

University of Toronto  
Department of Economics



Working Paper 670

Identification of Firms' Beliefs in Structural Models of Market  
Competition

By Victor Aguirregabiria

June 29, 2020

# Identification of Firms' Beliefs in Structural Models of Market Competition

Victor Aguirregabiria\*  
University of Toronto and CEPR

June 24th, 2020

## Abstract

Firms make decisions under uncertainty and differ in their ability to collect and process information. As a result, in changing environments, firms have heterogeneous beliefs on the behavior of other firms. This heterogeneity in beliefs can have important implications on market outcomes, efficiency, and welfare. This paper studies the identification of firms' beliefs using their observed actions – a *revealed preference and beliefs approach*. I consider a general structural model of market competition where firms have incomplete information and their beliefs and profits are nonparametric functions of decisions and state variables. Beliefs may be out of equilibrium. The framework applies both to continuous and discrete choice games and includes as particular cases models of competition in prices or quantities, auction models, entry games, and dynamic investment games. I focus on identification results that exploit a natural exclusion restriction in models of competition: an observable variable that affects a firm's cost (or revenue) but does not have a direct effect on other firms' profits. I present identification results under three scenarios — common in empirical IO — on the data available to the researcher.

**Keywords:** Non-equilibrium beliefs; Structural models of competition; Identification; Revealed beliefs approach..

**JEL codes:** C57, D81, D83, D84, L13.

Victor Aguirregabiria. Address: 150 St. George Street. Toronto, ON, M5S 3G7, Canada.  
Phone: (416) 978-4358. E-mail: [victor.aguirregabiria@utoronto.ca](mailto:victor.aguirregabiria@utoronto.ca)

---

\*I would like to thank comments from Arvind Magesan, Jean Francois Houde, Jeff Smith, Erhao Xie, Federico Zicenko, and participants at the 2019 *Canadian Economic Association* annual conference in Banff.

# 1 Introduction

A firm’s behavior depends on its beliefs about the actions of other firms in the same market. In a retail market, the choice of price to maximize profits depends on beliefs about competitors’ prices. In a procurement auction, a firm’s bid depends on its expectations about other firms’ bids. Managers form their beliefs under uncertainty on demand, costs, and competitors’ decisions. These managers and their firms differ in their abilities to collect and process information and, as a result, they are heterogeneous in their expectations. This heterogeneity in beliefs can have important implications on firms’ performance and survival in the market. Nevertheless, firms with different accuracy in beliefs can survive in the same industry, for the same reasons as they can coexist with different productivity.

The importance of firms’ heterogeneity in their ability to form expectations — and the possibility of biased or non-equilibrium beliefs — has been long recognized in economics, at least since the work of Simon (1958, 1959). However, in most fields in economics, the status quo has been to assume rational expectations. In empirical industrial organization (IO), some of the most commonly used structural models of oligopoly competition assume complete information, perfect certainty, and Nash equilibrium. For instance, this is the case in models of price competition with differentiated product (Berry, Levinsohn, and Pakes, 1995; Berry and Haile, 2014), and in empirical games of market entry (Bresnahan and Reiss, 1991; Ciliberto and Tamer, 2009). Though there is a substantial literature on structural models of incomplete information in empirical IO, it is mostly concentrated in auctions (Guerre, Perrigne, and Vuong, 2000; Athey and Haile, 2002), and in discrete choice games, both static (Seim, 2006; Sweeting, 2009; Bajari et al., 2010), and dynamic (Aguirregabiria and Mira, 2007; Igami, 2017). Empirical applications to models of quantity or price competition are not so common, though Armantier and Richard (2003) and Aryal and Zicenko (2019) are good exceptions.

Most empirical applications of games of incomplete information assume that firms have homogeneous beliefs that correspond to a Bayesian Nash equilibrium. Nevertheless, recent papers in structural IO relax equilibrium assumptions and present evidence of substantial heterogeneity and biases in firms’ beliefs. As one would expect, biased beliefs are more likely in new markets and after regulatory changes: for instance, after deregulation of the US telecommunication industry (Goldfarb and Xiao, 2011), the UK electricity market (Doraszelski, Lewis, and Pakes, 2018), the

Texas electricity spot market (Hortaçsu and Puller, 2008; and Hortaçsu, Luco, Puller, and Zhu, 2019), or the Washington State liquor market (Huang, Ellickson, and Lovett, 2018), and in the early years of the fast-food restaurant industry in UK (Aguirregabiria and Magesan, 2020). All these papers use a *revealed preference & beliefs approach* to identify the structural parameters in costs and demand together with firms' subjective beliefs. That is, a firm's observed behavior reveals information about its preferences (i.e., the structure of its profit function) and about its beliefs.<sup>1</sup>

Identification of beliefs using a *revealed preference & beliefs (RP&B)* approach requires restrictions either on profit or on beliefs functions. The papers mentioned above use different restrictions to identify subjective beliefs. In this paper, I present a systematic analysis of the joint identification of firms' beliefs and structural parameters in a general class of empirical games of market competition. The analysis introduces minimum restrictions on preferences (profits) and beliefs which are nonparametric functions of firms' actions and state variables. I investigate the identification of beliefs under very weak restrictions.

I present a framework where firms have incomplete information and their beliefs on competitors' behavior are unrestricted (nonparametric) probability functions on the space of competitors' actions and conditional on observable state variables. Beliefs may be out of equilibrium. Revenue and cost functions are also nonparametrically specified. The framework applies both to continuous and discrete choice games and includes as particular cases models of competition in prices or quantities, auction models, entry games, and dynamic investment games. I focus on identification results that exploit a natural exclusion restriction in models of competition: an observable variable that affects a firm's cost (or revenue) but does not have a direct effect on other firms' profits. Examples of this type of variable are firm-specific input prices, total factor productivity, or firm-specific state variables such as capital stock or the firm's incumbency status. I show the identification power of this exclusion restriction under several scenarios on the data available to the researcher: data only on firms' choices and state variables, as well as applications where the researcher has price and quantity data to identify the revenue function or the cost function. Identification conditions vary substantially when the model is static or dynamic, so I study separately these two cases.

This paper relates to several literatures in empirical IO and econometrics. As mentioned above,

---

<sup>1</sup>Using observed behavior to identify agents' subjective beliefs was already proposed by Frank Ramsey in his article titled "Truth and Probability" (Ramsey, 1926). Ramsey argues that probability is related to the knowledge that each individual possesses, leading to the notion of subjective probability or beliefs. Then, he explains how subjective beliefs can be inferred using observed behavior. On pages 174-175 in his article, Ramsey uses a simple example to illustrate the identification of beliefs from observed actions.

it belongs to a growing literature in empirical IO on structural models of market competition where firms have biased beliefs (see the references above).

In the econometrics of games, several papers have studied identification of models that allow for biased beliefs but impose other restrictions, such as *Level-k rationality* (An, 2017), *Cognitive Hierarchy* (Brown, Camerer, and Lovo, 2013), or *Rationalizability* (Aradillas-Lopez and Tamer, 2008; Kline, 2018). In this paper, I do not impose these restrictions and show that they are testable. An, Hu, and Xiao (2018) show the identification subjective expectations in single-agent dynamic structural models using a RP&B approach. The exclusion restriction that these authors use is quite different to the one that I present in this paper.

In experimental economics, a good number of papers have estimated agents' beliefs in games played in laboratory experiments, and have investigated *strategic uncertainty* (Van Huyck, Battalio, and Beil, 1990; Heinemann, Nagel, and Ockenfels, 2009). In Aguirregabiria and Xie (2020), we propose a simple to implement experimental design that generates an exclusion restriction that identifies agents' beliefs in a nonparametric model where agents can have of other-regarding preferences.

The identification of subjective beliefs in games also relates to the identification of structural games with multiple equilibria in the data. Recent contributions include De Paula and Tang (2012), Otsu, Pesendorfer, and Takahashi (2016), and Aguirregabiria and Mira (2019).

Finally, as I describe in section 3, the identification of firms' beliefs on competitors' strategies relates to the traditional IO literature on identification of the *nature of competition*, pioneered by Bresnahan (1982). In contrast to the traditional approach, the model in this paper acknowledges that firms' beliefs are endogenous objects that depend on all the state variables affecting demand and costs. I show that the exclusion restrictions typically used to identify the form of competition (or the so-called *conjectural variation parameters*) cannot identify firms' beliefs.

I have organized the rest of the paper as follows. Sections 2 and 3 present general frameworks – static and dynamic, respectively – that include as particular cases most models of competition in empirical applications in IO. Section 4 presents our main results on the identification of firms' beliefs using a RP&B approach. I summarize and conclude in section 5.

## 2 Static games

### 2.1 Basic framework

Consider  $N$  firms competing in a market. Firms are indexed by  $i$ . The profit function of firm  $i$  is  $\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$  where  $a_i \in \mathcal{A}$  is the action of firm  $i$ ,  $\mathbf{a}_{-i} = (a_j : j \neq i) \in \mathcal{A}^{N-1}$  is the vector with the actions of the other firms. The decision variable –  $a_i$  – can be either discrete or continuous, and it can represent – among other possibilities – a firm’s level of output, its price, the binary indicator of entry in a market, the firm’s number of stores, or its investment in R&D. The vector  $\mathbf{x} \in \mathcal{X}$  represents variables that are known by all the firms. The term  $\varepsilon_i \in \mathbb{R}$  is private information of firm  $i$ . For instance, this private information can be a component of the firm’s cost, or a private signal about the state of the demand. I denote  $\varepsilon_i$  as the ‘type’ of firm  $i$ . Firms’ types  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N)$  are drawn from a distribution with cumulative distribution function  $F(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N \mid \mathbf{x})$ . I use  $F_i(\varepsilon_i \mid \mathbf{x})$  to represent the distribution function of  $\varepsilon_i$ . The primitives of the game are  $\Pi$ ’s,  $\mathcal{A}$ ,  $\mathcal{X}$ , and  $F$ .

Firms simultaneously decide their actions to maximize their respective expected profits. Under the standard solution concept of Bayesian Nash Equilibrium (BNE), the primitives of the model are assumed common knowledge – that is, every firm knows that every firm knows ... these primitives. The model that I consider here does not impose this restriction. This model only assumes that each firm knows its own profit function, the vector of variables  $\mathbf{x}$ , and its private information  $\varepsilon_i$ . For instance, some firms may not know the distribution function  $F$  or the profit functions of other firms. Furthermore, the fact that  $\mathbf{x}$  is known to all the firms might not be common knowledge.

A firm does not know the private information of its competitors and therefore it does not know their actions. Firms form probabilistic beliefs about the actions of competitors. Let  $b_i(\mathbf{a}_{-i} \mid \varepsilon_i, \mathbf{x})$  be a probability density function that represents firm  $i$ ’s beliefs. This is a probability function in the space of the actions of firms other than  $i$  and conditional on firm  $i$ ’s information. We use  $B_i(\mathbf{a}_{-i} \mid \varepsilon_i, \mathbf{x})$  to denote the distribution function associated with the density  $b_i(\mathbf{a}_{-i} \mid \varepsilon_i, \mathbf{x})$ . Given its beliefs, a firm’s expected profit is:

$$\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}, b_i) = \int_{\mathbf{a}_{-i}} \Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) b_i(\mathbf{a}_{-i} \mid \varepsilon_i, \mathbf{x}) d\mathbf{a}_{-i}, \quad (1)$$

The integral is over the Lebesgue measure on  $\mathcal{A}^{N-1}$ , which can be either continuous or discrete. A firm chooses its strategy function  $\sigma_i(\varepsilon_i, \mathbf{x}, b_i)$  to maximize expected profits:

$$\sigma_i(\varepsilon_i, \mathbf{x}, b_i) = \arg \max_{a_i \in \mathcal{A}} \Pi_i^e(a_i, \varepsilon_i, \mathbf{x}, b_i). \quad (2)$$

It is convenient to represent a firm's strategy as a cumulative distribution function. Let  $p_i(a_i|\mathbf{x})$  be the probability density function of choice variable  $a_i$  conditional on  $\mathbf{x}$ , that is, the so called *conditional choice probability (CCP)* function. We use  $P_i(a_i|\mathbf{x})$  to denote the cumulative distribution function associated to  $p_i$ . According to the model, this distribution comes from firm  $i$ 's best response and satisfies the following equation. For any value  $a^0 \in \mathcal{A}$ ,

$$P_i(a^0|\mathbf{x}) \equiv \Pr(\sigma_i(\varepsilon_i, \mathbf{x}, b_i) \leq a^0 | \mathbf{x}; b_i) = \int 1\{\sigma_i(\varepsilon_i, \mathbf{x}, b_i) \leq a^0\} dF_i(\varepsilon_i|\mathbf{x}) \quad (3)$$

In this framework, the payoff functions  $\{\Pi_i\}$  and the distribution of private signals,  $F(\cdot|\mathbf{x})$ , are primitives of the model. The predictions of the model are the choice probabilities. The belief functions  $\{b_i\}$  are endogenous outcomes of the model. However, the model is incomplete in the sense that it does not specifies how these beliefs are determined. Instead, it specifies a general framework that includes as particular cases many different models for the determination of beliefs such as Bayesian Nash equilibrium, Cognitive Hierarchy models, and many others.

## 2.2 Main assumptions

I focus on models where a firm's action  $a_i$  is a single variable that can be either continuous or discrete. If the decision is continuous, then  $\mathcal{A} = \mathbb{R}$ . If the decision is discrete, then it is ordered and  $\mathcal{A} = \{0, 1, \dots, J\}$ , e.g., number of products, stores, etc.<sup>2</sup> The framework imposes some restrictions on the marginal profit function to guarantee that a firm's best response function is strictly monotonic in  $\varepsilon_i$ . For the rest of this subsection, I present assumptions such that: (i) these marginal conditions of optimality are necessary and sufficient such that they fully characterize a firm's optimal best response function; and (ii) we can obtain a simple characterization of the optimal decision rule using the *cumulative choice probability function*  $P_i(a_i|\mathbf{x})$ .

Let  $\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$  be the *marginal profit function*. Here the concept of *marginal profit* is broad and depends on the decision variables  $a_i$ . If the decision variable is continuous – such as output or price – the marginal profit is at the *intensive margin* and it corresponds to the mathematical concept of partial derivative:  $\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) \equiv \partial\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})/\partial a_i$ . If the decision variable is discrete – such as entry decision, number of stores, or number of products – the marginal profit is at the *extensive margin* and it corresponds to a difference function:  $\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) \equiv \Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) - \Pi_i(a_i - 1, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$ .

<sup>2</sup>Note that any binary choice (e.g., a market entry decision) is a particular case of ordered discrete choice variable.

The following assumption provides sufficient conditions for the monotonicity of a firm's best response function with respect to its type  $\varepsilon_i$ .

*ASSUMPTION 1. For any value of  $(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$ , the marginal profit function  $\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$  is: (A) additively separable in the private information  $\varepsilon_i$  such that*

$$\Delta\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) = \Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) - \varepsilon_i \quad ; \quad (4)$$

*(B)  $\Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x})$  is strictly decreasing in the own action  $a_i$ . For discrete choice models, and without loss of generality, I adopt the notational convention of  $\Delta\pi_i(0, \mathbf{a}_{-i}, \mathbf{x}) = +\infty$  and  $\Delta\pi_i(J+1, \mathbf{a}_{-i}, \mathbf{x}) = -\infty$ . ■*

The strict monotonicity of the marginal profit with respect to the own action – Assumption 1(B) – holds in most models of market competition. For models of competition in price or quantity, a downward sloping demand curve and a non-decreasing marginal costs are sufficient conditions for the monotonicity of the marginal profit with respect to the own action.

The additivity of the private information in assumption 1(A) is not without loss of generality. The following example presents a simple model to illustrate some restrictions imposed by this assumption.

*EXAMPLE 1.* Consider a model of Cournot competition in an homogeneous product industry. Variable  $a_i \in \mathbb{R}_+$  represents firm  $i$ 's amount of output. The inverse demand function is linear:  $p = \alpha - \beta \sum_{j=1}^N a_j$ . Parameters  $\alpha$  and  $\beta$  are the true demand parameters. Firms do not know these parameters. Instead, each firm receives a private and independent signal  $(\alpha_i, \beta_i)$  about the value of these parameters. The cost function of firm  $i$  is:  $\gamma_i a_i + \delta_i a_i^2$ , where  $\gamma_i$  and  $\delta_i$  are private information of this firm. Therefore, the profit function is  $\Pi_i = (\alpha_i - \beta_i \sum_{j=1}^N a_j) a_i - \gamma_i a_i - \delta_i a_i^2$ , and the marginal profit function is:

$$\Delta\Pi_i = \alpha_i - \beta_i \sum_{j=1}^N a_j - \beta_i a_i - \gamma_i - \delta_i a_i \quad (5)$$

This expression shows that the private information variables  $\alpha_i$  and  $\gamma_i$  enter additively in the marginal profit. However, this is not the case for the private information variables  $\beta_i$  and  $\delta_i$  which interact with the own output and with the output of other firms. Therefore, for this model,



Assumption 1(A) restricts private information to enter in the intercept of the demand curve ( $\alpha_i$ ) or/and in the linear term of cost function ( $\gamma_i$ ). It does not allow private information in the slope of the demand ( $\beta_i$ ) or in the quadratic term of the cost function ( $\delta_i$ ). ■

For all the identification results in this paper, Assumption 1(A) can be replaced with the weaker assumption that  $\Delta\Pi_i = \Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) - g_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) \varepsilon_i$  where function  $g_i(a_i, \mathbf{a}_{-i}, \mathbf{x})$  is positive valued and it is known to the researcher. For notational simplicity, I use Assumption 1(A) instead of this weaker version.<sup>3</sup>

Assumption 2 establishes the restriction that firms' types are independently distributed – *independent private values (IPV)*. This restriction is stronger than what we need to characterize best response functions and to obtain the identification results in this paper. However, it is convenient because it facilitates the derivations and proofs.

*ASSUMPTION 2. (A) The private information variables  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N)$  are independently distributed conditional on  $\mathbf{x}$ :  $F(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N | \mathbf{x}) = \prod_{i=1}^N F_i(\varepsilon_i | \mathbf{x})$ . (B) Every firm  $i$  knows that other firms' private information is independent of its own private information such that  $\mathbf{a}_{-i}$  and  $\varepsilon_i$  are independent conditional on  $\mathbf{x}$ . ■*

Assumption 2 implies that  $\varepsilon_i$  is not an argument of the beliefs function  $b_i(\mathbf{a}_{-i} | \varepsilon_i, \mathbf{x})$ . Firms have beliefs that are consistent with this independence: every firm is aware that – conditional on  $\mathbf{x}$  – the actions of the other firms are independent of its own type. For the rest of the paper, I represent a belief function as  $b_i(\mathbf{a}_{-i} | \mathbf{x})$ .

*ASSUMPTION 3. The private information  $\varepsilon_i$  is a continuous random variable with support the real line and with a cumulative distribution function conditional on  $\mathbf{x}$  –  $F_i(\varepsilon_i | \mathbf{x})$  – that is strictly increasing over the whole real line. ■*

---

<sup>3</sup>In a recent paper, Allen and Rehbeck (2019) show that the assumption of additive separability of unobservables provides identification of preferences and counterfactual behavior in a very general class of single-agent static decision models.

### 2.3 Characterization of firms' best response functions

Define the expected marginal profit (without including the private component  $\varepsilon_i$ ):

$$\Delta\pi_i^e(a_i, \mathbf{x}; b_i) \equiv \int_{\mathbf{a}_{-i}} \Delta\pi_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) b_i(\mathbf{a}_{-i}|\mathbf{x}) d\mathbf{a}_{-i}. \quad (6)$$

Consider the marginal condition of optimality. If  $a_i$  is firm  $i$ 's optimal choice, then: (i) for models with continuous decision variable,  $\Delta\pi_i^e(a_i, x; b_i) - \varepsilon_i = 0$ ; and (ii) for models with discrete decision variable,  $\Delta\pi_i^e(a_i + 1, x; b_i) < \varepsilon_i \leq \Delta\pi_i^e(a_i, x; b_i)$ , where – remember – I have adopted the convention that  $\Delta\pi_i(0, \mathbf{a}_{-i}, \mathbf{x}) = +\infty$  and  $\Delta\pi_i(J + 1, \mathbf{a}_{-i}, \mathbf{x}) = -\infty$ . Let  $\sigma_i(\varepsilon_i, x, b_i)$  be firm  $i$ 's best response function.

The following Proposition 1 establishes that the marginal condition is necessary and sufficient. This proposition also characterizes a firm's best response function in terms of the cumulative choice probability function.

*PROPOSITION 1. (A) Under Assumptions 1 and 2, the marginal condition of optimality is a necessary and sufficient condition for firm  $i$ 's best response, i.e.,  $\sigma_i(\varepsilon_i, \mathbf{x}, b_i) = a_i$  iff the marginal condition holds for value  $a_i$ . (B) Under Assumptions 1 to 3, for any value  $a^0$  in the choice set  $\mathcal{A}$ , we have the following relationship between the cumulative choice probability function  $P_i(a^0|\mathbf{x})$  and the expected marginal profit:*

$$P_i(a^0|\mathbf{x}) = F_i [\Delta\pi_i^e(a^0, \mathbf{x}; b_i) | \mathbf{x}]. \quad (7)$$

*This relationship is invertible such that we have:*

$$Q_i(a^0|\mathbf{x}) = \Delta\pi_i^e(a^0, \mathbf{x}; b_i), \quad (8)$$

*where  $Q_i(a^0|\mathbf{x}) \equiv F_i^{-1} [P_i(a^0|\mathbf{x})]$  and  $F_i^{-1}$  is the inverse of  $F_i$  (quantile function). ■*

The following examples illustrate this characterization of best response functions using four standard models of competition: Cournot, auctions, market entry, and number of stores, respectively.

*EXAMPLE 2 (Cournot competition).* Consider a Cournot game of quantity choice in an homogeneous product industry. Let  $a_i \in \mathbb{R}_+$  be firm  $i$ 's amount of output. The market demand function

is  $d(Q, \mathbf{x})$  where  $Q = \sum_{i=1}^N a_i$ , and a firm's marginal cost function is  $c_i(a_i, \mathbf{x}) + \varepsilon_i$ . The expected marginal profit function is  $\Delta\pi_i^e(a_i, \mathbf{x}; b_i) - \varepsilon_i$  where:

$$\Delta\pi_i^e(a_i, \mathbf{x}; b_i) = -c_i(a_i, \mathbf{x}) + \int_{\mathbf{a}_{-i}} a_i d\left(\sum_{j \neq i} a_j, \mathbf{x}\right) b_i(\mathbf{a}_{-i}|\mathbf{x}) d\mathbf{a}_{-i}. \quad \blacksquare \quad (9)$$

*EXAMPLE 3 (Market entry game).* Consider a game of market entry. Let  $a_i \in \{0, 1\}$  be the indicator of the event "firm  $i$  is active in the market". A firm's profit if not active in the market is zero,  $\Pi_i(0, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) = 0$ . If active in the market, a firm's profit is equal to the variable profit  $v_i(\mathbf{a}_{-i}, \mathbf{x})$  - i.e., revenue minus variable cost - minus the entry cost  $ec_i(\mathbf{x}) + \varepsilon_i$ . The marginal profit is the profit if active minus the profit if not active, which in this case is zero. Then, the expected marginal profit function is:

$$\Delta\pi_i^e(1, \mathbf{x}; b_i) - \varepsilon_i = -ec_i(\mathbf{x}) - \varepsilon_i + \sum_{\mathbf{a}_{-i}} v_i(\mathbf{a}_{-i}, \mathbf{x}) b_i(\mathbf{a}_{-i}|\mathbf{x}). \quad \blacksquare \quad (10)$$

*EXAMPLE 4 (Competition in number of stores or products).* Consider a competition game where  $a_i \in \{0, 1, \dots, J\}$  represents the number of products that the firm has in the market. Similarly as in the market entry game, a firm's profit is equal to the variable profit  $v_i(a_i, \mathbf{a}_{-i}, \mathbf{x})$  minus the fixed cost  $fc_i(a_i, \mathbf{x}) + a_i \varepsilon_i$ . In this case, the private information  $\varepsilon_i$  is associated to the increase in the fixed cost from managing one more product. Then, the expected marginal profit function is:

$$\Delta\pi_i^e(a_i, \mathbf{x}; b_i) - \varepsilon_i = -\Delta fc_i(a_i, \mathbf{x}) - \varepsilon_i + \sum_{\mathbf{a}_{-i}} \Delta v_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) b_i(\mathbf{a}_{-i}|\mathbf{x}), \quad (11)$$

with  $\Delta fc_i(a_i, \mathbf{x}) \equiv fc_i(a_i, \mathbf{x}) - fc_i(a_i - 1, \mathbf{x})$ , and  $\Delta v_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) \equiv v_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) - v_i(a_i - 1, \mathbf{a}_{-i}, \mathbf{x})$ . \blacksquare

*EXAMPLE 5 (Procurement auction).* This example is slightly different to the previous ones because the private information variable  $\varepsilon_i$  is not additive in the marginal condition of optimality. I use this example to illustrate how this condition is not necessary to obtain the characterization of the best response function that I use for the identification results in this paper. Consider a procurement auction where  $a_i \in \mathbb{R}$  represents firm  $i$ 's bid. The profit function is  $\Pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) = (a_i - c_i(\mathbf{x}) - \varepsilon_i) 1\{a_j > a_i \forall j \neq i\}$ , where  $c_i(\mathbf{x}) + \varepsilon_i$  is the cost and  $1\{\cdot\}$  is the indicator function such that  $1\{a_j > a_i \forall j \neq i\}$  is the indicator of the event "firm  $i$  has the lowest bid and wins the auction". The expected

profit function is:

$$\begin{aligned}\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) &= (a_i - c_i(\mathbf{x}) - \varepsilon_i) \int_{\mathbf{a}_{-i}} 1\{a_j > a_i \forall j \neq i\} b_i(\mathbf{a}_{-i}|\mathbf{x}) d\mathbf{a}_{-i} \\ &= (a_i - c_i(\mathbf{x}) - \varepsilon_i) W(a_i, \mathbf{x}, b_i)\end{aligned}\tag{12}$$

where  $W(a_i, \mathbf{x}, b_i) \equiv \int_{\mathbf{a}_{-i}} 1\{a_j > a_i \forall j \neq i\} b_i(\mathbf{a}_{-i}|\mathbf{x}) d\mathbf{a}_{-i}$  is firm  $i$ 's subjective probability of winning the auction given its beliefs. Therefore, expected marginal profit function is:

$$\Delta\pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) = W(a_i, \mathbf{x}, b_i) + (a_i - c_i(\mathbf{x}) - \varepsilon_i) \Delta W(a_i, \mathbf{x}, b_i),\tag{13}$$

where  $\Delta W(a_i, \mathbf{x}, b_i)$  represents the derivative of the subjective probability of winning with respect to the own bid  $a_i$ . Note that this expected marginal profit function is not additively separable in  $\varepsilon_i$ . However,  $\Delta\pi_i^e$  is strictly monotonic in  $a_i$  and  $\varepsilon_i$ , and this implies that the best response function is strictly monotonic in  $\varepsilon_i$ . More specifically, we have that  $\sigma_i(\varepsilon_i, \mathbf{x}, b_i) = a^0$  if and only if  $\varepsilon_i = a^0 - c_i(\mathbf{x}) + \frac{W(a^0, \mathbf{x}, b_i)}{\Delta W(a^0, \mathbf{x}, b_i)}$ . This implies that we can represent the best response function using the following formula for the conditional quantile function:

$$Q_i(a^0|\mathbf{x}) = a^0 - c_i(\mathbf{x}) + \frac{W(a^0, \mathbf{x}, b_i)}{\Delta W(a^0, \mathbf{x}, b_i)}. \quad \blacksquare\tag{14}$$

## 2.4 Common equilibrium restrictions in empirical applications

The framework presented above includes as particular cases most games of competition with incomplete information in empirical IO applications. A main difference is that most empirical applications have assumed that firms' beliefs satisfy some equilibrium restrictions. Different equilibrium concepts have been used in the literature. I present here the equilibrium concepts that have received more attention in empirical applications in IO.

All these equilibrium concepts assume that firms choose their best response strategies given their beliefs: that is, they impose the best response conditions described above. In addition, they restrict beliefs to satisfy some additional equilibrium restrictions. I describe below these additional restrictions.

(a) *Bayesian Nash Equilibrium (BNE) with independent private values.* This is the most commonly used solution concept in games of incomplete information in IO. It has received particular attention in auction games (e.g., Guerre, Perrigne, and Vuong, 2000; Athey and Haile, 2002), and in discrete choice games (e.g., Seim, 2006; Sweeting, 2009). It has been used also in empirical applications

of Cournot competition models with incomplete information (Armantier and Richard, 2003; Aryal and Zicenko, 2019).

A BNE can be described as  $N$  cumulative choice probability functions –  $\{P_i(a_i|\mathbf{x}) : i = 1, 2, \dots, N\}$  – satisfying the following conditions: (i) [best responses]  $P_i(a_i|\mathbf{x})$  satisfies the best response equation (7) given beliefs  $b_i$ ; and (ii) [rational beliefs] the cumulative beliefs function  $B_i$  is equal to the actual distribution function of the choices of the other firms conditional on  $\mathbf{x}$ :

$$B_i(\mathbf{a}_{-i}|\mathbf{x}) = \prod_{j \neq i} P_j(a_j|\mathbf{x}) \quad (15)$$

(b) *Cognitive Hierarchy (CH) and Level-K models.* These models propose equilibrium concepts where firms have biased beliefs, that is,  $B_i(\mathbf{a}_{-i} | \mathbf{x}) \neq \prod_{j \neq i} P_j(a_j|\mathbf{x})$ . They are based on the following ideas. Firms are heterogeneous in their beliefs and there is a finite number of belief types. That is,  $B_i(\mathbf{a}_{-i}|\mathbf{x})$  belongs to a finite family of  $K$  belief functions,  $\{B^{(k)}(\mathbf{a}_{-i}|\mathbf{x}) : k = 1, 2, \dots, K\}$ . Each of member of this family is a ‘belief type’. Belief types correspond to different levels of strategic sophistication and are determined by a hierarchical structure.

A firm type-0 has arbitrary believes  $B^{(0)}$ . In the Level-k model, a type-k firm believes that all the other firms are type-(k-1). Therefore, a type-k firm has beliefs:

$$B^{(k)}(\mathbf{a}_{-i}|\mathbf{x}) = \prod_{j \neq i} F_j \left( \Delta \pi_j^e(a_j, \mathbf{x}; B^{(k-1)}) \mid \mathbf{x} \right) \quad (16)$$

This recursive equation defines the belief functions for every type  $k$  between 1 and  $K$ . Note that the only unrestricted function is the beliefs function for type-0: the rest of the belief functions are known functions of  $B^{(0)}$  and the primitives of the model.

The CH model is more flexible than the Level-k model. In the CH model, a type-k firm believes that the other firms come from a probability distribution over types 0 to (k-1). This model has been estimated in IO applications in Goldfarb and Xiao (2011), Brown, Camerer, and Lovo (2013), and Hortaçsu, Luco, Puller, and Zhu (2019).

These models allow for some flexibility in beliefs. However, they still impose important restrictions. More specifically, they do not include BNE or rational beliefs as a particular case, and there is a small number of belief types –  $K$  is smaller than  $N$ , and typically 2 or 3 in actual applications.

(c) *Rationalizability* (Bernheim, 1984; Pearce, 1984). The concept of rationalizability imposes two simple restrictions on firms’ beliefs and behavior. First, every firm maximizes its own expected

profit given beliefs. And second, this rationality is common knowledge, i.e., every firm knows that every firm knows ... that all the firms are rational. The set of outcomes of the game that satisfy these conditions – the set of rationalizable outcomes – includes all the Bayesian Nash equilibria (BNE) of the game but also outcomes where players have biased beliefs. More specifically, in a game with multiple equilibria, each firm has beliefs that are consistent with a BNE, but these beliefs may not correspond to the same BNE: that is, a firm believes that they are playing equilibrium  $A$  and other firm believes that the equilibrium played is  $B$ .

In general, the set of rationalizable beliefs is larger than the set of  $BNE$  but substantially smaller than the set of all the possible beliefs. Therefore, the condition of common knowledge rationality imposes non trivial restrictions with respect to the model that I consider in this paper. I show that these additional restrictions are testable.

(d) *Correlated Bayesian Nash Equilibrium.* In recent work, Bergemann and Morris (2013, 2016) have introduced the solution concept of Bayesian Correlated Equilibrium (BCE). This solution is more robust than BNE, in the sense that it delivers all predictions compatible with BNE for any information structure within a wide class. Magnolfi and Roncoroni (2017) study inference based on the BCE solution concept. Their goal is to identify payoff parameters but they do not study the identification of beliefs. Their work illustrates a trade-off between robustness to assumptions about information structures and the ability to achieve point identification.

### 3 Dynamic games

In this section, I extend the previous framework to a dynamic game. Time is discrete and indexed by  $t$ . Now,  $\pi_{it}(a_{it}, \mathbf{a}_{-it}, \varepsilon_{it}, \mathbf{x}_t)$  represents the profit function of firm  $i$  at period  $t$ . The arguments of this function have the same interpretation as in section 2. Firms choose their actions at every period  $t$  to maximize their expected and discounted profits  $\mathbb{E}_t(\sum_{s=0}^{T-t} \beta_i^s \pi_{it+s})$ , where  $\beta_i$  is the firm's discount factor and  $T$  is the time horizon that can be finite or infinite.

I introduce an additional assumption that, in dynamic structural models, is typically described as the *conditional independence assumption*.

*ASSUMPTION 4.* (A) The vector of state variables  $\mathbf{x}_t$  follows a controlled Markov process with transition probability density function  $f_{xt}(\mathbf{x}_{t+1} | a_{it}, \mathbf{a}_{-it}, \mathbf{x}_t)$ . (B) The private information vari-

able  $\varepsilon_{it}$  is independently and identically distributed over time. ■

Assumption 4(B) implies that the private information variables are not serially correlated. This assumption is not innocuous. It rules out the possibility of firms using the history of other firms' decisions to learn about these firms' type. This type of learning is the focus of the Experience-Based equilibrium concept in Fershtman and Pakes (2012).

Every period, firms choose simultaneously their actions to maximize their intertemporal values. A firm's value at period  $t$  depends on the actions of other firms at that period and in the future. Firms form probabilistic beliefs about the actions of competitors, now and in the future. Let  $b_{it+s}^{(t)}(\mathbf{a}_{-i,t+s} \mid \mathbf{x}_{t+s})$  be a probability density function, in the space of the actions of firms other than  $i$ , that represents the beliefs of firm  $i$  at period  $t$  about the behavior of other players at period  $t+s$ . This representation of beliefs allows for general forms of beliefs updating. According to this notation,  $b_{it+s}^{(t+1)} - b_{it+s}^{(t)}$  represents the change (or update) from period  $t$  to period  $t+1$  in the beliefs that firm  $i$  has about the behavior of competitors at period  $t+s$ .

Given a firm's beliefs at period  $t$ ,  $b_i^{(t)} \equiv \{b_{i,t+s}^{(t)} : s \geq 0\}$ , its best response at period  $t$  is the solution of a single-agent dynamic programming (DP) problem. We can represent this DP problem using Bellman's principle. Let  $V_{it}^{\mathbf{b}_i^{(t)}}(\mathbf{x}_t, \varepsilon_{it})$  be the value function. Then,

$$V_{it}^{\mathbf{b}_i^{(t)}}(\mathbf{x}_t, \varepsilon_{it}) = \max_{a_{it} \in \mathcal{A}} \left\{ \int_{\mathbf{a}_{-it}} \left[ \Pi_{it}(a_{it}, \mathbf{a}_{-it}, \varepsilon_{it}, \mathbf{x}_t) + v_{it}^{\mathbf{b}_i^{(t)}}(a_{it}, \mathbf{a}_{-it}, \mathbf{x}_t) \right] b_{it}^{(t)}(\mathbf{a}_{-it} \mid \mathbf{x}_t) d\mathbf{a}_{-it} \right\} \quad (17)$$

where  $v_{it}^{\mathbf{b}_i^{(t)}}(a_{it}, \mathbf{a}_{-it}, \mathbf{x}_t)$  is the continuation value that has the following expression:

$$\beta \int V_{it+1}^{\mathbf{b}_i^{(t)}}(\mathbf{x}_{t+1}, \varepsilon_{it+1}) f_{xt}(\mathbf{x}_{t+1} \mid a_{it}, \mathbf{a}_{-it}, \mathbf{x}_t) d\mathbf{x}_{t+1} dF_i(\varepsilon_{it+1} \mid \mathbf{x}_t) \quad (18)$$

Given its beliefs, a firm chooses its strategy to maximize its value. That is, the best response strategy function,  $\sigma_{it}(\varepsilon_{it}, \mathbf{x}_t, b_i^{(t)})$ , is the maximand of the term in brackets  $\{.\}$  in equation (17). As in the static game, it is convenient to represent a firm's strategy as a cumulative distribution function, or as a quantile function.

*PROPOSITION 2.* *In the dynamic game, under Assumptions 1 to 4, for any value  $a^0$  in the choice set  $\mathcal{A}$ , we have the following relationship between the cumulative choice probability function  $P_{it}(a^0 \mid \mathbf{x}_t)$  and the marginal expected intertemporal profit:*

$$P_{it}(a^0 \mid \mathbf{x}_t) = F_i \left[ \Delta \pi_{it}^e(a^0, \mathbf{x}_t; b_{it}^{(t)}) + \Delta v_{it}^{e, \mathbf{b}_i^{(t)}}(a^0, \mathbf{x}_t; b_{it}^{(t)}) \mid \mathbf{x}_t \right], \quad (19)$$

where  $\Delta\pi_{it}^e(a^0, \mathbf{x}_t; b_{it}^{(t)}) \equiv \int_{\mathbf{a}_{-it}} \Delta\pi_{it}(a_{it}, \mathbf{a}_{-it}, \mathbf{x}_t) b_{it}^{(t)}(\mathbf{a}_{-it}|\mathbf{x}_t) d\mathbf{a}_{-it}$ , and  $\Delta v_{it}^{e, \mathbf{b}_i^{(t)}}(a^0, \mathbf{x}_t; b_{it}^{(t)}) \equiv \int_{\mathbf{a}_{-it}} \Delta v_{it}^{\mathbf{b}_i^{(t)}}(a_{it}, \mathbf{a}_{-it}, \mathbf{x}_t) b_{it}^{(t)}(\mathbf{a}_{-it}|\mathbf{x}_t) d\mathbf{a}_{-it}$ . This relationship is invertible such that we have:

$$Q_{it}(a^0|\mathbf{x}_t) = \Delta\pi_{it}^e(a^0, \mathbf{x}_t; b_{it}^{(t)}) + \Delta v_{it}^{e, \mathbf{b}_i^{(t)}}(a^0, \mathbf{x}_t; b_{it}^{(t)}), \quad (20)$$

where  $Q_{it}(a^0|\mathbf{x}_t) \equiv F_i^{-1}[P_{it}(a^0|\mathbf{x}_t)]$ . ■

In this framework, the sequence of beliefs  $b_i^{(t)} = \{b_{i,t+s}^{(t)} : s \geq 0\}$  is completely unrestricted. This framework contains as particular cases most solution concepts in dynamics games of competition with incomplete information. I present here some common cases.

(a) *Markov Perfect Equilibrium (MPE)*. This is the most commonly used solution concept in applications of dynamic games in empirical IO (Maskin and Tirole, 1988; Ericson and Pakes, 1995). Here we consider a version of MPE that allows for non-stationarity due to finite time horizon  $T$  or/and primitive functions  $\pi_{it}$  and  $f_{xt}$  that vary over time.

A MPE can be described as  $N$  sequences of cumulative choice probability functions –  $\{P_{it}(a_{it}|\mathbf{x}_t) : i = 1, 2, \dots, N; t \geq 1\}$  – satisfying the following conditions: (i) [best responses]  $P_{it}(a_{it}|\mathbf{x}_t)$  satisfies the best response condition given beliefs  $b_i^{(t)}$ ; and (ii) [rational beliefs] beliefs  $b_i^{(t)}$  are equal to the actual probability distribution of the choices of the other firms: for any  $t \geq 1, s \geq 0, \mathbf{a}_{-i} \in \mathcal{A}^{N-1}$ , and  $\mathbf{x} \in \mathcal{X}$ ,

$$b_{i,t+s}^{(t)}(\mathbf{a}_{-i}|\mathbf{x}) = \prod_{j \neq i} p_{j,t+s}(a_j|\mathbf{x}) \quad (21)$$

(b) *Dynamic equilibrium with Learning*. Bayesian, Adaptive, or other forms of learning. All these equilibrium concepts impose some restrictions on beliefs: both on the heterogeneity of firms' beliefs and on the evolution of beliefs over time. They are restricted versions of the general model presented above.



## 4 Identification

### 4.1 Data

This section presents results on the identification of beliefs in the previous general framework. I distinguish three scenarios for the data available to the researcher which are common in empirical IO applications.

**(a) Only firms' choice data.** The researcher has a sample of  $M$  local markets, indexed by  $m$ , where she observes firms' actions and state variables:  $\{a_{imt}, \mathbf{x}_{mt} : i = 1, 2, \dots, N; t = 1, 2, \dots, T^{data}\}$ . In empirical applications of market entry models, it is often the case that the researcher has only choice data – e.g., firms' entry/exit decision – and there is no direct information on firms' revenues or costs.

**(b) Choice data + revenue function.** In addition to data on firms' choices, the researcher may have data on some components of the profit function. In many IO applications, the researcher observes prices and quantities and can estimate the demand system. Given the demand system, the researcher knows the revenue function.

**(c) Choice data + revenue function + cost function.** Data on firms' marginal costs is rare but it is sometimes available (Hortaçsu and Puller, 2008; Hortaçsu et al., 2019). Marginal costs can be also obtained from the estimation of a production function if the dataset contains information on firms' output and input quantities, and input prices.

To incorporate in our framework the data that the researcher has on the revenue or cost functions, we distinguish these two components in the profit function. A firm's profit is equal to revenue minus cost:  $\pi_i = r_i - c_i$ . Accordingly, the marginal profit is equal to the marginal revenue minus the marginal cost:  $\Delta\pi_i = \Delta r_i - \Delta c_i$ . The economic interpretation of this marginal revenue and marginal cost depends on the particular decision variable of the model that can be continuous – e.g., quantity, price, investment – or discrete – e.g., entry, number of products.

For the identification analysis below, I consider that the researcher has a random sample with infinite markets:  $M \rightarrow \infty$ . This population level approach is standard in the literature on identifi-

ation. Given this infinite sample, the cumulative choice probability function  $P_i(a^0|\mathbf{x}^0)$  is identified for every firm  $i$  and at every point  $(a^0, \mathbf{x}^0)$  in the support  $\mathcal{A} \times \mathcal{X}$ . More precisely, by definition we have that  $P_i(a^0|\mathbf{x}^0) = \mathbb{E}(1\{a_{im} \leq a^0\} | \mathbf{x}_m = \mathbf{x}^0)$ , and the expectation  $\mathbb{E}(1\{a_{im} \leq a^0\} | \mathbf{x}_m = \mathbf{x}^0)$  is identified from our sample. For the rest of this section, I treat  $P_i$  as a known function.

## 4.2 The identification problem

Consider the static game. By Proposition 1, we have that:

$$P_i(a^0|\mathbf{x}) = F_i(\Delta r_i^e(a^0, \mathbf{x}, b_i) - \Delta c_i^e(a^0, \mathbf{x}, b_i) | \mathbf{x}) \quad (22)$$

where  $\Delta r_i^e(a^0, \mathbf{x}, b_i) \equiv \int \Delta r_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) b_i(\mathbf{a}_{-i}|\mathbf{x}) d\mathbf{a}_{-i}$  and  $\Delta c_i^e(a^0, \mathbf{x}, b_i) \equiv \int \Delta c_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) b_i(\mathbf{a}_{-i}|\mathbf{x}) d\mathbf{a}_{-i}$  are the (subjective) expected marginal revenue and expected marginal cost, respectively. Equation (22) summarizes all the restrictions that the model imposes on the distribution function  $P_i$ . The left-hand-side of this equation – the distribution  $P_i$  – is known to the researcher. The right-hand-side depends on the model primitives – the structural functions  $F_i$ ,  $\Delta r_i$ , and  $\Delta c_i$  – and on beliefs  $b_i$ . The model is fully (point) identified if the system of equations in (22) implies unique values for structural functions and beliefs.

It is clear that the model is strongly under-identified. While the number of restrictions – the dimension of the distribution  $P_i$  – is  $(|\mathcal{A}| - 1) |\mathcal{X}|$ , we have that only the dimension of the beliefs function  $b_i$  is  $(|\mathcal{A}|^{N-1} - 1) |\mathcal{X}|$  which is obviously larger than the number of restrictions. When  $F_i$ ,  $\Delta r_i$ , and  $\Delta c_i$  are unknown to the researcher, the under-identification is stronger. Despite the under-identification of the model, I show below that it possible to identify a function that only depends on beliefs. The identification of this "beliefs object" can be used to test the validity of different equilibrium concepts and restrictions on beliefs.

For the sake of simplicity, I first illustrate the identification results in a simple model of competition: a binary choice game with two firms. Furthermore, I assume that the probability distribution  $F_i$  is known to the researcher. Later, in section 4.4, I show that extend this identification result extends to models with: (i) multinomial and continuous choices; (ii) nonparametric specification  $F_i$ ; (iii) more than two players; and (iv) dynamic games.

## 4.3 Two-firms binary choice game

Consider a binary choice game of price competition between two firms:  $a_i = 0$  represents the choice of the low price (promotion price) and  $a_i = 1$  means the choice of high price (regular price).

Let  $q_i = d_i(a_i, a_{-i}, \mathbf{x})$  be the demand function for the product of firm  $i$ . and let  $C_i(q_i, \mathbf{x})$  be the cost as a function output. Therefore, using our notation, the revenue function is  $r_i(a_i, a_{-i}, \mathbf{x}) \equiv a_i d_i(a_i, a_{-i}, \mathbf{x})$ , and the cost function is  $c_i(a_i, a_{-i}, \mathbf{x}) = C_i(d_i(a_i, a_{-i}, \mathbf{x}), \mathbf{x})$ .<sup>4</sup>

Let  $P_i(0|\mathbf{x})$  – or in short  $P_i(\mathbf{x})$  – be the probability that firm  $i$  chooses the low price. And let  $b_i(0|\mathbf{x})$  – or in short  $b_i(\mathbf{x})$  – be this firm’s belief about the probability that the competitor chooses the low price. The marginal profit function is  $\Delta\pi_i(a_{-i}, \mathbf{x}) \equiv \pi_i(1, a_{-i}, \mathbf{x}) - \pi_i(0, a_{-i}, \mathbf{x})$ , that is, the difference between the profit with high price and with low price. Marginal profit is equal to marginal revenue minus marginal cost:  $\Delta\pi_i(a_{-i}, \mathbf{x}) = \Delta r_i(a_{-i}, \mathbf{x}) - \Delta c_i(a_{-i}, \mathbf{x})$ .

As established in Proposition 1, the best response equation can be represented as:

$$Q_i(\mathbf{x}) = \Delta\pi_i(0, \mathbf{x}) + b_i(\mathbf{x}) [\Delta\pi_i(1, \mathbf{x}) - \Delta\pi_i(0, \mathbf{x})] \quad (23)$$

with  $Q_i(\mathbf{x}) \equiv F_i^{-1}(P_i(\mathbf{x}))$ . Given this system of equations – for every value of  $\mathbf{x}$  – we are interested in the identification of marginal profits  $\Delta\pi_i(0, \mathbf{x})$  and  $\Delta\pi_i(1, \mathbf{x})$ , and the beliefs function  $b_i(\mathbf{x})$ . We are particularly interested in the identification of the beliefs function  $b_i(\mathbf{x})$ , or at least on the identification of an object or parameter that only depends on this belief function.

Without further restrictions, the model is under-identified. More specifically, the order condition for identification does not hold: for each value of  $\mathbf{x}$ , there is only one restriction – i.e., one value of  $Q_i(\mathbf{x})$  – but three unknowns,  $\Delta\pi_i(0, \mathbf{x})$ ,  $\Delta\pi_i(1, \mathbf{x})$ , and  $b_i(\mathbf{x})$ . Section 4.3.1 describes the identification of beliefs when the researcher has revenue and cost data. In sections 4.3.2 and 4.3.3, I present an exclusion restriction that identifies firms’ beliefs even with only choice data.

### 4.3.1 Identification with revenue and cost data

Suppose that the researcher knows the revenue function and the cost function: the dataset includes information on prices and quantities of inputs and outputs that can be used to identify demand and cost functions. This implies that the marginal profits  $\Delta\pi_i(0, \mathbf{x})$  and  $\Delta\pi_i(1, \mathbf{x})$  are known to the researcher. Therefore, under the condition that  $\Delta\pi_i(1, \mathbf{x}) - \Delta\pi_i(0, \mathbf{x}) \neq 0$ , equation (23) implies that the beliefs function is fully identified:

$$b_i(\mathbf{x}) = \frac{Q_i(\mathbf{x}) - \Delta\pi_i(0, \mathbf{x})}{\Delta\pi_i(1, \mathbf{x}) - \Delta\pi_i(0, \mathbf{x})} \quad (24)$$

---

<sup>4</sup>Even if a firm’s cost depends only on its own output, the cost as a function of prices depends both on the own price and competitors’ prices. This is simply because the quantity produced and sold by a firm depends on all the prices. In contrast, in a Cournot game, where the decision variable  $a_i$  represents a firm’s output, the cost function  $c_i(a_i, a_{-i}, \mathbf{x})$  does not depend on  $a_{-i}$ .

The identification condition  $\Delta\pi_i(a_{-i} = 1, \mathbf{x}) - \Delta\pi_i(a_{-i} = 0, \mathbf{x}) \neq 0$  is quite intuitive. A firm's observed behavior reveals information about the firm's beliefs only if beliefs have an effect on behavior, and this is the case only if other firms' actions affect the firm's profit, i.e., only if  $\Delta\pi_i(a_{-i} = 1, \mathbf{x}) - \Delta\pi_i(a_{-i} = 0, \mathbf{x}) \neq 0$ .

Given the identification of firms' beliefs, the researcher can test the validity of restrictions on beliefs with economic relevance.

(a) *Testing for unbiased beliefs.* We say that firm  $i$  has unbiased beliefs about the behavior of the other firm if  $b_i(\mathbf{x}) - p_{-i}(\mathbf{x}) = 0$  for every value of  $\mathbf{x}$ . Given the identification of  $b_i(\mathbf{x})$  and that  $p_{-i}(\mathbf{x})$  is known to the researcher, we can test the null hypothesis of a firm's unbiased beliefs.

(b) *Testing for BNE.* The concept of BNE imposes the restrictions that all the firms play best responses and have unbiased beliefs. Therefore, testing the null hypothesis of BNE is equivalent to test the joint restrictions  $b_1(\mathbf{x}) - p_2(\mathbf{x}) = 0$  and  $b_2(\mathbf{x}) - p_1(\mathbf{x}) = 0$  for every value of  $\mathbf{x}$ .

(c) *Testing for Rationalizability.* Given that the researcher knows firms' profit functions, she can construct the set of rationalizable beliefs, and then test if the identified beliefs –  $b_1(\mathbf{x})$  and  $b_2(\mathbf{x})$  – belong to this set. To construct the set of rationalizable beliefs we can use a simple iterative procedure as in Aradillas-Lopez and Tamer (2008). This iterative procedure exploits the property that the best response probability function  $F_i(\Delta\pi_i(0, \mathbf{x}) + b_i(\mathbf{x}) [\Delta\pi_i(1, \mathbf{x}) - \Delta\pi_i(0, \mathbf{x})])$  is strictly monotonic in the beliefs function  $b_i(\mathbf{x})$ . Suppose, without loss of generality, that  $\Delta\pi_i(1, \mathbf{x}) - \Delta\pi_i(0, \mathbf{x}) > 0$ . At iteration  $k$ , the set of level- $k$  rationalizable beliefs for firm 1 is  $[L_1^{(k)}(\mathbf{x}), U_1^{(1)}(\mathbf{x})]$  with

$$\begin{cases} L_1^{(k)}(\mathbf{x}) &= F_2(\Delta\pi_2(0, \mathbf{x}) + L_2^{(k-1)}(\mathbf{x}) [\Delta\pi_2(1, \mathbf{x}) - \Delta\pi_2(0, \mathbf{x})]) \\ U_1^{(k)}(\mathbf{x}) &= F_2(\Delta\pi_2(0, \mathbf{x}) + U_2^{(k-1)}(\mathbf{x}) [\Delta\pi_2(1, \mathbf{x}) - \Delta\pi_2(0, \mathbf{x})]) \end{cases} \quad (25)$$

And we have the symmetric expression for the set of level- $k$  rationalizable beliefs for firm 2.

### 4.3.2 Identification with revenue but not cost data

Suppose that the researcher has data that identifies the demand system and therefore the revenue function  $r_i(a_i, a_{-i}, \mathbf{x})$ . The cost function is unknown. The best response equation is:

$$Q_i(\mathbf{x}) = \Delta r_i(0, \mathbf{x}) - \Delta c_i(0, \mathbf{x}) + b_i(\mathbf{x}) [\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x}) + \Delta c_i(1, \mathbf{x}) - \Delta c_i(0, \mathbf{x})] \quad (26)$$

where  $Q_i(\mathbf{x})$ ,  $\Delta r_i(0, \mathbf{x})$ , and  $\Delta r_i(1, \mathbf{x})$  are known, and  $b_i(\mathbf{x})$ ,  $\Delta c_i(1, \mathbf{x})$ , and  $\Delta c_i(0, \mathbf{x})$  are unknown. This restriction cannot identify beliefs and cost functions. For any possible value of  $b_i(\mathbf{x})$ , there exist values of the marginal costs  $\Delta c_i(1, \mathbf{x})$  and  $\Delta c_i(0, \mathbf{x})$  such that the best response equation holds. Therefore, there exist infinite combinations of  $(b_i(\mathbf{x}), \Delta c_i(0, \mathbf{x}), \Delta c_i(1, \mathbf{x}))$  that satisfy the best response equation (26).

This identification problem is closely related to the traditional identification problem of the *nature of competition*, or the identification of collusive behavior (Bresnahan, 1982). Almost any observed behavior can be justified as one with "non-collusive beliefs" if we select the appropriate marginal cost function. The following Example describes connection in detail.

*EXAMPLE 6 (Firms' beliefs and conjectural variations).* In an influential paper, Bresnahan (1982) studies the identification of the *form (or nature) of competition* in a model with complete information. In a complete information game, the nature of competition can be described as a conjectural variation (CV) parameter. This CV parameter has similarities with our beliefs function, but there are also substantial differences between them. Our beliefs function is an endogenous object that varies with all the exogenous characteristics in the vector  $\mathbf{x}$  affecting demand and costs. CV parameters are typically interpreted as exogenously given and do not vary when demand or costs change. As I explain below, this has important implications on the identification of beliefs relative to the identification of CV parameters.

The best response equation in Bresnahan (1982) is similar as equation (26) but replacing the beliefs function  $b_i(\mathbf{x})$  with a parameter  $CV_i$  that is assumed invariant with  $\mathbf{x}$ . After the identification of the demand and marginal revenue functions, Bresnahan proposes an exclusion restriction that implies the identification of the parameter  $CV_i$ . I first describe this identification result using our notation, and then I show this assumption cannot provide identification of beliefs in our model of incomplete information.

*Bresnahan's identification of the nature of competition (CV).* Suppose that the vector of exogenous variables  $\mathbf{x}$  has two components  $(\tilde{\mathbf{x}}, z)$  where  $z$  is a variable that satisfies two conditions: (Bresnahan-*i*)  $z$  affects the marginal revenue function, or more precisely the function  $\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x})$  – that is, variation in  $z$  "rotates" the demand curve such that marginal revenue changes; and (Bresnahan-*ii*)  $z$  does not enter in the marginal cost function.

Consider the best response equation (26) but where the beliefs function  $b_i(\mathbf{x})$  is replaced with a

constant parameter  $CV_i$ . Let  $z^1$  and  $z^2$  be two different values of the special variable that "rotates" the demand curve. Consider the best response equation evaluated at two different points,  $(\tilde{\mathbf{x}}, z^1)$  and  $(\tilde{\mathbf{x}}, z^2)$ , and obtain the difference between these two equations. We get:

$$\begin{aligned} Q_i(\tilde{\mathbf{x}}, z^1) - Q_i(\tilde{\mathbf{x}}, z^2) &= \Delta r_i(0, \tilde{\mathbf{x}}, z^1) - \Delta r_i(0, \tilde{\mathbf{x}}, z^2) \\ &+ CV_i [\Delta r_i(1, \tilde{\mathbf{x}}, z^1) - \Delta r_i(0, \tilde{\mathbf{x}}, z^1) - \Delta r_i(1, \tilde{\mathbf{x}}, z^2) + \Delta r_i(0, \tilde{\mathbf{x}}, z^2)] \end{aligned} \quad (27)$$

Everything in this equation except parameter  $CV_i$  is known to the researcher. Furthermore, the identification assumption (ii) above implies that  $\Delta r_i(1, \tilde{\mathbf{x}}, z^1) - \Delta r_i(0, \tilde{\mathbf{x}}, z^1) - \Delta r_i(1, \tilde{\mathbf{x}}, z^2) + \Delta r_i(0, \tilde{\mathbf{x}}, z^2)$  is different to zero. Therefore, we can solve for  $CV_i$  to identify this parameter.

*This exclusion restriction does not work for the identification of beliefs.* In our model, the beliefs function  $b_i(\mathbf{x})$  is an endogenous object that depends on all the exogenous variables affecting demand or costs. Therefore, under Bresnahan's identification assumptions (i) and (ii), we have that right-hand-side of equation (27) becomes:

$$\begin{aligned} &\Delta r_i(0, \tilde{\mathbf{x}}, z^1) - \Delta r_i(0, \tilde{\mathbf{x}}, z^2) + [b_i(\tilde{\mathbf{x}}, z^1) - b_i(\tilde{\mathbf{x}}, z^2)] [\Delta c_i(1, \tilde{\mathbf{x}}) - \Delta c_i(0, \tilde{\mathbf{x}})] \\ &+ b_i(\tilde{\mathbf{x}}, z^1) [\Delta r_i(1, \tilde{\mathbf{x}}, z^1) - \Delta r_i(0, \tilde{\mathbf{x}}, z^1)] - b_i(\tilde{\mathbf{x}}, z^2) [\Delta r_i(1, \tilde{\mathbf{x}}, z^2) - \Delta r_i(0, \tilde{\mathbf{x}}, z^2)] \end{aligned} \quad (28)$$

This expression depends both on beliefs and costs and it cannot be used to separately identify one from the other. ■

***A useful exclusion restriction: firm-specific cost shifters*** The rest of this subsection presents an exclusion restriction that provides identification of firms' beliefs. Suppose that the vector  $\mathbf{x}$  contains a firm-specific variable that affects the marginal cost of a firm but not the marginal cost of its competitors. For instance, input prices – wages, prices of intermediate inputs – can have firm specific variation because long-term contracts, bargaining, internal labor markets, etc. Assumption 5 describes this condition more formally.

*ASSUMPTION 5.* Vector  $\mathbf{x}$  has the following elements  $(\tilde{\mathbf{x}}, z_1, z_2, \dots, z_N)$  where  $\tilde{\mathbf{x}}$  can affect the marginal revenues and marginal costs of all the firms in an unrestricted way, and each variable  $z_i$  is firm-specific and satisfies the following conditions: (A) firm  $i$ 's marginal cost (or/and marginal revenue) depends on  $z_i$ ; and (B) firm  $i$ 's marginal cost and marginal revenue do not depend on  $\mathbf{z}_{-i} \equiv \{z_j : j \neq i\}$ . ■

Under Assumption 5, it is possible to identify firm  $i$ 's beliefs using this firm's change in behavior when  $\mathbf{z}_{-i}$  varies. Proposition 3 establishes formally this result.

*PROPOSITION 3.* Let  $z_{-i}^1$ ,  $z_{-i}^2$ , and  $z_{-i}^3$  be three different values for the variable  $z_{-i}$ . Then, under Assumptions 1-3 and 5, the following equation holds:

$$\frac{b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^2)}{b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^3)} = \frac{Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^2)}{Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^3)} \quad (29)$$

such that an object that depends only on beliefs (left hand side) is identified using the firm's observed behavior (right hand side). ■

The term  $Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^2)$  captures the change in the behavior of firm  $i$  when  $z_{-i}$  varies from  $z_{-i}^1$  to  $z_{-i}^2$ : that is, the change in the probability that firm  $i$  charges a low price when the competitor's wage rate changes. Since variable  $z_{-i}$  does not affect firm  $i$ 's marginal revenue or marginal cost – Assumption 5(B) – the observed change in the pricing behavior in firm  $i$  should be because a change in its beliefs. The difference between the best-response equation at points  $(\tilde{\mathbf{x}}, z_i, z_{-i}^1)$  and  $(\tilde{\mathbf{x}}, z_i, z_{-i}^2)$  is:

$$Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^2) = [b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^2)] [\Delta\pi_i(1, \tilde{\mathbf{x}}, z_i) - \Delta\pi_i(0, \tilde{\mathbf{x}}, z_i)] \quad (30)$$

This difference is not sufficient to identify the beliefs parameter  $b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^2)$ . The reason, is that  $\Delta\pi_i(1, \tilde{\mathbf{x}}, z_i) - \Delta\pi_i(0, \tilde{\mathbf{x}}, z_i)$  depends on unknown marginal costs through the term  $\Delta c_i(1, \tilde{\mathbf{x}}, z_i) - \Delta c_i(0, \tilde{\mathbf{x}}, z_i)$ . However, we can also obtain the difference between the best-response equation at points  $(\tilde{\mathbf{x}}, z_i, z_{-i}^1)$  and  $(\tilde{\mathbf{x}}, z_i, z_{-i}^3)$  to get a similar equation as (30) but for  $Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^3)$ . Note that the term  $\Delta\pi_i(1, \tilde{\mathbf{x}}, z_i) - \Delta\pi_i(0, \tilde{\mathbf{x}}, z_i)$  is common between these equations. Therefore, we can cancel this unknown common term by obtaining the ratio between these two difference equations.

Equation (29) shows that the observed variation in the pricing behavior of firm  $i$  – when the competitor's input prices change – reveals information about this firm's beliefs. We can separate beliefs from the primitives in the profit function.

In some models, the cost function of a firm does not depend on the action of other firms. For instance, this is the case in Cournot models of quantity competition or in the entry games because, in these models, the cost function  $c_i$  is a "pure" cost function and not the composition of the true cost

function and the demand function. In these models, we have that  $\Delta c_i(1, \tilde{\mathbf{x}}, z_i) - \Delta c_i(0, \tilde{\mathbf{x}}, z_i) = 0$  such that  $\Delta \pi_i(1, \tilde{\mathbf{x}}, z_i) - \Delta \pi_i(0, \tilde{\mathbf{x}}, z_i) = \Delta r_i(1, \tilde{\mathbf{x}}) - \Delta r_i(0, \tilde{\mathbf{x}})$  that is known to the researcher. Therefore, we can identify the following beliefs parameter:

$$b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - b_i(\tilde{\mathbf{x}}, z_i, z_{-i}^2) = \frac{Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^1) - Q_i(\tilde{\mathbf{x}}, z_i, z_{-i}^2)}{\Delta r_i(1, \tilde{\mathbf{x}}) - \Delta r_i(0, \tilde{\mathbf{x}})} \quad (31)$$

Given the identification of these beliefs objects, we can implement tests for the null hypotheses of unbiased beliefs and BNE in a similar way as I have described above at the end of section 4.3.1.

### 4.3.3 Identification using only firms' choice data

The previous exclusion restriction can be applied to the identification of beliefs also when the researcher has not identified the revenue function. Proposition 3 applies also to this case. Similarly, we can use the identified beliefs parameters to test the null hypotheses of unbiased beliefs and BNE.

## 4.4 Extensions

Under Assumption 5 and the condition that  $F_i$  is known to the researcher, the identification result in Proposition 3 extends to models where the decision variable is (ordered) multinomial or continuous choices as long as the support of the the state variable  $z_i$  has at least as many points as the decision variable  $z_i$ . The proof is a bit more technical than Proposition 3 because it requires to represent the model in vector form and to show that a linear function of beliefs can be written as linear projection of the vector of quantiles. See Proposition 1 in Aguirregabiria and Magesan (2020).

For the other extensions, for simplicity, I consider the two-player binary choice game.

### 4.4.1 Identification in dynamic game

We can apply Proposition 2 to the two-player dynamic binary choice game to obtain the following expression for the quantile best response condition:

$$\begin{aligned} Q_{it}(\mathbf{x}_t) &= \Delta r_{it}(0, \mathbf{x}_t) - \Delta c_{it}(0, \mathbf{x}_t) + b_{it}^{(t)}(\mathbf{x}_t) [\Delta r_{it}(1, \mathbf{x}_t) - \Delta r_{it}(0, \mathbf{x}_t) - \Delta c_{it}(0, \mathbf{x}_t) + \Delta c_{it}(1, \mathbf{x}_t)] \\ &+ \Delta v_{it}^{\mathbf{b}_i^{(t)}}(0, \mathbf{x}_t) + b_{it}^{(t)}(\mathbf{x}_t) \left[ \Delta v_{it}^{\mathbf{b}_i^{(t)}}(1, \mathbf{x}_t) - \Delta v_{it}^{\mathbf{b}_i^{(t)}}(0, \mathbf{x}_t) \right] \end{aligned} \quad (32)$$

In this equation, the first line is identical to the static game, and the second line contains the continuation value and therefore the dynamics of the game.

Unfortunately, the exclusion in Assumption 5 is not sufficient for the identification of beliefs in the dynamic game. Without further assumptions, variable  $z_{-it}$  is a state variable that affects the



(marginal) continuation values of the two firms, even if this variable satisfies Assumption 5 such that it does not have a different effect on the marginal profit of firm  $i$  at period  $t$ . For instance, suppose that  $z_{-it}$  is the wage rate paid by firm  $-i$  – the competitor of firm  $i$ . Though the value of the competitors’ wage rate at period  $t$  does not have a direct effect on the contemporaneous marginal profit of firm  $i$  – once we account for the expected decision of the competitor – it can affect the continuation value as it affects future wages.

Assumption 5\* extends assumption 5 such that we can get identification of beliefs in dynamic games.

*ASSUMPTION 5\**. Vector  $\mathbf{x}_t$  has the following elements  $(\tilde{\mathbf{x}}_t, z_{1t}, z_{2t}, \dots, z_{Nt})$  which satisfy conditions (A) and (B) in Assumption 5. In addition, they also satisfy (C) the transition probability of the state variable  $z_{it}$  is such that  $z_{it+1}$  does not depend on  $(z_{it}, \mathbf{z}_{-it})$  once we condition on  $a_{it}$  and  $\tilde{\mathbf{x}}_t$ , i.e.,

$$\Pr(z_{it+1} \mid a_{it}, \tilde{\mathbf{x}}_t, z_{it}, \mathbf{z}_{-it}) = \Pr(z_{it+1} \mid a_{it}, \tilde{\mathbf{x}}_t). \quad \blacksquare \quad (33)$$

Assumption 5\*(C) holds in many applications of dynamic games in empirical IO. The incumbent status, capacity, capital stock, or product quality of a firm at period  $t - 1$  are state variables that enter in the firm’s payoff function at period  $t$  due to investment and adjustment costs. A firm’s payoff function at period  $t$  depends also on the competitors’ values of these variables at period  $t$ , but it does not depend on the competitors’ values of these variables at  $t - 1$ . Very importantly, Assumption 5\*(C) does not mean that firm  $i$  does not condition his behavior on the state variables  $\mathbf{z}_{-it}$ . Each firm conditions his behavior on all the state variables that affect the profit of a firm in the game, even if these variables are excluded from his own payoff.

More specifically, an important class of dynamic games that satisfies Assumption 5\*(C) consists of models where the transition rule for this state variable is  $z_{i,t+1} = a_{it}$ . It is clear that this transition rule is a particular case of equation (33). This is an important class of dynamic games that includes as particular cases games of market entry/exit, technology adoption, pricing with menu costs, and some dynamic games of quality or capacity competition, among others.

Of course, Assumption 5\*(C) rules out many candidates for exclusion restrictions that do satisfy conditions Assumption 5(A) and Assumption 5(B). As I have mentioned above, this is the case of

the competitor's input prices because they typically follow Markov processes where  $z_{i,t+1}$  depends on  $z_{it}$ .

The key implication of Assumption 5\*(C) is that the marginal continuation values  $\Delta v_{it}^{\mathbf{b}_i^{(t)}}(a_{-it}, \mathbf{x}_t)$  – which are defined as conditional on the competitor's decision  $a_{-it}$  – do not depend on the state variable  $z_{-it}$ : that is,  $\Delta v_{it}^{\mathbf{b}_i^{(t)}}(a_{-it}, \tilde{\mathbf{x}}_t, z_{it}, z_{-it}) = \Delta v_{it}^{\mathbf{b}_i^{(t)}}(a_{-it}, \tilde{\mathbf{x}}_t)$ . This state variable affects future expected continuation values  $\Delta v_{it}^{e, \mathbf{b}_i^{(t)}}(\mathbf{x}_t; b_{it}^{(t)})$  only through the firm's beliefs: that is,  $\Delta v_{it}^{e, \mathbf{b}_i^{(t)}}(\mathbf{x}_t; b_{it}^{(t)}) = \Delta v_{it}^{\mathbf{b}_i^{(t)}}(0, \tilde{\mathbf{x}}_t) + b_{it}^{(t)}(\tilde{\mathbf{x}}_t, z_{it}, z_{-it}) [\Delta v_{it}^{\mathbf{b}_i^{(t)}}(1, \tilde{\mathbf{x}}_t) - \Delta v_{it}^{\mathbf{b}_i^{(t)}}(0, \tilde{\mathbf{x}}_t)]$ . This property implies the identification of beliefs.

*PROPOSITION 4. Consider the two-player binary choice dynamic game under Assumptions 1-4 and 5\*. Then, the following equation holds:*

$$\frac{b_{it}(\tilde{\mathbf{x}}_t, z_{it}, z_{-it}^1) - b_{it}(\tilde{\mathbf{x}}_t, z_{it}, z_{-it}^2)}{b_{it}(\tilde{\mathbf{x}}_t, z_{it}, z_{-it}^1) - b_{it}(\tilde{\mathbf{x}}_t, z_{it}, z_{-it}^3)} = \frac{Q_{it}(\tilde{\mathbf{x}}_t, z_{it}, z_{-it}^1) - Q_{it}(\tilde{\mathbf{x}}_t, z_{it}, z_{-it}^2)}{Q_{it}(\tilde{\mathbf{x}}_t, z_{it}, z_{-it}^1) - Q_{it}(\tilde{\mathbf{x}}_t, z_{it}, z_{-it}^3)} \quad (34)$$

such that an object that depends only on beliefs (left hand side) is identified using the firm's observed behavior (right hand side). ■

Note that Proposition 4 does not impose any restriction on the evolution of beliefs over time. Therefore, the result is robust to very general forms of firms' learning. On the negative side, Proposition 4 establishes the identification of firms' beliefs about competitors contemporaneous behavior: that is, beliefs at period  $t$  about the opponents' behavior at period  $t$ . We cannot identify beliefs about the opponent's behavior in the future. However, Aguirregabiria and Magesan (2019) show that the identification of the evolution of these contemporaneous beliefs is enough for testing a very general class of models of learning and beliefs formation.

#### 4.4.2 Identification of beliefs with unknown distribution of private information

When the state variables  $z_i$  have continuous support, beliefs can be identified even when the researcher does not know the distribution  $F_i$ , that is nonparametrically specified.

*PROPOSITION 5. Consider the static binary choice game under Assumptions 1-3 and 5. Suppose that: (i) the distribution  $F_i$  is independent of  $z_i$  and  $z_{-i}$  but it may depend on  $\tilde{\mathbf{x}}$ ; (ii)  $z_i$  and  $z_{-i}$  are continuous random variables; (iii)  $P_i(\tilde{\mathbf{x}}, z_i, z_{-i})$  is strictly monotonic in  $z_i$  and  $z_{-i}$  and asymmetric*

in these two arguments, i.e., for  $z \neq z'$ , generically,  $P_i(\tilde{\mathbf{x}}, z, z') \neq P_i(\tilde{\mathbf{x}}, z', z)$ ; (iv) the researcher knows the revenue function; and (v) firm  $i$ 's marginal cost does not depend on  $a_{-i}$ . Let  $(z_i^A, z_{-i}^A)$  and  $(z_i^B, z_{-i}^B)$  be two arbitrary values of  $(z_i, z_{-i})$ . Then, there exist unique values  $z_{-i}^{AB*}$  and  $z_{-i}^{BA*}$  – which can be identified by the researcher – such that:  $z_{-i}^{AB*} \neq z_{-i}^{BA*}$ ;  $P_i(\tilde{\mathbf{x}}, z_i^A, z_{-i}^A) = P_i(\tilde{\mathbf{x}}, z_i^B, z_{-i}^{AB*})$ ;  $P_i(\tilde{\mathbf{x}}, z_i^B, z_{-i}^B) = P_i(\tilde{\mathbf{x}}, z_i^A, z_{-i}^{BA*})$ ; and the following condition holds:

$$\frac{b_i(\tilde{\mathbf{x}}, z_i^A, z_{-i}^A) - b_i(\tilde{\mathbf{x}}, z_i^A, z_{-i}^{BA*})}{b_i(\tilde{\mathbf{x}}, z_i^B, z_{-i}^B) - b_i(\tilde{\mathbf{x}}, z_i^B, z_{-i}^{AB*})} = - \frac{\Delta r_i(1, \tilde{\mathbf{x}}, z_i^A) - \Delta r_i(0, \tilde{\mathbf{x}}, z_i^A)}{\Delta r_i(1, \tilde{\mathbf{x}}, z_i^B) - \Delta r_i(0, \tilde{\mathbf{x}}, z_i^B)} \quad (35)$$

such that an object that depends only on beliefs (left hand side) is identified using the firm's observed behavior and revenue function. ■

Condition (iv) – firm  $i$ 's marginal cost does not depend on the other firm's actions – applies to Cournot, market entry, auctions, and many other models. Under this condition, the best response quantile equation becomes:

$$Q_i(\mathbf{x}) = \Delta r_i(0, \mathbf{x}) - \Delta c_i(0, \mathbf{x}) + b_i(\mathbf{x}) [\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x})] \quad (36)$$

Under conditions (ii) and (iii) in Proposition 5, for any value  $p$  in the interior of the image set of function  $P_i(\tilde{\mathbf{x}}, z_i^B, \cdot)$ , we can find a unique value  $z_{-i}^*$  that solves equation  $p = P_i(\tilde{\mathbf{x}}, z_i^B, z_{-i})$  with respect to  $z_{-i}$ . Given that the function  $P_i(\cdot)$  is identified, this value  $z_{-i}^*$  is also identified. Define  $z_{-i}^{AB*}$  as the unique value of  $z_{-i}$  that solves the equation  $P_i(\tilde{\mathbf{x}}, z_i^A, z_{-i}^A) = P_i(\tilde{\mathbf{x}}, z_i^B, z_{-i})$  with respect to  $z_{-i}$ , and similarly, define  $z_{-i}^{BA*}$  as the unique value of  $z_{-i}$  that solves the equation  $P_i(\tilde{\mathbf{x}}, z_i^B, z_{-i}^B) = P_i(\tilde{\mathbf{x}}, z_i^A, z_{-i})$  with respect to  $z_{-i}$ . Given the asymmetry of function  $P_i$  with respect to  $z_i$  and  $z_{-i}$  – condition (iii) – we have that  $z_{-i}^{AB*}$  and  $z_{-i}^{BA*}$  are two different values.

Since  $P_i(\tilde{\mathbf{x}}, z_i^A, z_{-i}^A) = P_i(\tilde{\mathbf{x}}, z_i^B, z_{-i}^{AB*})$  and the distribution function  $F_i$  is independent of  $(z_i, z_{-i})$  – condition (i) – the quantile values  $Q_i(\tilde{\mathbf{x}}, z_i^A, z_{-i}^A)$  and  $Q_i(\tilde{\mathbf{x}}, z_i^B, z_{-i}^{AB*})$  are also the same. Applying this condition to the best response quantile equation, we have that (omitting  $\tilde{\mathbf{x}}$  as an argument for notational simplicity):

$$\begin{aligned} 0 &= Q_i(z_i^A, z_{-i}^A) - Q_i(z_i^B, z_{-i}^{AB*}) \\ &= \{ \Delta r_i(0, z_i^A) - \Delta c_i(0, z_i^A) + b_i(z_i^A, z_{-i}^A) [\Delta r_i(1, z_i^A) - \Delta r_i(0, z_i^A)] \} \\ &\quad - \{ \Delta r_i(0, z_i^B) - \Delta c_i(0, z_i^B) + b_i(z_i^B, z_{-i}^{AB*}) [\Delta r_i(1, z_i^B) - \Delta r_i(0, z_i^B)] \} \end{aligned} \quad (37)$$

Similarly, we obtain the condition:

$$\begin{aligned}
0 &= Q_i(z_i^B, z_{-i}^B) - Q_i(z_i^A, z_{-i}^{BA*}) \\
&= \{ \Delta r_i(0, z_i^B) - \Delta c_i(0, z_i^B) + b_i(z_i^B, z_{-i}^B) [\Delta r_i(1, z_i^B) - \Delta r_i(0, z_i^B)] \} \\
&\quad - \{ \Delta r_i(0, z_i^A) - \Delta c_i(0, z_i^A) + b_i(z_i^A, z_{-i}^{BA*}) [\Delta r_i(1, z_i^A) - \Delta r_i(0, z_i^A)] \}
\end{aligned} \tag{38}$$

Adding up equations (37) and (38), we obtain the condition  $0 = [b_i(z_i^B, z_{-i}^B) - b_i(z_i^B, z_{-i}^{AB*})] [\Delta r_i(1, z_i^B) - \Delta r_i(0, z_i^B)] + [b_i(z_i^A, z_{-i}^A) - b_i(z_i^A, z_{-i}^{BA*})] [\Delta r_i(1, z_i^A) - \Delta r_i(0, z_i^A)]$ , and this condition implies equation (35) in Proposition 5.

## 5 Conclusions

Firms face substantial uncertainty about competitors' strategies. The assumption of complete information may be convenient in some empirical applications, but it ignores sources of strategic uncertainty that can be important to understand the observed variation in behavior between firms, across markets, and over time. Furthermore, in changing economic environments – e.g., new regulations, mergers, pandemics – the assumption that firms have unbiased beliefs can be very unrealistic and affect our estimates of structural parameters. Perhaps most importantly, heterogeneity in firms' beliefs can be an important source of misallocation and inefficiency in some industries.

In this paper, I have presented a framework where firms have incomplete information, face strategic uncertainty, and may have biased beliefs. I have shown that standard exclusion restrictions – which are present in most applications in empirical IO – provide identification of objects that depend only of firms' beliefs. The approach imposes minimal restrictions on beliefs and on structural functions. As such, the identified beliefs can be used to explore the determinants of firms' beliefs and their form of learning.

## References

- [1] Aguirregabiria, V. and A. Magesan (2019). Strategic uncertainty and players' learning in dynamic games. Manuscript. Department of Economics, University of Calgary.
- [2] Aguirregabiria, V. and A. Magesan (2020). Identification and estimation of dynamic games when players' beliefs are not in equilibrium. *The Review of Economic Studies*, 87(2), 582-625.
- [3] Aguirregabiria, V., and P. Mira (2007). Sequential estimation of dynamic discrete games. *Econometrica*, 75(1), 1-53.
- [4] Aguirregabiria, V., and Mira, P. (2019). Identification of games of incomplete information with multiple equilibria and unobserved heterogeneity. *Quantitative Economics*, 10(4), 1659-1701.
- [5] Aguirregabiria, V., and E. Xie (2020). Identification of non-equilibrium beliefs in games of incomplete information using experimental data. *Journal of Econometric Methods*. DOI <https://doi.org/10.1515/jem-2019-0029>
- [6] Allen, R., and J. Rehbeck (2019). Identification with additively separable heterogeneity. *Econometrica*, 87(3), 1021-1054.
- [7] An, Y. (2017). Identification of first-price auctions with non-equilibrium beliefs: A measurement error approach. *Journal of Econometrics*, 200, 326-343.
- [8] An, Y., Y. Hu, R. Xiao (2018). Dynamic decisions under subjective expectations: A structural analysis. Manuscript. Department of Economics. Johns Hopkins University.
- [9] Aradillas-Lopez, A., and E. Tamer (2008). The identification power of equilibrium in simple games. *Journal of Business & Economic Statistics*, 26, 261-283.
- [10] Armantier, O., and O. Richard (2003). Exchanges of cost information in the airline industry. *RAND Journal of Economics*, 461-477.
- [11] Aryal, G., and F. Zicenکو (2019). Empirical framework for Cournot oligopoly with private information. Manuscript. Department of Economics. University of Pittsburgh.
- [12] Asker, J., C. Fershtman, C., J. Jeon, and A. Pakes (2016). The competitive effects of information sharing. NBER Working Paper, No. 22836. National Bureau of Economic Research.
- [13] Athey, S. (2001). Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica*, 69(4), 861-889.
- [14] Athey, S., and K. Bagwell (2001). Optimal Collusion with Private Information. *RAND Journal of Economics*, 32(2), 428-465.
- [15] Athey, S., and P. Haile (2002). Identification of standard auction models. *Econometrica*, 70(6), 2107-2140.

- [16] Bajari, P., H. Hong, J. Krainer, and D. Nekipelov (2010). Estimating static models of strategic interactions. *Journal of Business & Economic Statistics*, 28, 469-482.
- [17] Bergemann, D., and S. Morris (2013). Robust predictions in games with incomplete information. *Econometrica*, 81(4), 1251-1308.
- [18] Bergemann, D., and S. Morris (2016). Bayes correlated equilibrium and the comparison of information structures in games. *Theoretical Economics*, 11(2), 487-522.
- [19] Bernheim, B. (1984). Rationalizable strategic behavior. *Econometrica*, 52, 1007-1028.
- [20] Berry, S. T., and P. Haile (2014). Identification in differentiated products markets using market level data. *Econometrica*, 82(5), 1749-1797.
- [21] Berry, S., Levinsohn, J., and Pakes, A. (1995). Automobile prices in market equilibrium. *Econometrica*, 63(4), 841-890.
- [22] Borkovsky, R., P. Ellickson, B. Gordon, V. Aguirregabiria, P. Gardete, P. Grieco, T. Gureckis, T. Ho, L. Mathevet, and A. Sweeting (2015). Multiplicity of equilibria and information structures in empirical games: challenges and prospects. *Marketing Letters*, 26, 115-125.
- [23] Bresnahan, T. (1982). The oligopoly solution concept is identified, *Economics Letters*, 10(1-2), 87-92.
- [24] Brown, A., C. Camerer, and D. Lovo (2013). Estimating structural models of equilibrium and cognitive hierarchy thinking in the field: The case of withheld movie critic reviews. *Management Science*, 59(3), 733-747.
- [25] Ciliberto, F., and E. Tamer (2009). Market structure and multiple equilibria in airline markets. *Econometrica*, 77(6), 1791-1828.
- [26] De Paula, A., and X. Tang (2012). Inference of signs of interaction effects in simultaneous games with incomplete information. *Econometrica*, 80(1), 143-172.
- [27] Doraszelski, U., G. Lewis, and A. Pakes (2018). Just starting out: Learning and equilibrium in a new market. *American Economic Review*, 108, 565-615.
- [28] Doraszelski, U. and M. Satterthwaite (2010). Computable Markov-perfect industry dynamics. *Rand Journal of Economics*, 41, 215-243.
- [29] Ericson, R., and A. Pakes (1995). Markov-perfect industry dynamics: A framework for empirical work. *Review of Economic Studies*, 62, 53-82.
- [30] Fershtman, C. and A. Pakes (2012). Dynamic games with asymmetric information: A framework for empirical work. *Quarterly Journal of Economics*, 127, 1611-1661.
- [31] Goldfarb, A., and M. Xiao (2011). Who thinks about the competition? Managerial ability and strategic entry in US local telephone markets. *American Economic Review*, 101, 3130-3161.

- [32] Guerre, E., I. Perrigne, and Q. Vuong (2000). Optimal nonparametric estimation of first-price auctions. *Econometrica*, 68(3), 525-574.
- [33] Heinemann F., R. Nagel, and P. Ockenfels (2009). Measuring strategic uncertainty in coordination games. *Review of Economic Studies*, 76, 181–221.
- [34] Hortaçsu, A., and S. Puller (2008). Understanding strategic bidding in multi-unit auctions: a case study of the Texas electricity spot market. *The RAND Journal of Economics*, 39(1), 86-114.
- [35] Hortaçsu, A., F. Luco, S. Puller, and D. Zhu (2019). Does strategic ability affect efficiency? Evidence from electricity markets. *American Economic Review*, 109(12), 4302-42.
- [36] Huang, Y., P. Ellickson, and M. Lovett (2018). Learning to Set Prices. Manuscript available at SSRN 3267701.
- [37] Igami, M. (2017). Estimating the innovator’s dilemma: Structural analysis of creative destruction in the hard disk drive industry, 1981–1998. *Journal of Political Economy*, 125(3), 798-847.
- [38] Kline, B. (2018). An empirical model of non-equilibrium behavior in games. *Quantitative Economics*, 9(1), 141–181.
- [39] Magnolfi, L., and C. Roncoroni (2017). Estimation of discrete games with weak assumptions on information. Manuscript, University of Wisconsin-Madison.
- [40] Maskin, E., and J. Tirole (1988). A theory of dynamic oligopoly, II: Price competition, kinked demand curves, and Edgeworth cycles. *Econometrica*, 3, 571–599.
- [41] Morris, S. and H. Shin (2002). Measuring strategic uncertainty. Manuscript, Princeton University.
- [42] Otsu, T., M. Pesendorfer, and Y. Takahashi (2016). Pooling data across markets in dynamic Markov games. *Quantitative Economics*, 7(2), 523-559.
- [43] Pearce, D. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica*, 52, 1029-1050.
- [44] Ramsey, F. (1926). Truth and Probability. In Ramsey, 1931, *The Foundations of Mathematics and other Logical Essays*, Ch. VII, p.156-198, edited by R.B. Braithwaite, London: Kegan, Paul, Trench, Trubner & Co., New York: Harcourt, Brace and Company.
- [45] Seim, K. (2006). An empirical model of firm entry with endogenous product-type choices. *The RAND Journal of Economics*, 37(3), 619-640.
- [46] Simon, H. (1958). The role of expectations in an adaptive or behavioristic model. In M. Bowman (ed.) *Expectations, uncertainty, and business behavior*. Social Science Research Council. New York.

- [47] Simon, H. (1959). Theories of decision-making in economics and behavioral science. *American Economic Review*, 49, 253-283.
- [48] Sweeting, A. (2009). The strategic timing incentives of commercial radio stations: An empirical analysis using multiple equilibria. *The RAND Journal of Economics*, 40(4), 710-742.
- [49] Van Huyck, R. Battalio, and R. Beil (1990). Tacit coordination games, strategic uncertainty, and coordination failure. *American Economic Review*, 80(1), 234–248.
- [50] Xie, E. (2018). Inference in games without Nash equilibrium: An application to restaurants competition in opening hours. Manuscript. Department of Economics, University of Toronto.