University of Toronto
Department of Economics

Working Paper 705

On Extending Stochastic Dominance Comparisons to Ordinal Variables and Generalising Hammond Dominance

By Gordon John Anderson and Teng Wah Leo

September 01, 2021

On Extending Stochastic Dominance Comparisons to Ordinal Variables and Generalizing Hammond Dominance

Gordon Anderson*
University of Toronto

Teng Wah Leo†
St. Francis Xavier University

1st September 2021

Abstract

Following the increasing use of discrete ordinal data for wellbeing analysis, this note builds on Hammond ($H-$) dominance concepts developed in Gravel et al. (2020) for discrete ordinal variables by observing and exploiting the fact that the coefficients associated with successive sums of cumulative distribution functions are Binomial coefficient functions of the order of dominance under consideration. Drawing first on notions of stochastic dominance relations for continuous variables to develop analogous concepts for discrete ordinal variables, it highlights the important limitation that increasing orders of dominance lead to loss of degrees of freedom which can be significant when the number of categories is low, as is common among ordered categorical variables, effectively bounding the maximum order of dominance. However, expanding on $H-$ dominance by utilizing the Binomial coefficients facilitates sequential consideration of higher orders of $H-$ dominance without this loss, thereby surmounting the limitation.

JEL Code: C14; I3
Keywords: Stochastic Dominance; Discrete Variables; Ordinal Variables; Hammond Transfers

*Department of Economics, University of Toronto. Email Address: `anderson@chass.utoronto.ca`

†**Corresponding Author**: Department of Economics, St. Francis Xavier University. Email Address: `tleo@stfx.ca`

1 Introduction

Absence of cardinality in discrete ordinal variables makes utility functions based on attributed cardinal scales difficult to interpret, hampering applications of stochastic dominance techniques (Kahneman and Krueger (2006); Schröder and Yitzhaki (2017); Bond and Lang (2019)). In an attempt to overcome these difficulties this note makes three contributions. By providing a simple formula for successive sums of cumulative distribution functions (SSCDF's) and using the objective function suggested by Gravel et al. (2020), it develops necessary and sufficient conditions for ordinal dominance, and so extends stochastic dominance techniques to the ordinal paradigm. It demonstrates that SSCDF's are linear functions of the probability density function (PDF), with coefficients that are binomial in form. Given the well known asymptotic normality of maximum likelihood estimates of discrete PDF's this linearity makes statistical inference a relatively simple exercise (Rao 2009). It also highlights a key limitation associated with discrete variables that possess a limited number of outcomes. An additional degree of freedom is lost with each successive order of dominance, resulting in a concomitant bound on the maximum order of dominance. However, building on *Hammond ($H-$) Dominance* as developed in Gravel et al. (2020), this limitation is surmounted by defining an alternative set of valuations linked to the Binomial coefficients, which together with the associated necessary and sufficient conditions provided, facilitate sequential examination of higher orders of $H-$dominance without the concomitant loss of degrees of freedom. The application of both definitions is illustrated in a simple application to self-reported health in China.

2 Stochastic Dominance for Discrete Ordinal Variables

Anderson and Leo (2021) provided the definition of stochastic dominance between two states for discrete cardinal variables, and the necessary and sufficient conditions for various orders of dominance. In the absence of cardinality, translation of those results to a discrete ordinal environment requires a viable objective function. Gravel et al. (2020) perceptively noted that for two samples of equal size, an ethical observer would prefer a state g to g' if the following were true:

$$\sum_{i=1}^{J} n_i^g \alpha_i \geq \sum_{i=1}^{J} n_i^{g'} \alpha_i \quad \Rightarrow \quad n\sum_{i=1}^{J} f_{i,g}\alpha_i \geq n\sum_{i=1}^{J} f_{i,g'}\alpha_i \tag{1}$$

where α_i is the subjective valuation for each category, n is the total number of observations, n_i^g is the number of observations in category i observed in state g, and $f_{i,g}$ for $i = 1, \ldots, J$ in state g is the probability density function (PDF). This provides a similar objective to expected utility comparisons.

Anderson and Leo (2021) show that successive sums of CDF's (a discrete analogue to integrated CDF's) have a particularly useful form, the associated lemma is reproduced below for reference. Denote the CDF as $F_j^{(1)} = \sum_{i=1}^{j} f_{i,g}$, and successive sums of the CDF (SSCDF) as $F_{j,g}^{(s)} = \sum_{i=1}^{j} F_{i,g}^{(s-1)}$. Then the following is true:

Lemma 1: *For order of successive sum $s \geq 1$, the expression for the SSCDF for discrete cardinal (or discrete ordinal) variables is,*

$$F_{j,g}^{(s)} = \sum_{i=1}^{j} \frac{[j-i+(s-1)]!}{(j-i)!(s-1)!} f_{i,g} \quad = \quad \sum_{i=1}^{j} \begin{pmatrix} [j-i+(s-1)] \\ s-1 \end{pmatrix} f_{i,g} = \boldsymbol{\beta}_j^{(s-1)\prime} \boldsymbol{f}_g^j \tag{2}$$

where $\boldsymbol{f}_g^j = [f_{1,g} \ldots f_{j,g}]'$.

thus the density coefficients for each order of SSCDF are just binomial coefficients, which are readily extended to the multidimensional case (Anderson and Leo 2021). Given the well known asymptotic normality of maximum likelihood estimates of $\boldsymbol{f}_g^j$ and the linearity of (2), inference at the various orders of dominance comparison is relatively straight forward (Rao 2009).

To illustrate the significance, observe that for two states g and g', where g's SSCDF is uniformly below g''s at s, then,

$$\begin{aligned} F_{j,g}^{(s)} = \boldsymbol{\beta}_j^{(s-1)\prime} \boldsymbol{f}_g^j \quad &\leq \quad \boldsymbol{\beta}_j^{(s-1)\prime} \boldsymbol{f}_{g'}^j = F_{j,g'}^{(s)} \\ \implies -\boldsymbol{\beta}_j^{(s-1)\prime} \boldsymbol{f}_g^j \quad &\geq \quad -\boldsymbol{\beta}_j^{(s-1)\prime} \boldsymbol{f}_{g'}^j \end{aligned}$$

which means that the associated binomial coefficients can be used as valuations in (1) by setting $\boldsymbol{\alpha} = -\boldsymbol{\beta}^{(s-1)}$. Indeed, similar to the cardinal framework, the signs of the finite differences between these coefficients could be used to define mutually exclusive permissable valuation sets associated with each order. The following lemma highlights the relationships between the finite differences of the binomial coefficients associated with each order of dominance, the proof of which is in appendix A.1.

Lemma 2: *For $\beta_i^{(s-1)}$, $i = 1, \ldots, J$, at order $s \geq 2$, define*

$$\begin{aligned} \Delta^1 \left(\beta_i^{(s-1)}\right)\Big|_{s=2} \quad &= \quad \left(\beta_{i+1}^{(s-1)} - \beta_i^{(s-1)}\right)\Big|_{s=2} \\ \Delta^2 \left(\beta_i^{(s-1)}\right)\Big|_{s=3} \quad &= \quad \left(\Delta^{(1)}\beta_{i+1}^{(s-1)} - \Delta^{(1)}\beta_i^{(s-1)}\right)\Big|_{s=3} \\ \vdots \quad &= \quad \vdots \end{aligned}$$

These differences can be written as

$$\Delta^{s-1}\left(\beta_i^{(s-1)}\right) \quad = \quad \frac{(-1)^{s-1}(s-1)!}{(J-i)[J-(i+1)]\ldots[J-(i+(s-2))]} \begin{pmatrix} J-i \\ s-1 \end{pmatrix} \geq 0 \tag{3}$$

for $s \geq 2$.

Further, the following lemma, proved in Anderson and Leo (2021) and similar to the continuous case covered in Davidson and Duclos (2000), indicates that, should the CDF of g be uniformly below that of g' at some order s upto outcome indexed $j < J$, then it will be uniformly below that of g' at a higher order s' for some k, $j < k < J$.

Lemma 3: *If distribution $F_{j,g} \leq F_{j,g'}$ at $s = 1$ with strict inequality somewhere in the sequence, then for some arbitrary k, $i \leq k \leq J$, $k > j$, $F^{(s)}_{k,g} \leq F^{(s)}_{k,g'}$ for a suitably selected large $s > 1$.*

Finally, to obtain the necessary and sufficient conditions, the following lemma on the general decomposition of equation (1) is required (Gravel et al. (2020) demonstrated it for the first order), the proof of which is in appendix A.2.

Lemma 4: *Define the d^{th} order finite difference of the coefficients in vector $\boldsymbol{\alpha}$ with typical element α_j, $j = 1, \ldots, J$, $\Delta^d \alpha_j = \Delta^{d-1}\alpha_{j+1} - \Delta^{d-1}\alpha_j$, for $d = 1, \ldots, J-j$. Then by repeated decomposition,*

$$\begin{aligned} \sum_{i=1}^{J} f_{i,g}\alpha_i &= \sum_{k=1}^{i} (-1)^{k-1} F^{(k-1)}_{J-(k-1),g} \Delta^{k-1}\alpha_{J-(k-1)} + \sum_{j=1}^{J-i} (-1)^i F^{(i)}_{j,g} \Delta^i \alpha_j \\ &= \sum_{k=1}^{J} (-1)^{k-1} F^{(k-1)}_{J-(k-1),g} \Delta^{k-1}\alpha_{J-(k-1)} \end{aligned} \tag{4}$$

Two important points have evaded analyst's attention when moving from the continuous to the discrete paradigm. Note that the first order decomposition of $\sum_{i=1}^{J} f_{i,g}\alpha_i$ can be written as,

$$\sum_{i=1}^{J} f_{i,g}\alpha_i = \alpha_J - \sum_{j=1}^{J-1} F^{(1)}_{j,g} \Delta^1 \alpha_j$$

so that, for the usual first order dominance restriction of increasing differences valuation ($\Delta^1 \alpha_i \geq 0$ for $i = 1, \ldots, J-1$), the necessary and sufficient condition for state g to be preferred to state g' is $F^{(1)}_{j,g} \leq F^{(1)}_{j,g'}$ $\forall j = 1, \ldots, J-1$, which excludes $j = J$. This is no surprise since the number of finite differences are bounded by the number of discrete outcomes. Expanding the expression to the second order,

$$\sum_{i=1}^{J} f_{i,g}\alpha_i = \alpha_J - F^{(1)}_{J-1}\Delta^1\alpha_{J-1} + \sum_{j=1}^{J-2} F^{(2)}_{j,g} \Delta^2 \alpha_j$$

so that, given the additional restriction that the increasing differences be at a decreasing rate (i.e. $\Delta^1\alpha_j \geq 0$ and $\Delta^2\alpha_j \leq 0$), the necessary and sufficient conditions for second order stochastic dominance for ordinal variables requires $F^{(1)}_{J-1,g} \leq F^{(1)}_{J-1,g'}$ and $F^{(2)}_{j,g} \leq F^{(2)}_{j,g'}$ $\forall j = 1, \ldots, J-2$. Observe the additional restriction on the finite difference at the second order removes the $(J-1)^{th}$ and J^{th} outcome, in essence giving them a weight of zero. In and of itself, this is not an issue until it is realized that ordinal variables are typically qualitative in nature, and have a limited number of responses, so that the common definition of stochastic dominance at increasing orders leads to necessary and sufficient conditions that reduce the number of *degrees of freedom* available to the researcher, so that in the limit $s \longrightarrow J-1$, we would be comparing $f_{1,g}$ versus $f_{1,g'}$ since $F^{(s)}_{1,g} = f_{1,g}$ for all s. Hence unlike the continuous paradigm, stochastic dominance comparisons beyond $s = J-1$ are not available.

Given the above results, a definition of stochastic dominance for discrete ordinal variables may be written as follows:

Definition 1: *For all integer $s = 1, \dots, J-1$, define the finite difference for the set of feasible valuations $\left(\alpha_1^{(s)}, \dots, \alpha_J^{(s)}\right)$ for J discrete ordinal categories as,*

$$\begin{aligned}
\Delta^1 \alpha_i^{(s)} &= \alpha_{i+1}^{(s)} - \alpha_i^{(s)}, \textit{for } i = 1, \dots, J-1 \\
\Delta^2 \alpha_i^{(s)} &= \Delta^1 \alpha_{i+1}^{(s)} - \Delta^1 \alpha_i^{(s)}, \textit{for } i = 1, \dots, J-2 \\
\vdots &= \vdots
\end{aligned}$$

where by lemma 4, the feasible set of valuations associated with each order of dominance is,

$$\mathcal{A}^{(s)} = \left\{ \left(\alpha_1^{(s)}, \dots, \alpha_J^{(s)}\right) \in \mathbb{R}^J \middle| (-1)^s \Delta^d \alpha_j^{(s)} \leq 0, \quad \forall d \leq s, \forall j = 1, \dots, J-s \right\} \tag{5}$$

Then an sth order ordinal stochastic dominance comparison between two states g versus g', each with density vector $\boldsymbol{f}_g^J$ and $\boldsymbol{f}_{g'}^J$ respectively, with g dominating g' with valuations $\alpha_j^{(s)} \in \mathcal{A}^{(s)}$, for $j = 1 \dots, J-s$, when

$$n \sum_{j=1}^{J} \alpha_j^{(s)} f_{j,g} \geq n \sum_{j=1}^{J} \alpha_j^{(s)} f_{j,g'} \tag{1.a}$$

$$\iff F_{j,g}^{(s)} \leq F_{j,g'}^{(s)}, \forall j = 1, \dots, J-s \tag{6.a}$$

$$, F_{j,g}^{(s)} < F_{j,g'}^{(s)}, \textit{for some } j = 1, \dots, J-s \tag{6.b}$$

$$\& \ F_{J-s',g}^{(s')} \leq F_{J-s',g'}^{(s')}, \forall s' = 1, \dots, s-1 \tag{6.c}$$

and the discrete ordinal preorder relation is denoted by $F_{J,g}^{(s)} \succ_O F_{J,g'}^{(s)}$

3 Extension of Hammond Dominance

As a resolution to the limitation on the order of dominance imposed by the limited number of higher order moment restrictions available, using Hammond (1976), Gravel et al. (2020) interpreted the valuations as importance weights allotted to the various outcomes. This facilitated determination of whether the net gain from deriving one distributional state from another was normatively achievable. Gravel et al.'s pertinent findings are noted in the following definitions.

Definition 2: ***(Increment)*** *A state g is obtained from state g' through an increment if there exists $j \in \{1, \dots, J-1\}$ such that:*

$$\begin{aligned}
n_h^g &= n_h^{g'} & &, \forall h \neq j \ \& \ j+1; \\
n_j^g &= n_j^{g'} - 1; & n_j^g &= n_{j+1}^{g'} + 1
\end{aligned}$$

which means that distribution g can be "constructed" in its likeness in g' through raising the mass in a higher ordered category. This "construction" could be similarly performed in the opposite direction.

Definition 3: ***(Decrement)*** *A state g is obtained from state g' through a decrement if and only if state g' can be obtained from state g through an increment in the sense as defined in definition 2.*

A Hammond (1976) transfer is then obtained through the combination of both increment and decrement:

Definition 4: ***(Hammond Transfer)*** *A state g is obtained from state g' through a Hammond transfer if there exist categories $1 \leq i < j \leq k < l \leq J$ such that:*

$$\begin{aligned} n_h^g = n_h^{g'} \qquad &, \forall h \neq i, j, k, l \\ n_i^g = n_i^{g'} - 1 \quad ; \quad & n_j^g = n_j^{g'} + 1 \\ n_k^g = n_k^{g'} + 1 \quad ; \quad & n_l^g = n_l^{g'} - 1 \end{aligned}$$

Hammond transfers differ from those of Pigou-Dalton since the former does not equate an increment to a similar decrement because in an ordinal scenario without cardinality, the net effects have no meaning. The set of values for α_i corresponding with definitions 2, 3, and 4 are as follows:

Definition 5: *State g is derived from state g' through increments as defined in definition 2 implies equation (1) for all lists of real numbered normative valuations $(\alpha_1, \ldots, \alpha_J) \in \mathcal{A}_F$, where*

$$\mathcal{A}_F = \left\{ (\alpha_1, \ldots, \alpha_J) \in \mathbb{R}^J \middle| \, \alpha_1 \leq \cdots \leq \alpha_J \right\} \tag{7}$$

thus requiring that $\Delta^1 \alpha_j \geq 0$ for $j = 1, \ldots, J-1$. Further,

Definition 6: *State g is derived from state g' through Hammond Transfers as defined in definition 4 implies equation (1) for all lists of real numbered normative valuations $(\alpha_1, \ldots, \alpha_J) \in \mathcal{A}_H$, where*

$$\mathcal{A}_H = \left\{ (\alpha_1, \ldots, \alpha_J) \in \mathbb{R}^J \middle| \, (\alpha_{i+1} - \alpha_i) \geq (\alpha_k - \alpha_{i+1}), \quad i = 1, \ldots, J-1 \right\} \tag{8.a}$$

It is the inequality in (8.a) that provides the necessary "curvature". As will be observed, $\boldsymbol{\alpha} = -\boldsymbol{\beta}^{(s)} \in \mathcal{A}_H$ for $s \geq J$, so that the Hammond Dominance defined by Gravel et al. (2020) can be extended and build upon. Finally, Hammond dominance is defined as follows:

Definition 7: *For the H function in state g,*

$$H_{J,g} = \sum_{j=1}^{J} a_j f_{j,g} \tag{9.a}$$

where $(a_1,\dots,a_J)$ *is defined as* $a_i = 2^{J-i}$*, so that* $(-a_1,\dots,-a_J)\in\mathcal{A}_H$*.*

$$n^g\sum_{j=1}^{J}(-a_j)f_{j,g} \geq n^{g'}\sum_{j=1}^{J}(-a_j)f_{j,g'} \tag{1.b}$$

$$\iff \qquad H_{j,g} \leq H_{j,g'} \;\; \forall j\in\{1,\dots,J\} \tag{9.b}$$

and we say state g $H-$*dominates* g'*, or* $H_{J,g}\succ_H H_{J,g}$

Gravel et al. (2020) showed that inequality (1) is true if and only if $H_{J,g}\succ_H H_{J,g'}$, and that state g is obtained through a finite sequence of *increments*, and/or *Hammond Transfers* from state g'. It is interesting to note that $\alpha_i = 2^{J-i}$ is the discrete equivalent of the exponential function e^x, since $\Delta^1 2^{J-i} = 2^{J-i}$. Tempting as it is to link definition 1 to $H-$dominance, it must be kept in mind that ordinal dominance stops at $s = J-1$, whereas $H-$dominance starts at $s \geq J$.

Since the binomial coefficients associated with the SSCDFs are decreasing at an increasing rate as s increases, they suggest themselves as candidates for α as defined in definition 7. Indeed, this is viable by the following lemma for $s\geq J$, the proof of which is in appendix A.3.

Lemma 5: *Define* $\boldsymbol{\alpha}^{(t)} = -\boldsymbol{\beta}_J^{(J-1+t)}$ *with typical element* $-\beta_i^{(J-1+t)}$ $i=1,\dots,J$*, and* $t = 0,1,2,\dots$*. Then* $\boldsymbol{\alpha}^{(t)}\in\mathcal{A}_H^{(t)}$*, where*

$$\mathcal{A}_H^{(t)} = \left\{\left(\alpha_1^{(t)},\dots,\alpha_J^{(t)}\right)\in\mathbb{R}^J \middle| \left(\alpha_{i+1}^{(t)}-\alpha_i^{(t)}\right)\geq\left(\alpha_k^{(t)}-\alpha_{i+1}^{(t)}\right),\;\;\forall 1\leq i<k\leq J-1\right\} \tag{8.b}$$

Then the set $\mathcal{A}_H$ makes continual refinement of the $\boldsymbol{\alpha}$ valuations possible if $H-$dominance at lower orders are not achieved, although not in exactly the same manner, or motivated by the expansion of the objective function in inequality (1). Nonetheless, statistically significant $H-$dominance can be achieved incrementally from lower to higher orders as formally stated below.

Definition 8: *Define* $\boldsymbol{\alpha}^{(t)}$ *as in lemma 5 given a choice of* t*. For the* H *function in state* g*,*

$$H_{J,g}^{(t)} = \sum_{j=1}^{J}\beta_j^{(J-1+t)}f_{j,g} = F_{J,g}^{(J+t)} \tag{10.a}$$

state g *is preferred to* g'*,*

$$n\sum_{j=1}^{J}\alpha_j^{(t)}f_{j,g} \geq n\sum_{j=1}^{J}\alpha_j^{(t)}f_{j,g'} \tag{1.c}$$

$$\iff \qquad H_{j,g}^{(t)} \leq H_{j,g'}^{(t)} \;\; \forall j=1,\dots,J-1 \tag{10.b}$$

and state g *is said to* $H-$*dominate* g' *at order* t*, or* $H_{J,g}^{(t)}\succ_H H_{J,g'}^{(t)}$*, for* $t=0,\dots,\infty$*.*

There are several points to note. Firstly, the modified $H-$dominance defined in definition 8 is based on a chosen $\boldsymbol{\alpha}^{(t)}$ at a specified t. It can be a very large number, and will not be limited by the number of ordered categories. Secondly, since the modified $H-$dominance is basically the SSCDF when $s \geq J$, the link between the objective function and the SSCDF is retained. Decomposing inequality (1.c) to refine the conditions for higher orders is no longer valid since decomposition will rely on lower orders of the SSCDF below J. Thirdly, stochastic dominance as defined by definition 1 implies $H-$dominance at all possible t by lemma 3. Finally, by lemma 1, both approaches of $H-$dominance and stochastic dominance of definition 1 have well known asymptotic sampling behaviours (Rao 2009) which facilitates statistical inference.

4 Empirical Illustration

The increasing urban−rural coastal−inland inequities in the provision of health care and insurance coverage since the economic reforms in China has been well documented (Grogan 1995; Zhang and Kanbur 2008), and it would be interesting to see if these disparities are reflected in peoples' perceptions of their health status. Here the use of both ordinal and $H-$dominance criteria are illustrated using individual self-reported health (SRH) status extracted from the Chinese General Social Survey (CGSS), a nationwide social survey project conducted by the Renmin University of China (Bian and Li 2012). The survey encompassed Chinese households across the country between 2003 to 2017, of which the years 2008, 2010, 2012, 2015, and 2017 are used here. The application demonstrates how dominance techniques can be used to examine potentially nonlinear changes in perceptions across regions, gender and time. There are five response categories with '1' being "*Very Unhealthy*", and '5' being "*Very Healthy*". The data is summarized in figure 1, with several notable features. Under arbitrary attribution of scale, the means for all groups across years have been stable at category '4' ("*Relatively Healthy*"), with the coastal communities having a greater tendency towards category '5'. A tendency towards poorer responses in the latter years suggesting a fall in perceptions of individual health across the country is also apparent. There also appears to be gender disparity throughout the country, with males having higher perceptions of personal health than females.

The rankings reported below through the Hasse diagrams of figures 2 and 3 were derived through statistical inference. For ordinal dominance of definition 1, each pairwise comparison is tested via this couplet of hypotheses:

$$\text{vs.} \begin{cases} H_0 : F^{(1)}_{j,g} \leq F^{(1)}_{j,g'} \\ H_1 : F^{(1)}_{j,g} > F^{(1)}_{j,g'} \end{cases}, \forall j = 1, \ldots, 4 \ \& \ \text{vs.} \begin{cases} H_0 : F^{(1)}_{j,g} \geq F^{(1)}_{j,g'} \\ H_1 : F^{(1)}_{j,g} < F^{(1)}_{j,g'} \end{cases}, \forall j = 1, \ldots, 4$$

so that dominance at the first order is achieved if either $F^{(1)}_{j,g} \leq F^{(1)}_{j,g'}$ or $F^{(1)}_{j,g} \geq F^{(1)}_{j,g'}$ with strict inequality at some $j = 1, \ldots, 4$. If both are statistically significant or insignificant, the test moves on to the next order. At the second order, the complete set of hypotheses are,

$$\text{vs.} \begin{cases} H_0 : F^{(2)}_{j,g} \leq F^{(2)}_{j,g'} \\ H_1 : F^{(2)}_{j,g} >^{(2)}_{j,g'} \end{cases} , \forall j = 1, \ldots, 3 \quad \& \text{ vs.} \begin{cases} H_0 : F^{(2)}_{j,g} \geq F^{(2)}_{j,g'} \\ H_1 : F^{(2)}_{j,g} < F^{(2)}_{j,g'} \end{cases} , \forall j = 1, \ldots, 3$$

$$\& \text{ vs.} \begin{cases} H_0 : F^{(1)}_{4,g} \leq F^{(1)}_{4,g'} \\ H_1 : F^{(1)}_{4,g} > F^{(1)}_{4,g'} \end{cases} \qquad \& \text{ vs.} \begin{cases} H_0 : F^{(1)}_{4,g} \geq F^{(1)}_{4,g'} \\ H_1 : F^{(1)}_{4,g} < F^{(1)}_{4,g'} \end{cases}$$

The hypothesis tests for $H-$dominance are in turn,

$$\text{vs.} \begin{cases} H_0 : H^{(0)}_{j,g} \leq H^{(0)}_{j,g'} \\ H_1 : H^{(0)}_{j,g} > H^{(0)}_{j,g'} \end{cases} , \forall j = 1, \ldots, 4 \ \& \text{ vs.} \begin{cases} H_0 : H^{(0)}_{j,g} \geq H^{(0)}_{j,g'} \\ H_1 : H^{(0)}_{j,g} < H^{(0)}_{j,g'} \end{cases} , \forall j = 1, \ldots, 4$$

These tests make use of the fact that the maximum likelihood estimator of $f_{j,g}$ is $\widehat{f}_{j,g} = \frac{\sum_{i=1}^{n} \mathbb{I}(x_{i,g}=j)}{n}$, and that $\sqrt{n}\left(\widehat{\boldsymbol{f}}_g - \boldsymbol{f}_g\right) \xrightarrow{a} N(\mathbf{0}, \boldsymbol{\Sigma}_g)$, where $\boldsymbol{f}_g = [f_{1,g} \ \ f_{2,g} \ \ \ldots \ \ f_{J,g}]'$, and $\boldsymbol{\Sigma}_g = \mathbf{diag}(\boldsymbol{f}'_g) - \boldsymbol{f}_g \boldsymbol{f}'_g$ (Rao 2009). If the sample sizes are not the same, the limiting bahavior is modified as $\sqrt{m}\left(\widehat{\boldsymbol{f}}_g - \boldsymbol{f}_g\right) \xrightarrow{a} N(\mathbf{0}, \boldsymbol{\Sigma}_g)$ for $m = \frac{n_g n_{g'}}{n_g + n_{g'}}$. Then for ordinal dominance,

$$\sqrt{n}\left(\widehat{F}^{(s)}_{j,g} - F^{(s)}_{j,g}\right) = \sqrt{n}\boldsymbol{\beta}^{(s-1)\prime}_j \left(\widehat{\boldsymbol{f}}^j_g - \boldsymbol{f}^j_g\right) \xrightarrow{a} N\left(0, \boldsymbol{\beta}^{(s-1)\prime}_j \boldsymbol{\Sigma}_{j,g} \boldsymbol{\beta}^{(s-1)}_j\right)$$

for $j = 1, \ldots, J$ and $s = 1, \ldots, J-1$, where $\boldsymbol{\beta}^{(s)}$ is the vector of the binomial coefficients, with $\boldsymbol{\beta}^{(s)}_j$ being its first j elements, $\boldsymbol{f}^j_g$ are the associated vector of densities, with variance $\boldsymbol{\Sigma}_{j,g} = \mathbf{diag}(\boldsymbol{f}^j_g) - \boldsymbol{f}^j_g \boldsymbol{f}^{j\prime}_g$. By extension, the limiting behavior of $H^{(t)}_{j,g}$ is,

$$\sqrt{n}\left(\widehat{H}^{(t)}_{j,g} - H^{(t)}_{j,g}\right) = \sqrt{n}\boldsymbol{\beta}^{(J-1+t)\prime}_j \left(\widehat{\boldsymbol{f}}^j_g - \boldsymbol{f}^j_g\right) \xrightarrow{a} N\left(0, \boldsymbol{\beta}^{(J-1+t)\prime}_j \boldsymbol{\Sigma}_{j,g} \boldsymbol{\beta}^{(J-1+t)}_j\right)$$

for $t = 0, 1, \ldots, \infty$.

The ranking results depicted in the Hasse diagrams of figures 2 and 3 reflect the impressions gained from figure 1. The rankings of figure 2 were achieved at the first order (at 10% level of significance), with the sole exception of the second order dominance of coastal females in 2015 over coastal females in 2010. Overall, it is clear that coastal males' perception of their health dominates their compatriots, followed by coastal females, then inland males and inland females respectively. The deteoriation of health perceptions over time across the spectrum of groups is equally evident.

Figure 1: Self-Reported Health Box Plot by Gender, Region & Year

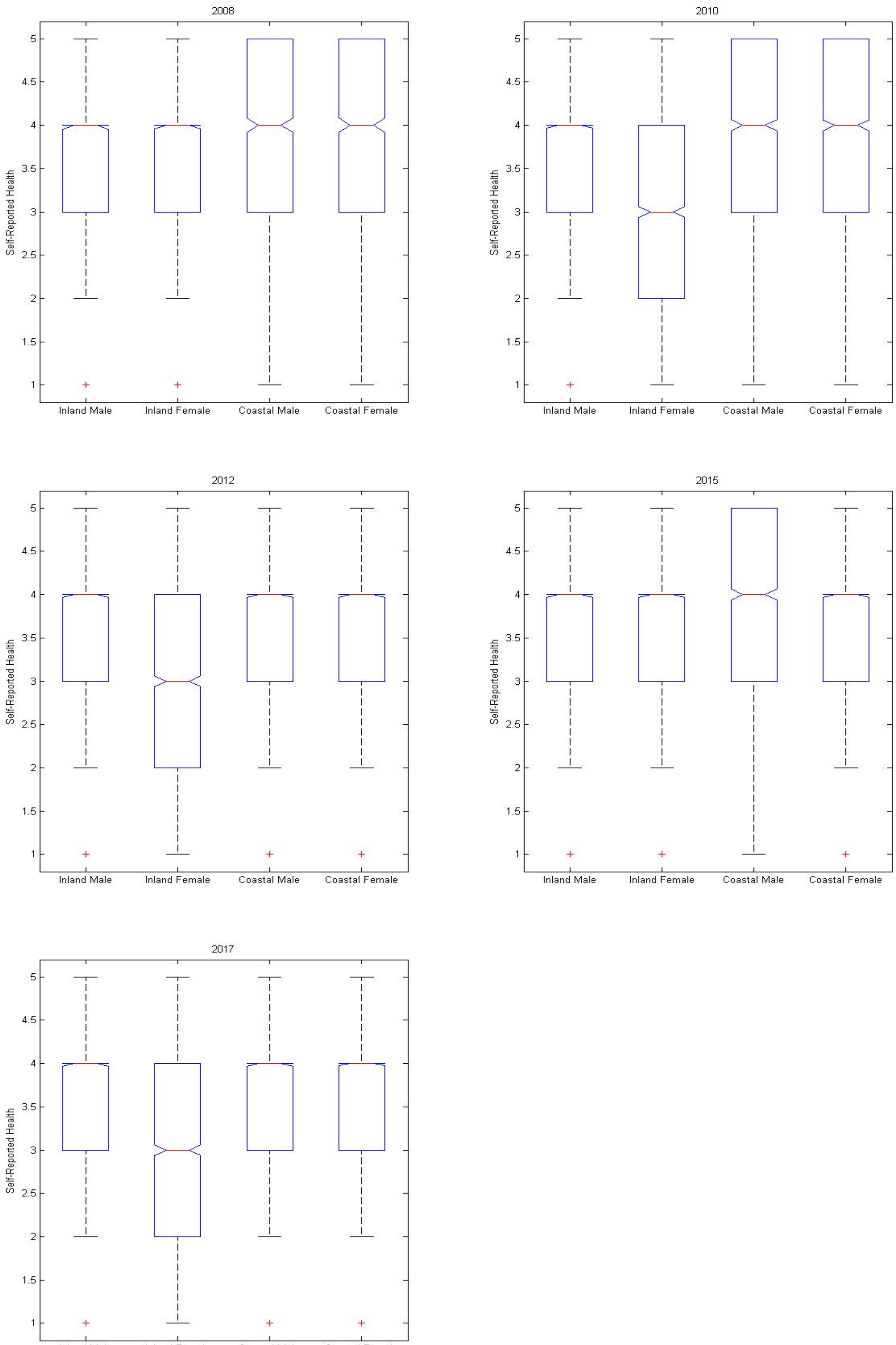


Figure 2: Hasse Diagram based on Ordinal Dominance

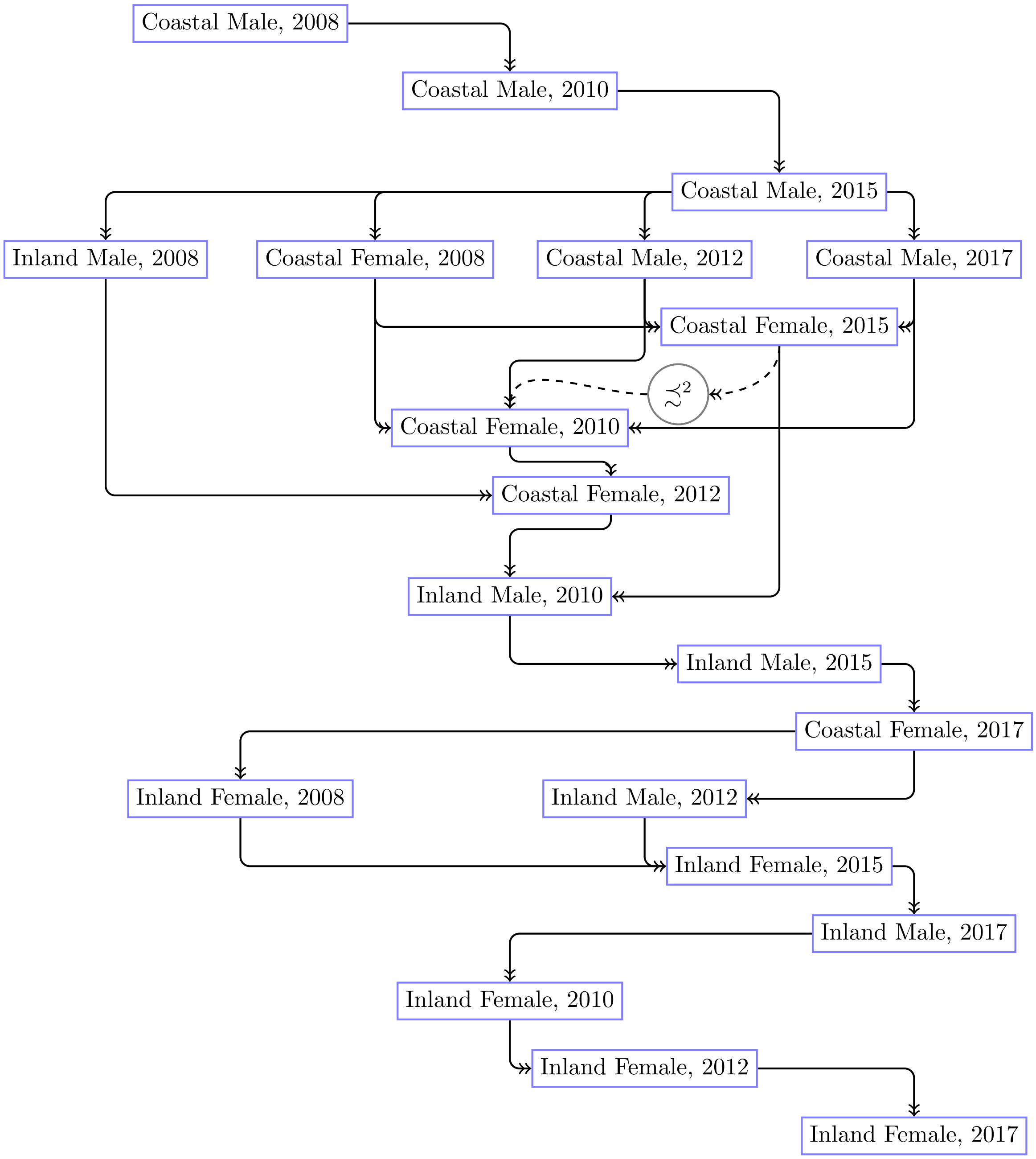
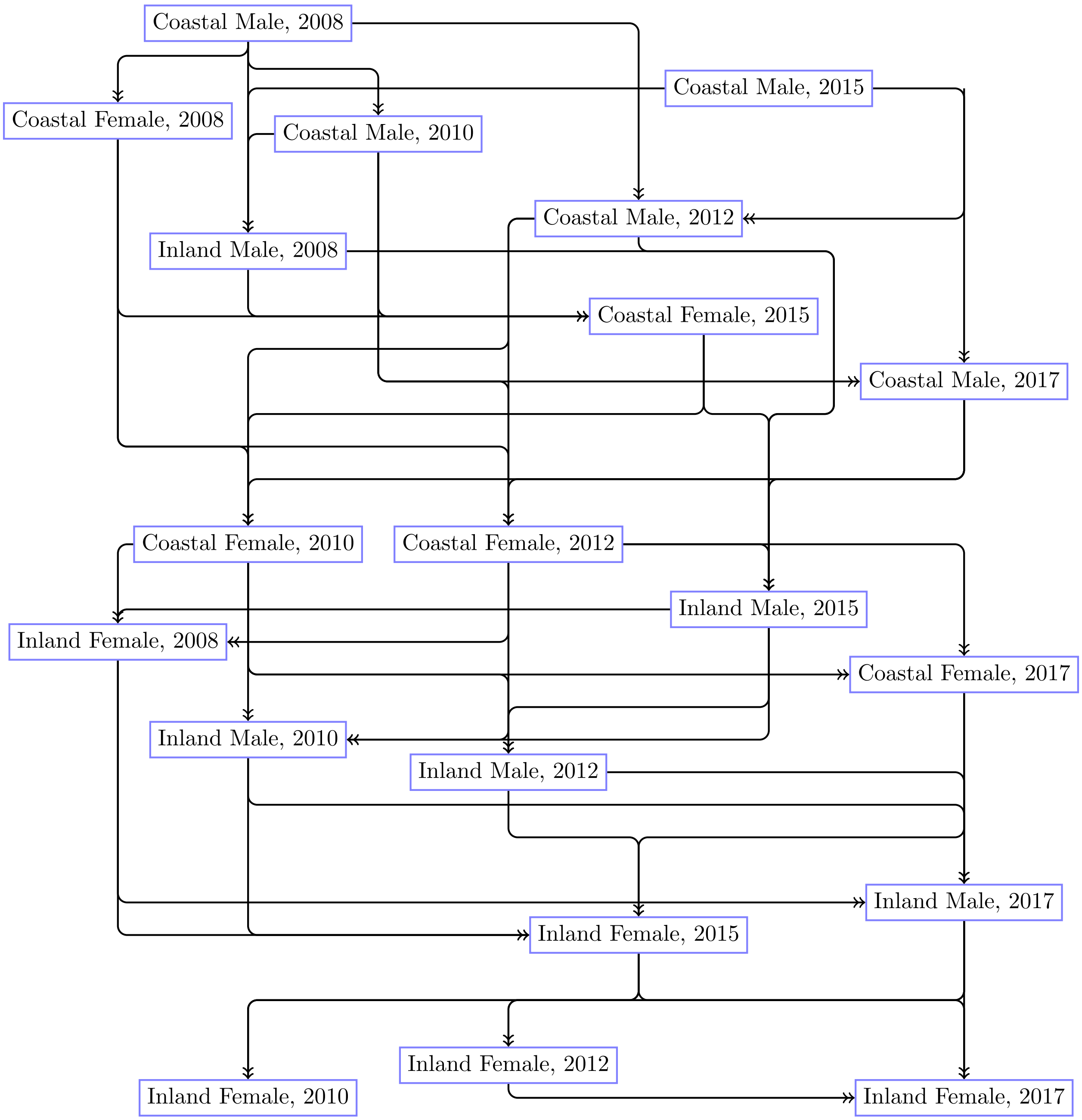


Figure 3: Hasse Diagram based on Hammond Dominance



Although most of the dominance relationships (at 10% level of significance) are maintained for Hammond Dominance in figure 3, there are significant increases in "no dominance" relationships, as suggested by the boxplots of figure 1, so that in consequence, there is a proliferation of "edges". Nonetheless, the observed dominance of coastal communities over those inland, males over females across geography, and deteoriation of perceived health over time remains. One may legitimately ask why the rankings of figure 2 are not replicated in figure

3, as would have been suggested by lemma 3. The relationships which became indeterminate occured where the first order dominance occured at a single category at the right tail, with the remainder categories being statistically not significant. With the change of weights, which is essentially what occurs moving from definition 1 to 8, the variance increase consequently diluted the initially observed statistical significance, causing the increase in ranking indeterminancy. Indeed, to achieve similar sets of ranking, the level of statistical significance needs to be relaxed to $15\% - 20\%$.

5 Conclusion

Building on work by Gravel et al. (2020) and Anderson and Leo (2021), two approaches of extending stochastic dominance concepts to discrete ordinal variables are proposed. The first translates well known stochastic dominance techniques to the ordinal discrete paradigm using the objective function suggested by Gravel et al. (2020) while using binomial coefficients associated with SSCDFs (Anderson and Leo 2021). This approach is however limited by the small number of distinct ordered categories commonly seen in discrete ordinal variables. The second approach builds on Gravel et al.'s $H-$dominance ideas, and proposes an alternative valuation system using the binomial coefficients associated with increasing orders of SSCDFs. This latter approach facilitates systematic higher order dominance examination. Given the well known statistical sampling theory for ordered categorical data, statistical inference at all orders of dominance is straightforward, and is demonstrated in the illustrative example for self-reported health in China.

A Mathematical Appendix

A.1 Proof of Lemma 2

First note that for first order stochastic dominance, all categories are equally weighted regardless of whether the variable is continuous cardinal, discrete cardinal or discrete ordinal, so that the difference is 0. For the second order, $s = 2$, all we need is for the first difference to be positive. So from lemma 1,

$$\beta_i^{(s-1)} = \binom{[J - i + (s-1)]}{s-1}$$

for $i \in \{1, \ldots, J\}$, keeping in mind that $\beta_i^{(s-1)}$ is strictly decreasing in i. Then for any adjacent coefficients, $\beta_i^{(s-1)}$ and $\beta_{i+1}^{(s-1)}$, their difference can be written as,

$$\begin{aligned}
\Delta^1 \beta_i^{(s-1)} = \beta_{i+1}^{(s-1)} - \beta_i^{(s-1)} &= \begin{pmatrix} [J-(i+1)+(s-1)] \\ s-1 \end{pmatrix} - \begin{pmatrix} [J-i+(s-1)] \\ s-1 \end{pmatrix} \\
\Rightarrow \Delta^1 \beta_i^{(s-1)}\Big|_{s=2} &= \begin{pmatrix} [J-(i+1)+(s-1)] \\ s-1 \end{pmatrix} \frac{-(s-1)}{J-i}\Bigg|_{s=2} \\
&= \begin{pmatrix} J-i \\ 1 \end{pmatrix} \frac{-1}{J-i} < 0
\end{aligned}$$

In turn when $s = 3$, the second order finite difference is,

$$\begin{aligned}
\Delta^2 \beta_i^{(s-1)} = \Delta^1 \beta_{i+1}^{(s-1)} - \Delta^1 \beta_i^{(s-1)} &= \left(\beta_{i+2}^{(s-1)} - \beta_{i+1}^{(s-1)}\right) - \left(\beta_{i+1}^{(s-1)} - \beta_i^{(s-1)}\right) \\
\Rightarrow \Delta^2 \beta_i^{(s-1)}\Big|_{s=3} &= \frac{(s-2)(s-1)}{[J-i][J-(i+1)]} \begin{pmatrix} [J-(i+2)+(s-1)] \\ s-1 \end{pmatrix}\Bigg|_{s=3} \\
&= \begin{pmatrix} J-i \\ 2 \end{pmatrix} \frac{2}{[J-i][J-(i+1)]} > 0
\end{aligned}$$

For $s = 4$

$$\begin{aligned}
\Delta^3 \beta_i^{(s-1)} &= \Delta^2 \beta_{i+1} - \Delta^2 \beta_i \\
\Rightarrow \Delta^3 \beta_i^{(s-1)}\Big|_{s=4} &= \frac{-(s-3)(s-2)(s-1)}{(J-i)[J-(i+1)][J-(i+2)]} \begin{pmatrix} [J-(i+3)+(s-1)] \\ s-1 \end{pmatrix}\Bigg|_{s=4} \\
&= \begin{pmatrix} J-i \\ 3 \end{pmatrix} \frac{-(3!)}{(J-i)[J-(i+1)][J-(i+2)]} > 0
\end{aligned}$$

This suggests that the d^{th} difference can be written as,

$$\Delta^d \beta_i^{(s-1)} = \frac{(-1)^d (s-d)\ldots(s-2)(s-1)}{(J-i)[J-(i+1)]\ldots[J-(i+(d-1))]} \begin{pmatrix} [J-(i+d)+(s-1)] \\ s-1 \end{pmatrix}$$

So that the $(d+1)^{th}$ difference is,

$$\begin{aligned}
&\Delta^{d+1} \beta_i^{(s-1)} = \Delta^d \beta_{i+1}^{(s-1)} - \Delta^d \beta_i^{(s-1)} \\
= &\frac{(-1)^d (s-d)\ldots(s-2)(s-1)}{[J-(i+1)][J-(i+2)]\ldots[J-((i+1)+(d-1))]} \begin{pmatrix} [J-(i+1+d)+(s-1)] \\ s-1 \end{pmatrix} \\
&- \frac{(-1)^d (s-d)\ldots(s-2)(s-1)}{(J-i)[J-(i+1)]\ldots[J-(i+(d-1))]} \begin{pmatrix} [J-(i+d)+(s-1)] \\ s-1 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^d(s-d)\dots(s-2)(s-1)}{(J-i)[J-(i+1)][J-(i+2)]\dots[J-(i+(d-1))]} \left(\begin{array}{c} [J-(i+(d+1))+(s-1)] \\ s-1 \end{array} \right) \\
&\quad \times \left[\frac{(J-i)-[J-(i+d)+(s-1)]}{J-(i+d)} \right] \\
&= \frac{(-1)^d(s-d)\dots(s-2)(s-1)}{(J-i)[J-(i+1)][J-(i+2)]\dots[J-(i+(d-1))]} \left(\begin{array}{c} [J-(i+(d+1))+(s-1)] \\ s-1 \end{array} \right) \\
&\quad \left[\frac{(-1)[s-(d+1)]}{J-(i+d)} \right]
\end{aligned}
$$

which is as required and this completes the proof by induction for the differences. In turn, we may now write the $(s-1)^{th}$ finite difference for the s^{th} order dominance as,

$$
\begin{aligned}
\Delta^d \beta_i^{(s-1)} \Big|_{d=s-1} &= \frac{(-1)^d(s-d)\dots(s-2)(s-1)}{(J-i)[J-(i+1)]\dots[J-(i+(d-1))]} \left(\begin{array}{c} [J-(i+d)+(s-1)] \\ s-1 \end{array} \right) \Bigg|_{d=s-1} \\
&= \left(\begin{array}{c} J-i \\ s-1 \end{array} \right) \frac{(-1)^{s-1}(s-1)!}{(J-i)[J-(i+1)]\dots[J-(i+(s-2))]}
\end{aligned}
$$

for $s \geq 2$. $\blacksquare$

A.2 Proof of Lemma 4

The method of decomposition which is essentially the discrete version of integration by parts was also used in Gravel et al. (2020) in the ordinal variable case, and is here expanded. We start with the second order since the first was already shown by Gravel et al. (2020). Stating the first order expansion:

$$
\sum_{i=1}^{J} f_{i,g}\alpha_i = \quad \textstyle\sum_{j=1}^{J} \alpha_j f_g(x_j) \quad = \alpha_J - \displaystyle\sum_{j=1}^{J-1} F_{j,g}^{(1)} \Delta^1 \alpha_j
$$

This process can be repeated so that,

$$
\begin{aligned}
&\sum_{j=1}^{J-1} F_{j,g}^{(1)}(\alpha_{j+1}-\alpha_j) \\
&= \left\{ \begin{array}{ll}
 & F_{1,g}^{(1)}\Delta^1\alpha_1 \\
+ & F_{2,g}^{(1)}\Delta^1\alpha_1 + F_{2,g}^{(1)}[\Delta^1\alpha_2 - \Delta^1\alpha_1] \\
+ & F_{3,g}^{(1)}\Delta^1\alpha_1 + F_{3,g}^{(1)}[\Delta^1\alpha_2 - \Delta^1\alpha_1] + F_{3,g}^{(1)}[\Delta^1\alpha_3 - \Delta^1\alpha_2] \\
+ & \dots \\
+ & F_{J-1,g}^{(1)}\Delta^1\alpha_1 + F_{J-1,g}^{(1)}[\Delta^1\alpha_2 - \Delta^1\alpha_1] + \\
 & F_{J-1,g}^{(1)}[\Delta^1\alpha_3 - \Delta^1\alpha_2] + \dots + F_{J-1,g}^{(1)}[\Delta^1\alpha_{J-1} - \Delta^1\alpha_{J-2}]
\end{array} \right.
\end{aligned}
$$

so that

$$
\begin{aligned}
&= \left\{ \begin{array}{ll}
& F^{(1)}_{J-1,g}\Delta^1\alpha_1 \\
+ & \left(F^{(1)}_{J-1,g} - F^{(1)}_{1,g}\right)[\Delta^1\alpha_2 - \Delta^1\alpha_1] \\
+ & \left[F^{(1)}_{J-1,g} - \left(F^{(1)}_{1,g} + F^{(1)}_{2,g}\right)\right][\Delta^1\alpha_3 - \Delta^1\alpha_2] \\
+ & \ldots \\
+ & \left[F^{(1)}_{J-1,g} - \left(\sum_{j=1}^{J-2} F^{(1)}_{j,h}\right)\right][\Delta^1\alpha_{J-1} - \Delta^1\alpha_{J-2}]
\end{array} \right. \\
&= F^{(1)}_{J-1,g}\Delta^1\alpha_{J-1} - \sum_{j=1}^{J-2} F^{(2)}_{j,g}\Delta^2\alpha_j
\end{aligned}
$$

where $F^{(2)}_{j,g} = \sum_{k=1}^{j} F^{(1)}_{k,g}$. So that,

$$
\sum_{i=1}^{J} f_{i,g}\alpha_i = \alpha_J - F^{(1)}_{J-1,g}\Delta^1\alpha_{J-1} + \sum_{j=1}^{J-2} F^{(2)}_{j,g}\Delta^2\alpha_j
$$

More generally,

$$
\begin{aligned}
\sum_{i=1}^{J} f_{i,g}\alpha_i &= \alpha_J + \sum_{k=2}^{i}(-1)^{k-1}F^{(k-1)}_{J-(k-1),g}\Delta^{k-1}\alpha_{J-(k-1)} + \sum_{j=1}^{J-i}(-1)^i F^{(i)}_{j,g}\Delta^i\alpha_j \\
&= \alpha_J + \sum_{k=2}^{J}(-1)^{k-1}F^{(k-1)}_{J-(k-1),g}\Delta^{k-1}\alpha_{J-(k-1)}
\end{aligned}
$$

which is the discrete analogue of Ekern's (1980) result.■

A.3 Proof of Lemma 5

We need to show the inequality $(\alpha_{i+1} - \alpha_i) \geq (\alpha_k - \alpha_{i+1})$ for all $i, j, k \in \{1, \ldots, J\}$, where $1 < i < j < k \leq J$. First define,

$$
\alpha_i = \left\{ \begin{array}{ll}
-\begin{pmatrix} [j - i + (s-1)] \\ s-1 \end{pmatrix} & \text{for } i \in \{1, \ldots, j\} \\
0 & \text{for } i \in \{j+1, \ldots, J\}
\end{array} \right.
$$

Then for $i \geq j+1$,

$$
\alpha_{i+1} - \alpha_i = 0 - 0 = \alpha_k - \alpha_{i+1}
$$

Similarly, for $i = j$, then

$$
\begin{aligned}
\alpha_{i+1} - \alpha_i = \alpha_{j+1} - \alpha_j &= 0 + 1 \\
&> 0 = \alpha_k - \alpha_{j+1} = \alpha_k - \alpha_{i+1}
\end{aligned}
$$

Finally, for $i < j$

$$\begin{aligned}
\alpha_{i+1} - \alpha_i &= -\binom{[j-(i+1)+(s-1)]}{s-1} + \binom{[j-i+(s-1)]}{s-1} \\
&= -\binom{[j-(i+1)+(s-1)]}{s-1}\left[1 - \frac{[j-i+(s-1)]}{(j-i)}\right] \\
&= \binom{[j-(i+1)+(s-1)]}{s-1}\frac{s-1}{j-i} \\
&\begin{matrix} \geq \\ < \end{matrix} \left\{ \binom{[j-(i+1)+(s-1)]}{s-1} = \alpha_k - \alpha_{i+1} \begin{cases} \text{if } s \geq j \\ \text{if } s < j \end{cases} \right.
\end{aligned}$$

where the last inequality couplet follows since what we need is for the inequality to be true for all i, j, and since $\max_i\{j-i\} = j-1$, the result follows. Thus it shows that for any $j \leq J-1$, the inequality $(\alpha_{i+1} - \alpha_i) \geq (\alpha_j - \alpha_{i+1})$ holds if $s \geq j$, or when we consider order of dominance $s \geq J$. ■

References

Anderson, G. and Leo, T. W. (2021). Sufficient Conditions for J^{th} Order Stochastic Dominance for Discrete Cardinal Variables, and Their Formulae. Working Paper, University of Toronto, Department of Economics.

Bian, Y. and Li, L. (2012). The Chinese General Social Survey (2003-8): Sample Designs and Data Evaluation. *Chinese Sociological Review*, 45(1), 70–97.

Bond, T. N. and Lang, K. (2019). The sad truth about happiness scales. *Journal of Political Economy*, 127(4), 1629–1640.

Davidson, R. and Duclos, J.-Y. (2000). Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality. *Econometrica*, 68, 1435–1464.

Ekern, S. (1980). Increasing Nth Degree Risk. *Economics Letters*, 6, 329–333.

Gravel, N., Magdalou, B., and Moyes, P. (2020). Ranking Distributions of an Ordinal Variable. *Economic Theory*. https://doi.org/10.1007/s00199-019-01241-4.

Grogan, C. M. (1995). Urban Economic Reform and Access to Health Care Coverage in the People's Republic of China. *Social Science and Medicine*, 41(8), 1073–1084.

Hammond, P. J. (1976). Equity, Arrow's conditions and Rawls's Difference Principle. *Econometrica*, 44(4), 793–804.

Kahneman, D. and Krueger, A. B. (2006). Developments in the Measurement of Subjective Well-Being. *Journal of Economic Perspectives*, 20(1), 3–24.

Rao, C. (2009). *Linear Statistical Inference and its Applications*. 2 edn. New York: Wiley.

Schröder, C. and Yitzhaki, S. (2017). Revisiting the Evidence for Cardinal Treatment of Ordinal Variables. *European Economic Review*, 92, 337–358.

Zhang, X. and Kanbur, R. (2008). Spatial Inequality in Education and Health Care in China. *China Economic Review*, 16(2), 189–204.