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Multilateral and Multidimensional Wellness Measurement in the Absence of Cardinal Measure: Health, Loneliness, Ageing and Gender in 21st Century China.

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Very Preliminary Please do not quote.

Abstract.

Comparing the wellbeing of groups using self reported measures of wellbeing can be challenging. The scale dependency of many summary statistics applied to arbitrary Cantril scales attached to ordinal categorical data can engender a lack of coherency in results based upon alternative, equally valid scales. Furthermore, the conditions under which results will be robust across alternative scales seldom prevail in practice. Here scale independent methods for the multilateral and multidimensional wellness measurement and comparison of groups are proposed and implemented in a study of the health-loneliness-aging-gender nexus in 21st century China. The results indicate that poor health and loneliness appears to increase with age, though not monotonically. Improved health status is always associated with better un-loneliness outcomes and improved un-loneliness status is always associated with better health outcomes. While a large portion of the population are not affected by loneliness, of those who are, ill health is generally more likely to be reported. With regard to the health - loneliness joint distribution, generally, males enjoy better joint outcomes than their female counterparts in almost every comparison and urban dwellers enjoy better outcomes than their rural counterparts.

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Introduction.

In reporting a substantial increase in the use of self assessed wellbeing measures in recent years, Kahneman and Krueger (2006) stressed the importance of recognizing that such measures represent an individuals' perceptions of experiences rather than their actual utility as conventionally conceived by economists[3]. Evaluating the subjective wellbeing of groups often relies upon summary statics applied to scales devised for and attached to assessment categories (Cantril 1965, de Jong Gierveld and Kamphuis 1985, Russell 1996 are early examples), happiness gradient assessment studies, happiness-income-growth studies, health assessment studies and happiness and inequality studies are all cases in point[4]. The scale dependency inherent in such approaches can present challenges when making multilateral group comparisons since, as noted in Schroder and Yitzhaki (2017), Liddell and Kruschke (2018) and Bond and Lang (2019), the choice of scale is arbitrary and the results are not robust to the use of alternative, equally valid, scale choices[5]. While Allison and Foster (2004) and Apouey (2007) were aware of the issue in specific univariate inequality and bi-polarization settings, it has not been addressed in a more general multilateral wellbeing measurement environment. Furthermore, the problem is exacerbated in multidimensional environments since they usually involve arbitrarily assigned dimension weights for aggregation purposes (Klugman, Rodríguez and Choi 2011).

Responses to questions like "How would you assess your health status – 1:Poor, 2:Fair, 3:Good, 4:Very Good or 5:Excellent ?" or "How Lonely do you Feel - 1:Very, 2:Quite a bit, 3:Somewhat, 4:A little, 5:Not at all?" are fundamentally ordinal in nature[6]. While it is clear that each successive state or level is to be preferred to its predecessor (i.e. the ordering is monotonic increasing), the numbers attached to each level for the purpose of analysis are at best, arbitrary in nature. There is no reason to think that "Very Good" has hedonically twice the value or import of "Fair". The number allocation "1:Poor, 3:Fair, 6:Good, 10:Very Good, 15:Excellent" constitutes an equally valid, but none-the-less arbitrary, numerical ordering, the numbering process could equally be additive, geometric or hypergeometric, indeed it need not be systematic at all as long as it is monotonic increasing. In effect any number system accorded the respective categories has the status of an assumption about the relative value of categories and different assumptions usually result in materially different results. This means that frequently employed objects like location measures, measures of variation around them and the weighted sums thereof employed in multidimensional analyses in such artificially cardinalized ordinal environments are equally

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[3] Indeed, utility is an unbounded concept, one can always enjoy more or less utility in the next period, whereas self assessed wellbeing is usually recorded on a numerically bounded segment of the real line so situations can arise whereby an agent cannot experience more or less wellbeing in a subsequent period.

[4] The means, medians, quantiles, coefficients of variation, Gini coefficients variously employed in Helliwell, Layard, Sachs 2012, 2018, Easterlin 1973, 1995, Deaton 2008, Clark, Frijters Shields 2008, DeNeve, Ward, Keulenaer, van Landeghem, Kavetsos, Norton 2018, Newell, Girgis, Sanson-Fisher, Savolainen 1999, Ferrer-i-Carbonell, Ramos 2014, are all scale dependent constructs.

[5] Indeed, the issue is not confined to self reported wellbeing data but prevails whenever ordered categorical information is used as a proxy for latent continuous variates e.g. when educational status is used as a proxy for latent embodied human capital.

[6] One of the most common quality of life - health assessment measures Ware and Gandek (1998)

arbitrary in nature and any conclusions drawn would not necessarily be robust to any other arbitrarily chosen, but equally valid, cardinality assumptions.

The problem is one of scaling. Since the summary statistics employed (and functions of them such as inequality, poverty and polarization measures) are inherently scale dependent, different arbitrarily chosen scales can result in different conclusions, highlighting the ambiguity inherent in any set of comparisons. Schroder and Yitzhaki (2017) derive the conditions under which such ordering ambiguity would not prevail and go on to suggest that the conditions seldom obtain in practice, Bond and Lang (2019) offer empirical verification that this is indeed the case in a variety of examples. What is required are measures that reflect subjective wellbeing differences between groups in many dimensions which are not scale dependent. This will not evade the ambiguity problem that is inherent in the data but it will take the arbitrariness of scaling choice out of the analysis and provide rankings and orderings that are not the consequence of a chosen scale. Here such scale and functional form independent, continuous, consistent ordering instruments with independent of irrelevant alternative and non-ambiguity properties, together with measures of the extent of ambiguity inherent in a collection of groups, are proposed, developed and exemplified in a Chinese health and loneliness study.

Degrees of loneliness and health are integral to human wellbeing, especially amongst older populations (Gerst-Emerson and Jayawardhana 2015, Ong, Uchino, and Wethington 2016). A substantial literature on risk factors for loneliness in older adults (synthesized in Pinquart and Sörensen 2003) identified many health-related risk factors including gender, marital status, location and socioeconomic resources which have been substantiated in more recent large-scale, studies of older adults (Nicolaisen and Thorsen 2014, Perissinotto, Stojacic and Covinsky 2012). In a Chinese study Luo and Waite (2014) recorded about 28% of older Chinese adults as reporting feeling lonely, with lonely adults facing increased risks of dying over the subsequent years. Some of the effect was explained by social and health behaviors, but most of the effect was explained by health outcomes.

In this study the ordering techniques that are developed in section 1 are extended to multivariate environments and higher order comparisons in section 2. Section 3 considers the axiomatic foundations of the indices that are not obvious by construction. The techniques are applied to survey data drawn from the China Health and Retirement Longitudinal Study (CHARLS) 2013 follow up to a 2011 baseline study in Section 4 and Section 5 draws some conclusions. To anticipate the results based on groupings determined by age, gender, partner status, and urban rural location, the “scale free” indices indicate that, in spite of a low level of health and loneliness inequality, there is not a great deal of ambiguity in the system. While only a small portion of the population appear to be lonely, when they are, they are more likely to report ill health. Indeed, increased loneliness promotes ill health promotes increased loneliness appears to be the prevailing pattern at most levels. With regard to the health-loneliness joint distribution, generally, males enjoy better health and un-loneliness outcomes than their female counterparts in almost every comparison and urban dwellers enjoy better outcomes than their rural counterparts.

Section 1. First Order Univariate Comparisons, The Basic Idea.

To fix ideas, imagine the members of a given group A each had an equal chance of each of 5 ordered health outcomes, with a discrete uniform probability density function (i.e. 0.2 for each level so $p_i = 0.2\ for\ i = 1,..,5$), then the cumulative distribution function (CDF) of group A would report 0.2 as Poor, 0.4 as Fair or worse, 0.6 as Good or worse, 0.8 as Very Good or worse and the whole group as Excellent or worse (since the CDF (F_j) is given by compounding the probabilities so that $F_j = \sum_{i=1}^{j} p_i\ for\ j = 1,..,5$) [7]. Suppose further that a group B had a 30% chance of poor health 20% chances of Fair, Good and Very Good health and a 10% chance of Excellent Health and Group C had a 10% chance of Poor Health, 20% chances of Fair, Good and Very Good health and a 30% chance of Excellent Health. For the purpose of discussion, Table 1 presents the CDF's of the three groups plus a group B* that is very similar to group B and a group C* that is very similar to group C. In comparing A and B groups, note that a randomly selected individual from B would have at least as great or greater probability of the same or worse outcome than a similarly randomly selected member of group A at every level, alternatively put, the group B member has a lesser chance of better outcomes at all levels than does the member from group A.

Table 1. Comparisons of Cumulative Distribution (CDF's).

Group	Poor	Fair	Good	Very Good	Excellent	UD1
A	0.2	0.4	0.6	0.8	1.0	0.5
B	0.3	0.5	0.7	0.9	1.0	0.012
B*	0.31	0.49	0.7	0.9	1.0	0.012
C	0.1	0.3	0.5	0.7	1.0	0.988
C*	0.09	0.31	0.5	0.7	1.0	0.988
Upper Envelope	0.31	0.50	0.70	0.90	1.0	0.000
Lower Envelope	0.09	0.30	0.50	0.70	1.0	1.000

Essentially Group A is unambiguously better placed than Group B in terms of its overall Health outcomes and, if each individual in the calculus had equal weight, Group B could unequivocally be deemed to have worse Health Wellbeing than Group A. There is in effect a stochastic dominance relationship (Levy 1998, Yalonetzky 2013, Whang 2019) between Group A and Group B probability distributions. When A's CDF is everywhere less than or equal to B's CDF i.e.:

$$F_i^A \leq F_i^B\ for\ i = 1,..,5, with\ strict\ inequality\ somewhere \qquad [1]$$

A is said to stochastically dominate B at the first order. In such a case, if the self assessment levels had true cardinal value x, any Wellbeing Value Function $U(x)$ with $\frac{dU(x)}{dx} > 0$ would have an expected value under A's distribution that is at least as great as its expected value under B's

[7] The following discussion is relevant for any finite number of ordered outcomes, without loss of generality the discourse here confines itself to 5.

distribution. That the result holds for any such Wellbeing Value Function in the class means that it is unambiguous and unequivocal. However, the luxury of cardinality (or a fully specified wellbeing value function which attaches a real number to each level of wellbeing) is not available, here the Wellbeing Value Function is implicitly revealed as "a greater chance of a better outcome". Non the less, a sense of "Wellbeing distance" between the two groups could be gleaned from adding the differences in probability over the outcome levels.

Letting F_i^j be the value of the $j'th$ Groups CDF at the $i'th$ assessment level for i= Poor, Fair, Good, Very Good, Excellent, and j=A, B, B*, C, C*, then a measure of the extent to which A is preferable to B is given by CDif(A,B), the cumulative differences, where:

$$\text{CDif(A, B)} = \sum_i \left(F_i^B - F_i^A\right)$$

Noting that 1-F_i^B (1-F_i^A) is the chance of having a better outcome than the i'th level in the B (A) group, this index has the interpretation of measuring, in an equally weighted sum, the extent to which group A has better chances of better outcomes than group B. According to Table 1, it would take the value 0.4. Similar arguments can be made regarding the comparison of Group A with Group C but this time Group A's distribution is everywhere at least as high as group C's so that Group C first order dominates Group A, and again the ordering is unequivocal so that Group C's outcomes are preferred to Group A's outcomes which in turn are preferred to group B's outcomes. Note that $\text{Cdif(C, B)} = \sum_i \left(F_i^B - F_i^C\right) = 0.8 > \text{CDif(A, B)} = \sum_i \left(F_i^B - F_i^A\right) = 0.4$ provides quantifiable measures of the relative benefits of C over B as opposed to A over B which accords with the ordering above. The important point to note is that these measures do not depend upon the value of x or any such scale, they would retain the same value whatever values of x were attributed to the various levels, they are in effect measures of distance in probability which abstract from scaling but non-the-less reflect an ordering of expected wellbeing.

Note that while C* dominates A (and consequently B) and B* is dominated by A (and consequently C), there is no such dominance relationship between distributions C and C* (or B and B*). C's CDF is sometimes above and sometimes below that of C* (similarly with B and B*), hence there is some ambiguity as to which is the most (least) preferred distribution in these particular comparisons. Thus in Table 1 there is no uniquely best or uniquely worst Group and the CDif measure is no longer an unequivocal measure of superiority or inferiority between groups, evaluating the relative merits of B and B* or C and C* is problematic. However, exploiting this idea of distance further, let $F_{UENV,i}$ be the Upper Envelope of the collection of CDF's under comparison, which is given by:

$$F_i^{UENV} = \max_j\left(F_i^j\right) \, for \;\; i = \text{Poor, Fair, Good, Very Good, Excellent}\,; j = A, B, B^*, C, C^*$$

F_i^{UENV} (line 6 in Table 1) is also a CDF and corresponds to the synthetic worst outcome distribution that could be contrived by combining the worst aspects of all the groups (If there were a group that was uniquely dominated by all other groups in the collection, its CDF would coincide with $F_{i,UENV}$). As such, it is first order stochastically dominated by all other groups, and a measure of the merit of any group j relative to this synthetic worst case scenario is given by:

$$\mathrm{CDif(UENV, j)} = \sum_i \left(F_i^{UENV} - F_i^j\right)$$

As an index, its lower bound is 0 (which will arise when a particular group CDF coincides with the upper envelope) and in theory its upper bound is the number of assessment categories less 1, so if a standardized index confined to the unit interval is required, one could divide by the number of outcome categories less 1[8]. However, this turns out to be a little extreme, in effect it assumes a worst case scenario group with all its members in the lowest assessment category. A more palatable alternative would be to consider the lower envelope $F_{i,LENV}$ where:

$$F_i^{LENV} = \min_j\left(F_i^j\right) \, for \;\; i = \text{Poor, Fair, Good, Very Good, Excellent}; j = A, B, B^*, C, C^*$$

This is also a CDF (line 7 of Table 1) which corresponds to the synthetic best possible outcome distribution that could be contrived by combining the best aspects of all the groups (If there were a group that uniquely dominated all other groups in the collection, its CDF would coincide with F_i^{LENV}). Then $\mathrm{CDif(j, LENV)}$ where:

$$\mathrm{CDif(j, LENV)} = \sum_i \left(F_i^j - F_i^{LENV}\right)$$

would correspond to the distance of group j from the synthetic best case scenario, in essence a measure of how "bad" group j was relative to the synthetic best case scenario.

Then $UD1(\mathrm{j})$, a first order relative wellbeing index could be contemplated where:

$$0 \leq \mathrm{UD1(j)} = \frac{\sum_i \left(F_i^{UENV} - F_i^j\right)}{\sum_i \left(F_i^{UENV} - F_i^{LENV}\right)} \leq 1 \qquad [2]$$

This index, a discrete ordinal distribution analogue of the First Order Utopia-Dystopia index (Anderson, Post and Whang 2019, Anderson and Leo 2020) which was developed for continuous distributions, has many advantages. It can be shown to satisfy many of the axioms required of wellbeing indices (Sen 1987, 1995) as it is continuous (at least piecewise), independent of scale and functional form and consistent (if a distribution is more preferred the statistic yields a larger value), it is normalized and has an independence of irrelevant alternatives property (Anderson and Leo 2020). In addition, unlike most other normalized statistics, it attains the value 0 (1) only when a particular group has unambiguously the worst (best) outcomes. Its asymptotic distribution is readily obtained (See Appendix) which facilitates inference and furthermore, it is readily extended to multidimensional, multilateral comparisons obviating many of the weighting issues associated with a multidimensional environment.

The Upper and Lower envelopes of sub groupings in the collection also have an additional, very useful application. Suppose two mutually exclusive collections of groups A and B where A = $\left\{F_i^j, j = 1,..k, i = 1,..,5\right\}$ and B = $\left\{F_i^j, j = k+1,..K, i = 1,..,5\right\}$ and define $F_i^{UENV,A}$ as:

[8] At the extreme if all members of group j were in the highest category and all members of any other group were in the lowest category the maximum value of CDif(UENV,j) would be K=1 in a K category situation.

$$F_i^{UENV,A} = \max_{j=1,..,k}\left(F_i^j\right) for\ \ i = \text{Poor, Fair, Good, Very Good, Excellent}$$

and define $F_i^{LENV,B}$ as:

$$F_i^{LENV,B} = \min_{j=k+1,..,K}\left(F_i^j\right) for\ i = \text{Poor, Fair, Good, Very Good, Excellent}$$

Then if:

$$F_i^{LENV,B} \geq F_i^{UENV,A}\ for\ \ i = \text{Poor, Fair, Good, Very Good, Excellent}$$

All members of the A collection of groups unambiguously dominate all members of the B collection of groups (essentially a separating hyperplane can be established between the two collections). Collection A can then be considered to be better placed since any randomly selected individual from it has a better chances of better outcomes than a correspondingly randomly selected individual from group B collection.

To examine whether or not a separating hyperplane could be established between $F_i^{LENV,B}$ and $F_i^{UENV,A}$ (or any pair of CDF's), consider comparing $\left|\sum_{i=1}^{5}\left(F_i^{LENV,B} - F_i^{UENV,A}\right)\right|$ with $\sum_{i=1}^{5}\left|\left(F_i^{LENV,B} - F_i^{UENV,A}\right)\right|$ note that:

$$\left|\sum_{i=1}^{5}\left(F_i^{LENV,B} - F_i^{UENV,A}\right)\right| \leq \sum_{i=1}^{5}\left|\left(F_i^{LENV,B} - F_i^{UENV,A}\right)\right|$$

Equality will occur when $\left(F_i^{LENV,B} - F_i^{UENV,A}\right) \geq 0\ \forall\ i$ or $\left(F_i^{LENV,B} - F_i^{UENV,A}\right) \leq 0\ \forall\ i$ that is to say, equality will occur when there is a first order dominance relationship between the two distributions. This suggests, for all $F_i^{LENV,B}, F_i^{UENV,A}$ $i = 1,..,5$ such that $F_i^{LENV,B} \neq F_i^{UENV,A}\ for\ some\ i = 1,..,5$ a Surface Separation Index SS, of the form:

$$-1 \leq SS = \frac{\sum_{i=1}^{5}\left(F_i^{LENV,B} - F_i^{UENV,A}\right)}{\sum_{i=1}^{5}\left|\left(F_i^{LENV,B} - F_i^{UENV,A}\right)\right|} \leq 1$$

Which attains -1 or 1 if and only if there is surface separation. SS will equal 1 when $F_i^{UENV,A}$ first order dominates $F_i^{LENV,B}$ and will attain -1 when $F_i^{UENV,A}$ is first order dominated by $F_i^{LENV,B}$.

Section 2. Multidimensionality and Higher Order Comparisons.

In the present case of two dimensioned Health and Loneliness wellbeing, the joint impact of health and loneliness outcomes on wellbeing has to be evaluated in terms of the joint probability density function (JPDF) and its joint cumulative distribution function counterpart JCDF. With respect to the JPDF, $p_{i,k}^j$ is now the probability that an agent randomly selected from the $j'th$ group is in the $i'th$ health and $k'th$ loneliness categories respectively, where $p_{i,k}^j \geq$

$0\ and\ \sum_{i=1}^{5}\sum_{k=1}^{5}\ p_{i,k}^{j} = 1$. Its corresponding JCDF has a typical element $F_{i,k}^{j}\ for\ i,j = 1,..,5$ where $0 \leq F_{i,k}^{j} = \sum_{i*=1}^{i}\sum_{k*=1}^{k}\ p_{i*,k*}^{j} \leq 1$, (essentially compounding probabilities across 2 dimensions) in this case $F_{i,k}^{j}$ corresponds to the probability that a randomly selected agent from group j has a health outcome no greater than the $i'th$ category and a loneliness outcome no greater than the $k'th$ category. In this context, if health and loneliness outcomes are temporarily endowed with the cardinality measures x and y respectively, and a Wellbeing Value Function $U(x,y)$ with $\frac{dU(x,y)}{dx} \geq 0, \frac{dU(x,y)}{dy} \geq 0\ and\ \frac{d^2U(x,y)}{dxdy} \leq 0$ is posited, a necessary and sufficient condition for $E(U(x,y)|A) \geq E(U(x,y)|B)$ is that Group A outcomes stochastically dominate Group B outcomes (Atkinson and Bourguignon 1982) which demands:

$$F_{i,k}^{A} \leq F_{i,k}^{B}\ for\ i = 1,..,5\ and\ k = 1,..,5\ with\ strict\ inequality\ somewhere.$$

Note that, in this case, Group A's outcomes would be unambiguously preferred to Group B's for any Wellbeing Value Function in the specified class. In particular, since $U(x,y)$ is a function which implicitly weights dimensions x and y, this would hold for any weighting scheme[9] consistent with the specified class of Wellbeing Value Functions.

This leads quite naturally to a 2 dimensioned analogue of the aforementioned Utopia-Dystopia index for 2 dimensioned ordinal variates UD2 which may be written as:

$$0 \leq \mathrm{UD2(j)} = \frac{\sum_k\sum_i\left(F_{i,k}^{UENV} - F_{i,k}^{j}\right)}{\sum_k\sum_i\left(F_{i,k}^{UENV} - F_{i,k}^{LENV}\right)} \leq 1 \qquad [3]$$

Where:

$$F_{i,k}^{UENV}\ \{F_{i,k}^{LENV}\} = \max_j\left(F_{i,j}^{j}\right)\ \ \left\{\min_j\left(F_{i,j}^{j}\right)\right\}\ \ for\ \ i =$$
$\text{Poor, Fair, Good, Very Good, Excellent}\ and\ k = \text{Very lonely, Quite a bit lonely, Somewhat lonely, A little lonely, Not at all lonely.}$

Checking for Ambiguity.

When [1] does not hold the situation is ambiguous, there is uncertainty as to whether group A or group B is in the preferred position. The absolute value of the surface separation index can be used to establish an "unambiguous difference" between any two distributions that are not identical and the average surface separation in a collection of distributions will provide evidence of the extent of ambiguity inherent in the system. Let the surface separation index between groups k an j be $SS_{k,j}$, then $0 < \left|SS_{k,j}\right| = \frac{\left|\sum_{i=1}^{5}\left(F_i^k - F_i^j\right)\right|}{\sum_{i=1}^{5}\left|\left(F_i^k - F_i^j\right)\right|} \leq 1$, groups k an j will be unambiguously different if $\left|SS_{k,j}\right| = 1$ so that AI, an index of the extent ambiguity in a collection of K groups indexed $k = 1,..,K$ is given by[10]:

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[9] The choice of which has been the source of some controversy in wellbeing measurement, see for example Klugman et. al. (2011) and references therein.

[10] An importance weighted version of the statistic is possible which is given by:

$$AI = 1 - \frac{2}{K(K-1)} \sum_{k=2}^{K} \sum_{j=1}^{k-1} \left|SS_{k,j}\right|$$

where w_k is the relative importance of the k'th group (for example its relative population size) where $w_k > 0$ and $\sum_{k=1}^{K} w_k = 1$.

In the univariate cardinal measure case, when [1] doesn't hold (SS < 1) researchers check for 2nd or higher order dominance conditions involving integrals of CDF's. These higher order dominance relations reflect additional concavity conditions on the space of admissible Wellbeing Value Functions which reflect concerns for inequality and poorness (successive integration levels place increasing weight on outcomes in the lower end of the outcome spectrum). Accordingly, Atkinson and Bourguignon (1982) go on to discuss the somewhat more complex but intuitively similar 2nd order dominance conditions in the two dimensional case. These involve integrals of CDF's which are not possible without the luxury of cardinal measure, however the approach can be mimicked by discretely compounding the CDF's to develop a Joint Compounded Cumulative Density Function (JCCDF) which, in the present context, has a typical element $CF_{i,k}^{j}$ for $i, j = 1, .., 5$ where:

$$CF_{i,k}^{j} = \sum_{i*=1}^{i} \sum_{k*=1}^{k} F_{i*,k*}^{j}.$$

Again this leads quite naturally to a 2 dimensioned analogue of a Second Order Utopia-Dystopia index for 2 dimensioned ordinal variates 2UD2 which may be written as:

$$0 \leq \mathrm{2UD2(j)} = \frac{\sum_k \sum_i \left(CF_{i,k}^{UENV} - CF_{i,k}^{j}\right)}{\sum_k \sum_i \left(CF_{i,k}^{UENV} - CF_{i,k}^{LENV}\right)} \leq 1$$

Where:

$$CF_{i,k}^{UENV} = \max_{j}\left(CF_{i,k}^{j} : \mathrm{j} = \mathrm{A}, A^{*}, \mathrm{B}, \mathrm{C}, C^{*}\right)$$

And

$$CF_{i,k}^{LENV} = \min_{j}\left(CF_{i,k}^{j} : \mathrm{j} = \mathrm{A}, A^{*}, \mathrm{B}, \mathrm{C}, C^{*}\right)$$

$$for \;\; i = \text{Poor, Fair, Good, Very Good, Excellent:}$$

$$k = v\text{ery lonely, Quite a bit lonely, Somewhat lonely, A little lonely, Not at all lonely}$$

The intuition behind these second order ordinal comparators is that they are no longer an equally weighted sum of CDF differences, they now attach more weight to, and thus focus more attention upon, probability differences at the lower end of the preference spectrum. So, if discrepancies between groups with respect to the worst outcomes are of greater concern, 2nd or higher order comparators may be appropriate. Again, surface separation may be established by the proximity of SSC to 1 where:

$$0 < \left|SSC_{k,j}\right| = \frac{\left|\sum_{i=1}^{5}\left(CF_{i}^{k} - CF_{i}^{j}\right)\right|}{\sum_{i=1}^{5}\left|\left(CF_{i}^{k} - CF_{i}^{j}\right)\right|} \leq 1$$

$$AIW = 1 - \frac{2}{1 - \sum_{k=1}^{K} w_k^2} \sum_{k=2}^{K} \sum_{j=1}^{k-1} w_k w_j \left|SS_{k,j}\right|$$

Section 3. Axiomatic Foundations of Indices.

To facilitate interpretation and comparability, the Economic Wellbeing Literature develops indices on an axiomatic foundation (Sen 1995, Gravel Magdalou and Moyes 2020 are examples). Wellbeing analysts seek usually indices that obey certain axioms like Continuity, Scale Independence, Coherency, Normalization, Monotonicity and Inequality Sensitivity. Continuity of the index is usually required with respect to the variable x (usually income or consumption) which in the present case is not continuously measured, however these indices are piecewise continuous in the variable p. Scale Independence is required to secure independence from monotonic translations of the wellbeing function, note that scaling measures do not appear in any formulae here (the indices are scale independent by construction) so independence is secured. Coherency (if group g's distribution is preferred to group h's distribution then indices should reflect that i.e. $UD(g) > UD(h)$) will be satisfied if the indices can be shown to be monotonic increasing in category ordering. Normalization requires that indices values are bounded on the unit interval which is the case with respect to the normalized intervals. Monotonicity requires that indices are sensitive to category ordering so that ceteris paribus, if an agent enjoys a better outcome the index should increase. This is obviously related to the coherency property, but has special import in the case of ordered categories. Finally, Inequality Sensitivity is an ethically grounded principle of transfers property (Dalton 1920) that requires measures be sensitive to dispersion of wellbeing so that, for a given level of aggregate wellbeing, the society that has it more equally shared is the society that is preferred. In the current situation adherence to the last two properties needs to be demonstrated.

To fix ideas suppose there exist S ordered categories indexed s=1,..,S in line with their ordering so that category s is preferred to category t when s > t. The probability density function (PDF) defines the chance that a randomly selected agent resides in a particular category so the categories have associated probabilities $p_s \geq 0 \; where \; \sum_{s=1}^{S} p_s = 1$ and the corresponding associated cumulative distribution function (CDF) is given by:

$$F_j = \sum_{i=1}^{j} p_i \; for \; j = 1,..,S$$

The Cumulative CDF is given by:

$$CF_j = \sum_{i=1}^{j} F_i = \sum_{i=1}^{j} \sum_{k=1}^{i} p_k \; for \; j = 1,..,S$$

Suppose G groups indexed $g = 1,..,G$, their respective PDF's, CDF's and Cumulative CDF's are identified with a second subscript g (so that $p_{s,g}$ is the probability that a randomly selected individual in group g is in category s). Define the upper envelope of the collection of group CDF's as F_i^{UENV}

$$F_s^{UENV} = \max_g (F_{s,g}) \; for \;\; s = 1,..,S. \;\; g = 1,..,G$$

And the upper envelope of the group cumulative CDF's

$$CF_s^{UENV} = \max_g\left(CF_{s,g}\right) \, for \;\; s = 1,..,S. \;\; g = 1,..,G$$

The un-normalized first and second order Utopia-Dystopia indices[11] for the g'th group is given by:

$$\mathrm{UD1(UENV, g)} = \sum_s \left(F_s^{UENV} - F_{s,g}\right)$$

$$\mathrm{UD2(UENV, j)} = \sum_s \left(CF_s^{UENV} - CF_{s,g}\right)$$

Monotonicity property – if an agent transfers to a higher category the index should increase. Working with nUD1 where n is the population of the $g'th$ group and assume that group is not part of the upper envelope (if it were things just net out). $np_{s,g}$ corresponds to the number of people in the $g'th$ group in the $s'th$ category and $nF_{s,g}$ corresponds to the number of people in the $s'th$ category or lower. Suppose that an agent moves from category s to category s+k and let superscript A denote the "after the move" index. Note that:

$np_{t,g}^A = np_{t,g} \, for \; all \; t < s, t = s + l, l = 1, ., k \; and \; t > s + k;$

$and \; np_{s,g}^A = np_{s,g} - 1; \, np_{s+k,g}^A = np_{s+k,g} + 1.$

It follows that:

$nF_{t,g}^A = nF_{t,g} \, for \; all \; t < s; \; nF_{s+l,g}^A = nF_{s+l,g} - 1 \, for \; l = 0,..,k-1 \; and \; nF_{t,g}^A = nF_{t,g}$ $for \; all \; t > s + k.$

So that $n\mathrm{UD1(UENV, g)}^A = n\mathrm{UD1(UENV, g)} + k => \mathrm{UD1(UENV, g)}^A > \mathrm{UD1(UENV, g)}$

For any $s < S - k$:

Furthermore:

$nCF_{t,g}^A = nCF_{t,g} \, for \; all \; t < s; \; nCF_{s+i,g}^A = nCF_{s+i,g} - i; \, for \; all \; i = 0, ., k \; and \; nCF_{s+k+i,g}^A = nCF_{s+k+i,g} - k \, for \; i = 0,..,S - k - s$

So that $n\mathrm{UD2(UENV, g)}^A = n\mathrm{UD2(UENV, g)} + k! + (S - s - k)k => \mathrm{UD2(UENV, g)}^A > \mathrm{UD2(UENV, g)}$

As for distribution sensitivity, the requirement is that movement to an adjacent improved category for someone in the lower end of the category spectrum should result in a greater increase in the index than a similar movement for someone in the upper end of the category spectrum. Whilst this is not the case for UD1 it is the case for UD2 since $n\mathrm{UD2(UENV, g)}^A -$

[11] To satisfy normalization axioms UD1 and UD2 are divided by $\sum_s (F_s^{UENV} - F_s^{LENV})$ and $\sum_s (CF_s^{UENV} - CF_s^{LENV})$ respectively

nUD2(UENV, g) diminishes with increasing s so that a transfer in the upper end of the spectrum results in a smaller increase than a similar transfer at the lower end of the spectrum.

Section 4. Results.

This study employs survey data drawn from the China Health and Retirement Longitudinal Study (CHARLS) 2013 follow up to a 2011 baseline study. Within each sampled household, a Main Respondent (MR), defined to be a family member who was at least 45 years of age, was not a nanny, short-term worker, or visitor to the household, and had sufficient knowledge about the household, was identified. Both the MR and the MR's spouse (if any and present at the time of the survey) were recruited to be respondents for the baseline survey answering questions about their health and loneliness. The sample size after eliminating incomplete records is 13593.

Initially, to get a sense of the loneliness – health relationship, cardinality of the Self Reported assessments measures is presumed and reduced form regressions for S=L (Loneliness) and S=H (Health) indices are examined:

$$y_{i,S} = \alpha + \beta_{1,S}GD_i + \beta_{2,S}UD_i + \beta_{3,S}AGE_i + \beta_{4,S}EDU_i + \beta_{5,S}SS_i +$$

$$\left(\beta_{6,S}AGE_i + \beta_{7,S}EDU_i + \beta_{8,S}SS_i\right) * GD_i + \left(\beta_{9,S}AGE_i + \beta_{10,S}EDU_i + \beta_{11,S}SS_i\right) * UD_i + \varepsilon_{i,S}$$

Table 2. Reduced Form Equations

	Loneliness (1 : not at all 5 : very)				Health (1: good 5 poor)			
	Beta	std dev	std.norm	P(|Z|>0)	Beta	std dev	std.norm	P(|Z|>0)
const	1.2336	0.0812	15.1980	1.0000	3.0828	0.1118	27.5788	1.0000
gender GD	0.0220	0.0989	0.2229	0.5882	0.2978	0.1362	2.1863	0.9856
urban UD	-0.0512	0.1177	-0.4346	0.6681	-0.1914	0.1621	-1.1808	0.8812
AGE	0.0009	0.0012	0.8161	0.7928	0.0079	0.0016	4.9506	1.0000
education (EDU)	-0.0269	0.0060	-4.4987	1.0000	-0.0382	0.0082	-4.6362	1.0000
single status (SS)	0.3860	0.0305	12.6407	1.0000	-0.0483	0.0421	-1.1479	0.8745
AGE .*GD	0.0004	0.0015	0.2853	0.6123	-0.0024	0.0020	-1.1816	0.8813
EDU .* GD	-0.0055	0.0072	-0.7560	0.7752	-0.0037	0.0100	-0.3708	0.6446
SS .* GD	-0.0116	0.0373	-0.3097	0.6216	0.0577	0.0514	1.1232	0.8693
AGE .* UD	-0.0020	0.0017	-1.1395	0.8728	-0.0006	0.0024	-0.2371	0.5937
EDU .* UD	0.0260	0.0083	3.1235	0.9991	0.0176	0.0115	1.5364	0.9378
SS .*UD	-0.1553	0.0440	-3.5337	0.9998	-0.0847	0.0605	-1.3995	0.9192
σ^2 (R^2)	0.5437		(0.0445)		1.0311		(0.0245)	

GD:1=female 0=male. UD 1=urban 0=rural Age is a 4 category variable <50, 50-60, 60-70 and >70 EDU is a 10 category variable SS=1 if single 0 otherwise.

In line with the studies cited in the introduction, Loneliness appears to be negatively correlated with education (the more educated the less lonely) in Rural China, positively related to Single Status (less so in urban China) and gender effects appear to be weak in this dimension. With respect to Poor Health, it appears to be positively related to gender (being female), positively related to age, negatively related to education (less so in Urban China)

Predictions from these equations are used (in a simultaneous regression format) in a regression of loneliness on health of the form:

$$\hat{y}_{i,L} = \theta_1 + \theta_2 \hat{y}_{i,H} + \omega_i$$

The result of which Table 3 reveals as a strong positive relationship between loneliness and poor health.

Table 3. Loneliness – Health Poorness Structural Equation (Predicted Loneliness on predicted poor health regression).

	θ	std dev	std.norm	P(|Z|>0)
Constant	-0.6206	0.0247	-25.1680	1.0000
Predicted health poorness	0.5423	0.0071	76.4716	1.0000
σ^2 (R^2)	0.0177		(0.3008)	

However, these results are predicated upon an arbitrary presumption of cardinality in the health, loneliness and education measures which could well change with the presumption of a different, but equally valid cardinality structure. The question is, what can be said in the absence of any arbitrarily assigned cardinality?

Turning to measures of health and loneliness wellbeing that do not rely upon cardinality, Table 4 reports the overall joint probability and resulting cumulative distribution functions of loneliness and health poorness in the sampled population.

Table 4 Overall Joint Distributions and (standard errors).

	Joint Probability Density Function					Joint Cumulative Distribution Function				
Loneliness	Health					Health				
	Poor	Fair	Good	Very Good	Excellent	Poor	Fair	Good	Very Good	Excellent
Very	0.0077	0.0065	0.0021	0.0004	0.0001	0.0077	0.0142	0.0163	0.0167	0.0168
	(0.0008)	(0.0007)	(0.0004)	(0.0002)	(0.0001)	(0.0008)	(0.0010)	(0.0011)	(0.0011)	(0.0011)
Quite a lot	0.0066	0.0081	0.0042	0.0008	0.0002	0.0143	0.0289	0.0352	0.0364	0.0367
	(0.0007)	(0.0008)	(0.0006)	(0.0002)	(0.0001)	(0.0010)	(0.0014)	(0.0016)	(0.0016)	(0.0016)
Somewhat	0.0074	0.0120	0.0060	0.0011	0.0004	0.0218	0.0483	0.0605	0.0629	0.0636
	(0.0007)	(0.0009)	(0.0007)	(0.0003)	(0.0002)	(0.0013)	(0.0018)	(0.0020)	(0.0021)	(0.0021)
A little	0.0177	0.0310	0.0260	0.0058	0.0020	0.0394	0.0970	0.1352	0.1434	0.1460
	(0.0011)	(0.0015)	(0.0014)	(0.0007)	(0.0004)	(0.0017)	(0.0025)	(0.0029)	(0.0030)	(0.0030)
Not at all	0.1165	0.3064	0.2840	0.1048	0.0422	0.1560	0.5200	0.8422	0.9552	1.0000
	(0.0028)	(0.0040)	(0.0039)	(0.0026)	(0.0017)	(0.0031)	(0.0043)	(0.0031)	(0.0018)	(0.0000)

Note the large portion of the population (80%) that are not affected by loneliness (similar to the Luo and Waite 2014 study), of those who are, ill health is generally more likely to be reported (note greater density in lower left quadrant of the pdf than the upper right quadrant). Less than 5% of the population report excellent health and no loneliness.

Health status conditional Loneliness status probability density and cumulative distribution functions (i.e. Given the individuals j'th health status level, what is their chance of the i'th level loneliness status?) and the corresponding loneliness conditioned Health distributions are reported in Tables 5a and 5b. These distributions reveal a great deal about the health and loneliness nexus. What is revealed is consistent first order dominance relationships[12] between the distributions across the conditioning spectra so that a higher status Health (Un-Loneliness) conditioned

Table 5a. Health status conditioned loneliness distributions.

	Health Status conditioned pdf					Health Status conditioned cdf				
Loneliness Status	Poor	Fair	Good	Very Good	Excellent	Poor	Fair	Good	Very Good	Excellent
Very	0.0494	0.0179	0.0065	0.0035	0.0022	0.0494	0.0179	0.0065	0.0035	0.0022
Quite a lot	0.0423	0.0223	0.0130	0.0071	0.0045	0.0917	0.0402	0.0195	0.0106	0.0067
Somewhat	0.0475	0.0330	0.0186	0.0097	0.0089	0.1392	0.0732	0.0381	0.0203	0.0156
A little	0.1135	0.0852	0.0807	0.0514	0.0445	0.2527	0.1584	0.1188	0.0717	0.0601
Not at all	0.7473	0.8418	0.8812	0.9283	0.9399	1.0000	1.0000	1.0000	1.0000	1.0000
Health Probs	0.1559	0.3640	0.3223	0.1129	0.0449	0.1559	0.3640	0.3223	0.1129	0.0449

	Cumulative density differences and (Approximate Standard Errors)			
	Fair vs Poor	good vs fair	very good vs Good	Excellent vs very good
Very	0.0315 (0.0147)	0.0114 (0.0090)	0.0030 (0.0058)	0.0013 (0.0044)
Quite a lot	0.0515 (0.0202)	0.0207 (0.0139)	0.0089 (0.0099)	0.0039 (0.0076)
Somewhat	0.0660 (0.0250)	0.0351 (0.0187)	0.0178 (0.0137)	0.0047 (0.0108)
A little	0.0943 (0.0328)	0.0396 (0.0282)	0.0471 (0.0239)	0.0116 (0.0203)

Table 5b. Loneliness conditioned health distributions.

	Loneliness Status conditioned pdf					Loneliness Status conditioned cdf				
Health Staus	Very	Quite a lot	Some-what	A little	Not at all	Very	Quite a lot	Some-what	A little	Not at all
Poor	0.4583	0.3317	0.2751	0.2145	0.1364	0.4583	0.3317	0.2751	0.2145	0.1364
Fair	0.3869	0.4070	0.4461	0.3758	0.3588	0.8452	0.7387	0.7212	0.5903	0.4952
Good	0.1250	0.2111	0.2230	0.3152	0.3326	0.9702	0.9498	0.9442	0.9055	0.8278
Very Good	0.0238	0.0402	0.0409	0.0703	0.1227	0.9940	0.9900	0.9851	0.9758	0.9505
Excellent	0.0060	0.0101	0.0149	0.0242	0.0494	1.0000	1.0000	1.0000	1.0000	1.0000
Lonely Probs	0.0168	0.0199	0.0269	0.0825	0.8539	0.0168	0.0199	0.0269	0.0825	0.8539

	Cumulative density differences and (Approximate Standard Errors)			
	Quite a lot vs Very	Somewhat vs Quite	Somewhat vs Little	Not at all vs Little
Poor	0.1266 (0.0396)	0.0566 (0.0375)	0.0606 (0.0350)	0.0781 (0.0309)
Fair	0.1065 (0.0329)	0.0175 (0.0362)	0.1309 (0.0384)	0.0951 (0.0405)
Good	0.0204 (0.0160)	0.0056 (0.0183)	0.0387 (0.0215)	0.0777 (0.0276)
Very Good	0.0040 (0.0073)	0.0049 (0.0091)	0.0093 (0.0113)	0.0253 (0.0153)

cumulative Loneliness (Health) distribution is everywhere below a lower Health (Un-Loneliness) status conditioned Loneliness (Health) cumulative distribution. What this implies is that a higher

[12] First Order Dominance requires the complete absence of significantly negative elements in the cumulative density differences columns in Tables 5a and 5b.

level health status outcome results in a dominating health conditioned loneliness outcome distribution and improved non-loneliness status results in a dominating loneliness conditioned health outcome distribution. What this implies is that improved health status is always associated with better loneliness outcomes and improved loneliness status is always associated with better health outcomes, note that this assessment has been achieved without resort to arbitrary attribution of cardinality.

The Health Status – Loneliness Connection.

Much can be gleaned from examining the health - loneliness status connection across cohorts. Aside from age, research has identified gender, marital status and urban/rural location as risk factors in the Health and Loneliness connection. 4 age groupings {<50, 50 – 59, 60 – 69, ≥ 70}, single/partner status, gender, urban/rural status are employed to form 32 mutually exclusive and exhaustive groups for a more detailed analysis of these factors. The Overlap (Anderson, Linton and Whang 2012) between two probability density functions measures the extent of commonality between the two distributions. In the discrete case, for any two PDF's p_m, q_m defined over M possible outcomes m = 1,..,M, the overlap OV[13] is defined as:

$$OV = \sum_{m=1}^{M} \min(p_m, q_m)$$

The overlap between a groups health and loneliness marginal distributions reveals the extent to which its poor health is associated with its loneliness (in the population overall it is 0.1910). If it were 1, then poor health status would uniquely identify the corresponding loneliness status (and vice versa), everyone in a given health category would also be in the corresponding loneliness category. If it were 0 then the two marginal distributions are segmented and there would be no one in a given health category who would be in the corresponding loneliness category (so for example someone with the poorest (best) health category would not experience extreme (least) loneliness)). The higher the value of the overlap, the greater is the association between poor health and loneliness.

Table 6 reports the poor health – loneliness overlap connection. In short, with the exception of young single urban males, the health and un-loneliness association is stronger (higher overlaps) for single status individuals relative to their partnered counterparts (lower overlaps) and it appears to be higher in rural environments than corresponding urban environments, the weakest association (lowest overlaps) appear in the youngest groups.

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[13] Assuming $p_m \neq q_m$ $for\ all\ m$, letting $p_m^* = p_m$ if $p_m < q_m$ and $q_m^* = q_m$ if $q_m < p_m$ and let $p = \sum_{m=1}^{M} p_m^*$ and $q = \sum_{m=1}^{M} q_m^*$, Asymptotic Standard Errors for OV are readily calculated in the discrete case as:

$$\sqrt{\frac{1}{n}(p(1-p) + q(1-q))}$$

Which for Table 6 has an upper bound of 0.03.

Table 6.

Age	single	gender	urban	health-loneliness Pdf overlap	Rank
1	0	0	0	0.1563	8
2	0	0	0	0.1608	9
3	0	0	0	0.1801	15
4	0	0	0	0.1693	10
1	1	0	0	0.1842	17
2	1	0	0	0.3567	32
3	1	0	0	0.3179	30
4	1	0	0	0.3073	27
1	0	1	0	0.1828	16
2	0	1	0	0.1729	13
3	0	1	0	0.1717	12
4	0	1	0	0.1862	18
1	1	1	0	0.2435	21
2	1	1	0	0.3110	29
3	1	1	0	0.2918	25
4	1	1	0	0.3109	28
1	0	0	1	0.1495	7
2	0	0	1	0.1432	6
3	0	0	1	0.1321	4
4	0	0	1	0.0905	2
1	1	0	1	0.0000	1
2	1	0	1	0.3043	26
3	1	0	1	0.2895	24
4	1	0	1	0.3421	31
1	0	1	1	0.1700	11
2	0	1	1	0.1402	5
3	0	1	1	0.1102	3
4	0	1	1	0.1786	14
1	1	1	1	0.2069	19
2	1	1	1	0.2805	23
3	1	1	1	0.2476	22
4	1	1	1	0.2279	20

To get a sense of rankings, and for comparison purposes, average health $\mu_h$and average non-loneliness μ_l represent the two extremes of a Wellbeing Value Function which attaches zero weight to loneliness in the former case and zero weight on health in the latter case. The middle ground is represented by $\sqrt{\mu_{UL}\mu_H}$, but recall these are based upon an arbitrary choice of scaling and are scale dependent, so for comparison purposes, the first order Utopia-Dystopia two dimensioned index (Equation [3]) is reported along side them together with their ranks in Table 7. The first thing to note is the lack of consonance between the scale independent UD2 index and the three scale dependent sample mean based indices. The former is robust to any scaling structure and to dimension weighting structures whereas the latter is not. The loneliest, health poorest individual is unequivocally and unambiguously by all measures a single rural female in her 60's and the most un-loneliest (there doesn't appear to be a good antonym for loneliness in the lexicon!) healthiest individual is an urban partnered male in his 40's. Generally, loneliness and ill health increase with age, are more prevalent in Rural China and more prevalent in females.

Table 7. First Order Wellbeing Indices and Rankings.

Age	single	gender	urban	μ_T	μ_H	$\sqrt{\mu_{UL}\mu_H}$	UD2	R(μ_{UL})	R(μ_H)	R($\sqrt{\mu_{UL}\mu_H}$)	R(UD2)
1	0	0	0	4.8414	2.7241	3.6316	0.7467	7	8	6	6
2	0	0	0	4.8181	2.6625	3.5816	0.7044	10	11	9	11
3	0	0	0	4.8006	2.5383	3.4908	0.6586	12	23	17	13
4	0	0	0	4.7756	2.4160	3.3967	0.5817	15	28	24	19
1	1	0	0	4.8158	2.6316	3.5599	0.7147	11	15	12	10
2	1	0	0	4.3758	2.6815	3.4255	0.2896	29	10	22	28
3	1	0	0	4.4154	2.5487	3.3546	0.3154	27	22	25	27
4	1	0	0	4.3716	2.5505	3.3391	0.2531	30	21	26	29
1	0	1	0	4.7957	2.5578	3.5023	0.6593	13	19	16	12
2	0	1	0	4.7621	2.4247	3.3980	0.5693	16	27	23	20
3	0	1	0	4.7273	2.3340	3.3217	0.5215	17	29	28	21
4	0	1	0	4.7071	2.3180	3.3032	0.4963	18	31	29	22
1	1	1	0	4.5391	2.4522	3.3363	0.3520	25	25	27	26
2	1	1	0	4.3816	2.4417	3.2709	0.1948	28	26	30	30
3	1	1	0	4.2796	2.2188	3.0815	0.0223	32	32	32	32
4	1	1	0	4.3202	2.3225	3.1676	0.1129	31	30	31	31
1	0	0	1	4.8972	2.9439	3.7970	0.8612	4	2	1	2
2	0	0	1	4.9005	2.8398	3.7305	0.8442	3	3	3	3
3	0	0	1	4.8964	2.7513	3.6703	0.7999	5	4	5	4
4	0	0	1	4.8922	2.5690	3.5451	0.7431	6	18	14	7
1	1	0	1	5.0000	2.7500	3.7081	0.9043	1	5	4	1
2	1	0	1	4.5217	2.6522	3.4630	0.3740	26	14	19	25
3	1	0	1	4.5789	2.6053	3.4539	0.4819	24	16	20	23
4	1	0	1	4.6579	3.0263	3.7545	0.6548	20	1	2	14
1	0	1	1	4.8250	2.7150	3.6194	0.7274	9	9	7	8
2	0	1	1	4.8351	2.6598	3.5861	0.7168	8	12	8	9
3	0	1	1	4.9011	2.5847	3.5592	0.7640	2	17	13	5
4	0	1	1	4.7857	2.5357	3.4836	0.6335	14	24	18	15
1	1	1	1	4.6552	2.6552	3.5157	0.5839	21	13	15	18
2	1	1	1	4.6463	2.7439	3.5706	0.5945	22	6	11	16
3	1	1	1	4.6190	2.5524	3.4336	0.4571	23	20	21	24
4	1	1	1	4.6838	2.7353	3.5793	0.5883	19	7	10	17

Table 8 reports the first and second order Utopia-Dystopia indices for the multi-variate multilateral comparisons together with their respective standard errors (the indices have an asymptotically normal distribution so that inference is feasible). The striking feature here is that the ordering doesn't change much when greater weight is placed upon poorer outcomes in the comparison process. No group moves more than two places in the ordering and 20 of the 32 groups do not change places at all, in particular the best joint outcome group (young single urban males) and the worst joint outcome group (single rural females in their 60's) retain their positions, but neither group is unambiguously best or worst. Generally, males outrank their female counterparts in almost every comparison and Urban dwellers outrank rural counterparts. There appears to be very little separation between subgroupings and there is no unambiguously best or worst group.

Table 8. 1st and 2nd order comparisons using Multivariate Utopia-Dystopia Indices.

Age	single	gender	urban	UD2	s.e.(UD2)	Rank UD2	2UD2	s.e.(2UD2)	Rank 2UD2
1	0	0	0	0.7467	0.0926	6	0.7340	0.0946	7
2	0	0	0	0.7044	0.0249	11	0.6959	0.0519	11
3	0	0	0	0.6586	0.0248	13	0.6498	0.0503	14
4	0	0	0	0.5817	0.0306	19	0.5850	0.0704	17
1	1	0	0	0.7147	0.0213	10	0.7002	0.0383	10
2	1	0	0	0.2896	0.0455	28	0.2873	0.1066	28
3	1	0	0	0.3154	0.0389	27	0.2940	0.0863	27
4	1	0	0	0.2531	0.0418	29	0.2353	0.0946	29
1	0	1	0	0.6593	0.0251	12	0.6523	0.0537	13
2	0	1	0	0.5693	0.0311	20	0.5668	0.0700	19
3	0	1	0	0.5215	0.0314	21	0.5214	0.0707	21
4	0	1	0	0.4963	0.0323	22	0.4935	0.0721	22
1	1	1	0	0.3520	0.0416	26	0.3549	0.0966	25
2	1	1	0	0.1948	0.0469	30	0.1992	0.1105	30
3	1	1	0	0.0223	0.0475	32	0.0174	0.1090	32
4	1	1	0	0.1129	0.0448	31	0.1042	0.1031	31
1	0	0	1	0.8612	0.0159	2	0.8541	0.0212	2
2	0	0	1	0.8442	0.0159	3	0.8321	0.0252	3
3	0	0	1	0.7999	0.0190	4	0.7893	0.0359	4
4	0	0	1	0.7431	0.0228	7	0.7389	0.0488	6
1	1	0	1	0.9043	0.0095	1	0.9204	0.0107	1
2	1	0	1	0.3740	0.0389	25	0.3417	0.0808	26
3	1	0	1	0.4819	0.0318	23	0.4673	0.0624	23
4	1	0	1	0.6548	0.0353	14	0.6584	0.0852	12
1	0	1	1	0.7274	0.0247	8	0.7136	0.0492	8
2	0	1	1	0.7168	0.0251	9	0.7047	0.0538	9
3	0	1	1	0.7640	0.0190	5	0.7547	0.0385	5
4	0	1	1	0.6335	0.0270	15	0.6221	0.0540	15
1	1	1	1	0.5839	0.0253	18	0.5592	0.0433	20
2	1	1	1	0.5945	0.0297	16	0.5789	0.0596	18
3	1	1	1	0.4571	0.0385	24	0.4532	0.0899	24
4	1	1	1	0.5883	0.0354	17	0.5935	0.0799	16

Finally, to get a sense of the distributional inequality in the collection of groups the Multilateral Transvariation, Distributional Gini and Ambiguity coefficients are provided in Table 9. The

Overlap is also related to Gini's Transvariation (Gini 1915) GT, (GT = 1-OV), generalized versions of which can be employed to study the extent of distributional variation in the collection of distributions (Anderson, Linton, Pittau, Whang and Zelli 2020) via Multilateral Transvariation (MGT) and Distributional Gini (DGINI) coefficients. They indicate a considerable amount of overlap (i.e. similarity) in the probability density functions with a greater amount of definition in their corresponding cumulative densities which results in a fairly low level of ambiguity which diminishes with the higher order of stochastic comparison (as it should, given the more restrictive family of Utility functions it represents).

Table 9.

	Multilateral Transvariation	Distributional Gini	Ambiguity Index 1	Ambiguity Index 2
Coefficient	0.0613	0.2120	0.2047	0.1908

Conclusions.

The scale dependence of most summary statistics and the arbitrary nature of Cantril type scales applied to ordinal categorical data poses an ambiguity problem for analysts when ranking and ordering groups. Different, but equally valid, scale choices can produce conflicting, ambiguous results which are not robust to choice of scale. The conditions required of comparator group distributions, which have been shown by Schroder and Yitzhaki (2017) to be necessary for robustness and the absence of ambiguity, seldom appear to hold in practice (Bond and Lang 2019). Here ranking, ordering and inequality measures for ordinal categorical data together with a measure of the ambiguity inherent in the data have been proposed and implemented in a study of health and loneliness in China. Subgroups defined by age, gender, partner status and urban/rural location were compared and contrasted. For comparison purposes, scale dependent group means were compared with a comparable robust scale independent measure and some notable differences in ranking were apparent.

As for the robust scale independent results, while the overall health and loneliness connection is not strong, the poor health and loneliness association appears strongest for single status individuals in rural environments and weakest in the youngest groups some substantive and robust scale independent conclusions can be drawn. Health conditioned un-loneliness status distributions and loneliness conditioned health status distributions imply that improved health status is always associated with better un-loneliness outcomes and improved un-loneliness status is always associated with better health outcomes. While a large portion of the population are not affected by loneliness, of those who are, ill health is generally more likely to be reported. With regard to the health loneliness joint distribution, generally, males enjoy better health and un-loneliness outcomes than their female counterparts in almost every comparison and urban dwellers outrank their rural counterparts. There appears to be very little separation between subgroupings (the distributional Gini coefficient is low) and there is no unambiguously best or worst group. However, the loneliest, health poorest individual is unequivocally and unambiguously by all measures a single rural female in her 60's and the un-loneliest healthiest individual is an urban partnered male in his 40's. Generally, loneliness and ill health increase with age, are more prevalent in Rural China and more prevalent in females.

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Appendix.

Assume independent subgroups indexed k=1,..,K with the i’th level self reported happiness level probability $p_{i,k}$ $i = 1,..,m$ stacked in the m x 1 vector $\underline{p_k}$ which is multinomial with a variance

$$V\left(\underline{p_k}\right) = \begin{pmatrix} p_1 & 0 & . & 0 \\ 0 & p_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & p_m \end{pmatrix} - \begin{pmatrix} p_1^2 & p_1p_2 & . & p_1p_m \\ p_2p_1 & p_2^2 & . & p_2p_m \\ . & . & . & . \\ p_mp_1 & p_mp_2 & . & p_m^2 \end{pmatrix}$$

Given the m x m dimensioned integrating matrix D, where:

$$D = \begin{pmatrix} 1 & 0 & . & 0 \\ 1 & 1 & . & 0 \\ . & . & . & . \\ 1 & 1 & . & 1 \end{pmatrix}$$

$\underline{F_k}$, the CDF of the k'th group is such that $\underline{F_k} = D\underline{p_k}$ with variance $DV\left(\underline{p_k}\right)D'$

Generally, $\underline{F_k} - \underline{F_l}$ will have a variance $V\left(\underline{F_k} - \underline{F_l}\right) = D\left(V\left(\underline{p_k}\right) + V\left(\underline{p_l}\right) - 2COV\left(\underline{p_k}, \underline{p_l}\right)\right)D'$

and $D\left(\underline{F_k} - \underline{F_l}\right)$ will have a variance $DV\left(\underline{F_k} - \underline{F_l}\right)D'$. Note that $COV\left(\underline{p_k}, \underline{p_l}\right) = 0$ with subgroup independence, however when considering either $\underline{F}^{UENV} - \underline{F_k}$ or $\underline{F_k} - \underline{F}^{LENV}$ this will not necessarily be the case because the two vectors under comparison may have common elements (essentially when an element in $\underline{F_k}$ is a component of the corresponding frontier.

If the respective envelopes do not contain elements of $\underline{F_k}$, independence and zero covariance will prevail since the envelopes will be made up of elements from distributions that are independent of $\underline{F_k}$[14]. However, letting $\underline{p}^{ENV}$ be the PDF implied by the upper or lower envelope, if the envelopes and $\underline{F_k}$ have elements in common, the respective rows and columns of the variance-covariance matrix will be 0 since it will be the case that $D2COV\left(\underline{p_k}, \underline{p}^{ENV}\right)D' = D\left(V\left(\underline{p_k}\right) + V\left(\underline{p}^{ENV}\right)\right)D'$ for those particular rows and columns and zero elsewhere and the Variance- covariance matrix in this case will thus be $D\left(V\left(\underline{p_k}\right) + V\left(\underline{p}^{ENV}\right)\right)D'$ with the corresponding rows and columns set to 0.

Typically, first and second order Utopia-Dystopia indices work with scaled sums of the differences which, letting d be the m dimensioned unit vector, are of the form:

$d'\left(\underline{F}^{UENV} - \underline{F_k}\right)$ and $d'D\left(\underline{F}^{UENV} - \underline{F_k}\right)$

With respective variances:

$d'V\left(\underline{F}^{UENV} - \underline{F_k}\right)d$ and $d'DV\left(\underline{F}^{UENV} - \underline{F_k}\right)D'd$.

For scaled indices where the scaling factors are $s_1 = d'\left(\underline{F}^{UENV} - \underline{F}^{LENV}\right)$ and $s_2 = d'D\left(\underline{F}^{UENV} - \underline{F}^{LENV}\right)$ respectively the corresponding variances would be:

[14] Similarly when $\underline{F}^{UENV}$ and $\underline{F}^{LENV}$ are being compared independence will prevail since by definition they will not have elements in common unless all distributions have a common element or elements which for most applications is unlikely.

$d'V(\underline{F}^{UENV} - \underline{F}_k)d/s_1^2$ and $d'DV(\underline{F}^{UENV} - \underline{F}_k)D'd/s_2^2$.

When estimates of the underlying $p_i's$ are maximum likelihood estimates asymptotic normality of the sums and differences can be claimed (Rao 1973) so that, based upon a null hypothesis of no difference:

$$\sqrt{T}d'(\underline{F}^{UENV} - \underline{F}_k) \sim N(0, d'V(\underline{F}^{UENV} - \underline{F}_k)d)$$

And

$$\sqrt{T}d'D(\underline{F}^{UENV} - \underline{F}_k) \sim N(0, d'DV(\underline{F}^{UENV} - \underline{F}_k)D'd)$$

Where T is the appropriate sample size factor.