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Misallocation Effects of Labor Market Frictions

By Stanislav Rabinovich and Ronald Wolthoff

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Stanislav Rabinovich† Ronald Wolthoff‡

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Abstract

We theoretically study misallocation of labor in a heterogeneous-firm model with imperfectly directed search. Some workers can direct their search, while others are uninformed about the location of wage offers ex ante and are assigned to job openings randomly. The main result is that too many workers apply to high-productivity firms, relative to the social optimum. This occurs because too many firms take advantage of their market power, attracting only random searchers. Because it is the low-productivity firms that do so, this induces all the directed searchers to concentrate at the high-productivity firms, a "flight-to-quality" phenomenon. Improvements in information have ambiguous effects on worker allocation, wages, and worker utility. A minimum wage can increase employment and welfare by reallocating workers across firms. With an endogenous entry choice, policy design meets with a tradeoff in balancing the misallocation inefficiency and a standard entry externality.

Keywords: Directed search, random search, labor markets, minimum wage, misallocation.

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†UNC Chapel Hill. Email: srabinov@email.unc.edu.

‡University of Toronto. Email: ronald.p.wolthoff@gmail.com.

1 Introduction

Do labor market frictions lead to an inefficient allocation of workers across jobs? Do policy interventions mitigate or exacerbate these distortions? These classic economic questions have returned to the forefront of economists' attention, thanks to recent evidence on rising labor market power and the ongoing debates about the desirability of policies such as the minimum wage. It is well-known that labor market frictions distort *aggregate* employment, and that the minimum wage has the potential to mitigate this inefficiency: this is a robust prediction of both the textbook monopsony model (e.g. Manning, 2011), and the search and matching literature that we build on here (e.g. Hosios, 1990). The focus on aggregates, however, potentially misses important distributional effects. If heterogeneous firms do not take advantage equally of their market power, employment will be distorted more at some firms than others, leading to misallocation of labor. Policy interventions, in turn, may affect firms unequally as well, inducing reallocation of workers across jobs. Understanding such allocative effects requires combining heterogeneity with labor market frictions.

In this paper we theoretically analyze misallocation of labor in a parsimonious framework with two key ingredients: *heterogeneous firms* and *partially directed search*. The labor market is frictional in the tradition of the competitive search literature: firms post wages, understanding that higher-wage job openings will be filled more quickly. Firms differ in their productivity. Workers differ in whether they can direct their search. A fraction of workers are directed searchers, who choose which job openings to apply for, understanding that higher posted wages attract more competing applicants. The remaining fraction are random searchers, who are assigned to vacancies randomly; hence their only decision is whether to accept or reject the posted wage. This modeling assumption can be interpreted as differences in information about available job openings. An alternative interpretation is differences in mobility, whereby some workers have lower costs of moving for a high-wage job than others. Firms in this economy will face a choice between attracting only random searchers or attracting both kinds of searchers. This friction will constrain both the socially efficient allocation and the equilibrium allocation of workers across firms of different productivities.[1]

Our first and main result is that, in equilibrium, too many workers are employed at high-productivity firms, compared to the social optimum. This contrasts with the intuitive prediction that lack of information results in too many workers at low-productivity firms. Both the constrained efficient allocation and the decentralized equilibrium in this environment are characterized by a threshold rule: firms above some productivity threshold attract

[1]The idea that either information costs or mobility costs lead to market power, and constrain efficient allocation of workers across firms, has a long tradition in the history of thought: see e.g. Pigou (1932), Robinson (1933), Manning (2003).

both random and directed searchers, whereas firms with productivity below this threshold attract only random searchers. However, the equilibrium productivity threshold is always higher than the socially efficient one. In other words, too many firms take advantage of their market power in equilibrium. These firms forego attracting directed searchers so as to extract more surplus from the random searchers. This induces too many directed searchers to queue up at the higher-productivity firms. As a result, there is a misallocation of labor away from the middle and toward the top of the productivity distribution, a "flight to quality" type phenomenon. As we discuss below, this flight to quality is a manifestation of a novel externality that would be absent if search were either purely directed or purely random.

Our second result is that improvements in information - captured by the fraction of directed searchers among workers - have ambiguous effects on worker allocation and wages. An increase in the fraction of directed searchers may lower wages and increase congestion at high productivity firms, and is more likely to do so when firms' productivity is very dispersed. In fact, we show that the effect of information on worker utility, allocation and wages depends on a tradeoff between the dispersion of productivity and the curvature of the matching technology. In particular, such a perverse effect of information would be impossible under homogeneous productivity. One intriguing implication of our finding is that, to the extent that innovations such as online job search led to improvements in information about available job openings, they need not have led to an increase in wages.

Our third result concerns the effect of the minimum wage. A suitably chosen minimum wage does not bind for the high-productivity firms that attract directed searchers, but binds for the low-productivity firms attracting random searchers only. The latter firms are then induced to pay higher wages, thereby attracting workers away from the high-productivity firms. This reallocation of workers is efficiency-improving since it alleviates the congestion at high-productivity firms; in fact, we show that an appropriately chosen minimum wage restores the constrained-efficient allocation. While the previous literature has focused overwhelmingly on the aggregate employment effects of the minimum wage, we emphasize its allocative effects. Notably, the minimum wage *does* affect employment in our framework, even in the absence of an extensive entry margin; in fact, the minimum wage just described raises employment. However, the reasons for this are distinct from the conventional narrative. In the standard monopsony model, mandating higher wages results in more workers willing to work because of an upward-sloping labor supply curve. In our model, mandating higher-wages results in a reallocation of workers from firms with many applicants to firms with few applicants; this raises employment in spite of total labor supply remaining constant.

Finally, we extend the model to allow endogenous entry choice by firms. In this case, the misallocation inefficiency described above coexists with a standard congestion externality from entry. Not surprisingly, a single policy instrument, such as a minimum wage, is insufficient to correct both inefficiencies simultaneously. However, we show that the combination of an appropriate hiring subsidy and a minimum wage can restore the constrained-efficient allocation.

1.1 Relationship to literature

Our paper builds on, and contributes to, the existing literature on imperfectly directed search, such as Lester (2011), Lentz and Moen (2017), Shi (2018), or Cheremukhin et al. (2020). Our model environment follows Lester (2011), in which a fraction of workers are directed searchers, while the remaining workers are random searchers. The model in Lester (2011) can be thought of as a special case of ours, with homogeneous firms; we provide a formal discussion of this in Section 4.3.[2] In this setting, like ours, the fraction of firms that choose to attract only random searchers differs from what would be socially efficient. However, in Lester (2011) all the firms are identical, and hence the planner would set the measure of firms attracting only random searchers to zero. In fact, because firms are homogeneous, the fraction of random searchers among workers does not affect the constrained-efficient allocation. We contribute to this literature by introducing heterogeneity among firms. As a result of this important modification, the information friction constrains the social planner as well, and in general there is a non-zero measure of firms exclusively targeting random searchers even at the social optimum. Constrained inefficiency is now driven by a suboptimal allocation of workers across firms of differing productivities.

Our paper is directly relevant for work on labor misallocation induced by search frictions. This literature has recognized that frictions can lead to an inefficient composition of jobs or inefficient allocation of workers among them. For example, in Bertola and Caballero (1994), Acemoglu (2001), and Davis (2001), too many workers are allocated to low-productivity firms: there, search is random, and the key friction leading to inefficiency is akin to an investment holdup problem in the labor market. In Acemoglu and Shimer (1999) and Golosov et al. (2013), search is directed, but workers are risk-averse; these papers show that there will likewise be misallocation towards low-productivity firms if workers cannot insure against the risk of not finding a job. In Galenianos et al. (2011), as in our work, it is market power, driven by a finite number of firms, that leads to misallocation of workers, namely too many workers being employed at low-productivity firms. In our paper, where market power is

[2] See also Bethune et al. (2020) for an application to a monetary economy. Both Lester (2011) and Bethune et al. (2020) are set in the context of a product market rather than a labor market, and hence agents are buyers and sellers rather than workers and firms, but this distinction is inconsequential except for the interpretation.

instead driven by imperfectly directed search, we show that there is misallocation toward *high-productivity* firms, contrary to the previous literature. Notably, such a result would not arise when search is either fully random or fully directed. We also share with Galenianos et al. (2011) the prediction that a binding minimum wage reallocates workers towards low-productivity firms; however, in our setting this reallocation can be welfare-improving. More broadly, we believe that our results provide a new perspective on misallocation. Since too many workers apply to high-productivity firms, average output per employed worker is *higher* that in the constrained-efficient outcome. Misallocation in our model manifests itself in the aggregate as low employment, not low productivity.

Finally, our framework and results extend existing analysis of congestion externalities in search and matching, emanating from Hosios (1990) and generalized by work such as Albrecht et al. (2010), Masters (2015), Julien and Mangin (2017), Mukoyama (2019), and Julien and Mangin (2019); all of these papers study versions of the Hosios condition in a random search environment. The standard congestion externality identified in the literature stems from the fact that a firm does not internalize the effect of its entry on the hiring probabilities of other firms. The misallocation inefficiency found in our paper is the result of a novel externality. When a firm restricts its employment in order to exploit random searchers, it foregoes hiring directed searchers, but does not internalize the congestion these directed searchers cause at other firms. In the extension with endogenous entry, this misallocation inefficiency is combined with the standard congestion externality from entry, leading to a model that, as we argue in Section 5, can be thought of as generalizing the work of Albrecht et al. (2010), Masters (2015), and Julien and Mangin (2017) to a partially directed search environment.

This paper is organized as follows. After describing the environment in Section 2, we first characterize the planner's solution in Section 3. Section 4 presents the main results of the paper: we characterize the decentralized equilibrium (4.1) and show that the equilibrium features misallocation (4.2), that information has ambiguous effects (4.4), and that a minimum wage can be welfare-improving (4.5). Section 5 considers an extension with endogenous entry by firms. Finally, Section 6 concludes.

2 Environment

We consider a static model. There is a measure 1 of firms, indexed by $j \in [0,1]$, each with one vacancy.[3] Firms are heterogeneous in productivity: we denote the productivity of firm j by $y(j)$, which we assume to be continuous and strictly increasing in j. On its support, the cumulative distribution of productivity F is therefore given by $F(x) = \sup\{j | y(j) \leq x\}$,

[3]Section 5 considers an extension with endogenous entry of firms.

which is continuous and strictly increasing in x.[4]

There is a measure 1 of workers, all initially unemployed. A fraction $\psi \in [0,1]$ of workers are random searchers, who will be assigned randomly across all the vacancies. The remaining fraction $1-\psi$ are directed searchers, who can choose which vacancy to target. Matching works as follows. If the searcher-vacancy ratio at a particular vacancy is λ, the vacancy gets filled with probability $m(\lambda)$, and each worker applying to that vacancy has a probability $m(\lambda)/\lambda$ of being matched. The matching function $m(\lambda)$ satisfies the standard assumptions $m(\lambda) \leq 1$, $m' > 0, m'' < 0$, as well as $m(0) = 0$, $\lim_{\lambda \to 0} m'(\lambda) = \infty$ and $\lim_{\lambda \to \infty} m'(\lambda) = 0$. For future reference, we also define the elasticity $\epsilon(\lambda) = \lambda m'(\lambda)/m(\lambda)$, and the function $g(\lambda) = m(\lambda) - \lambda m'(\lambda)$. By the assumptions on m, we see that g satisfies $g'(\lambda) > 0$ and $g(\lambda) < m(\lambda)$. A worker matched with firm j produces $y(j)$; unmatched workers and firms produce 0.

Note that if $\psi = 0$, this is a standard competitive search environment. However, with $\psi > 0$, there will be at least ψ workers at each vacancy. Hence, the queue length at each vacancy satisfies $\lambda \geq \psi$. At the other extreme, $\psi = 1$, we have a standard random search environment in which $\lambda = 1$ always.

There are multiple ways to interpret the partial randomness of search. One is in terms of differential information: random searchers could be viewed as workers uninformed about the posted wages of the various job openings. Alternatively, one could interpret some workers as more mobile across firms than others. The idea that frictions such as imperfect information or imperfect mobility lead to market power - and impede efficient allocation of labor - has a long tradition in economics; see e.g. Pigou (1932), Robinson (1933), Manning (2003). Crucially, information or mobility frictions that constrain individual workers will also constrain the social planner - an insight that will be important below.

3 Planner's problem

The planner chooses the distribution of workers across posted vacancies. Thus, the planner's problem can be written as choosing $\lambda(j)$ for every $j \in [0,1]$ so as to maximize

$$\int_0^1 m(\lambda(j))\, y(j)\, dj. \tag{1}$$

[4] The homogeneous-productivity economy can be dealt with by considering the limiting case as $y(0) \to y(1)$. We discuss the homogeneous-productivity case formally in Section 4.3.

The planner maximizes (1) subject to two constraints. First, there is a resource constraint, which says that the total measure of workers at all the vacancies must add up to 1:

$$\int_0^1 \lambda(j)\, dj = 1. \tag{2}$$

Second, and crucially, the planner must respect the randomness of search for some workers. This means that the planner must assign the ψ random searchers randomly across all the posted vacancies, thereby assigning at least ψ workers to each vacancy. Hence, the random search constraint states

$$\lambda(j) \geq \psi \quad \forall j \in [0,1]. \tag{3}$$

Let η be the Lagrange multiplier on (2), and let $\mu(j)\, dj$ be the Lagrange multiplier on (3) for each j. The first-order condition for $\lambda(j)$ can be written as

$$\mu(j) = \eta - m'(\lambda(j))\, y(j). \tag{4}$$

When constraint (3) does not bind, we have $m'(\lambda(j))\, y(j) = \eta$ and $\mu(j) = 0$, yielding $\lambda(j) = (m')^{-1}(\eta/y(j))$. Whenever constraint (3) binds, we have $\lambda(j) = \psi$ and $\mu(j) = \eta - m'(\psi)\, y(j) > 0$. We can therefore write

$$\lambda_p(j) = \max\left\{\psi, (m')^{-1}(\eta/y(j))\right\}, \tag{5}$$

where the subscript p denotes the planner's allocation throughout. Moreover, $\eta - m'(\psi)\, y(j)$ is clearly decreasing in j. Therefore, the constraint binds for all j below some threshold j_p and does not bind for j above it. This threshold is given by $j_p = \inf\{j : m'(\psi)\, y(j) \geq \eta\}$; if $j_p > 0$, it is the unique solution to $m'(\psi)\, y(j_p) = \eta$. We can then rewrite (5) as

$$\lambda_p(j) = \begin{cases} \psi, & j < j_p \\ (m')^{-1}(\eta/y(j)), & j \geq j_p \end{cases} \tag{6}$$

The queue length $\lambda_p(j)$ thus defined is non-increasing in η for each j, and non-decreasing in j. The multiplier η is then pinned down as the unique value for which the resource constraint holds with equality, i.e.

$$j_p\psi + \int_{j_p}^1 (m')^{-1}(\eta/y(j))\, dj = 1. \tag{7}$$

Uniqueness of the solution follows from the fact that the integral is strictly decreasing in η. This can be summarized as

Lemma 1. *The constrained-efficient allocation is characterized by a number η, a threshold $j_p \in [0,1]$ and a function $\lambda_p(\cdot)$ satisfying (5), (6), and (7). There exists a solution to this system, and it is unique.*

Proof. See Appendix A.1. □

Note that the continuity of $y(j)$ implies the continuity of $\lambda_p(j)$, which will be important below in the comparison to the market equilibrium.

Next, we consider the conditions under which constraint (3) in fact binds. The following result shows that this requires productivity dispersion and the fraction of random searchers to be sufficiently large.

Lemma 2. *A necessary and sufficient condition for $j_p > 0$ is*

$$\int_0^1 (m')^{-1}\left(m'(\psi)\,\frac{y(0)}{y(j)}\right) dj > 1. \tag{8}$$

Proof. See Appendix A.2. □

Intuitively, the partial randomness of search captured by (3) is more likely to severely constrain the social planner when productivity is very dispersed. In fact, consider the limiting case with homogeneous productivity. The social planner would never want to assign different queue lengths to firms with the same productivity, due to the concavity of the matching function; with constant productivity, the social planner would therefore assign the same $\lambda = 1$ to all firms, and (3) clearly would not bind. Further, even when productivity is dispersed, the random search constraint (3) would not bind for a low enough ψ.

4 Equilibrium

We now analyze the decentralized equilibrium and show how and in what respects it differs from the planner's allocation. Each firm decides what wage to post. The $1-\psi$ directed searchers observe all the posted wages and decide to which firm to apply. As is standard in competitive search theory, we restrict attention to symmetric applications strategies. The ψ random searchers are assigned to vacancies randomly. The combination of these choices determines the queue length at each firm, and hence its profits, as a function of the wage it posted.

The definition of equilibrium requires us to specify the queue length $\lambda^*(w)$ attracted by a firm as a function of the wage w it posts, even for wages that are not posted in equilibrium. This is specified as follows. If a firm posts a wage w and attracts a queue length λ, the utility of a directed searcher applying to that firm is $\frac{m(\lambda)}{\lambda}w$. Define the *market utility* $\mathcal{U}$ to be the maximum utility across all submarkets that a directed searcher can obtain:

$$\mathcal{U} \equiv \max_{w,\lambda} \frac{m(\lambda)}{\lambda} w, \tag{9}$$

where the maximization is performed over all the submarkets w, λ active in equilibrium. While $\mathcal{U}$ is an equilibrium object, each firm takes it as given when deciding what wage to post. In particular, each firm understands that, if the wage-queue combination it offers provides utility of less than $\mathcal{U}$, then it will not attract any directed searchers, and its queue length must therefore equal ψ. As for random searchers, their only decision is whether to accept or reject the posted wage of the firm they meet; this decision simply constrains wages to be non-negative due to individual rationality. Each firm j maximizes its profits taking this worker behavior into account.

Definition 1. *An equilibrium consists of a market utility $\mathcal{U}$, a function $\lambda^*(w) \geq \psi$, and a function $w(j)$ that satisfy:*

1. *Worker optimization:*

$$\frac{m(\lambda^*(w))}{\lambda^*(w)} w \leq \mathcal{U} \ \forall w \tag{10}$$

and

$$\frac{m(\lambda^*(w))}{\lambda^*(w)} w < \mathcal{U} \implies \lambda^*(w) = \psi \tag{11}$$

2. *Firm optimization: for each $j \in [0,1]$,*

$$w(j) \in \arg\max_{w' \geq 0} m(\lambda^*(w'))(y(j) - w') \tag{12}$$

3. *Market clearing:*

$$\int_0^1 \lambda^*(w(j))\, dj = 1 \tag{13}$$

The first two items formalize the optimizing behavior of workers and firms described above. The market-clearing condition (13) is the analogue of (2), stating that the total measure of workers adds up to one. The resulting equilibrium *allocation* consists of an assignment of queue length to each firm, $\lambda_d(j)$, satisfying $\lambda_d(j) = \lambda^*(w(j))$.

4.1 Characterization, existence and uniqueness

We now characterize the equilibrium allocation $\lambda_d(j)$. Consider a firm's choice of what wage to post. If firm j offers less than the market utility $\mathcal{U}$, it will attract random searchers only, therefore receiving profits $m(\psi)(y(j)-w)$. From this it is easy to conclude that a firm that chooses not to attract directed searchers will offer a wage of zero, and its profits are therefore

$$\pi^R(j) = m(\psi)\, y(j)\,. \tag{14}$$

On the other hand, if firm j would like to attract some directed searchers, it solves the problem

$$\pi^D(j) = \max_{w,\lambda} m(\lambda)(y(j)-w) \tag{15}$$

subject to

$$\frac{m(\lambda)}{\lambda} w \geq \mathcal{U}. \tag{16}$$

It is easy to see that (16) will bind. Solving for w using the binding constraint (16) and substituting into (15), we get the maximization problem

$$\pi^D(j) = \max_{\lambda} m(\lambda)\, y(j) - \lambda\mathcal{U}; \tag{17}$$

the solution satisfies

$$m'(\lambda)\, y(j) = \mathcal{U}, \tag{18}$$

and the maximized profit is therefore

$$\pi^D(j) = g\left((m')^{-1}(\mathcal{U}/y(j))\right) y(j)\,, \tag{19}$$

recalling that $g(\lambda) = m(\lambda) - \lambda m'(\lambda)$. Comparing (19) to (14), we conclude firm j weakly prefers to attract directed searchers if and only if

$$g\left((m')^{-1}(\mathcal{U}/y(j))\right) \geq m(\psi) \tag{20}$$

Since m is concave and $y(j)$ is continuous and strictly increasing, the left-hand side of (20) is strictly increasing in j. This means that there is a unique threshold

$$j_d = \inf\left\{ j : g\left((m')^{-1}(\mathcal{U}/y(j))\right) \geq m(\psi) \right\}, \tag{21}$$

which, at an interior ($j_d > 0$) equilibrium, solves $g\left((m')^{-1}(\mathcal{U}/y(j_d))\right) = m(\psi)$. The equilibrium queue length of firm j, denoted by $\lambda_d(j)$, then satisfies

$$\lambda_d(j) = \begin{cases} \psi, & \text{if } j < j_d, \\ (m')^{-1}(\mathcal{U}/y(j))\,, & \text{if } j \geq j_d. \end{cases} \tag{22}$$

Finally, the market utility $\mathcal{U}$ of directed searchers is pinned down by the market clearing condition

$$j_d\psi + \int_{j_d}^{1} (m')^{-1} (\mathcal{U}/y(j))\, dj = 1. \tag{23}$$

To summarize, we have

Lemma 3. *The decentralized equilibrium allocation is characterized by a market utility $\mathcal{U}$, a threshold j_d, and a function $\lambda_d(j)$ satisfying (21), (22) and (23). There exists a decentralized equilibrium, and it is unique.*

Proof. See Appendix A.3. ☐

An immediate corollary, which will be important below, is that the equilibrium queue length $\lambda_d(j)$ is necessarily discontinuous at j_d, as long as $j_d > 0$. This follows directly since $g(\lambda) < m(\lambda)$ for any λ, and so $g(\lambda) = m(\psi)$ requires $\lambda > \psi$.

Corollary 1. *If $0 < j_d < 1$, the equilibrium queue length $\lambda_d(j)$ is discontinuous at $j = j_d$.*

This result has a very clear and interesting economic interpretation. Intuitively, suppose that firm j_d is indifferent between posting a wage of 0 (hence attracting a queue length of ψ) and posting a strictly positive wage high enough to attract directed searchers. Since the latter entails a discrete increase in the wage, indifference requires a discrete increase in the queue length.

4.2 Constrained inefficiency

We next compare the equilibrium to the constrained efficient allocation. To understand the mechanism leading to the main result below, consider the tradeoff for a firm $j > j_p$, for whom $\lambda_p(j) > \psi$. In order for the equilibrium to be efficient, $\lambda_p(j)$ must also be the equilibrium queue length faced by such a firm; but then it must hold that $g(\lambda_p(j)) \geq m(\psi)$, since otherwise the firm would prefer to forego attracting directed searchers and exploit the random searchers instead. If the queue length $\lambda_p(j)$ prescribed by the social planner is not high enough, a firm that should efficiently be attracting directed searchers may wish not to do so in equilibrium. This logic leads to the main inefficiency result.

Proposition 1. *Assume $\psi \in (0,1)$. Let $\eta, \lambda_p(\cdot)$ be the constrained efficient allocation, with the corresponding threshold j_p. Let $\mathcal{U}, \lambda_d(\cdot)$ be the decentralized equilibrium allocation, with the corresponding threshold j_d. Then the following hold: (i) $\mathcal{U} \leq \eta$, (ii) $j_d \geq j_p$, and (iii) $\lambda_d(j) \geq \lambda_p(j)$ for all $j \geq j_d$. Moreover, if either (8) holds or $g(1) < m(\psi)$, then the allocation is constrained inefficient and (i)-(iii) hold strictly.*

Proof. See Appendix A.4. □

The productivity threshold for attracting directed searchers is at least as high in the market equilibrium as the constrained-efficient one. Moreover, this inequality is strict—and hence the equilibrium is inefficient—if either the constrained efficient threshold is interior, or the fraction of random searchers is high enough, or both.

Suppose that (8) holds; in this case, the constrained-efficient threshold j_p is strictly positive, i.e. the partial randomness of search in fact constrains the social planner. The constrained-efficient queue length $\lambda_p(j_p)$ is clearly continuous at j_p. Now, consider the equilibrium queue length $\lambda_d(j)$, given by (22). Unlike the planner's $\lambda_p(j)$, the equilibrium queue length is necessarily discontinuous at the threshold j_d by Corollary 1, as discussed above. In other words, there is a jump in the queue length at the threshold j_d. This immediately shows that the equilibrium allocation is constrained-inefficient. Next, we argue that this implies $j_d > j_p$. The key technical point is that, for any $j \geq \max\{j_p, j_d\}$, we must have $m'(\lambda_d(j))/m'(\lambda_p(j)) = \mathcal{U}/\eta$, or, in other words, the equilibrium and planner's queue lengths cannot intersect for $j \geq \max\{j_p, j_d\}$. For market clearing to hold, i.e. for the average equilibrium queue length to still equal 1 despite the jump at j_d, it must be that the threshold is strictly greater than the planner's threshold j_p.

Alternatively, suppose that the constrained-efficient allocation is a corner solution: $j_p = 0$. In this case, the constrained-efficient allocation has all the firms attracting a queue length strictly greater than ψ; but note that the lowest queue length must satisfy $\lambda_p(0) \leq 1$; otherwise, the resource constraint would be violated.[5] Then, if $g(1) < m(\psi)$, this allocation is inconsistent with equilibrium, since the lowest-productivity firm would prefer not to attract directed searchers. As a result, if $g(1) < m(\psi)$, the equilibrium must have $\lambda_d(0) = \psi$ and hence $j_d > 0$, by the argument made above that $\lambda_p(j)$ and $\lambda_d(j)$ do not cross.

Consider now the implications of this inefficiency for the distribution of workers across firms. If $j_d > j_p$, too many firms attract a queue length of ψ rather than a strictly higher queue length. In other words, too few firms attract random searchers only rather than attracting some directed searchers. Market clearing implies that the firms that do attract directed searchers necessarily attract more of them than socially optimal, hence $\lambda_d(j) > \lambda_p(j)$ for all $j \geq j_d$. This main result is illustrated in Figure 1, which plots both $\lambda_p(j)$ and $\lambda_d(j)$, for the case of an interior j_p. The planner's queue length $\lambda_p(j)$ is continuous everywhere, flat at ψ for $j \leq j_p$ and strictly increasing thereafter. The equilibrium $\lambda_d(j)$ is continuous in j for all $j \neq j_d$, but jumps from ψ to some $\Lambda_d = (m')^{-1}(\mathcal{U}/y(j_d)) > \psi$ at j_d, and stays strictly above $\lambda_p(j)$ thereafter.

[5] In fact, we must have $\lambda_p(0) < 1$ strictly, except in the limiting case of homogeneous productivity, considered in section 4.3, for which $\lambda_p(j) = 1\ \forall j$.

Intuitively, in equilibrium, too many firms choose to attract only random searchers in order to extract the full surplus from those random searchers. As a result, all the directed searchers are allocated to a smaller subset of firms, leading to a higher queue length at those firms. Since it is the low-productivity firms that choose to attract random searchers and the high-productivity firms that continue to attract directed searchers, this results in too many workers employed at the high-productivity firms, relative to the social optimum. Note that this misallocation of labor manifests itself as lower employment, not lower productivity. In fact, average output per employed worker is higher in equilibrium than under the efficient allocation, since more searchers queue up at higher-productivity firms.

The misallocation of labor can be interpreted as the result of a novel congestion externality. When a firm decides whether or not to attract directed searchers, it does not internalize the effect this has on congestion at other firms. Specifically, when low-productivity firms forego attracting directed searchers by posting a low wage, they do not internalize that these directed searchers will now apply for higher-productivity firms, driving up congestion at those firms. Such an externality would, of course, be absent if search were fully directed ($\psi = 0$), since firms then fully internalize the effect of their posted wage on congestion through market utility. Such an externality would also be absent if search were fully random ($\psi = 1$), because in this case the wage is not allocative: what is crucial is that directed searchers are driven away when a firm decides to exploit random searchers.

The inefficiency can also be interpreted in terms of a labor demand distortion. Recall that in a standard directed search framework, the market utility $\mathcal{U}$ can be interpreted as the shadow price of labor, as is evident from the formulation in (17). When search is purely directed, we have $\mathcal{U} = \eta$; the shadow price of labor is equal to the planner's shadow value. When search is imperfectly directed, firms can hire random searchers, so they perceive labor as less costly, and we have $\mathcal{U} < \eta$. Thus, in equilibrium, directed searchers are "too cheap," and hence the firms that do hire directed searchers demand too many of them.

4.3 Comparison to the homogeneous-productivity case

It is instructive to compare our results to the homogeneous-productivity environment, which is the special case handled in the previous literature, such as Lester (2011) and Bethune et al. (2020). Suppose that $y(j) = \overline{y}$ for all j. This is equivalent to a productivity distribution putting probability 1 on $\overline{y}$, and can be thought of as the limiting case of our model as $y(0) \to y(1)$.[6] In this case, the constrained-efficient allocation has $j_p = 0$ and $\lambda_p(j) = 1$ for all j (in particular, (8) trivially does not hold). This is because, by the concavity of

[6] The analysis below closely follows Lester (2011); we include it for completeness and make no claims of originality here.

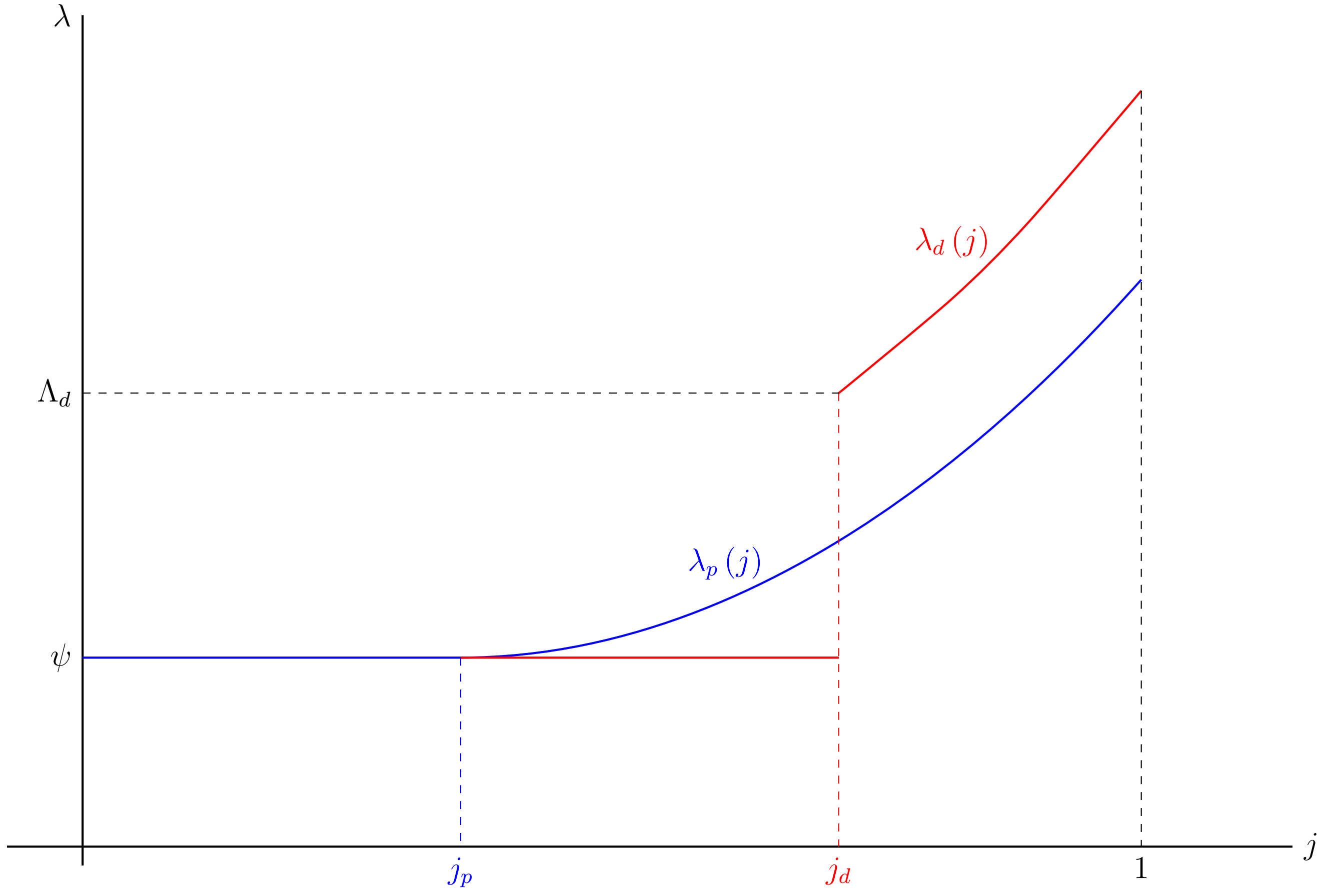

Figure 1: Equilibrium ($\lambda_d(j)$) vs. efficient allocation ($\lambda_p(j)$).

the matching technology, the social planner would have all the workers searching in one submarket, and that submarket has queue length strictly larger than ψ. In other words, if all the firms are identical, the partial randomness of search is immaterial for the social planner.

To characterize equilibrium behavior, suppose first that the equilibrium has all the firms efficiently choosing identical queue lengths, $\lambda_d(j) = 1$. Each firm then attracts directed searchers, getting profits of $g(1)\overline{y}$. In order for this to be optimal, we must have $g(1) \geq m(\psi)$. If $g(1) < m(\psi)$, however, this cannot be an equilibrium, and the unique equilibrium must have firms randomizing between attracting directed searchers and not doing so. Because of homogeneous productivity, all the firms attracting directed searchers will have the same queue length, denoted Λ_d. Since the equilibrium has identical-productivity firms randomizing, we must have the indifference condition $g(\Lambda_d) = m(\psi)$. Without loss of generality, assume that firms with $j < j_d$ have $\lambda(j) = \psi$, and firms with $j \geq j_d$ have $\lambda(j) = \Lambda_d$; the threshold j_d must then satisfy the market-clearing condition $j_d\psi + (1 - j_d)\Lambda_d = 1$. This yields

Corollary 2. *In the homogeneous-productivity case, the equilibrium takes one of two forms:*

1. *If $m(\psi) \leq g(1)$, the equilibrium allocation is constrained-efficient: $j_d = 0$ and $\lambda(j) = 1$ for all j.*

2. *If $m(\psi) > g(1)$, the equilibrium allocation is constrained-inefficient; $\lambda(j) = \psi$ for $j < j_d$ and $\lambda(j) = \Lambda_d$ for $j \geq j_d$, where*

$$j_d = \frac{\Lambda_d - \psi}{\Lambda_d - 1} \tag{24}$$

and $g(\Lambda_d) = m(\psi)$.

With homogeneous firms, $g(1) \geq m(\psi)$ is both necessary and sufficient for constrained efficiency. Intuitively, when ψ is high, firms are indifferent between attracting directed searchers and exploiting random searchers; this leads to identical firms choosing different queue lengths, which is socially suboptimal. This can be thought of as a limiting case of our misallocation result.

4.4 Comparative statics with respect to ψ

The analysis leading to Proposition 1 has established that imperfectly directed search leads to a misallocation of labor, which has a "flight to quality" nature: too many workers queue up at high-productivity firms. A natural conjecture is that an improvement in information, i.e. a decrease in ψ, would mitigate this flight to quality. We will show that the opposite may be true: an improvement in information may lower wages and increase congestion at high-productivity firms. We provide conditions for this to occur, which suggest that such a reversal is more likely when there is a lot of heterogeneity. We focus on the effect of ψ on market utility, $\mathcal{U}$. Since $\lambda_d(j) = (m')^{-1}(\mathcal{U}/y(j))$ for $j \geq j_d$, there is a direct mapping from $\mathcal{U}$ to the queue length; in particular a higher $\mathcal{U}$ is equivalent to lower queue length at high-productivity firms. Moreover, under the natural condition that the elasticity of the matching function $\epsilon(\lambda)$ is decreasing in λ, a higher $\mathcal{U}$ implies higher wages. The analysis will therefore have implications for, e.g., the response of wages to improvements in information technology.

We focus throughout on the case where $y(j)$ is continuously differentiable in j, and parameters are such that the equilibrium is interior, i.e. $j_d > 0$. In this case, the threshold firm j_d is exactly indifferent between attracting directed searchers and not doing so:

$$g\left(\left(m'\right)^{-1}\left(\mathcal{U}/y\left(j_d\right)\right)\right) = m\left(\psi\right) \tag{25}$$

Together, the indifference condition (25) and the market-clearing condition (23) constitute a system of two equations in two unknowns, uniquely determining market utility $\mathcal{U}$ and threshold j_d. Differentiating this system with respect to ψ yields:

Proposition 2. *Suppose that $y(j)$ is continuously differentiable in j. We have $\frac{dj_d}{d\psi} \geq 0$ always, with strict inequality unless $j_d = 0$; and*

$$\frac{d\mathcal{U}}{d\psi} > 0 \iff \frac{m'\left(\psi\right)\left(\Lambda_d - \psi\right)}{m\left(\Lambda_d\right) - m\left(\psi\right)} < \frac{j_d y'\left(j_d\right)}{y\left(j_d\right)} \tag{26}$$

where $g\left(\Lambda_d\right) = m\left(\psi\right)$.

Proof. See Appendix A.5. □

An improvement in information unambiguously lowers the measure of firms who attract only random searchers. However, it has an ambiguous effect on the market utility of directed searchers. To understand why, note that $\mathcal{U} = m'\left(\Lambda_d\right) y\left(j_d\right)$, where Λ_d is the queue length of the threshold firm j_d, satisfying $g\left(\Lambda_d\right) = m\left(\psi\right)$. An improvement in information, i.e. a decrease in ψ, lowers Λ_d, raising $m'\left(\Lambda_d\right)$. However, it also lowers j_d and hence $y\left(j_d\right)$. These have opposing effects on $\mathcal{U}$. Intuitively, an improvement in information draws more firms into attracting directed searchers, but the marginal firms that get drawn into this market are lower-productivity. Since market utility is pinned down by the productivity of the marginal firm, the latter effect lowers market utility. In other words, it is the marginal firm that "prices" a directed searcher. Which effect dominates depends on the relative size of the effect on Λ_d and $y\left(j_d\right)$, which in turn depends on the tradeoff, shown in (26), between the concavity of the matching function and the dispersion of productivity. An improvement in information lowers market utility if the productivity of the threshold firm changes much more than the queue length of the threshold firm, which happens when productivity is very dispersed.

Note than improvement in information necessarily cannot decrease market utility in the homogeneous-productivity case.

Corollary 3. *Suppose $y(j) = \overline{y}$ for all j. Then:*

1. *If $m\left(\psi\right) \leq g\left(1\right)$, then $\frac{d\mathcal{U}}{d\psi} = 0$.*
2. *If $m\left(\psi\right) > g\left(1\right)$, then $\frac{d\mathcal{U}}{d\psi} < 0$.*

This is precisely the result from Lester (2011). That paper also showed that this result can be reversed in the finite-agent version of the economy, where an improvement in information can indeed reduce wages. Here, we have shown that an improvement in information can reduce wages in the economy with a continuum of agents, if firms are sufficiently heterogeneous. The channel behind this reversal is different. In Lester (2011), the key intuition is that an increase in the number of informed workers would raise firms' incentives to post high wages, but would also raise the degree of competition among the informed workers. The former was shown to always dominate in the continuum economy, but not in the finite economy. Here, we show that the effect of information is ambiguous in the continuum economy with heterogeneous firms: an improvement in information raises firms' incentive to attract directed searchers, but lowers the productivity of the marginal firm attracting directed searchers.

Our results imply that an improvement in information, captured by an increase in the fraction of directed searchers, may lower wages and raise congestion at high-productivity firms. Moreover, we argued that such an outcome is more likely if there is a lot of heterogeneity. In the context of recent labor market trends, our results suggest that improvements in information technology, such as online job search, are not necessarily at odds with the observed fall in the labor share, especially in light of the high and growing productivity dispersion.

4.5 Policy implications: the minimum wage

The inefficiency identified in our analysis naturally raises the question of whether simple policy interventions can improve worker allocation and welfare. Our focus on the minimum wage is motivated by its prominence in policy debates surrounding growing employer market power. In particular, it is well known that employer market power can easily reverse theoretical predictions regarding the effects of the minimum wage on *total* employment (see e.g. Stigler, 1946; Bhaskar et al., 2002; Manning, 2011). Here, we identify a complementary mechanism through which the minimum wage may mitigate the *misallocation* of workers across firms.

Consider the effect of introducing a minimum wage w_{min}. We will focus on parameters such that (8) holds. The definition of equilibrium is largely unchanged from Definition 1, except that firms' profit maximization in Equation (12) is now performed subject to the constraint $w' \geq w_{\text{min}}$. There are two cases to consider. A minimum wage below a threshold will bind for firms attracting only random searchers (who would otherwise pay a wage of zero), but will not bind for firms attracting directed searchers. In this case, the equilibrium queue length $\lambda^*(w)$ and market utility $\mathcal{U}$ are such that $\lambda^*(w_{\min}) = \psi$ and $\frac{m(\lambda^*(w_{\min}))}{\lambda^*(w_{\min})} w_{\min} < \mathcal{U}$.On the other hand, if the minimum wage is above the threshold, it will also bind for at least some firms attracting directed searchers. In this case, we have $\lambda^*(w_{\min}) \geq \psi$ (strict if the minimum wage is strictly above the threshold) and $\frac{m(\lambda^*(w_{\min}))}{\lambda^*(w_{\min})} w_{\min} = \mathcal{U}$. This is formalized in the following intuitive result:

Lemma 4. *A minimum wage $w_{\min}$ is binding for some firms attracting directed searchers if and only if $w_{\min} \geq \epsilon(\psi)\, y(j_p)$.*

Proof. See Appendix A.6. □

The threshold $\epsilon(\psi)\, y(j_p)$ is the "shadow" wage for the marginal firm attracting directed searchers in the constrained-efficient allocation. Intuitively, suppose that some of the firms attracting directed searchers pay the minimum wage; such firms would have a queue length strictly higher than ψ. Since a firm attracting only random searchers would still have to pay the minimum wage, there are no firms that attract only random searchers. In other words, the minimum wage is so high that the presence of ψ random searchers is immaterial for the equilibrium allocation.[7] In particular, this means that the minimum wage "overshoots" the planner's shadow wage.

For this reason, we focus on the case when $w_{\min} < \epsilon(\psi)\, y(j_p)$ and characterize the effects of a raise in the minimum wage in this range. We first observe that the equilibrium still has the cutoff property, whereby firms below some j_d attract random searchers only. Since the minimum wage does not bind for any firm attracting directed searchers, the profit of such a firm looks the same, taking market utility $\mathcal{U}$ as given. Therefore, we have

$$\pi^D(j) = g\left((m')^{-1}(\mathcal{U}/y(j))\right) y(j) \tag{27}$$

On the other hand, if a firm chooses to only attract random searchers, it must pay them at least the minimum wage $w_{\min}$. Its profit is therefore

$$\pi^R(j) = m(\psi)(y(j) - w_{\min}) \tag{28}$$

Applying the envelope theorem to $\pi^D(j)$, we obtain

$$\frac{d}{dj}\left[\pi^D(j) - \pi^R(j)\right] = m\left((m')^{-1}(\mathcal{U}/y(j))\right) - m(\psi) > 0 \tag{29}$$

[7] In fact, under such a minimum wage the equilibrium allocation to the equilibrium of an economy with $\psi = 0$.

This implies that $\pi^D(j) \geq \pi^R(j)$ if and only if $j \geq j_d$, where the unique cutoff j_d satisfies

$$g\left((m')^{-1}(\mathcal{U}/y(j_d))\right)y(j_d) = m(\psi)(y(j_d) - w_{\min}) \tag{30}$$

As before, the queue length is determined by

$$\lambda_d(j) = \begin{cases} \psi, & j \leq j_d \\ (m')^{-1}(\mathcal{U}/y(j)), & j > j_d \end{cases} \tag{31}$$

The equilibrium is therefore pinned down by (30), (31), and the market-clearing condition.

We now consider the effect of raising the minimum wage. Taking as given the market utility, this has no effect on the expression in (27), but lowers the profits of a firm attracting only random searchers, given by (28). In equilibrium, this will induce more firms to attract directed searchers, lowering j_d and raising $\mathcal{U}$. This gives the following result:

Proposition 3. *Suppose there is a minimum wage $w_{\min} < y(j_p)\epsilon(\psi)$. The unique equilibrium is characterized by (30), (31), and the market-clearing condition (23). As long as it does not surpass $y(j_p)\epsilon(\psi)$, an increase in $w_{\min}$ (i) lowers j_d, (ii) raises $\mathcal{U}$, (iii) raises employment, and (iv) raises total welfare.*

Proof. See Appendix A.7. □

Intuitively, a small enough minimum wage is non-binding for firms already attracting directed searchers; however, it forces firms attracting solely random searchers to pay a higher wage. This lowers the opportunity cost of attracting directed searchers, inducing more firms to do so. The minimum wage thus reallocates some workers from firms with very high productivity to firms with medium productivity. Because there was too much congestion at high-productivity firms, this reallocation increases employment and welfare. It is worth noting that a minimum wage that is set too high will be efficiency-reducing. In particular, a minimum wage above $\epsilon(\psi)y(j_p)$ will bind for directed searchers, and therefore will result in inefficiently high queue lengths for low-productivity firms. In fact, the above analysis directly implies that the optimal minimum wage is precisely $\epsilon(\psi)y(j_p)$, which achieves the constrained-efficient outcome:

Corollary 4. *The constrained-efficient outcome is achieved by setting $w_{\min} = \epsilon(\psi)y(j_p)$. This minimum wage is increasing in ψ.*

Proof. See Appendix A.8. □

The last part of the claim, that the efficiency-restoring minimum wage is increasing in ψ, follows directly from the fact (shown in Appendix A.8) that $\epsilon(\psi)\, y(j_p)$ is increasing in ψ. Intuitively, a higher ψ raises the "shadow price" of an additional directed searcher, implying that the equilibrium wage paid to the marginal directed searcher needs to rise as well.

5 Endogenous entry

In the previous analysis, the productivity distribution of firms was taken as exogenously given. While this simplified exposition, there are a number of reasons to consider the firms' entry decision. First, the social planner, who faces a lower bound constraint on the queue of each operating firm, might prefer to shut down some low-productivity firms rather than assigning a queue of random searchers to them. Second, policies such as the minimum wage plausibly affect the extensive margin as well as the allocation of workers across firms.

In this section, we show the robustness of our findings by extending the model with an endogenous choice for firms whether to operate. We show that this gives rise to a standard entry externality in equilibrium, as low-productivity firms do not internalize that they crowd out hiring at high-productivity firms when they enter. Given a level of entry, however, the misallocation inefficiency that we highlight above continues to operate in the same way as before.

To formalize this, we continue to assume that there is a measure 1 of firms indexed by $j \in [0,1]$, with productivity $y(j)$ which is continuous and strictly increasing in j. Each firm decides whether or not to operate, at cost $\kappa > 0$. This endogenously determines the measure of vacancies, v. The rest of the environment is unchanged. In particular, since a fraction ψ of the workers are random searchers, the queue length at any firm cannot be less than ψ/v.

5.1 Planner's problem

The planner chooses the set of firms that operate and the allocation of workers among those firms. It is simple to show that, with regard to entry, the planner will follow a threshold rule, where firm j enters if and only if $j \geq j_p^*$; the total measure of vacancies is therefore $v = 1 - j_p^*$. Thus, the planner's problem is to choose j_p^* and $\lambda(j)$ for each $j \in \left[j_p^*, 1\right]$ to maximize

$$\int_{j_p^*}^{1} \left[m(\lambda(j))\, y(j) - \kappa\right] dj \tag{32}$$

subject to the constraints

$$\int_{j_p^*}^{1} \lambda(j)\, dj = 1 \tag{33}$$

and

$$\lambda(j)\left(1-j_p^*\right) \geq \psi \quad \forall j \geq j_p^* \tag{34}$$

To guarantee an interior solution for the measure of entering firms, we assume throughout that $m(\psi)\, y(0) < \kappa < y(1)$. To make the partial randomness of search relevant, we will also make an assumption that guarantees that the marginal entrant will attract only random searchers, i.e. that constraint (34) will bind for some entering firms.

Assumption 1. *$\int_n^1 (m')^{-1}\left(m'(\psi)\frac{y(n)}{y(j)}\right) dj > 1$, where n is the solution to $g\left(\frac{\psi}{1-n}\right) y(n) = \kappa$.*

This assumption amounts to stating that productivity is sufficiently dispersed, and ψ is sufficiently large, relative to the size of the entry cost κ.

As before, the constraint (34) will bind for j below some threshold j_p and will not bind for j above it. Under Assumption 1, we will have $j_p > j_p^*$, as the following result establishes.

Lemma 5. *Under Assumption 1, the constrained-efficient allocation is characterized by numbers η, j_p^*, $j_p > j_p^*$, and a function $\lambda_p(j)$ defined on $j \in \left[j_p^*, 1\right]$, satisfying:*

$$m'\left(\psi_p^*\right) y\left(j_p\right) = \eta, \ \text{where } \psi_p^* = \psi / \left(1 - j_p^*\right). \tag{35}$$

$$\lambda_p(j) = \begin{cases} \psi_p^*, & \text{if } j \in \left[j_p^*, j_p\right) \\ (m')^{-1}\left(\eta / y(j)\right), & \text{if } j \in \left[j_p, 1\right]. \end{cases} \tag{36}$$

$$\int_{j_p^*}^1 \lambda_p(j)\, dj = 1 \tag{37}$$

$$g\left(\psi_p^*\right) y\left(j_p^*\right) - \kappa = \psi_p^* m'\left(\psi_p^*\right)\left(\mathbb{E}\left(\min\left\{y(j), y\left(j_p\right)\right\} | j \geq j_p^*\right) - y\left(j_p^*\right)\right). \tag{38}$$

Proof. See Appendix A.9. □

Given j_p^*, conditions (35)-(37) characterize the socially optimal $\lambda_p(j)$, j_p, and η, similarly to the exogenous entry case described in Lemma 1. Condition (35) pins down the threshold j_p for attracting directed searchers. Condition (36) characterizes the queue length for every firm that operates, and Condition (37) is the resource constraint.

The new equation is Condition (38), which states that the marginal benefit of adding an additional firm equals the marginal cost. The marginal cost of an additional entrant is κ. The marginal benefit is the expected output of that extra firm, adjusted for the crowding-out of hiring by other firms due to congestion in the matching function; this is captured by the term $g\left(\psi_p^*\right) y\left(j_p^*\right)$. Because of heterogeneity, however, there is an additional cost: when the marginal firm is added, that firm crowds out hiring by firms that are *more productive*. This is captured by the right-hand side term $\psi_p^* m'\left(\psi_p^*\right)\left(\mathbb{E}\left(\min\left\{y(j), y\left(j_p\right)\right\} | j \geq j_p^*\right) - y\left(j_p^*\right)\right)$.

If search were purely random, we would have $j_p = 1$ and hence this term would become $\psi_p^* m'\left(\psi_p^*\right)\left(\mathbb{E}\left(y\left(j\right)|j \geq j_p^*\right) - y\left(j_p^*\right)\right)$; we would be concerned about the firm with the lowest productivity crowding out the average productivity. This crowding out effect is only operative on random searchers, however, and so the marginal firm would never cause congestion for any firm above j_p; hence the adjustment term $\min\left\{y\left(j\right), y\left(j_p\right)\right\}$. This is similar to the constrained efficiency condition in search models with heterogeneity considered by Albrecht et al. (2010), Masters (2015), and Julien and Mangin (2017); the difference is that those papers restricted attention to the purely random search case.[8]

5.2 Equilibrium

We now consider the decentralized equilibrium. Each firm j decides whether or not to enter and, conditional on entering, whether to attract directed searchers or not. Since the profits of a firm are strictly increasing in j, firms will follow a threshold rule for entering: firms will enter if and only if $j \geq j_d^*$. Conditional on entering, the problem of a firm is the same as in the exogenous-entry case, except that ψ is replaced by $\psi_d^* = \psi/\left(1 - j_d^*\right)$. We can establish the following equilibrium characterization:

Lemma 6. *Under Assumption 1, the equilibrium allocation is characterized by numbers $\mathcal{U}$, j_d^*, $j_d > j_d^*$, and a function $\lambda_d\left(j\right)$ defined on $j \in [j_d^*, 1]$, satisfying:*

$$g\left(\left(m'\right)^{-1}\left(\mathcal{U}/y(j_d)\right)\right) = m\left(\psi_d^*\right), \text{ where } \psi_d^* = \psi/\left(1 - j_d^*\right) \tag{39}$$

$$\lambda_d\left(j\right) = \begin{cases} \psi_d^*, & \text{if } j \in [j_d^*, j_d) \\ \left(m'\right)^{-1}\left(\mathcal{U}/y\left(j\right)\right), & \text{if } j \in [j_d, 1]. \end{cases} \tag{40}$$

$$\int_{j_d^*}^{1} \lambda_d\left(j\right) dj = 1 \tag{41}$$

$$m\left(\psi_d^*\right) y\left(j_d^*\right) - \kappa = 0 \tag{42}$$

Proof. See Appendix A.10. □

Conditions (39)-(41) are analogous to the equilibrium characterization for the exogenous-entry case. Condition (42) is a free-entry condition, which says that the profit of the marginal firm equals the entry cost. One can solve these conditions in two steps. First, the free-entry condition determines the equilibrium threshold for entry j_d^*, and therefore ψ_d^*, in isolation

[8] These papers consider heterogeneity on the worker side rather than the firm side, but this difference is less consequential for the main insight.

from other equilibrium outcomes. Second, given this level of entry, the solution for $\mathcal{U}$, j_d and $\lambda_d(j)$ can be obtained in the same way as in Lemma 3.

Comparison of the equilibrium conditions (39)-(42) to the efficiency conditions (35)-(38) reveals that the decentralized equilibrium is inefficient on two margins. First, the firms' free entry condition in equilibrium differs from the planner's optimal entry condition. Second, conditional on a level of entry, the equilibrium conditions for the allocation of workers across firms are inefficient as well. This is formalized in the following result:

Proposition 4. *Suppose Assumption 1 holds. Then equilibrium entry is too high: $j_d^* < j_p^*$. Furthermore, consider the allocation problem*

$$\max_{\lambda(j)} \int_{j_d^*}^{1} m(\lambda(j))\, y(j)\, dj \quad s.t. \quad \int_{j_d^*}^{1} \lambda(j)\, dj = 1 \quad and \quad \lambda(j) \geq \psi_d^* \tag{43}$$

The solution satisfies $\lambda_o(j) = \max\left\{\psi_d^*, (m')^{-1}\left(m'(\psi_d^*)\frac{y(j_o)}{y(j_d)}\right)\right\}$, with $j_d > j_o$ and $\lambda_d(j) > \lambda_o(j)\ \forall j \geq j_d$.

Proof. See Appendix A.11. □

The inefficiency is now the combination of a standard externality from excessive entry and the misallocation of labor conditional on entry. The first part of the proposition states that entry is excessive relative to the social optimum. This is a classic entry externality well known in the literature, which occurs because an individual firm does not internalize either the crowding-out effect on other firms or the fact that those firms are more productive. The second part of the proposition states that, taking as given the entry threshold j_d^*, it is possible to reallocate workers across firms so as to improve welfare relative to the decentralized equilibrium. This misallocation of labor is exactly the same as the one emphasized in the exogenous-entry model: within the firms that entered, too many applicants concentrate at the higher-productivity firms. This result therefore illustrates that the misallocation inefficiency we have highlighted earlier is robust, and in fact reappears in much the same form when entry is endogenous.

5.3 Policy interventions

The presence of a two-fold externality in the endogenous entry case raises the question of whether a single policy instrument, such as a minimum wage, can still restore efficiency. In this section, we discuss the effects of introducing a minimum wage on each of the margins of adjustment, and the resulting effects on welfare.

As before, we focus on the case in which the minimum wage is low enough to not bind for firms attracting directed searchers. Such a minimum wage will lower profits for firms attracting random searchers only, which now affects not only the allocation of workers across firms but also the entry of firms into the labor market in the first place. We start with analyzing the effect on equilibrium entry, as it can be pinned down in isolation from the other equilibrium outcomes. The introduction of the minimum wage $w_{\min}$ changes firms' free entry condition, i.e. Condition (42) in Lemma 6, to $m\left(\psi/\left(1-j_d^*\right)\right)\left(y\left(j_d^*\right)-w_{\min}\right)-\kappa=0$. Total differentiation yields $dj_d^*/dw_{\min}\geq 0$, with equality if and only if $y'\left(j_d^*\right)\to\infty$. That is, a marginal increase in the minimum wage reduces the number of firms that operate in equilibrium, except in the limit case in which there is zero mass of firms around the original entry cutoff j_d^*.

With respect to the allocation of workers across firms, there are now two opposing effects from the introduction of a minimum wage. As in the baseline model, there is a direct effect: the minimum wage lowers the opportunity cost of attracting directed searchers, which, all else equal, will lower the productivity threshold for attracting directed searchers. In addition, however, endogenous entry creates an indirect effect: the reduction in entry increases the queue length at firms that attract random searchers only, which, all else equal, will increase the productivity threshold for attracting directed searchers. Formally, Condition (39) in Lemma 6—which states that the threshold firm j_d should be indifferent between attracting random and directed searchers— now becomes $g\left(\left(m'\right)^{-1}\left(\mathcal{U}/y\left(j_d\right)\right)\right)y\left(j_d\right)=m\left(\psi_d^*\right)\left(y\left(j\right)-w_{\min}\right)$. As a result, the minimum wage can have ambiguous effects on market utility and the allocation of workers across firms.

In what follows, we establish that a minimum wage alone does not suffice to restore efficiency. To show this, consider first the optimal entry condition. As explained above, a higher minimum wage leads to lower entry, therefore mitigating the excessive entry externality. Inspection of (42) shows that the socially optimal entry threshold can be implemented by setting the minimum wage $w_{\min}$ such that $m\left(\psi/\left(1-j_p^*\right)\right)\left(y\left(j_p^*\right)-w_{\min}\right)=\kappa$. Combining this with (38) shows that this minimum wage must satisfy

$$w_{\min}=\epsilon\left(\psi/\left(1-j_p^*\right)\right)\mathbb{E}\left(\min\left\{y\left(j\right),y\left(j_p\right)\right\}|j\geq j_p^*\right)\tag{44}$$

This is very similar to the heterogeneity-adjusted Hosios condition characterized by Julien and Mangin (2017) and Masters (2019), who consider purely random search. Efficient entry is achieved by setting the wage paid by the marginal entrant firm so as to internalize the congestion it inflicts on other firms. With homogeneous firms, this would amount to setting the worker's output share to the elasticity of the matching function, $\epsilon\left(\psi_p^*\right)$. With heterogeneous firms, an additional adjustment needs to be made for the fact that the marginal entrant crowds out firms more productive than itself, leading to the expression in (44); consequently, the required minimum wage is strictly larger than $\epsilon\left(\psi/\left(1-j_p^*\right)\right) y\left(j_p^*\right)$. Under purely random search with heterogeneous firms, as considered in Albrecht et al. (2010), Masters (2015), Julien and Mangin (2017), or Masters (2019), we would have $\psi = 1$, $j_p = 1$, and the corrective minimum wage would then take the form $w_{\min} = \epsilon\left(1/\left(1-j_p^*\right)\right) \mathbb{E}\left(y\left(j\right)|j \geq j_p^*\right)$. In fact, under purely random search, such a minimum wage alone would be sufficient to restore efficiency, since entry was the only margin being distorted. In our environment with partially directed search, a marginal entrant only causes a congestion externality on firms between j_p^* and j_p, and the minimum wage implementing the efficient entry threshold takes account of this fact. Moreover, at that minimum wage, there is a misallocation of labor across the entering firms, precisely as in Proposition 1 and Proposition 4. To see this most clearly, observe that – given j_p^* – Corollary 4 implies that the efficient allocation of labor across firms would be implemented by setting the minimum wage to

$$
\begin{aligned}
w_{\min} &= \epsilon\left(\psi/\left(1-j_p^*\right)\right) y\left(j_p\right) && (45)\\
&> \epsilon\left(\psi/\left(1-j_p^*\right)\right) \mathbb{E}\left(\min\left\{y\left(j\right), y\left(j_p\right)\right\}|j \geq j_p^*\right) && (46)
\end{aligned}
$$

This means that the minimum wage achieving efficient entry is strictly smaller than the minimum wage that – given efficient entry – would achieve efficient allocation of workers across firms. Setting the minimum wage to (46) would implement efficient entry, $j_d^* = j_p^*$, but there would be misallocation of workers across operating firms. Setting the minimum wage to (45) would drive too many firms out of the market. A single minimum wage cannot simultaneously fix both inefficiency margins.

Because now two externalities operate in the market, it is not surprising that a single policy instrument no longer suffices to implement the planner's outcome. Nonetheless, a minimum wage can still play an important role in improving welfare. For example, as the following result shows, the combination of an appropriate corrective hiring subsidy and a minimum wage can restore constrained efficiency.

Corollary 5. *If Assumption 1 holds, the constrained-efficient outcome is achieved by setting* $w_{\min} = \epsilon\left(\psi/\left(1-j_p^*\right)\right) y\left(j_p\right)$ *and a hiring subsidy*

$$\mathcal{X} = \epsilon\left(\psi/\left(1-j_p^*\right)\right)\left[y\left(j_p\right) - \mathbb{E}\left(\min\left\{y\left(j\right), y\left(j_p\right)\right\}|j \geq j_p^*\right)\right] \quad (47)$$

The proof follows directly from the reasoning above. Suppose that firms enter if and only if $j \geq j_p^*$; setting the minimum wage at $w_{\min} = \epsilon\left(\psi/\left(1-j_p^*\right)\right) y\left(j_p\right)$ would guarantee efficient allocation of workers across firms. It then remains to set a hiring subsidy so that profits for firm j_p^* are zero. This hiring subsidy is equal to the difference between (45) and (46).

6 Conclusion

We have developed a model with partially directed search and used it to study misallocation of labor. Inefficiency is driven by imperfect ability of workers to direct their search. Importantly, the same inability to perfectly direct search also constrains the social planner; nonetheless, the equilibrium allocation differs from the planner's due to a novel externality.

The framework naturally leads to the interesting result that too many workers concentrate at high-productivity firms. This result serves as a cautionary note against treating misallocation of resources as synonymous with low productivity. The misallocation highlighted here manifests itself instead as suboptimally low employment. Furthermore, we show that improvements in information may decrease wages and amplify the "flight to quality." Interpreted in light of recent trends, this suggests that growing prevalence of online job search need not be inconsistent with a phenomenon like wage stagnation. Finally, our analysis points to a potential of policy interventions, such as the minimum wage, to improve welfare by reallocating workers across firms. Our findings are of relevance to the recent debates concerning the consequences of employer market power in the US and the desirability of labor market regulation, as well as to labor markets in developing countries, where information frictions are rampant.

There are a number of directions for future research. First, we have assumed that the fraction of directed searchers is exogenously given. If workers can decide, at a cost, whether or not to direct their search, potential strategic complementarities can arise between workers' investments in information and firms' wage posting decisions, possibly leading to multiple equilibria. Moreover labor market policies can now affect employment, output and welfare not only directly, but also by changing the incentives to acquire information. Second the framework presented here features heterogeneous productivity on the firm side only. An extension to two-sided heterogeneity would allow for studying the implications of imperfectly directed search for worker-firm sorting, which (as shown by Eeckhout and Kircher (2010) in a purely directed search setting) crucially depends on the matching technology. Finally, our analysis has been theoretical: to this end, the model is deliberately parsimonious and stylized. Quantifying the model's implications for employment, welfare, measured matching efficiency, and the effects of policies is a promising but challenging research agenda, which would surely require a dynamic model, most likely extended to allow on-the-job search. A key challenge is identifying the degree to which search is directed; Lentz and Moen (2017) represent progress in this dimension.

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A Proofs

A.1 Proof of Lemma 1

Proof. Because $j_p = \inf\{j : m'(\psi)\, y(j) \geq \eta\}$, the left-hand side of resource constraint (7) is a function of a single variable, η. Existence follows because the left-hand side is continuous in η, approaches ψ as $\eta \to \infty$, and approaches infinity as $\eta \to 0$. To prove uniqueness, we will show that the left-hand side is strictly decreasing in η. Suppose $\eta' > \eta$; then the corresponding j_p' and j_p satisfy $j_p' \geq j_p$, and so

$$\begin{aligned}
j_p'\psi + \int_{j_p'}^{1} (m')^{-1}(\eta'/y(j))\, dj &< j_p'\psi + \int_{j_p'}^{1} (m')^{-1}(\eta/y(j))\, dj \\
&= j_p\psi + \int_{j_p}^{1} (m')^{-1}(\eta/y(j))\, dj \\
&\quad + \int_{j_p}^{j_p'} \left(\psi - (m')^{-1}(\eta/y(j))\right) dj \\
&\leq j_p\psi + \int_{j_p}^{1} (m')^{-1}(\eta/y(j))\, dj,
\end{aligned}$$

where the last inequality used the fact that $\psi < (m')^{-1}(\eta/y(j))$ for $j > j_p$. □

A.2 Proof of Lemma 2

Proof. Recall that $(m')^{-1}(\cdot)$ is strictly decreasing. Therefore, $(m')^{-1}(m'(\psi)\, y(0)/y(j)) \geq \psi$ for any j, which implies

$$\int_0^1 \max\left\{\psi, (m')^{-1}\left(m'(\psi)\frac{y(0)}{y(j)}\right)\right\} dj = \int_0^1 (m')^{-1}\left(m'(\psi)\frac{y(0)}{y(j)}\right) dj.$$

Furthermore, η solves the resource constraint $\int_0^1 \max\left\{\psi, (m')^{-1}(\eta/y(j))\right\} dj = 1$. Since the left-hand side of the resource constraint is strictly decreasing in η, this implies that

$$\int_0^1 \max\left\{\psi, (m')^{-1}\left(m'(\psi)\frac{y(0)}{y(j)}\right)\right\} dj > 1$$

if and only if $\eta > m'(\psi)\, y(0)$, which in turn is equivalent to $j_p > 0$. □

A.3 Proof of Lemma 3

Proof. The proof is similar to the proof of Lemma 1. Because j_d satisfies (21), the left-hand side of the market-clearing condition (23) is a function of a single variable, $\mathcal{U}$. Existence follows because the left-hand side is continuous in $\mathcal{U}$, approaches ψ as $\mathcal{U} \to \infty$, and approaches infinity as $\mathcal{U} \to 0$. To prove uniqueness, we will show that the left-hand side of (23) is strictly decreasing in $\mathcal{U}$. Note that (21) defines j_d as a decreasing function of $\mathcal{U}$. Suppose $\mathcal{U}' > \mathcal{U}$; the corresponding j_d' and j_d then satisfy $j_d' \geq j_d$, and so

$$\begin{aligned}
j_d'\psi + \int_{j_d'}^{1} (m')^{-1} (\mathcal{U}'/y(j))\, dj &< j_d'\psi + \int_{j_d'}^{1} (m')^{-1} (\mathcal{U}/y(j))\, dj \\
&= j_d\psi + \int_{j_d}^{1} (m')^{-1} (\mathcal{U}/y(j))\, dj \\
&\quad + \int_{j_d}^{j_d'} \left(\psi - (m')^{-1} (\mathcal{U}/y(j))\right) dj \\
&\leq j_d\psi + \int_{j_d}^{1} (m')^{-1} (\mathcal{U}/y(j))\, dj.
\end{aligned}$$

The last inequality has used the fact that, for $j > j_d$, we have $g\left((m')^{-1}(\mathcal{U}/y(j))\right) > m(\psi)$; since $m(\lambda) > g(\lambda)\ \forall\lambda$, this implies $m\left((m')^{-1}(\mathcal{U}/y(j))\right) > m(\psi)$ and therefore $(m')^{-1}(\mathcal{U}/y(j)) > \psi$. □

A.4 Proof of Proposition 1

Proof. We first consider the case of an interior solution. Suppose that (8) holds, so that $j_p > 0$. We will show that $\mathcal{U} < \eta$. The proof is by contradiction. Suppose $\mathcal{U} \geq \eta$. Because $(m')^{-1}(\cdot)$ is strictly decreasing, resource constraints (7) and (23) imply $j_d \leq j_p$. That is, there exists a j such that $g\left((m')^{-1}(\mathcal{U}/y(j))\right) \geq m(\psi)$ but $(m')^{-1}(\eta/y(j)) \leq \psi$. However, by the assumption $\mathcal{U} \geq \eta$,

$$\begin{aligned}
g\left((m')^{-1}(\mathcal{U}/y(j))\right) &\leq g\left((m')^{-1}(\eta/y(j))\right) \\
&< m\left((m')^{-1}(\eta/y(j))\right),
\end{aligned}$$

where the first inequality follows from the fact that $(m')^{-1}(\cdot)$ is strictly decreasing, and the second inequality follows from $g(\lambda) < m(\lambda)$. As a result, $g\left((m')^{-1}(\mathcal{U}/y(j))\right) \geq m(\psi)$ implies $m\left((m')^{-1}(\eta/y(j))\right) > m(\psi)$, which yields the desired contradiction. Therefore, $\mathcal{U} < \eta$. The resource constraints then imply $j_d > j_p$. Furthermore, for any $j \geq j_d$, we have $\lambda_p(j) = (m')^{-1}(\eta/y(j))$ and $\lambda_d(j) = (m')^{-1}(\mathcal{U}/y(j))$, so $\mathcal{U} < \eta$ implies $\lambda_d(j) > \lambda_p(j)$.

Next, consider the case when (8) does not hold, and so the constrained-efficient allocation has $j_p = 0$. In this case, a necessary and sufficient condition for the equilibrium to be constrained-efficient is $g(\lambda_p(0)) \geq m(\psi)$. If $g(\lambda_p(0)) < m(\psi)$, then in equilibrium we must have $\lambda_d(0) = \psi$ and therefore $j_d > 0$. This implies $\mathcal{U} < \eta$, since, as already argued above, $\mathcal{U} \geq \eta$ necessitates $j_d \leq j_p$ by the resource constraints. Since $\mathcal{U} < \eta$, the definitions of $\lambda_p(j)$ and $\lambda_d(j)$ imply $\lambda_d(j) > \lambda_p(j)$ for all $j \geq j_d$. Finally, note that the resource constraint (7) requires $\lambda_p(0) < 1$, which means $g(1) < m(\psi)$ implies $g(\lambda_p(0)) < m(\psi)$ and hence constrained inefficiency. □

A.5 Proof of Proposition 2

Proof. The indifference condition (25) and the market-clearing condition (23) constitute a system of two equations in two unknowns, $\mathcal{U}$ and j_d, therefore pinning down $\mathcal{U}$ and j_d as implicit functions of ψ. Totally differentiating (25 and (23) with respect to ψ, we get, respectively,

$$\Lambda_d \mathcal{U} \frac{y'(j_d)}{y(j_d)} \times \frac{dj_d}{d\psi} - \Lambda_d \times \frac{d\mathcal{U}}{d\psi} = y(j_d)\, m'(\psi) \tag{48}$$

and

$$(\Lambda_d - \psi) \times \frac{dj_d}{d\psi} - \mathcal{A} \times \frac{d\mathcal{U}}{d\psi} = j_d, \tag{49}$$

where $\Lambda_d \equiv \lambda_d(j_d)$ is the unique solution to $g(\Lambda_d) = m(\psi)$, and

$$\mathcal{A} = \int_{j_d}^{1} \frac{1}{y(j)\, m''\left((m')^{-1}(\mathcal{U}/y(j))\right)} dj < 0 \tag{50}$$

Applying Cramer's rule to (48)-(49), we get

$$\frac{dj_d}{d\psi} = \frac{j_d \Lambda_d - y(j_d)\, m'(\psi)\, \mathcal{A}}{\Lambda_d(\Lambda_d - \psi) - \Lambda_d \mathcal{U} \frac{y'(j_d)}{y(j_d)} \mathcal{A}} \tag{51}$$

and

$$\frac{d\mathcal{U}}{d\psi} = \frac{\Lambda_d \mathcal{U} \frac{y'(j_d)}{y(j_d)} j_d - (\Lambda_d - \psi)\, y(j_d)\, m'(\psi)}{\Lambda_d(\Lambda_d - \psi) - \Lambda_d \mathcal{U} \frac{y'(j_d)}{y(j_d)} \mathcal{A}} \tag{52}$$

The denominator of the above expressions is positive. Therefore, (51) immediately implies $\frac{dj_d}{d\psi} > 0$; and (52) implies that $\frac{d\mathcal{U}}{d\psi} > 0$ if and only if

$$\Lambda_d \mathcal{U} \frac{y'(j_d)}{y(j_d)} j_d > (\Lambda_d - \psi)\, y(j_d)\, m'(\psi) \tag{53}$$

Finally, we use the fact that $\mathcal{U} = m'(\Lambda_d)\, y(j_d)$, and, by definition of Λ_d, $\Lambda_d m'(\Lambda_d) = m(\Lambda_d) - m(\psi)$. This means that (53) is equivalent to

$$\frac{m'(\psi)(\Lambda_d - \psi)}{m(\Lambda_d) - m(\psi)} < \frac{j_d y'(j_d)}{y(j_d)}, \tag{54}$$

which is precisely condition (26). □

A.6 Proof of Lemma 4

Step 1: If $w_{\min}$ does not bind for firms attracting directed searchers, then $w_{\min} < \epsilon(\psi)\, y(j_p)$.

Proof. Suppose that $w_{\min}$ does not bind for firms attracting directed searchers. This means that the inefficiency result of Proposition 1 applies, so $j_d > j_p$ and $\mathcal{U} < \eta$. This means that firm j_p strictly prefers not to attract directed searchers, and therefore

$$\begin{aligned} m(\psi)(y(j_p) - w_{\min}) &> g\left((m')^{-1}(\mathcal{U}/y(j_p))\right) y(j_p) && (55)\\ &> g\left((m')^{-1}(\eta/y(j_p))\right) y(j_p) && (56)\\ &= g(\psi)\, y(j_p) && (57)\\ &= m(\psi)(1 - \epsilon(\psi))\, y(j_p), && (58) \end{aligned}$$

where the second line follows from $\mathcal{U} < \eta$ and the concavity of m; the third line follows from the fact that $\eta = m'(\psi)\, y(j_p)$; and the last line follows from the definition of g. The above implies $w_{\min} < \epsilon(\psi)\, y(j_p)$. □

Step 2: If $w_{\min} < \epsilon(\psi)\, y(j_p)$, it does not bind for firms attracting directed searchers.

Proof. Suppose that the minimum wage binds for at least some firms who attract directed searchers. This means that the smallest queue length obtaining in equilibrium, which we denote by λ_0, must satisfy

$$\frac{m(\lambda_0)}{\lambda_0} w_{\min} = \mathcal{U} \tag{59}$$

It then follows that $\lambda_d(y(j)) = \max\left\{\lambda_0, (m')^{-1}(\mathcal{U}/y(j))\right\}$ must satisfy

$$\lambda_d(j) = \begin{cases} \lambda_0, & j \le j_0 \\ (m')^{-1}(\mathcal{U}/y(j)), & j > j_0 \end{cases} \tag{60}$$

where the threshold j_0 satisfies $m'(\lambda_0)\, y(j_0) = \mathcal{U}$. Finally, $\mathcal{U}$ must satisfy the modified market-clearing condition

$$\int_0^1 \max\left\{\lambda_0, (m')^{-1}(\mathcal{U}/y(j))\right\} dj = 1 \tag{61}$$

Note that (59) defines $\mathcal{U}$ as a decreasing function of λ_0, and (61) defines $\mathcal{U}$ as an increasing function of λ_0, so that the equilibrium $\mathcal{U}$ and λ_0, and hence j_0, are uniquely determined for any $w_{\min}$. Next, if $\lambda_0 = \psi$, (61) immediately gives $\mathcal{U} = \eta$ (planner's Lagrange multiplier), with the corresponding threshold necessarily equal to $j_0 = j_p$. This means that, in order to have $\lambda_0 = \psi$ and $j_0 = j_p$, the minimum wage must then be equal to

$$\frac{\psi}{m(\psi)}\eta = \frac{\psi}{m(\psi)}m'(\psi)\,y(j_p) = \epsilon(\psi)\,y(j_p) \tag{62}$$

This also proves the claim in Corollary Corollary 4 that $w_{\min} = \epsilon(\psi)\,y(j_p)$ implements the constrained-efficient allocation. Finally, since $\lambda_0 \geq \psi$, (61) also implies that $\mathcal{U} \geq \eta$, but then the corresponding minimum wage must satisfy

$$\frac{\lambda_0}{m(\lambda_0)}\mathcal{U} \geq \frac{\psi}{m(\psi)}\eta = \epsilon(\psi)\,y(j_p) \tag{63}$$

where the inequality transpires because $\lambda/m(\lambda)$ is increasing in λ. This proves that a minimum wage strictly less than $\epsilon(\psi)\,y(j_p)$ cannot be binding for firms attracting directed searchers. □

A.7 Proof of Proposition 3

Proof. Consider a minimum wage $w_{\min} < \epsilon(\psi)\,y(j_p)$, which binds for firms attracting random searchers only, and does not bind for firms attracting any directed searchers. The equilibrium is fully characterized by a market utility $\mathcal{U}$ for directed searchers, and a threshold j_d above which firms attract directed searchers, satisfying the indifference condition (30) and the market clearing condition (23).

Proof of (i) and (ii). We first analyze the effect of $w_{\min}$ on j_d and $\mathcal{U}$. Totally differentiating (30) with respect to $w_{\min}$, we obtain

$$-m(\psi) = (m(\Lambda_d) - m(\psi))\,y'(j_d)\,\frac{dj_d}{dw_{\min}} - \Lambda_d\frac{d\mathcal{U}}{dw_{\min}}, \tag{64}$$

where $\Lambda_d \equiv \lambda_d(j_d)$ is the solution to $m(\psi)(y(j_d) - w_{\min}) = g(\Lambda_d)\,y(j_d)$. Totally differentiating the market clearing condition (23) with respect to $w_{\min}$, we obtain

$$\frac{dj_d}{dw_{\min}} = \frac{d\mathcal{U}}{dw_{\min}} \times \frac{1}{\Lambda_d - \psi} \int_{j_d}^{1} \frac{1}{y(j)\, m''(y(j))} dj \tag{65}$$

Combining (64) with (65) gives

$$\frac{d\mathcal{U}}{dw_{\min}} = m(\psi) \left[\Lambda_d - \left(\frac{m(\Lambda_d) - m(\psi)}{\Lambda_d - \psi} \right) y'(j_d) \int_{j_d}^{1} \frac{1}{y(j)\, m''(y(j))} dj \right]^{-1} > 0, \tag{66}$$

from which (65) immediately implies $\frac{dj_d}{dw_{\min}} < 0$.

Proof of (iii). We now turn to characterizing the effect of the minimum wage on aggregate employment, which is given by

$$\mathcal{E} = j_d m(\psi) + \int_{j_d}^{1} (m')^{-1} \left(\frac{\mathcal{U}}{y(j)} \right) dj \tag{67}$$

Denote $\lambda_d(j) = (m')^{-1} \left(\frac{\mathcal{U}}{y(j)} \right)$ for notational convenience. Differentiating (67) with respect to $w_{\min}$ and using (65), we get

$$\begin{aligned} \frac{d\mathcal{E}}{dw_{\min}} &= \frac{dj_d}{dw_{\min}} \times (m(\psi) - m(\Lambda_d)) + \frac{d\mathcal{U}}{dw_{\min}} \times \int_{j_d}^{1} \frac{m'(\lambda_d(j))}{y(j)\, m''(\lambda_d(j))} dj & (68) \\ &= \frac{d\mathcal{U}}{dw_{\min}} \times \int_{j_d}^{1} \frac{1}{y(j)\, m''(\lambda_d(j))} \left[m'(\lambda_d(j)) - \frac{m(\Lambda_d) - m(\psi)}{\Lambda_d - \psi} \right] dj & (69) \\ &> 0. & (70) \end{aligned}$$

The last line follows from the concavity of m, since $m'' < 0$ and, for all $j \geq j_d$,

$$m'(\lambda_d(j)) \leq m'(\Lambda_d) < \frac{m(\Lambda_d) - m(\psi)}{\Lambda_d - \psi} \tag{71}$$

Proof of (iv). A similar argument applies to welfare, which is given by

$$\mathcal{W} = m(\psi) \int_{0}^{j_d} y(j)\, dj + \int_{j_d}^{1} (m')^{-1} \left(\frac{\mathcal{U}}{y(j)} \right) y(j)\, dj \tag{72}$$

Differentiating (72) with respect to $w_{\min}$, we get

$$\frac{d\mathcal{W}}{dw_{\min}} = \frac{dj_d}{dw_{\min}} \times \left(m(\psi) - m(\Lambda_d)\right) y(j_d) + \frac{d\mathcal{U}}{dw_{\min}} \times \int_{j_d}^{1} \frac{m'(\lambda_d(j))}{m''(\lambda_d(j))} dj \tag{73}$$

$$= \frac{d\mathcal{U}}{dw_{\min}} \times \int_{j_d}^{1} \frac{1}{y(j)\, m''(\lambda_d(j))} \left[m'(\lambda_d(j))\, y(j) - \frac{m(\Lambda_d) - m(\psi)}{\Lambda_d - \psi} y(j_d) \right] dj \tag{74}$$

$$> 0. \tag{75}$$

The last inequality follows since $m'' < 0$ and

$$m'(\lambda_d(j))\, y(j) = \mathcal{U} = m'(\Lambda_d)\, y(j_d) < \frac{m(\Lambda_d) - m(\psi)}{\Lambda_d - \psi} y(j_d) \tag{76}$$

by the concavity of m and the definition of $\lambda_d(j)$.

□

A.8 Proof of Corollary 4

Proof. The result that the minimum wage $w_{\min} = \epsilon(\psi)\, y(j_p)$ implements the constrained-efficient allocation follows directly from the proof of Lemma 4, which is provided in section A.6 above (see Equation (62)). It remains to show that $\epsilon(\psi)\, y(j_p)$ is increasing in ψ. Note that $\epsilon(\psi)\, y(j_p) = \frac{\psi}{m(\psi)}\eta$, where η is the solution to the resource constraint $\int_0^1 \max\left\{\psi, (m')^{-1}(\eta/y(j))\right\} dj = 1$. It then follows that η is increasing in ψ by the resource constraint, and $\frac{\psi}{m(\psi)}$ is increasing in ψ by the assumptions on m. This completes the proof. □

A.9 Proof of Lemma 5

Proof. The social planner is maximizing (32) subject to the constraints (33) and (34). Let η be the Lagrange multiplier on (33), and let $\mu(j)\, dj$ be the Lagrange multiplier on (34) for each j. The first-order condition for $\lambda(j)$ is then

$$\mu(j) = \eta - m'(\lambda(j))\, y(j), \tag{77}$$

which leads, as in the exogenous entry case, to the solution

$$\lambda_p(j) = \max\left\{\psi/\left(1 - j_p^*\right), (m')^{-1}(\eta/y(j))\right\}. \tag{78}$$

Define $j_p = \inf\left\{j : m'\left(\psi/\left(1-j_p^*\right)\right) y\left(j\right) \geq \eta\right\}$. Guessing that $j_p > j_p^*$, which we will verify below, immediately gives (35)-(37). Next, the first-order condition for j_p^* reads

$$m\left(\lambda_p\left(j_p^*\right)\right) y\left(j_p^*\right) - \kappa = \lambda_p\left(j_p^*\right)\eta - \int_{j_p^*}^1 \lambda_p\left(j\right)\mu\left(j\right) dj \tag{79}$$

Under the assumption that $j_p > j_p^*$, we must have $\lambda\left(j_p^*\right) = \psi_p^* = \psi/\left(1-j_p^*\right)$ and $\eta = m'\left(\psi_p^*\right) y\left(j_p\right)$. Moreover, this gives $\mu\left(j\right) = \frac{1}{1-j_p^*} m'\left(\psi_p^*\right) \max\left\{0, \left(y\left(j_p\right) - y\left(j\right)\right)\right\}$, so that (79) becomes

$$m\left(\psi_p^*\right) y\left(j_p^*\right) - \kappa = \psi_p^* m'\left(\psi_p^*\right) \times \frac{1}{1-j_p^*}\int_{j_p^*}^1 \min\left\{y\left(j\right), y\left(j_p\right)\right\} dj \tag{80}$$

Subtracting $\psi_p^* m'\left(\psi_p^*\right) y\left(j_p^*\right)$ from both sides gives (38).

Finally, we verify that $j_p > j_p^*$. Suppose the contrary. This would imply that (34) never binds. Then the solution to the planner's problem, from the above first-order conditions, must satisfy

$$\eta = m'\left(\lambda_p\left(j_p^*\right)\right) y\left(j_p^*\right) \tag{81}$$

and $\lambda_p\left(j\right) = \left(m'\right)^{-1}\left(m'\left(\lambda_p\left(j_p^*\right)\right)\frac{y\left(j_p^*\right)}{y\left(j\right)}\right)$ for all $j \geq j_p^*$, together with

$$\kappa = g\left(\lambda_p\left(j_p^*\right)\right) y\left(j_p^*\right) \geq g\left(\frac{\psi}{1-j_p^*}\right) y\left(j_p^*\right). \tag{82}$$

The latter implies $j_p^* \leq n$, where n solves $g\left(\frac{\psi}{1-n}\right) y\left(n\right) = \kappa$. Note that $\int_s^1 \left(m'\right)^{-1}\left(m'\left(\psi\right)\frac{y(s)}{y(j)}\right) dj$ is decreasing in s. By Assumption 1,

$$\begin{aligned}
1 &< \int_n^1 \left(m'\right)^{-1}\left(m'\left(\psi\right)\frac{y\left(n\right)}{y\left(j\right)}\right) dj && (83)\\
&\leq \int_{j_p^*}^1 \left(m'\right)^{-1}\left(m'\left(\psi\right)\frac{y\left(j_p^*\right)}{y\left(j\right)}\right) dj && (84)\\
&< \int_{j_p^*}^1 \left(m'\right)^{-1}\left(m'\left(\frac{\psi}{1-j_p^*}\right)\frac{y\left(j_p^*\right)}{y\left(j\right)}\right) dj && (85)\\
&\leq \int_{j_p^*}^1 \left(m'\right)^{-1}\left(m'\left(\lambda_p\left(j_p^*\right)\right)\frac{y\left(j_p^*\right)}{y\left(j\right)}\right) dj && (86)
\end{aligned}$$

contradicting the resource constraint (37). This completes the proof. □

A.10 Proof of Lemma 6

Proof. For any given j_d^*, previous results imply that the equilibrium must be characterized by a market utility $\mathcal{U}$, and a threshold rule for the queue length, such that firms with $j \geq j_d$ attract random searchers, and firms with $j < j_d$ do not. Suppose that $j_d > j_d^*$, which we will later verify. In other words, the marginal entrant attracts only random searchers. This immediately gives the equilibrium conditions (39) - (41), very much the same as in Lemma 3 for the exogenous entry case. Next, by the assumption that the marginal entrant attracts only random searchers, the zero profit condition for the marginal entrant yields (42).

It remains to verify that the marginal entrant attracts only random searchers in equilibrium. Suppose not. Then the equilibrium conditions would imply

$$\mathcal{U} = m'\left(\lambda_d\left(j_d^*\right)\right) y\left(j_d^*\right) \tag{87}$$

and $\lambda_d(j) = (m')^{-1}\left(m'\left(\lambda_d\left(j_d^*\right)\right)\frac{y(j_d^*)}{y(j)}\right)$ for all $j \geq j_d^*$, together with

$$\kappa = g\left(\lambda_p\left(j_d^*\right)\right) y\left(j_d^*\right) \geq g\left(\frac{\psi}{1-j_d^*}\right) y\left(j_d^*\right), \tag{88}$$

But then, Assumption (1) implies that the equilibrium allocation contradicts market clearing, as in Appendix A.9. □

A.11 Proof of Proposition 4

Proof. First, we show that entry is inefficiently high in equilibrium. This follows since

$$m\left(\psi_p^*\right) y\left(j_p^*\right) - \kappa > g\left(\psi_p^*\right) y\left(j_p^*\right) - \kappa > 0 = m\left(\psi_d^*\right) y\left(j_d^*\right) - \kappa,$$

and $m\left(\psi/\left(1-j\right)\right) y\left(j\right)$ is increasing in j. This immediately establishes that $j_d^* < j_p^*$. Next, consider the problem of choosing $\lambda(j)$ to maximize $\int_{j_d^*}^1 m\left(\lambda\left(j\right)\right) y\left(j\right) dj$ subject to the constraints $\lambda(j) \geq \psi_d^*$ and $\int_{j_d^*}^1 \lambda(j)\, dj = 1$. Similarly to Lemma 1, the solution satisfies $\lambda_o(j) = \max\left\{\psi_d^*, (m')^{-1}\left(\eta_o/y\left(j\right)\right)\right\}$, where $\eta_o = m'\left(\psi_d^*\right) y\left(j_o\right)$ for some j_o and

$$\int_{j_d^*}^1 \max\left\{\psi_d^*, (m')^{-1}\left(\eta_o/y\left(j\right)\right)\right\} dj = 1. \tag{89}$$

This system uniquely pins down j_o and η_o. The proof that $j_d > j_o$, and $\lambda_d(j) > \lambda_o(j)$ for $j \geq j_d$, is then identical to the proof of Proposition 1 in Appendix A.4. □