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Abstract

In response to the increasing use of discrete cardinal data with limited numbers of outcomes, Stochastic Dominance Theory is here extended to facilitate its application. Formulae, convenient for analysis, along with necessary and sufficient conditions for different orders of dominance are derived which reveal some key facts which have eluded general attention. In this paradigm, there is a loss of degrees of freedom as the dominance order increases with a concomitant upper bound to the order of dominance that can be considered, both engendered by the restrictions on finite differences between utility functions and the limited number of outcomes. Simple formulae for computing successive sums of cumulative distributions are found, and the relationship between lower and higher order dominance is proven in this discrete cardinal case.

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1 Introduction

This note extends the common concepts of stochastic dominance for continuous variables to that for discrete cardinal variables, highlighting the differences, and providing the set of simple formulae for developing their statistical properties. The two contributions are as follows: Firstly, it provides a decomposition of the difference in expected (von Neumann-Morgenstern type) utility that facilitates derivation of necessary and sufficient conditions for the j^{th} order stochastic dominance between states. In the process, it demonstratess that with each successive order of dominance considered, the comparison of the successive sums of cumulative distribution functions (SSCDFs) loses one additional right tail outcome, and consequently one degree of freedom. This limits the order of dominance that can be considered when, as is common among discrete cardinal variables, the number of outcomes is limited. Secondly, a convenient formula for calculating SSCDF's is provided which simplifies inference, and analogous to the continuous paradigm, the existence of a dominance relation in the left tail of the distribution implies higher order dominance over the whole range is proved for the discrete cardinal paradigm.

2 Discrete Cardinal Variable Stochastic Dominance

Consider J discrete cardinal outcomes indexed $j = 1, \ldots, J$, across states g and g', such that $x_{1,g} < x_{2,g} < \cdots < x_{J,g}$ for $x_{j,g} \in \mathcal{X} \subset \mathbb{R}$. Denote the probability density as $f(x_{j,g}) \equiv f_{j,g}$, and cumulative distribution function (CDF) as $F^{(1)}(x_{j,g}) \equiv F^{(1)}_{j,g} = \sum\limits_{i=1}^{j} f_{i,g}$ for state g, and successive sums of CDFs (SSCDFs) as $F^{(s)}_{j,g} = \sum\limits_{i=1}^{j} F^{(s-1)}_{i,g}$. Define the set of utility functions $u \in \mathcal{U}^{(1)}$ such that their first order finite differences are increasing, $\Delta^1 u(x_j) = u(x_{j+1}) - u(x_j) \geq 0$ $\forall j = 1, \ldots, J-1$. Observe that there is no equidistant constraint on the elements of $\mathcal{X}$. All that is required is that they have cardinality. Given that $F^{(1)}_{J,g} = \sum_{j=1}^{J} f_{j,g} = 1$, note that the expected utility $\mathbf{E}_g\, u(x)$ under any $u \in \mathcal{U}^{(1)}$ may be decomposed as:

$$\begin{aligned}
\mathbf{E}_g\, u(x) &= \sum_{j=1}^{J} u(x_j) f_g(x_j) \\
&= F^{(1)}_{J,g} u(x_J) - \sum_{j=1}^{J-1} F^{(1)}_{j,g} (u(x_{j+1}) - u(x_j)) \\
&= u(x_J) - \sum_{j=1}^{J-1} F^{(1)}_{j,g} \Delta^1 u(x_j)
\end{aligned}$$

This implies that for $\mathbf{E}_g\, u(x) - \mathbf{E}_{g'}\, u(x) \geq 0$, given $\Delta^1 u(x_j) \geq 0$, the neccessary and sufficient condition is for $F^{(1)}_{j,g} - F^{(1)}_{j,g'} \leq 0$ $\forall j = 1, \ldots, J-1$, which is well known. Importantly, this condition only considers $J-1$ differences, with the derivative of the utility function replaced by a first order finite difference. Although exclusion of $F^{(1)}_{J,g}$ does not affect the analysis since $F^{(1)}_{J,g} = 1$, the exclusion continues with successive orders of dominance, so that $J-1$ is excluded at second order, $J-2$ at the third order and so on.

To facilitate comparison at higher orders, the following lemma states the results of successive repetition of the decomposition, with the proof in Appendix 5.1.

Lemma 1: *Define the d^{th} order finite difference for $u \in \mathcal{U}^{(d)}$ as $\Delta^d u(x_i) = \Delta^{d-1}u(x_{i+1}) - \Delta^{d-1}u(x_i)$, such that $(-1)^d \Delta^d u(x_i) \leq 0 \ \forall d = 1, \ldots, J-i$, and $\forall i = 1, \ldots, J-1$. Then by repeated decomposition,*

$$\begin{aligned}
\boldsymbol{E}_g\, u(x) &= u(x_J) + \sum_{k=2}^{i} (-1)^{k-1} F^{(k-1)}_{J-(k-1),g} \Delta^{k-1} u(x_{J-(k-1)}) + \sum_{j=1}^{J-i} (-1)^i F^{(i)}_{j,g} \Delta^i u(x_j) \qquad (1)\\
&= u(x_J) + \sum_{k=2}^{J} (-1)^{k-1} F^{(k-1)}_{J-(k-1),g} \Delta^{k-1} u(x_{J-(k-1)})
\end{aligned}$$

which is the discrete analogue of Ekern's (1980) result. Therefore, the difference in expected utility between two states g and g', for $u \in \mathcal{U}^{(i)}$, has the following expression:

$$\mathbf{E}_g\, u(x) - \mathbf{E}_{g'}\, u(x) = \begin{cases} & \sum_{k=2}^{i} (-1)^{k-1} \left(F^{(k-1)}_{J-(k-1),g} - F^{(k-1)}_{J-(k-1),g'} \right) \Delta^{k-1} u(x_{J-(k-1)}) \\ + & \sum_{j=1}^{J-i} (-1)^i \left(F^{(i)}_{j,g} - F^{(i)}_{j,g'} \right) \Delta^i u(x_j) \end{cases} \qquad (2)$$

This facilitates obtaining all the necessary and sufficient conditions associated with higher order dominance up to order $J-1$. In other words, unlike the continuous case where dominance at the n^{th} order can theoretically go on *ad infinitum*, the discrete case is bounded by the number of possible outcomes J.

To illustrate the use of lemma 1, the necessary and sufficient conditions for second order discrete cardinal stochastic dominance are derived. Note that,

$$\mathbf{E}_g\, u(x) - \mathbf{E}_{g'}\, u(x) = \left(F^{(1)}_{J-1,g'} - F^{(1)}_{J-1,g} \right) \Delta^1 u(x_{J-1}) + \sum_{j=1}^{J-2} \left(F^{(2)}_{j,g} - F^{(2)}_{j,g'} \right) \Delta^2 u(x_j)$$

so that for $u \in \mathcal{U}^{(2)}$, $\Delta^1 u(x_j) \geq 0$ and $\Delta^2 u(x_j) \leq 0$, the necessary and sufficient conditions are, $F^{(2)}_{j,g} - F^{(2)}_{j,g'} \leq 0 \ \forall j = 1, \ldots, J-2$, with strict inequality for some j, and importantly $F^{(1)}_{J-1,g} - F^{(1)}_{J-1,g'} \leq 0$. Although the signs associated with these results have been proven alternatively by Quirk and Saposnik (1962) and Hadar and Russell (1969), and are widely known, an important point has been overlooked. Observe crucially that the sum of terms involving $\left(F^{(2)}_{j,g} - F^{(2)}_{j,g'} \right)$ is upto and including $J-2$, so that in effect the last two realizations are given zero weight, a point that has evaded notice in translating dominance from the continuous to discrete paradigm. Indeed, each increment in order of comparison imposes a zero weight restriction on an additional right tail outcome. This is due to the additional finite difference sign restriction, which are in essence "*curvature*" restrictions on the utility function.

A similar set of necessary and sufficient conditions can be drawn for the third order dominance case that parallels Whitmore (1970) in the continuous variable case. For the third order,

$$\mathbf{E}_g\, u(x) - \mathbf{E}_{g'}\, u(x) = \begin{cases} & \left(F^{(1)}_{J-1,g'} - F^{(1)}_{J-1,g}\right)\Delta^1 u(x_{J-1}) + \left(F^{(2)}_{J-2,g} - F^{(2)}_{J-2,g'}\right)\Delta^2 u(x_{J-2}) \\ + & \sum_{j=1}^{J-3}\left(F^{(3)}_{j,g'} - F^{(3)}_{j,g}\right)\Delta^3 u(x_j) \end{cases}$$

As in the second order case, given $\Delta^1 u(x_j) \geq 0$, $\Delta^2 u(x_j) \leq 0$, and now in addition $\Delta^3 u(x_j) \geq 0$, the necessary and sufficient conditions are $F^{(3)}_{j,g} - F^{(3)}_{j,g'} \leq 0\ \forall j = 1,\ldots,J-3$ with strict inequality at some j, $F^{(2)}_{J-2,g} - F^{(2)}_{J-2,g'} \leq 0$ and $F^{(1)}_{J-1,g} - F^{(1)}_{J-1,g'} \leq 0$. Observe now that the comparison $\left(F^{(3)}_{j,g} - F^{(3)}_{j,g'}\right)$ is upto and includes the $(J-3)^{th}$ outcome. In the limit, at $s = J-1$, the necessary and sufficient condition becomes $F^{(1)}_{J-1,g} - F^{(1)}_{J-1,g'} \leq 0; F^{(2)}_{J-2,g} - F^{(2)}_{J-2,g'} \leq 0; \ldots; F^{(J-1)}_{1,g} - F^{(J-1)}_{1,g'} \leq 0$, where the final condition makes the comparison between the density of the first outcome, which is just requiring $f_{1,g} - f_{1,g'} \leq 0$.

These findings may be consolidated as a definition for the j^{th} order stochastic dominance for discrete variables exhibiting cardinality.

Definition 1: *Let $x_1,\ldots,x_J \in \mathcal{X}$ be discrete cardinal variables. Denote*

$$\mathcal{U}^{(s)} = \left\{u : x_1,\ldots,x_J \to \mathbb{R} \,|\, (-1)^d \Delta^d u(x_j) \leq 0, \forall d = 1,\ldots,J-j,\ \ \forall j = 1,\ldots,J\right\} \tag{3}$$

Then $\forall u \in \mathcal{U}^{(s)}$ for some $s \leq J$, state g stochastically dominates g', in the sense of

$$\boldsymbol{E}_g\, u(x) = \sum_{j=1}^{J} u(x_j) f_g(x_j) \quad \geq \quad \sum_{j=1}^{J} u(x_j) f_{g'}(x_j) = \boldsymbol{E}_{g'}\, u(x) \tag{4}$$

$$\iff F^{(s)}_{j,g} \leq F^{(s)}_{j,g'},\quad \forall j = 1,\ldots,J-s \tag{4.a}$$

$$,\ F^{(s)}_{j,g} < F^{(s)}_{j,g'},\quad \textit{for some } j = 1,\ldots,J-s \tag{4.b}$$

$$\&\ F^{(s')}_{J-s',g} \leq F^{(s')}_{J-s',g'},\ \textit{for } s' = 1,\ldots,s-1 \tag{4.c}$$

then state g is said to be preferred to g'.

It is tempting to think that these ideas may be directly applied to discrete ordinal variables that are usually qualitative in nature. It would be incorrect, as noted in Gravel et al. (2020). Indeed with limited number of responses, any increase in order of comparison, even at the second order quickly runs out of degrees of freedom. For example, typical blood pressure or Body-Mass Index or Self-Reported Health classifications have limited responses of between four to five, so that at the second order, the researcher would be comparing two or three outcomes.

3 Formulae for Successive Sums of Discrete CDF

Due to the discrete nature, the successive sum of the distribution function (SSCDF) needs to be developed since the incomplete moment formula of Davidson and Duclos (2000) is no longer appropriate. The formula is provided below with the proof in appendix 5.2.

Lemma 2: *For order of successive sum $s \geq 1$, the expression for the SSCDF for cardinal discrete (and ordinal discrete) variables is,*

$$F_{j,g}^{(s)} = \sum_{i=1}^{j} \frac{[j-i+(s-1)]!}{(j-i)!(s-1)!} f_{i,g} \quad = \quad \sum_{i=1}^{j} \begin{pmatrix} [j-i+(s-1)] \\ s-1 \end{pmatrix} f_{i,g} = \boldsymbol{\beta}_j^{(s-1)\prime} \boldsymbol{f}_g^j \tag{5}$$

where $F_{j,g}^{(s)} \equiv F_g^{(s)}(x_j)$ and $\boldsymbol{f}_g^j = [f_{1,g} \ldots f_{j,g}]'$.

Observe that the s^{th} order SSCDF is a linear function of the probability density function (PDF), where the coefficient associated with the density is a binomial coefficient, which is readily extended to the multidimensional case. Let there be m dimensions, and denote the number of categories in each of the m dimensions as $\{j_1, \ldots, j_m\}$. Then (5) can be modified as follows,

$$F_{j_1,\ldots,j_m,g}^{(s)} = \sum_{i_1=1}^{j_1} \cdots \sum_{i_m=1}^{j_m} \begin{pmatrix} [j_1 - i_1 + (s-1)] \\ s-1 \end{pmatrix} \cdots \begin{pmatrix} [j_m - i_m + (s-1)] \\ s-1 \end{pmatrix} f_{i_1,\ldots,i_m,g} \tag{6}$$

Given this linearity, and the well known asymptotic properties of the maximum likelihood estimator of the discrete PDF, statistical inference is easily performed following Rao (2009).

Further, by the following lemma, the proof of which is in appendix 5.3, similar to the continuous cardinal variable case (Davidson and Duclos 2000), when there is dominance at order s in the left tail of the distribution, discrete cardinal stochastic dominance will prevail over the whole range at some sufficiently larger value of s.

Lemma 3: *If distribution $F_{j,g} \leq F_{j,g'}$ at $s = 1$ with strict inequality somewhere in the sequence, then for some arbitrary k, $i \leq k \leq J$, $k > j$, $F_{k,g}^{(s)} \leq F_{k,g'}^{(s)}$ for a suitably selected large $s > 1$.*

4 Conclusion

The framework for applying the tools of stochastic dominance analysis at a given order s to discrete cardinal variables has been developed. The provision of formulae for defining the necessary and sufficient conditions, and quick computation of successive sums of cumulative distributions useful in the analysis, revealed the incremental loss of degrees of freedom as the order of dominance considered increases. This idiosyncrasy, engendered by the finite number of outcomes common among discrete variables, has evaded notice in the literature, and puts a limit on the maximum order of dominance that can be considered.

5 Mathematical Appendix

5.1 Proof of Lemma 1

The method of decomposition, essentially the discrete version of integration by parts used in Gravel et al. (2020) in the ordinal variable case, is here expanded. The first order decomposition may be written as:

$$\begin{aligned}
\mathbf{E}_g(x) &= \sum_{j=1}^{J} u(x_j) f_g(x_j) \\
&= \begin{cases} & f_{1,g}u(x_1) \\ + & f_{2,g}u(x_1) + f_{2,g}[u(x_2) - u(x_1)] \\ + & f_{3,g}u(x_1) + f_{3,g}[u(x_2) - u(x_1)] + f_{3,g}[u(x_3) - u(x_2)] \\ + & \dots \\ + & f_{J,g}u(x_1) + f_{J,g}[u(x_2) - u(x_1)] + \\ & f_{J,g}[u(x_3) - u(x_2)] + \cdots + f_{J,g}[u(x_J) - u(x_{J-1})] \end{cases}
\end{aligned}$$

so that

$$\begin{aligned}
\mathbf{E}_g(x) &= \begin{cases} & F_{J,g}^{(1)} u(x_1) \\ + & \left(F_{J,g}^{(1)} - f_{1,g}\right)[u(x_2) - u(x_1)] \\ + & \left[F_{J,g}^{(1)} - (f_{1,g} + f_{2,g})\right][u(x_3) - u(x_2)] \\ + & \dots \\ + & \left[F_{J,g}^{(1)} - \left(\sum_{j=1}^{J-1} f_{j,h}\right)\right][u(x_J) - u(x_{J-1})] \end{cases} \\
&= F_{J,g}^{(1)} u(x_J) - \sum_{j=1}^{J-1} F_{j,g}^{(1)} \Delta^1 u(x_j) = u(x_J) - \sum_{j=1}^{J-1} F_{j,g}^{(1)} \Delta^1 u(x_j)
\end{aligned}$$

Where the last equality follows since $F_{J,g}^{(1)} = 1$, and observe that $F_{1,g}^{(1)} = F_{1,g}^{(2)} = \cdots = F_{1,g}^{(J-1)} = f_{1,g}$ as required. This process can be repeated so that,

$$\begin{aligned}
&\sum_{j=1}^{J-1} F_{j,g}^{(1)}(u(x_{j+1}) - u(x_j)) \\
= &\begin{cases} & F_{1,g}^{(1)} \Delta^1 u(x_1) \\ + & F_{2,g}^{(1)} \Delta^1 u(x_1) + F_{2,g}^{(1)}[\Delta^1 u(x_2) - \Delta^1 u(x_1)] \\ + & F_{3,g}^{(1)} \Delta^1 u(x_1) + F_{3,g}^{(1)}[\Delta^1 u(x_2) - \Delta^1 u(x_1)] + F_{3,g}^{(1)}[\Delta^1 u(x_3) - \Delta^1 u(x_2)] \\ + & \dots \\ + & F_{J-1,g}^{(1)} \Delta^1 u(x_1) + F_{J-1,g}^{(1)}[\Delta^1 u(x_2) - \Delta^1 u(x_1)] + \\ & F_{J-1,g}^{(1)}[\Delta^1 u(x_3) - \Delta^1 u(x_2)] + \cdots + F_{J-1,g}^{(1)}[\Delta^1 u(x_{J-1}) - \Delta^1 u(x_{J-2})] \end{cases}
\end{aligned}$$

$$
= \left\{ \begin{array}{ll} & F^{(1)}_{J-1,g}\Delta^1 u(x_1) \\ + & \left(F^{(1)}_{J-1,g} - F^{(1)}_{1,g}\right)\left[\Delta^1 u(x_2) - \Delta^1 u(x_1)\right] \\ + & \left[F^{(1)}_{J-1,g} - \left(F^{(1)}_{1,g} + F^{(1)}_{2,g}\right)\right]\left[\Delta^1 u(x_3) - \Delta^1 u(x_2)\right] \\ + & \ldots \\ + & \left[F^{(1)}_{J-1,g} - \left(\sum_{j=1}^{J-2} F^{(1)}_{j,h}\right)\right]\left[\Delta^1 u(x_{J-1}) - \Delta^1 u(x_{J-2})\right] \end{array} \right.
$$

$$
= F^{(1)}_{J-1,g}\Delta^1 u(x_{J-1}) - \sum_{j=1}^{J-2} F^{(2)}_{j,g}\Delta^2 u(x_j)
$$

where $F^{(2)}_{j,g} = \sum_{k=1}^{j} F^{(1)}_{k,g}$. So that,

$$
\mathbf{E}_g(x) = u(x_J) - F^{(1)}_{J-1,g}\Delta^1 u(x_{J-1}) + \sum_{j=1}^{J-2} F^{(2)}_{j,g}\Delta^2 u(x_j)
$$

More generally,

$$
\begin{aligned}
\mathbf{E}_g(x) &= u(x_J) + \sum_{k=2}^{i}(-1)^{k-1}F^{(k-1)}_{J-(k-1),g}\Delta^{k-1}u(x_{J-(k-1)}) + \sum_{j=1}^{J-i}(-1)^i F^{(i)}_{j,g}\Delta^i u(x_j) \\
&= u(x_J) + \sum_{k=2}^{J}(-1)^{k-1}F^{(k-1)}_{J-(k-1),g}\Delta^{k-1}u(x_{J-(k-1)})
\end{aligned}
$$

which is the discrete analogue of Ekern's (1980) result.■

5.2 Proof of Lemma 2

First, for $s = 2$,

$$
F^{(2)}_{j,g} = \sum_{i=1}^{j} F^{(1)}_{i,g} = \sum_{i=1}^{j}\sum_{k=1}^{i} f_{k,g} = \sum_{i=1}^{j}(j-i+1)f_{i,g} \tag{7}
$$

For $s = 3$,

$$
\begin{aligned}
F^{(3)}_{j,g} = \sum_{i=1}^{j} F^{(2)}_{i,g} &= \sum_{i=1}^{j}\sum_{k=1}^{i} F^{(1)}_{k,g} = \sum_{i=1}^{j}(j-(i-1))F^{(1)}_{i,g} \\
&= \sum_{i=1}^{j}(j-(i-1))\sum_{k=1}^{i} f_{k,g} = \sum_{i=1}^{j}\frac{(j-i+1)(j-i+2)}{2}f_{i,g}
\end{aligned} \tag{8}
$$

For $s = 4$,

$$\begin{aligned} F^{(4)}_{j,g} = \sum_{i=1}^{j} F^{(3)}_{i,g} &= \sum_{i=1}^{j}\sum_{k=1}^{i} F^{(2)}_{k,g} = \sum_{i=1}^{j}(j-(i-1))F^{(2)}_{i,g} \\ &= \sum_{i=1}^{j}(j-(i-1))\sum_{k=1}^{i}(i-(k-1))f_{k,g} \\ &= \sum_{i=1}^{j}\frac{[j-i+1][j-i+2][j-i+3]}{3!}f_{i,g} \end{aligned} \tag{9}$$

Thus suggesting,

$$\begin{aligned} \Rightarrow F^{(s)}_{j,g} &= \sum_{i=1}^{j}\frac{[j-i+(s-1)]!}{(j-i)!(s-1)!}f_{i,g} \\ &= \sum_{i=1}^{j}\begin{pmatrix} [j-i+(s-1)] \\ s-1 \end{pmatrix} f_{i,g} \end{aligned}$$

Then,

$$\begin{aligned} F^{(s+1)}_{j,g} &= \sum_{i=1}^{j} F^{(s)}_{i,g} = \sum_{i=1}^{j}\sum_{k=1}^{i} C^{i-k+(s-1)}_{s-1} f_{k,g} \\ &= \sum_{i=1}^{j}\sum_{k=i}^{j} C^{k-i+(s-1)}_{s-1} f_{i,g} \\ &= \sum_{i=1}^{j}\left[\frac{(s-1)!+[1+(s-1)]!+\cdots+\frac{[j-i+(s-1)]!}{(j-i)!}}{(s-1)!}\right] f_{i,g} \\ &= \sum_{i=1}^{j}\left[1+s+\frac{(1+s)s}{2}+\frac{(2+s)(1+s)s}{3!}+\cdots+\frac{[(j-i-1+s)\dots s]}{(j-i)!}\right] f_{i,g} \\ &= \sum_{i=1}^{j}\left[\begin{pmatrix} s-1 \\ s-1 \end{pmatrix}+\begin{pmatrix} 1+(s-1) \\ s-1 \end{pmatrix}+\cdots+\begin{pmatrix} j-i+(s-1) \\ s-1 \end{pmatrix}\right] f_{i,g} \\ &= \sum_{i=1}^{j}\begin{pmatrix} j-i+s \\ s \end{pmatrix} f_{i,g} = \sum_{i=1}^{j} C^{j-i+s}_{s} f_{i,g} \end{aligned}$$

where the last equality follows from the recursive use of the Pascal's Formula and noting that $C^s_s = C^{s-1}_{s-1}$, and the result follows by induction. ■

5.3 Proof of Lemma 3

Suppose g dominates g' at order $s = 1$ for $i \le j < J$, that is $F_{j,g} \le F_{j,g'}$ with strict inequality in that sequence. We need to show that for some arbitrary k, $j < k \le J$, that g dominates

g' up to k for an arbitrarily large s.

Firstly, the first order dominance implies that,

$$(F_{j,g'} - F_{j,g}) \geq \;\; \Rightarrow \;\; \sum_{i=1}^{j} (f_{i,g'} - f_{i,g}) = \beta \geq 0$$

We wish to know at which s would the following be true,

$$\left(F_{k,g'}^{(s)} - F_{k,g}^{(s)} \right) \geq 0$$

for an arbitrary $k > j$, which can be written as,

$$\sum_{i=1}^{k} \frac{[k - i + (s-1)]!}{[k-i]!(s-1)!} (f_{i,g'} - f_{i,g})$$

The sum may be split in two. The first part of sum can be written as,

$$\begin{aligned}
&\sum_{i=1}^{j} \frac{[k - i + (s-1)]!}{[k-i]!(s-1)!} (f_{i,g'} - f_{i,g}) \\
\geq \;\; & \frac{[k - j + (s-1)]!}{[k-j]!(s-1)!} \sum_{i=1}^{j} (f_{i,g'} - f_{i,g}) \\
= \;\; & \frac{\beta [k - j + (s-1)]!}{[k-j]!(s-1)!}
\end{aligned}$$

The second part of the sum may be bounded as,

$$\begin{aligned}
&\left| \sum_{i=j+1}^{k} \frac{[k - i + (s-1)]!}{[k-i]!(s-1)!} (f_{i,g'} - f_{i,g}) \right| \\
\leq \;\; & \sum_{i=j+1}^{k} \frac{[k - i + (s-1)]!}{[k-i]!(s-1)!} \quad (\text{since } |f_{i,g'} - f_{i,g}| \leq 1) \\
\leq \;\; & \frac{[k - (j+1) + (s-1)]!(k-j)}{[k-(j+1)]!(s-1)!}
\end{aligned}$$

Putting the two parts together,

$$\begin{aligned}
&\sum_{i=1}^{k} C_{s-1}^{k-i+(s-1)} (f_{i,g'} - f_{i,g}) \\
\geq \;\; & \frac{\beta [k - j + (s-1)]!}{[k-j]!(s-1)!} - \frac{[k - (j+1) + (s-1)]!(k-j)}{[k-(j+1)]!(s-1)!} \\
= \;\; & \frac{[k - (j+1) + (s-1)]!}{[k-(j+1)]!(s-1)!} \left[\frac{\beta [k - j + (s-1)]}{k-j} - (k-j) \right]
\end{aligned}$$

Therefore, we can choose s such that,

$$s \geq \frac{(k-j)^2}{\beta} - (k-j-1) > 0$$

and dominance at $i = k$ is achieved for order s.■

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