hold: $u_{N-1,L}(\hat{n}) + c_L \geq u_{N-l-1,H}(\hat{n}) + c_H$ and $u_{N-1,L}(\hat{n}) + c_L < u_{N-l,H}(\hat{n}) + c_H$. We have argued above that $u_{\hat{n},L}(\hat{n}) + c_L \geq u_{\hat{n}-l,H}(\hat{n}) + c_H$ is satisfied. Using this property, we obtain:

$$
\begin{aligned}
u_{N-1,L}(\hat{n}) + c_L &= \beta(N-1-\hat{n}; \bar{\mu}_2(\hat{n}))\bar{p}_2(\hat{n}) + (1 - \beta(N-1-\hat{n}; \bar{\mu}_2(\hat{n})))(u_{\hat{n},L}(\hat{n}) + c_L) \\
&\geq \beta(N-1-\hat{n}; \bar{\mu}_2(\hat{n}))\bar{p}_2(\hat{n}) + (1 - \beta(N-1-\hat{n}; \bar{\mu}_2(\hat{n})))(u_{\hat{n}-l,H}(\hat{n}) + c_H) \\
&= u_{N-l-1,H}(\hat{n}) + c_H,
\end{aligned}
$$

which establishes the first inequality. To prove the second inequality, $u_{N-1,L}(\hat{n}) + c_L < u_{N-l,H}(\hat{n}) + c_H$, it is sufficient to notice that $u_{\hat{n},L}(\hat{n}) = \beta(1, \bar{\mu}_1(\hat{n}))(\bar{p}_1(\hat{n}) - c_L - u_{\hat{n}-1,L}(\hat{n})) + u_{\hat{n}-1,L}(\hat{n})$ holds (low types are indifferent between sending their $\hat{n}$-th application to the first or second pooling market). With $\beta(n; \cdot) = \beta(n-1; \cdot) + (1 - \beta(n-1; \cdot))\beta(1; \cdot)$, we can follow the same steps as in (41-42), Section A.3, to establish that $u_{N-1,L}(\hat{n}) + c_L < u_{N-l,H}(\hat{n}) + c_H$ holds. We thus have $\gamma(p) = 1$ for all $p \in (\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n}))$.

Given $\gamma(p) = 1$ for $p \in (\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n})) \cup (\bar{p}_2(\hat{n}), p_H(\hat{n}))$, the associated profits for firms are weakly below k as long as $\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n}) \geq v_L - k$ is satisfied (see the argument in Section A.3 following (30)). Given the assumption $v_H > \hat{v}_H$, we can choose $\hat{n}$ sufficiently large, and hence $\bar{\gamma}(\hat{n})$ sufficiently close to σ, such that $\bar{p}_1(\hat{n}) \geq v_L - k$

application of high types.

holds. By construction, we have $\bar{p}_2(\hat{n}) > \bar{p}_1(\hat{n})$, hence $\bar{p}_2(\hat{n}) \geq v_L - k$ holds as well.

Putting these pieces together allows us to conclude that for N sufficiently large, we can find a threshold $\hat{n}_0$ sufficiently high such that there is an equilibrium with two pooling markets for each switching point $\hat{n} \in \{\hat{n}_0, N-1\}$.

Expected payoffs. We are now ready to prove the statement in the proposition. Fix ε arbitrarily close to zero and let δ_1, δ_2 be a pair of positive numbers such that

$$\delta_1(v_H - v_L) + \frac{\delta_2}{1-\delta_2}k \leq \varepsilon.$$

Since, as shown earlier, $\lim_{\hat{n}\to+\infty} \bar{\mu}_2(\hat{n}) = +\infty$ and $\lim_{\hat{n}\to+\infty} \bar{\gamma}(\hat{n}) = \sigma$, we can find a value for $\hat{n}$ such that $\bar{\gamma}(\hat{n}) < \sigma + \delta_1$ and $1 - e^{-\bar{\mu}_2(\hat{n})} > 1 - \delta_2$. In what follows we fix then the number of applications sent to pooling market 1 to be equal to a value of $\hat{n}$ such that these inequalities are satisfied. As $N \to +\infty$, the number of applications sent to the first pooling market is then fixed to $\hat{n} - l$, while the number of applications sent to the second pooling market tends to infinity. We want to show that we can find N large enough so that (15) holds.

Using the inequalities $\bar{\gamma}(\hat{n}) < \sigma + \delta_1$ and $1 - e^{-\bar{\mu}_2(\hat{n})} > 1 - \delta_2$ together with the free-entry condition imposed by (48) yields:

$$\bar{p}_2(\hat{n}) > (\sigma + \delta_1)v_L + (1 - (\sigma + \delta_1))v_H - \frac{k}{1-\delta_2}.$$

The level of total surplus attained by workers in equilibrium satisfies the following:

$$
\begin{aligned}
& \sigma u_{N,L}(\hat{n}) + (1-\sigma) u_{N,H}(\hat{n}) \\
> \; & \sigma u_{N,L}(\hat{n}) + (1-\sigma) u_{N-l,H}(\hat{n}) \\
= \; & \sigma \left[\beta(N-\hat{n};\bar{\mu}_2(\hat{n}))(\bar{p}_2(\hat{n}) - c_L) + (1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))u_{\hat{n},L}(\hat{n})\right] \\
& +(1-\sigma)\left[\beta(N-\hat{n};\bar{\mu}_2(\hat{n}))(\bar{p}_2(\hat{n}) - c_H) + (1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))u_{\hat{n}-l,H}(\hat{n})\right] \\
= \; & \beta(N-\hat{n};\bar{\mu}_2(\hat{n}))(\bar{p}_2(\hat{n}) - \sigma c_L - (1-\sigma)c_H) \\
& +(1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))(\sigma u_{\hat{n},L}(\hat{n}) + (1-\sigma)u_{\hat{n}-l,H}(\hat{n})) \\
\geq \; & \beta(N-\hat{n};\bar{\mu}_2(\hat{n}))\left(\sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k - \delta_1(v_H - v_L) - \frac{\delta_2}{1-\delta_2}k\right) \\
& +(1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))(\sigma u_{\hat{n},L}(\hat{n}) + (1-\sigma)u_{\hat{n}-l,H}(\hat{n})) \\
\geq \; & \beta(N-\hat{n};\bar{\mu}_2(\hat{n}))\left(\sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k - \varepsilon\right) \\
& +(1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))(\sigma u_{\hat{n},L}(\hat{n}) + (1-\sigma)u_{\hat{n}-l,H}(\hat{n}))
\end{aligned}
$$

Since $\hat{n}$ is fixed, $\bar{\mu}_2(\hat{n})$ is bounded and $\beta(N-\hat{n};\bar{\mu}_2(\hat{n}))$ tends to 1 as $N \to +\infty$ (workers send infinitely many applications to a market with a finite effective queue length). We thus have

$$
\lim_{N\to+\infty} (\sigma u_{N,L}(\hat{n}) + (1-\sigma)u_{N,H}(\hat{n})) \geq \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k - \varepsilon.
$$

□

B.3 Proof of Proposition 6

Let $c_H \in (v_L - k, \sigma v_L + (1-\sigma)v_H - k)$ and consider the candidate equilibrium we constructed in the proof of Proposition 3, with m as the number of applications low types send to the pooling market and m' as the number of applications high types send to the pooling market. For any $m, m' \geq 1$ there exists a value of the wage $\bar{p}(m,m')$, queue length $\bar{\mu}(m,m')$ and effective fraction of low types $\bar{\gamma}(m,m')$ in the pooling market satisfying (24) and (29).

Next, we impose the following condition on m, m' : for any m, let m' be determined as follows

$$
m' = \arg\max\{\tilde{m} \geq 0 : \beta(m-\tilde{m};\bar{\mu}(m,\tilde{m}))\bar{p}(m,\tilde{m}) + (1-\beta(m-\tilde{m},\bar{\mu}(m,\tilde{m})))(u^*_{N-m,L} + c_L) \geq c_H\}. \tag{49}
$$

This condition will play a role analogous to (11) (never satisfied when $c_H > v_L - k$). As we show in the next paragraph, a solution to (49) always exists provided N, m are sufficiently large so that $\bar{p}(m, m') > c_H$.

This follows from the fact that, for any sequence of values $m, m' \to \infty$, with $m-m'$ bounded (converging to some number greater or equal than 1), we have $\bar{\gamma}(m, m') \to \sigma$. If in addition the switching point $N - m \to \infty$ we have $\bar{\mu}(m, m') \to \infty$ and then also $\bar{p}(m, m') \to \sigma v_L + (1-\sigma)v_H - k$. Hence for $m, m', N-m$ sufficiently large and $\frac{m'}{m}$ sufficiently close to 1 we have

$$\bar{\gamma}(m, m')v_L + (1 - \bar{\gamma}(m, m'))v_H - k > c_H \tag{50}$$

and also, since $\sigma v_L + (1-\sigma)v_H - k > c_H$, $\bar{p}(m, m') > c_H$.

No profitable deviations. Next, we verify that firms have no incentives to deviate. For wages p in the interval $(p^*_{N-m,L}, \bar{p})$, we can follow steps (32-36), replacing l with $N-m$, to establish that $\gamma(p) = 1$ and hence no deviation to wages in this range is profitable. The analogous condition to (37) is $u^*_{N-m-1,L} + c_L < c_H + u_{n-1,H}$ for all $n = 1, ..., N$, which follows from

$$u^*_{N-m-1,L} + c_L < v_L - k < c_H,$$

and holds then for all $N - m$. Hence the switching point to the pooling market $N - m$ can now take an arbitrarily large value.

Consider next wages $p \in (\bar{p}, p_H)$, Since m' satisfies (49), we have

$$u_{N-m',L} + c_L = \beta(m - m'; \bar{\mu})\bar{p} + (1 - \beta(m - m', \bar{\mu}))(u^*_{N-m,L} + c_L) \geq c_H.$$

Hence, proceding similarly as in (38), we obtain:

$$\begin{aligned} u_{N-1,L} + c_L &= \beta(m'-1; \bar{\mu})\bar{p} + (1 - \beta(m'-1; \bar{\mu}))(u_{N-m',L} + c_L) \\ &\geq \beta(m'-1; \bar{\mu})\bar{p} + (1 - \beta(m'-1; \bar{\mu}))c_H \\ &= u_{m'-1,H} + c_H, \end{aligned}$$

the analogue of condition (39) in our candidate equilibrium, saying that the reservation utility for the last application sent to the pooling market is greater for the low than for the high types.

The analogue of (40) in our candidate equilibrium is $u_{N-1,L} + c_L < u_{m',H} + c_H$, requiring that the reservation utility for the last application sent by low types to the pooling market is smaller than the one for the first application sent by high types to a high quality market. Since m' is the largest value of $\tilde{m}$ satisfying the inequality in (49), we have

$$u_{N-m'-1,L}+c_L = \beta(m-(m'+1);\bar{\mu})\bar{p}+(1-\beta(m-(m'+1),\bar{\mu}))(u^*_{N-m,L}+c_L) < c_H. \quad (51)$$

We proceed then similarly as in (42) to obtain:

$$\begin{aligned} u_{N-1,L} + c_L &= \beta(m'-1;\bar{\mu})\bar{p} + (1-\beta(m'-1;\bar{\mu}))(u_{N-m',L} + c_L) \\ &= \beta(m';\bar{\mu})\bar{p} + (1-\beta(m';\bar{\mu}))(u_{N-m'-1,L} + c_L) \\ &< \beta(m';\bar{\mu})\bar{p} + (1-\beta(m';\bar{\mu}))c_H \\ &= u_{m',H} + c_H, \end{aligned}$$

where the inequality sign follows from (51). This establishes the analogue of (40) we intended to show.

Having shown these properties, we can follow the steps of the proof of Proposition 3, conditions (43-47), to show that $\gamma(p) = 1$ for all $p \in (\bar{p}, p_H)$. To show that no deviation to a wage in this interval is profitable it remains then to show that $\eta(\mu(p))(v_L - p) \leq k$ holds for $\mu(p)$ satisfying $(\mu(p), p) \in I_L(u_{N-1,L}, u_{N,L})$. This is true since $\bar{p} \geq v_L - k$, always holds here, as $\bar{p} > c_H$ and $c_H > v_L - k$.

The non profitability of deviations to wages $p < p^*_{N-m,L}$ and $p > p_H$ follows then directly by the same argument as in the proof of Proposition 3.

Expected payoffs. In the next and final step, we use a similar argument as in the proof of Proposition 5, taking the switching point for low types to the pooling market large enough. Fix ε arbitrarily close to zero and let δ be a positive number such that

$$\frac{\delta}{1-\delta}k < \varepsilon. \quad (52)$$

Recalling that $\mu^*_{n-1,L} \to +\infty$ as $n \to +\infty$, let the low types' switching point to the pooling market $N - m$ be the smallest number κ satisfying $1 - e^{-\mu^*_{\kappa,L}} \geq 1 - \delta$. For δ small, this condition implies $\bar{p} \geq c_H$ as long as N is sufficiently large. Having set $N - m = \kappa$, we can write all equilibrium variables as a function of N. For any N, the number of applications low types send to the pooling market is $m = N - \kappa$ and the number of applications high types send to the pooling market, $m'(N-\kappa)$, is determined by (49).

We consider then $N \to +\infty$. Since $(\bar{\mu}(N-\kappa), \bar{p}(N-\kappa))$ lies on the indifference curve associated to the κ-th application of the low types, as $N \to +\infty$ both $\bar{\mu}(N-\kappa)$ and $\bar{p}(N-\kappa)$ tend to a finite limit. This implies that also $m - m' = N - \kappa - m'(N-\kappa)$ has a finite limit as $N \to +\infty$.[24] Hence $\lim_{N\to+\infty} m'(N-\kappa) = +\infty$ and $\lim_{N\to+\infty} \frac{m'(N-\kappa)}{N-\kappa} = 1$. Also $\lim_{N\to+\infty} \bar{\gamma}(N-\kappa) = \sigma$.

Using the above properties, we want to show that we can find N large enough so that (16) holds. Since L-type workers send their κ+1-th application to the pooling market which features a higher effective queue length than their κ-th application, sent to a low market, we have

$$\eta(\bar{\mu}(m)) > \eta(\mu_{\kappa,L}) \geq 1 - \delta.$$

Together with the free-entry condition $\eta(\bar{\mu}(m))(\bar{\gamma}(m)v_L + (1-\bar{\gamma}(m))v_H - \bar{p}(m)) = k$, this implies:

$$\lim_{N\to+\infty} \bar{p}(N-\kappa) = \sigma v_L + (1-\sigma)v_H - \lim_{N\to+\infty} \frac{k}{\eta(\bar{\mu}(N-\kappa))} \geq \sigma v_L + (1-\sigma)v_H - \frac{k}{1-\delta}.$$

Taking then the limit of the expression of total surplus in equilibrium, as $N \to \infty$, we obtain:

$$\begin{aligned}
& \lim_{N\to+\infty} (\sigma u_{N,L}(N) + (1-\sigma)u_{N,H}(N)) \\
\geq\ & \lim_{N\to+\infty} (\sigma u_{N,L}(N) + (1-\sigma)u_{N-\kappa,H}(N-\kappa)) \\
=\ & \lim_{N\to+\infty} \begin{pmatrix} \sigma \left[\beta(N-\kappa);\bar{\mu}(N-\kappa))(\bar{p}(N-\kappa) - c_L) + (1-\beta(N-\kappa);\bar{\mu}(N-\kappa)))u^*_{\kappa,L}(N-\kappa)\right] \\ +(1-\sigma)\beta(N-\kappa;\bar{\mu}(N-\kappa))(\bar{p}(N-\kappa) - c_H) \end{pmatrix} \\
=\ & \lim_{N\to+\infty} \bar{p}(N-\kappa) - \sigma c_L - (1-\sigma)c_H \\
\geq\ & \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k - \frac{\delta}{1-\delta}k \\
>\ & \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k - \varepsilon
\end{aligned}$$

[24] Take m, N large enough so that a solution to (49) exists. Let then $N \to +\infty$ so that also $m = N - \kappa \to +\infty$. The solution for m' obtained from (49) is such that $m - m'$ is either unchanged or decreases.

where we used $\lim_{N\to+\infty}\beta(N-\kappa);\bar{\mu}(N-\kappa))=1$ and, in the last inequality, condition (52). Hence this proves that (16) is satisfied. $\square$

Appendix C Equilibrium Definition

Definition 2. *An equilibrium is a measure of vacancies ϕ, a distribution of wages F, application distributions (G_L, G_H), effective queue lengths $\mu(p)$, and effective queue compositions $\gamma(p)$ such that*

1. *For any $n \in \{1,\ldots,N\}$ and $p \in \mathcal{F}$, $\lambda_{n,L}(p)$ satisfies*

$$\phi\int_0^p \lambda_{n,L}(p')\,dF(p') = \sigma G_{n,L}(p)$$

   and $\lambda_{n,H}(p)$ satisfies

$$\phi\int_0^p \lambda_{n,H}(p')\,dF(p') = (1-\sigma)G_{n,H}(p).$$

2. *For any $i \in \{L,H\}$, $n \in \{1,\ldots,N\}$, and $p \in \mathcal{F}$, $\mu_{n,i}(p)$ satisfies*

$$\mu_{n,i}(p) = \lambda_{n,i}(p)\int_{\mathcal{F}^{n-1}}\prod_{j=n+1}^{N}\left(1-\frac{1-e^{-\mu(p_j)}}{\mu(p_j)}\right)d\overline{G}_{-n,i}(\mathbf{p}_{-n};p).$$

3. *For any $p \in \mathcal{F}$, $\mu(p)$ satisfies*

$$\mu(p) = \sum_{n=1}^{N}\sum_{i=L,H}\mu_{n,i}(p).$$

4. *For any $p \in \mathcal{F}$, $\gamma(p)$ must satisfy*

$$\gamma(p) = \frac{\sum_{n=1}^{N}\mu_{n,L}(p)}{\mu(p)}.$$

5. *For any* $i \in \{L, H\}$ *and* $n \in \{1, \ldots, N\}$, *every* $p \in$ *supp* $G_{n,i}$ *solves*

$$u_{n,i} = \frac{1 - e^{-\mu(p)}}{\mu\left(p\right)}\left(p - c_i - u_{n-1,i}\right) + u_{n-1,i}.$$

6. *For any* $p \in \mathcal{P} \backslash \mathcal{F}$, $\mu\left(p\right)$ *solves*

$$u_{n,i} \geq \frac{1 - e^{-\mu(p)}}{\mu\left(p\right)}\left(p - c_i - u_{n-1,i}\right) + u_{n-1,i} \tag{53}$$

   with weak inequality for any (n, i), *and with equality for at least one* (n, i) *if* $\mu(p) > 0$.

7. *For any* $p \in \mathcal{P} \backslash \mathcal{F}$, $\gamma\left(p\right)$ *satisfies*

$$\begin{cases} \gamma\left(p\right)\mu\left(p\right) = 0 & \text{if (53) holds with strict inequality for } i = L \text{ and all } n \\ \left(1 - \gamma\left(p\right)\right)\mu\left(p\right) = 0 & \text{if (53) holds with strict inequality for } i = H \text{ and all } n \end{cases}$$

8. *Any* $p \in \mathcal{F}$ *solves*

$$\left(1 - e^{-\mu(p)}\right)\left[\gamma\left(p\right)v_L + \gamma\left(p\right)v_H - p\right] = \pi^* \equiv \max_{p'}\left(1 - e^{-\mu(p')}\right)\left[\gamma\left(p'\right)v_L + \gamma\left(p'\right)v_H - p'\right].$$

9. $\phi \geq 0$ *and* $\pi^* \leq k$, *with complementary slackness.*