On Unit Free Assessment of The Extent of Multilateral Distributional Variation

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Abstract

Multilateral comparison of outcomes drawn from multiple groups pervade the social sciences and measurement of their variability, usually involving functions of respective group location and scale parameters, is of intrinsic interest. However, such approaches frequently mask more fundamental differences that more comprehensive examination of relative group distributional structures reveal. Indeed, in categorical data contexts, location and scale based techniques are no longer feasible without artificial and questionable cardinalization of categories. Here, Ginis' Transvariation measure is extended and employed in providing quantitative and visual multilateral comparison tools in discrete, continuous, categorical, univariate or multivariate settings which are particularly useful in paradigms where cardinal measure is absent. Two applications, one analyzing Eurozone cohesion in terms of the convergence or divergence of constituent nations income distributions, the other, drawn from a study of aging, health and income inequality in China, exemplify their use in a continuous and categorical data environment.

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1 Introduction

Since the path breaking work of R.A. Fisher (Fisher 1932, 1935), multilateral comparisons of grouped outcomes have become ubiquitous in the social and physical sciences rendering measurement of their collective variation of intrinsic interest. Equality of opportunity and mobility literatures compare outcomes of distinct circumstance groups (e.g. Arrow, Bowles and Durlauf 2000; Herrnstein and Murray 1994; Peragine, Palmisano and Brunori 2014; Roemer 1998; Weymark 2003). Wellbeing measures for ordering a collection of societies based upon functions of their respective income and inequality levels are discussed in Blackorby and Donaldson (1978). In finance, financial returns of a collection of portfolios are compared on a combined mean-variance basis (Markowitz, 1952; Bali, Brown, and Demirtas, 2013; Banz, 1981; Basu 1983; Jegadeesh 1990). In treatment effect, event and matching study and policy evaluation literatures (Angrist and Krueger, 2001) assessment is based upon comparisons of conditional means across outcome states. Appropriate unit free measures of collective between group variability would be helpful to all of these literatures.

With respect to assessing the extent of the variability in a collection of numbers, there are basically three unit free approaches: relative ranges (relative range and inter quantile ranges are examples); relative average distance from some location measure (e.g. the coefficient of variation or Theil's entropic measures, Theil 1967, Maassoumi 1986, Maassoumi, Racine and Stengos 2007), and average relative mean difference over all possible pairs – the Gini coefficient (Gini 1921). Each approach has a purpose with their specific advantages and challenges. Range type measures are easily computed and capture the potential span of differences but fail to reflect the extent of differences with respect to subgroups within the interior of the collection of numbers, that is to say they are not subgroup decomposable. Relative average difference measures in accounting for the difference from the average of each element in the collection reflect the extent of differences within the collection much better and, like the closely associated ANOVA technique, are often subgroup decomposable. However, Sen (1995) and Yitzhaki (2003) argue that measures of average absolute differences such as the Gini and Absolute Gini coeffcient (Gini 1921, Yitzhaki 1983, Chakravarty 1988), present a more comprehensive measure of the totality of differences than variance-based measures. Unfortunately, when analyzing subgroups, unlike variance-based measures, Gini-type measures are not subgroup decomposable (Bourguignon 1979) except in exceptional circumstances

(Mookherjee and Shorrocks 1982).

Carneiro, Hansen and Heckman (2002, 2003), in the context of treatment effects models, highlight a problem with these approaches, successfully demonstrating that using such summary statistics to explore distributional variation over collections of populations can be misleading.[1] Somewhat trivially, in a collection of distributions with identical means, difference in means tests would have zero power against more general distributional differences. The point is that considering a subset of conditional moments ignores important information about distributional differences in moments outside the subset that creates a "veil of ignorance" that is only countervailed by comparing subgroup distributions in their entirety across their complete range. Moreover, such analyses are not feasible in ordered categorical data contexts encountered in emerging subjective wellbeing measurement literatures (Kahneman and Krueger 2007) without arbitrary assignment of cardinal scales to ordinal categories. Unfortunately arbitrary scale assignment is not a solution because of the scale dependency problem (Schroder and Yitzhaki 2017, Liddell and Kruschke 2018, Bond and Lang 2019) and, since objects like the range, coefficient of variation and Gini coefficients are monotone scale dependent, this carries over to inequality measurement.

Here, Gini's Bilateral Distributional Transvariation (Gini 1916, 1959) is extended to multilateral environments in generating new measures (together with their respective asymptotically normal standard errors) which are distributional analogues of the three basic measures of the extent of variation in a collection of numbers. The measures, which record the relative distributional inequality in collections of discrete, continuous, categorical and potentially multivariate distributions, are in the respective forms of a Multilateral Transvariation (MGT) statistic, a distributional coefficient of variation (DCV) and a Distributional Gini (DisGini) coefficient, all come in population weighted and unweighted forms.

As illustrative applications, the new tools are employed in two situations. One analyzes Eurozone cohesion in terms of the convergence or divergence of its constituent income distributions in the 21st Century. The other, in a study of aging, and health and income inequality in China, exemplifies its use in a multivariate categorical data environment.

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[1]In the context of growth and convergence models, Durlauf and Quah (2002) make a similar point in noting that, if within subgroup variation grows sufficiently fast relative to average income variation, distributions will increasingly overlap and become more alike, effectively converging regardless of increasing subgroup mean variation (which records divergence).

In the following, Section 2 introduces three new instruments, together with their asymptotically normal standard errors, for assessing multilateral distributional differences. Section 2.1 develops the Multilateral Transvariation measure MGT. A natural generalization to K distributions of Gini's Bilateral Transvariation it was originally introduced in Anderson, Linton and Thomas (2017), here its sampling distribution is also developed. One problem with MGT is that it does not yield a sense of difference from the "average" distribution. Section 2.2 introduces DCV, an alternative index, analogous to the coefficient of variation, which measures relative distributional differences from the "average distribution". DCV is particularly useful in studying convergence and divergence issues and prompts new concepts of universal convergence or divergence (usefully visualized in radar charts) whereby all groups are simultaneously converging or diverging in concert. Tests for such universality are provided. A problem with both MGT and DCV is that they do not fully reflect the many bi-lateral functional differences and similarities within the collection of distributions. To overcome this limitation section 2.3 introduces DisGini for measuring the totality of bilateral similarities or differences in a collection of distributions much like a Gini coefficient does with respect to a collection of numbers. Population weighted and unweighted versions of the statistics are provided. Extensions and some properties of the measures are explored in Section 3. Section 4 reports the main results of two exemplifying applications. The first, a study of the progress of the Eurozone income distribution, addresses the question of increasing commonality in the income distributions of the Eurozone's constituent nations. The second, in exemplifying the efficacy of DisGini in ordered categorical data contexts, examines the progress of health-income inequalities as they relate to the aging process. Conclusions are drawn in Section 6.

2 Multilateral Transvariation

2.1 MGT: Generalizing Gini's Transvariation measure

In his original transvariation measure GT, Gini (1916, 1959) provided a measure of the difference between two distributions[2] which, for two distributions $f_i(x)$, $f_j(x)$ whose support[3] is confined to $\mathbb{R}^+$, can be defined, following Anderson, Linton and Thomas

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[2]See Pittau and Zelli (2017) for an overview of Gini's original concepts of transvariation.

[3]Since translation to discrete and categorical paradigms is straightforward, discussion is confined to the continuous paradigm for brevity purposes. Furthermore since the Gini coefficient has problems with negative values of x (Manero 2017), discussion is confined to distributions defined on the positive orthant for comparison purposes. It should be noted that the MGT and DisGini measures proposed

(2017), as follows:

$$\mathrm{GT}_{ij} = \frac{1}{2} \int_0^\infty |f_i(x) - f_j(x)| \, dx =$$
$$= \frac{1}{2} \int_0^\infty [\max(f_i(x), f_j(x)) - \min(f_i(x), f_j(x))] \, dx. \quad (1)$$

Since $0 \le \int_0^\infty |f_i(x) - f_j(x)| \, dx \le 2$, pre-multiplying by 0.5 yields a statistic that will be 0 when the two distributions are identical and 1 when they have mutually exclusive support. Note that, by definition, $\mathrm{GT}_{ij} = \mathrm{GT}_{ji}$, furthermore it has a one to one relationship with distributional overlap OV_{ij} measuring the extent of commonality between the two distributions (Anderson, Linton and Whang 2012), which is given by:

$$\mathrm{OV}_{ij} = \int_0^\infty \min(f_i(x), f_j(x)) \, dx. \quad (2)$$

Essentially GT = 1-OV.

Generalizing equation (1) to K distributions indexed $k = 1, \cdots, K$, suggests contemplating a Multilateral Gini Transvariation measure (MGT), defined as follows:

$$\mathrm{MGT} = \frac{1}{K} \int_0^\infty [\max(f_1(x), f_2(x), \cdots, f_K(x)) -$$
$$\min(f_1(x), f_2(x), \cdots, f_K(x))] \, dx. \quad (3)$$

As in the bilateral comparison, when the distributions have mutually exclusive support MGT = 1, when the distributions are identical MGT = 0.

A weighted version of MGT, MGTW is also possible, and has the form:

$$\mathrm{MGTW} = \int_0^\infty [\max(w_1 f_1(x), w_2 f_2(x), ., w_K f_K(x)) -$$
$$- \min(w_1 f_1(x), w_2 f_2(x), ., w_K f_K(x))] \, dx \quad (4)$$

where w_k are possible weights associated to the distributions f_k, $k = 1, \cdots, K$. When the K distributions are regarded as subgroups of an overall distribution, w_k are the proportions associated to each density function.

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here are not subject to this difficulty and are well defined on all support types.

2.1.1 Estimation and standard errors of MGT

Non-parametric estimation of MGT facilitates analysis of the collection of distributions over their full range revealing the extent of their similarity and differentness without reliance on the limited purview of summary statistics or visual perceptions.

In the case of discrete and categorical variables, estimation of category membership probabilities and their sampling distributions is straightforward following Rao (1973). Suppose C categories Γ_c indexed $c = 1, \cdots, C$ with a C vector of category membership probabilities p with typical element p_c and let x be an n vector of independent observations with typical element x_i so that $p_c = P(x_i \in \Gamma_c)$. Then, $\hat{p}_c$, the estimate of p_c, may be obtained by letting $z_{i,c} = 1$ when $x_i \in \Gamma_c$ and 0 otherwise, so $\widehat{p}_c = \frac{1}{n} \sum_{i=1}^{n} z_{i,k}$ and $\hat{p}$ the estimator of the vector will be such that:

$$\widehat{p} \sim_{asymp} N \left(\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_C \end{pmatrix} ; \frac{1}{n} \begin{pmatrix} p_1(1-p_1) & -p_1 p_2 & \cdots & -p_1 p_C \\ -p_2 p_1 & -p_2(1-p_2) & \cdots & -p_2 p_C \\ \vdots & \vdots & \vdots & \vdots \\ -p_C p_1 & -p_C p_2 & \cdots & p_C(1-p_C) \end{pmatrix} \right) \tag{5}$$

Computation of the various transvariation measures will follow the same pattern as the continuous distribution scenario.

For continuous distributions when kernel estimates of $f_k(x)$, $k = 1, \cdots, K$ are available, standard errors can also be derived.

Letting $f_k(x)$, $k = 1, .., K$, be continuous density functions with closed and bounded support $[a, b]$, and assume independent random samples from the k'th population X_{kh}, $h = 1, .., T_k$. We define the kernel estimates:

$$\widehat{f}_k(x) = \frac{1}{T_k} \sum_{h=1}^{T_k} \mathbb{K}_b(x - X_{k,h}), \tag{6}$$

where $\mathbb{K}$ is a (potentially d dimensioned multivariate) kernel with $\mathbb{K}_b(.) = \mathbb{K}(./b)/b^d$, where b is a positive bandwidth sequence. The estimated K distribution unweighted multilateral transvariation index is of the form[4]

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[4] Letting $\widehat{p}_{c,k}$ be the estimated membership probability of the c'th category in the k'th distribution, then the categorical analogue is given by:

$$\frac{1}{g(K)} \sum_{c=1}^{C} \left(\max_k(\hat{p}_{c,1}, \hat{p}_{c,2}, \cdots, \hat{p}_{c,K}) - \min_k(\hat{p}_{c,1}, \hat{p}_{c,2}, \cdots, \hat{p}_{c,K}) \right)$$

$$\widehat{\theta}_{KT} = \frac{\left\{ \int_a^b \max \left(\widehat{f}_1(x), \widehat{f}_2(x), \ldots, \widehat{f}_K(x) \right) dx - \int_a^b \min \left(\widehat{f}_1(x), \widehat{f}_2(x), \ldots, \widehat{f}_K(x) \right) dx \right\}}{g_{KT}(K)}$$

$$= \widehat{\theta}_{KTU} - \widehat{\theta}_{KTL},$$

where

$$\widehat{\theta}_{KTU} = \frac{\int_a^b \max \left(\widehat{f}_1(x), \widehat{f}_2(x), \ldots, \widehat{f}_K(x) \right) dx}{g_{KT}(K)},$$

$$\widehat{\theta}_{KTL} = \frac{\int_a^b \min \left(\widehat{f}_1(x), \widehat{f}_2(x), \ldots, \widehat{f}_K(x) \right) dx}{g_{KT}(K)},$$

and $g_{KT}(K)$ is a known linear function of K, the number of distributions in question.

Considering equation (6), assume for simplicity that the contact set (the set of values for which the densities are equal) is of Lebesgue measure zero. Define the sets $CK_{i,*}$ and $CK^{i,*}$:

$$CK_{i,*} = \{ x : f_i(x) < f_j(x) \, \forall j = 1, \ldots, K, j \neq i \} \quad \text{and}$$

$$CK^{i,*} = \{ x : f_i(x) > f_j(x) \, \forall j = 1, \ldots, K, j \neq i \} .$$

Let $p_{kU} = \Pr \left(X_k \in CK^{i,*} \right)$ and $p_{kL} = \Pr \left(X_k \in CK_{i,*} \right)$, and note that $CK_{i,*} \cap CK^{i,*} = \emptyset$ so that $p_{kUL} = \Pr \left(X_k \in CK^{i,*} \cap CK_{i,*} \right) = 0$.

Then, under standard regularity conditions (see, e.g., Assumptions (A1) and (A3) of Anderson, Linton and Whang (2012)), we have:

$$\widehat{\theta}_{KTU} - \theta_{KTU} = \frac{1}{g_{KT}(K)} \sum_{k=1}^{K} \int_{CK^{k,*}} \left(\widehat{f}_k(x) - E \left(\widehat{f}_k(x) \right) \right) dx + r_T, \qquad (7)$$

where r_T is generic notation for a remainder term that is of smaller order in probability (r_T may be different from expression to expression).

For notational convenience assume $T_k = T$ for all k and independent sampling over $k = 1, \ldots, K$. Following Lemma A.6 of Anderson, Linton and Whang (2012), the asymptotic variance of $\widehat{\theta}_{KTU}$ is given by:

$$AVAR \left(\widehat{\theta}_{KTU} \right) = \frac{1}{(g_{KT}(K))^2} \frac{1}{T} \sum_{k=1}^{K} p_{kU} (1 - p_{kU}) . \qquad (8)$$

Equation (8) can be consistently estimated by:

$$\frac{1}{(g_{KT}(K))^2} \frac{1}{T} \sum_{k=1}^{K} \widehat{p_{kU}} (1 - \widehat{p_{kU}}) ,$$

where

$$\widehat{p_{kU}} = \frac{1}{T} \sum_{h=1}^{T} 1\left(X_{kh} \in CK^{k,*}\right). \tag{9}$$

Similarly,

$$\widehat{\theta}_{KTL} - \theta_{KTL} = \frac{1}{g_{KT}(K)} \sum_{k=1}^{K} \int_{CK_{i,*}} \left(\widehat{f}_k(x) - E\left(\widehat{f}_k(x)\right)\right) dx + r_T. \tag{10}$$

The asymptotic variance of $\widehat{\theta}_{KTL}$ is given by

$$AVAR\left(\widehat{\theta}_{KTL}\right) = \frac{1}{(g_{KT}(K))^2} \frac{1}{T} \sum_{k=1}^{K} p_{kL}\left(1 - p_{kL}\right), \tag{11}$$

which can be consistently estimated by

$$\frac{1}{(g_{KT}(K))^2} \frac{1}{T} \sum_{k=1}^{K} \widehat{p_{kL}}\left(1 - \widehat{p_{kL}}\right),$$

where

$$\widehat{p_{kL}} = \frac{1}{T} \sum_{h=1}^{T} 1\left(X_{kh} \in CK_{k,*}\right). \tag{12}$$

Combining (7) and (10) toghether, we have

$$\widehat{\theta}_{KT} - \theta_{KT} = \frac{1}{g_{KT}(K)} \sum_{k=1}^{K} \int_{CK^{i,*}} \left(\widehat{f}_k(x) - E\left(\widehat{f}_k(x)\right)\right) dx$$

$$- \frac{1}{g_{KT}(K)} \sum_{k=1}^{K} \int_{CK_{i,*}} \left(\widehat{f}_k(x) - E\left(\widehat{f}_k(x)\right)\right) dx + r_T.$$

The asymptotic variance of $\widehat{\theta}_{KT}$ is given by

$$AVAR\left(\widehat{\theta}_{KT}\right) = \frac{1}{(g_{KT}(K))^2} \frac{1}{T} \sum_{k=1}^{K} \{p_{kU}\left(1 - p_{kU}\right) + p_{kL}\left(1 - p_{kL}\right) + 2p_{kU}p_{kL}\},$$

using the fact that $p_{kUL} = \Pr\left(X_k \in CK^{i,*} \cap CK_{i,*}\right) = 0$. It can be consistently estimated by

$$\frac{1}{(g_{KT}(K))^2} \frac{1}{T} \sum_{k=1}^{K} \{\widehat{p_{kU}}\left(1 - \widehat{p_{kU}}\right) + \widehat{p_{kL}}\left(1 - \widehat{p_{kL}}\right) + 2\widehat{p_{kU}}\widehat{p_{kL}}\}.$$

The distributional properties of MGTW can be derived as above by working with $w_k \widehat{f}_k(x)$ in place of $\widehat{f}_k(x)$ and modifying $g_{KT}(K)$ accordingly as in (4).

One problem with the multilateral transvariation measure is its maximum-minimum nature. Like the range statistic for a collection of numbers which does not reflect

differences in objects in the mid range, the MGT does not reflect the many bi-lateral functional differences and similarities camouflaged by just considering extreme density values. Indeed, it is in essence the distributional analogue of the relative range measure of a collection of numbers wherein the relative locations of interior and low weight members have little or no impact on its value. An alternative which is in effect an aggregation of all distributional differences from the average distribution or what will be referred to as the Distributional Coefficient of Variation (DCV) is introduced in the next section.

2.2 DCV: A Distributional Coefficient of Variation

The collection of K subgroups indexed $k = 1, \cdots, K$ with respective distributions $f_k(x)$ may be considered in the context of individual distributions being components within a mixture $f(x)$ representing the overall population distribution:

$$f(x) = \sum_{k=1}^{K} w_k f_k(x), \sum_{k=1}^{K} w_k = 1 \text{ and } w_k \geq 0 \text{ for all } k \tag{13}$$

where w_k are weights reflecting the importance of the component within the population. So, for example, $f(x)$ may refer to a societal income distribution with $f_k(x)$ being the income distribution of the k-th constituency and w_k its relative population size. Alternatively, from a representative agent or treatment effect perspective, the distributions describing outcomes of particular groups could be compared directly, without reference to their relative importance in the collection, in which case w_k would be set to $1/K$ for all k.

OV_{ko}, the distributional overlap between the k'th subgroup distribution and the overall mixture is such that:

$$OV_{ko} = \int_0^{\infty} \min\left(f_k(x), f(x)\right) dx \tag{14}$$

The corresponding subgroup/overall transvariation is related to the overlap measure as follows:

$$GT_{ko} = 1 - OV_{ko}.$$

Then DCV, the weighted average of subgroup-overall distribution transvariations, may then be written as:

$$DCV = \frac{1}{\left(1 - \sum_{k=1}^{K} w_k^2\right)} \sum_{k=1}^{K} w_k GT_{ko} = \frac{1}{\left(1 - \sum_{k=1}^{K} w_k^2\right)} \sum_{k=1}^{K} w_k (1 - OV_{ko}). \tag{15}$$

Note that when subgroup distributions are identical they will be identical to their weighted sum so that $\text{GT}_{ko} = 0$ for all k and DCV=0. When the subgroups have mutually exclusive support $\text{GT}_{ko} = 1 - w_k$ so that DCV=1.

This magnitude can be visualized in a radar chart whose spokes are the respective subgroup/overall distribution transvariations. The center of the chart corresponds to zero transvariation where all subgroups have identical distributions. The area of the polygon formed by joining the points of spokes is a representation of the aggregated extent of differences of the individual subgroups from the "average", unfortunately it is not independent of the ordering of the spokes. However, if polygon A, representing the inequality measure in period A, is everywhere inside polygon B representing inequality in period B, then A corresponds to an unequivocal or comprehensive reduction of inequality over period B in the sense that all subgroup distributions are closer to the mean distribution in A than they are in B.

In a convergence-divergence setting this suggests a notion of universal convergence with all groups tending toward the population or average distribution, in a wellbeing measurement setting it suggests the idea of a comprehensive reduction in inequality with all groups converging to the overall norm. This may be examined statistically by noting that the respective vectors of estimated spokes in A and B, $\widehat{GT}_O^A$ and $\widehat{GT}_O^B$, are respectively approximately distributed

$$\text{N} \sim \left(GT_O^A, \frac{diag(GT_O^A) - GT_O^A \cdot GT_O^{A'}}{n} \right) \quad \text{and} \quad \text{N} \sim \left(GT_O^B, \frac{diag(GT_O^B) - GT_O^B \cdot GT_O^{B'}}{n} \right)$$

and testing the joint hypothesis:

$$\text{H}_0 : GT_O^A - GT_O^B > 0, \ \text{against} \ \text{H}_1 : GT_O^A - GT_O^B \leq 0$$

or vice versa using the Maximum Modulus Distribution (Stoline and Ury 1979).

As with the Sen (1995) and Yitzhaki (2003) critiques of mean deviation measures, DCV still does not reflect the full panoply of distributional differences between groups. However a "Distributional" Gini Coefficient will.

2.3   DisGini: The "Distributional" Gini Coefficient

An examination of the subgroup decomposability of the standard Gini coefficient provides an insight into the development of the DisGini coefficient. Bourguignon (1979) demonstrated that Gini's relative mean difference coefficient (Gini 1921) is not subgroup decomposable, but it can be seen to be the sum of three components, WGini, BGini and NSF, as follows:

$$\text{Gini} = \sum_{k=1}^{K} w_k^2 \frac{\mu_k}{\mu} G_k + \frac{1}{2\mu} \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j |\mu_i - \mu_j| +$$

$$\frac{1}{2\mu} \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \int_0^{\infty} f_i(y) \int_y^{\infty} f_j(x)(x-y) dx dy = \tag{16}$$

$$= \text{WGini} + \text{BGini} + \text{NSF}.$$

Here WGini is a weighted sum of subgroup Ginis'. BGini is a term equivalent to a between group Gini coefficient of subgroup means measuring the relative inequality of distributional locations. It is a statistic often used in its unweighted form to measure distributional differences in equality of opportunity and convergence analyses (Peragine, Palmisano and Brunori 2014). NSF is a non-segmentation factor reflecting the extent to which subgroup distributions overlap or are not segmented.[5]

BGini suggests the possibility of a Gini-like coefficient reflective all aspects of distributional differences above and beyond simple differences in location, from hereon referred to as a distributional Gini coefficient or DisGini. To explore the connection note that since $\mu = \int x f(x) dx = \int (1 - F(x)) \, dx$, BGini may be written:

$$\text{BGini} = \frac{\sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j |\int (F_i(x) - F_j(x)) \, dx|}{2 \int (1 - F(x)) \, dx}$$

Generally, $\mu_j - \mu_i = \int_0^{\infty} (F_i(x) - F_j(x)) \, dx \leq \int_0^{\infty} |F_i(x) - F_j(x)| \, dx$, however when distribution j first order dominates distribution i, equality prevails and, when the relationship prevails for all pairs whose indices reflect the ordering, observe that:[6]

$$\text{BGini} = \frac{\sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \int |F_i(x) - F_j(x)| \, dx}{2 \int (1 - F(x)) \, dx} = \frac{\sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \text{GTF}_{ij}}{\int (1 - F(x)) \, dx},$$

where $\text{GTF}_{ij} = 0.5 \int |F_i(x) - F_j(x)| \, dx$ can be interpreted as the bilateral transvariation applied to the cumulative density functions.

To fully explore distributional differences consider instead:

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[5]Note that when sub-distributions have mutually exclusive closed and bounded support, NSF disappears, hence the Mookherjee and Shorrocks (1982) result. On the geometric interpretation of NSF see Lambert and Aronson (1993). This term can be also used to calculate a Gini based "segmentation index" SI which reflects the extent of segmentation in the collection of distributions where SI=1-NSF/Gini (Anderson et al. 2018; 2019)

[6]For example suppose one intersection point at $0 \leq a \leq \infty$, where $F_i(x) \geq F_j(x)$ for $x < a$ and $F_i(x) < F_j(x)$ for $x > a$, then $\int_0^{\infty} |F_i(x) - F_j(x)| dx = \int_0^{a} (F_i(x) - F_j(x)) \, dx - \int_a^{\infty} (F_i(x) - F_j(x)) \, dx = \int_0^{\infty} (F_i(x) - F_j(x)) \, dx - 2 \int_a^{\infty} (F_i(x) - F_j(x)) \, dx > \int_0^{\infty} (F_i(x) - F_j(x)) \, dx$ since $2 \int_a^{\infty} (F_i(x) - F_j(x)) \, dx$ is negative.

$$\text{DisGini} = \frac{1}{\varphi} \sum_{i=1}^{K} \sum_{j=1}^{K} 0.5 \int_{0}^{\infty} w_i w_j |f_i(x) - f_j(x)| dx = \frac{1}{\varphi} \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \text{GT}_{ij} \qquad (17)$$

Where φ is a scaling parameter. Note the term $\int_{0}^{\infty} w_i w_j |f_i(x) - f_j(x)| dx$ may be written as "$w_i w_j 2\text{GT}_{i,j}$" which is twice Gini's Transvariation of sub distributions $f_i(x)$ and $f_j(x)$, multiplied by the product of the respective population shares. Given the relationship (2) between GT and the overlap measure OV, (17) may be written as:

$$\text{DisGini} = \frac{1}{\varphi} \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j (1 - \text{OV}_{ij})$$

Which, letting c be a K element column vector of ones, may be written in matrix form:

$$\frac{1}{\varphi} c' \begin{bmatrix} 0 & w_1 w_2 (1 - OV_{12}) & \dots & w_1 w_K (1 - OV_{1K}) \\ w_2 w_1 (1 - OV_{21}) & 0 & \dots & w_2 w_K (1 - OV_{2K}) \\ \vdots & \vdots & \ddots & \vdots \\ w_K w_1 (1 - OV_{K1}) & w_K w_2 (1 - OV_{K2}) & \dots & 0 \end{bmatrix} c \qquad (18)$$

Consider a typical element $w_i w_j (1 - \text{OV}_{ij})$, when $i = j$ the element will be zero, also when $f_i(x) = f_j(x)$ for all x (i.e. subgroups i and j have identical distributions), the term will be 0. It follows that when all subgroups have identical distributions, expression (18) will be 0 since all of the elements are non-negative this will constitute a lower bound for DisGini.

Now consider the situation where all of the respective subgroup income distributions have mutually exclusive support, i.e. the subgroups are completely segmented so that for all $i \neq j$ and a given x, $f_i(x) \geq 0 \Rightarrow f_j(x) = 0$ and $f_j(x) \geq 0 \Rightarrow f_i(x) = 0$. This corresponds to the mixture distribution situation where there is no distributional overlap between any constituency pairing, thus Gini's Transvariation would be at a maximum value of 1.

In this case (17) may be written:

$$\frac{1}{\varphi} c' \begin{bmatrix} 0 & w_1 w_2 & \dots & w_1 w_K \\ w_2 w_1 & 0 & \dots & w_2 w_K \\ \vdots & \vdots & \ddots & \vdots \\ w_K w_1 & w_K w_2 & \dots & 0 \end{bmatrix} c = \frac{1}{\varphi} \sum_{k=1}^{K} w_k (1 - w_k) = \frac{1 - \sum_{k=1}^{K} w_k^2}{\varphi}$$

If the scaling parameter φ is set to $(1 - \sum_{k=1}^{K} w_k^2)$ then DisGini will always fall in the interval [0,1] and be equal to 1 when there is complete distributional inequality

in terms of complete segmentation of the constituency distributions. It follows that DisGini may finally be written as:

$$\text{DisGini} = \frac{1}{(1 - \sum_{k=1}^{K} w_k^2)} \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j (1 - \text{OV}_{ij}) = \frac{1}{(1 - \sum_{k=1}^{K} w_k^2)} \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \text{GT}_{ij}.$$
(19)

If comparison of the distributions without subgroup weighting is desired, as in the aforementioned representative agent type scenarios, simply set $w_i = \frac{1}{K}$ for all $i = 1, \cdots, K$.

2.3.1 Estimation and standard errors of DisGini

Estimation of the Distributional Gini Index (DisGini or DG) over K distributions is of the form:

$$\widehat{\theta}_{DG} = \frac{1}{g_{DG}(K)} \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \left\{ 2 - \int_a^b \min\left(\widehat{f}_i(x), \widehat{f}_j(x)\right) dx \right\},$$
(20)

where $\widehat{f}_k(x)$ are kernel estimates of $f_k(x)$, $k = 1, \cdots, K$, $g_{DG}(K)$ is a known function of K and the w_i's are also assumed known. This may be written as

$$\widehat{\theta}_{DG} = \frac{1}{g_{DG}(K)} \left(2K^2 - \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \left\{ \int_a^b \min\left(\widehat{f}_i(x), \widehat{f}_j(x)\right) \right\} dx \right).$$

So, for the distributional properties of $\widehat{\theta}_{DG}$ attention can be focussed upon:

$$\widehat{\theta}_{OV} = \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \left\{ \int_a^b \min\left(\widehat{f}_i(x), \widehat{f}_j(x)\right) dx \right\} = \sum_{i=1}^{K} \sum_{j=1}^{K} w_i w_j \left\{ \widehat{\theta}_{i,j} \right\},$$
(21)

where $\widehat{f}_k(x)$ are defined as in (6).

Considering the $\widehat{\theta}_{i,j}$, for simplicity assume independent samples of T observations and that the contact sets are of measure 0. Define the sets $C_{i,j}$ $i, j = 1, .., K$ $i \neq j$ as:

$$C_{i,j} = \{x : f_i(x) < f_j(x)\}.$$

Then

$$\widehat{\theta}_{i,j} - \theta_{i,j} = \int_{C_{i,j}} \left(\widehat{f}_i(x) - E\left(\widehat{f}_i(x)\right)\right) dx + \int_{C_{j.i}} \left(\widehat{f}_j(x) - E\left(\widehat{f}_j(x)\right)\right) dx + r_T$$

and thus

$$\widehat{\theta}_{OV} = \sum_{i=1}^{K}\sum_{j=1}^{K} w_i w_j \left(\widehat{\theta}_{i,j} - \theta_{i,j}\right) =$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} w_i w_j \left(\int_{C_{i,j}} \left(\widehat{f}_i(x) - E\left(\widehat{f}_i(x)\right)\right) dx + \int_{C_{j,i}} \left(\widehat{f}_j(x) - E\left(\widehat{f}_j(x)\right)\right) dx\right) + r_T.$$

Turning to its asymptotic variation, define:

$$p_{i:ij} = P\left(X_i{\in}C_{i,j}\right) \text{ and } p_{ij} = \Pr\left(X_i{\in}C_{i,j}{\cap}X_j{\in}C_{j,i}\right).$$

Then generally,

$$AVAR\left(\widehat{\theta}_{i,j}\right) = \frac{1}{T}\left(p_{i:ij}\left(1 - p_{i:ij}\right) + p_{j:ji}\left(1 - p_{j:ji}\right) + 2\left(p_{ij} - p_{i:ij}p_{j:ji}\right)\right),$$

which may simplify with independent sampling. However even if X_i and X_j are independent $\widehat{\theta}_{i,j}$ and $\widehat{\theta}_{k,l}$ will be dependent if they have one subscript in common so that $ACOV\left(\widehat{\theta}_{i,j}, \widehat{\theta}_{k,l}\right) \neq 0$ when there is a commonality in subscripts. All such terms need to be considered so that a threefold summation is required involving probabilities of sets of the form:

$$C_{i,j}{\cap}C_{i,k} = \{x : f_i(x) < \min\left(f_j(x), f_k(x)\right)\}.$$

Ultimately:

$$AVAR\left(\widehat{\theta}_{OV}\right) = \frac{1}{T}\sum_{i=1}^{K}\sum_{j>i} w_i^2 w_j^2 \left(\Pr\left(X_i{\in}C_{ij}\right) - \Pr\left(X_i{\in}C_{ij}\right)^2\right) +$$

$$+ \frac{2}{T}\sum_{i=1}^{K}\sum_{j>i}\sum_{k>j>i} w_i^2 w_j w_k \left(\Pr\left(X_i{\in}C_{ij}{\cap}C_{ik}\right) - \Pr\left(X_i{\in}C_{ij}\right)\Pr\left(X_i{\in}C_{ik}\right)\right),$$

which may be consistently estimated by replacing the population quantities by their sample analogues.

3 Further Extensions

3.1 Multivariate Distributions and Higher Order Integrals

An interesting feature of MGT, DCV and DisGini, is that they can handle multivariate distributions of discrete, continuous or categorical forms (or mixtures of both)

which can be a challenge for the standard coefficients. Simply write (3), (15) and (17) respectively as:

$$\text{MGT} = \frac{1}{K} \iint \int_0^\infty \left[\max \left(f_1 \left(x, y, z \right), f_2 \left(x, y, z \right), .., f_K \left(x, y, z \right) \right) - \right.$$
$$\left. \min \left(f_1 \left(x, y, z \right), f_2 \left(x, y, z \right), .., f_K \left(x, y, z \right) \right) \right] dx dy dz,$$

$$\text{DCV} = \frac{1}{\left(1 - \sum_{k=1}^K w_k^2 \right)} \sum_{k=1}^K w_k \left(1 - \iint \int_0^\infty \min \left(f_k(x, y, z), f(x, y, z) \right) dx dy dz \right)$$

and

$$\text{DisGini} = \frac{1}{\varphi} \sum_{i=1}^K \sum_{j=1}^K w_i w_j \text{GT}_{ij} =$$
$$\frac{0.5}{\varphi} \sum_{i=1}^K \sum_{j=1}^K \iint \int_0^\infty w_i w_j |f_i(x, y, z) - f_j(x, y, z)| dx dy dz.$$

Formulae (4),(15) and (19) then follow directly. In addition, by replacing the $f_i(x)$ with $F_i^h(x)$ where $F_i^h(x) = \int_0^x F_i^{(h-1)}(z) dz$ in (3), (15) or (17) and adjusting the normalizing parameter accordingly, multilateral variation of higher order integrals of distribution functions could be contemplated reflecting the classic stochastic dominance criteria for more restrictive wellbeing structures (see Anderson, Post and Whang, 2020). All of which is are matters for future research.

3.2 Axiomatic properties of inequality indices

These indices provide a complete ordering of collections of distributions with respect to their differentness, as such they satisfy some popular axioms in the inequality literature. Anonymity, sometimes referred to as the symmetry axiom, requires that the index not depend on who the groups are, so that groups switching places should not affect the index, which is the case for these indices.

Scale invariance (multiplying x by a constant does not influence the index) and translation invariance (adding a constant to each x does not influence the index) are both satisfied by distributions in the sense that they do not change their relative shape and thus they are satisfied by the indices.

The normalization axioms (the index is bounded between 0 and 1 with 0 corresponding to complete equality and 1 complete inequality) have already been shown to be satisfied by MGT, DCV and DisGini.

Replication Invariance, an axiom requiring that a measure be unaffected by universally scaling up the population size is also satisfied by these measures.

When sub-distributions are posited to be the atomistic equivalents of the sub-distributions employed in Duclos, Esteban and Ray (2004) and subjected to the same transformations, they comply with the polarization axioms posed therein. Finally it should also be noted that although the Gini coefficient has problems with negative incomes (see Manero 2017), it is not a problem for MGT, DCV and DisGini coefficients.

3.3 Expanding the number of groups

It is of interest to understand how the DisGini coefficient is affected by the expansion of the number of groups under consideration. Suppose K original groups indexed $k = 1, \cdots, K$ possess a coefficient DisGini_K and contemplate the addition of a group (indexed $K + 1$) which yields a coefficient DisGini_{K+1}, then it can be shown that DisGini will increase or diminish as it is exceeded by or exceeds the weighted sum of the new group's transvariations with respect to the existing groups in the analysis (see Appendix A).

4 Two empirical examples

4.1 Household income distributions in the Eurozone

Milanovic (2011) noted that growing divergence between constituencies within a federation can be a catalyst for the deterioration of its cohesion and the recent rise of economic nationalism in Europe has given cause for concern regarding the European Unions coherence (Krastev 2014, Webber 2018, Lindberg 2019). Formed to promote commonality of wellbeing among its constituents, there is interest in seeing whether the European nations household income distributions are converging. The growth and convergence literature suggests that variation of average incomes across constituencies is of interest since it speaks directly to the question of whether the distribution of income across economies is becoming more or less equitable (Quah 1993). However, deterioration of cohesiveness has much to do with the extent to which economic wellbeing differs across constituencies, the sense in which such differences are perceived by

agents within those constituencies and the relative importance of those constituencies. In this context, cohesiveness is more than just a matter of whether or not constituencies have similar average incomes, it is more a matter of whether or not they have common income distributions. When member nations are equally unequal with relatively similar income levels and distributions, there is a commonality of situation among member constituents which promotes cohesion, whereas a more divisive and alienated situation arises when such inequalities and income levels are not so equally shared in a more segmented society. To illustrate the efficacy of the new techniques, a study of the 21st century evolution of household income inequality in the Eurozone is performed. What emerges is a collection of distributions that result in a Eurozone with an increasingly unequal overall income distribution comprised of an increasingly similar (i.e. convergent) collection of unweighted distributions that, when population weighted, become divergent as a collection.

Viewed as an entity, the overall Eurozone household income distribution $f(x)$ is a mixture of the household income distributions $f_k(x)\, k = 1, .., K$ of its K constituent nations where the weights w_k correspond to relative population sizes (see equation 13).

Stochastic processes are frequently used to rationalize distributional structures and Gibrat's Law of Proportional Effects and some of its modifications (Gabaix 1999, Reed 2001) have been foundational in providing a theoretical rationale for expecting increasing income inequality. The Law posits that household incomes in subgroup k follow a stochastic process which in its simplest form in period t, has the form:

$$x_{k,t} = (1 + \delta_{k,t})\, x_{k,t-1}$$

where $\delta_{k,t}$ is a random variable with mean δ_k (which is small relative to one in absolute value) and variance σ_k^2. The law predicts that, given a starting value x_0 and letting $X = \ln(x)$, after T periods X_{kT} will have a mean equal to $X_0 + T(\delta_k + 0.5\sigma_k^2)$ and variance equal to $T\sigma_k^2$, respectively i.e. log income variation that grows through time. Following Modigliani and Brumberg (1954), classical economic models of income (Hall 1978) use this idea to predict increasingly unequal income distributions (Battistin, Blundell, and Lewbel 2009, Blundell and Preston 1998, Browning and Lusardi 1996). When applied to the $k = 1, \cdots, K$ constituent societies in the Eurozone, clearly different configurations of pairs (δ_k, σ_k^2) for $k = 1, \cdots, K$ will yield collections of distributions that could be converging or diverging, segmenting or increasingly overlapping, becoming more or less equal in distribution.

The weighted and unweighted versions of the Multilateral Transvariation and Distributional Gini statistics can yield insights into the progress of such distributional inequalities over the era, tending toward 0 as distributions converge and tending toward 1 as they segment or diverge. The weighted versions give insight into distributional differences of the Eurozone as an entity, with small populations given low weight and large populations high weight. The unweighted versions can be construed as a representative agent model recording the juxtaposition of nation income distributions directly without respect to their relative importance or impact in the overall Eurozone income distribution.

The data source is the European Union Survey on Income and Living Conditions (EU-SILC)[7]. To analyze the evolution of the Euro area income distribution over time, four temporally equi-spaced waves, 2006, 2009, 2012 and 2015 were chosen. Since data for Malta are only available from the 2008 wave, this country is excluded from analysis leaving 18 Euro zone countries. Income is the total household net disposable annual income (in thousands Euro) obtained by aggregation of all income sources from all household members net of direct taxes and social contributions.[8] All households are weighted by cross-sectional weights. Assuming consumption economies of scale in cohabitation, incomes are age and size-adjusted using the modified-OECD equivalence scale. Given significant disparities in the cost of living between countries, the PPP index for the household final consumption expenditure is used to adjust household incomes.

As an entity, the Eurozone had overall household income Gini coefficients of 0.305, 0.313, 0.317, 0.335 for the years 2006, 2009, 2012 and 2015 respectively, suggesting ever increasing household income disparities in the area over the period.

In the light of concerns regarding European disintegration, questions arise as to the extent to which such inequalities are equally shared across its various nations, thus considering the constituent nations as subgroups of the overall Eurozone income distribution[9]. To get some sense of the nation based distributional changes in the

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[7]Version estatCROS 2019ki9, released in May 2019. EU-SILC is a harmonized household-level survey that is a collection of annual national surveys of socio-economic conditions of individuals and households in EU countries.

[8]The income reference period refers to the previous year, consequently analysis with EU-SILC files actually refers to 2005-2014.

[9]An alternative approach would consider a transnational decomposition based upon latent household income classes that transcend nation boundaries. These latent classes can be identified by a semiparametric mixture distribution analysis (see Anderson, Pittau, Zelli and Thomas 2018).

Eurozone, Table 1 shows the overall Gini coefficient and the decomposition results over the period 2006–2015.

The BGini, the relative mean absolute difference of country means, being a function of the relative locations of distributions yields insight into their progress over the period. Noting that, while the within group component WGini is growing steadily but relatively slowly over the period, consistent with Gibrat's law, there appears to be a substantial increase in between nation inequality over the period reflecting a sustained divergence of national average household incomes over the period in line with the overall Gini coefficient for the Eurozone which reflects potentially increasing inequality as perceived by Eurozone member nations. However, recalling the Durlauf and Quah (2002) and Carneiro, Hansen and Heckman (2002, 2003) veil of ignorance concerns, be aware that these results are based on summary statistic comparisons (essentially differences in means and progressions in variances) which veil overall distributional differences.

Table 1: Gini overall inequality in the EuroArea and its decomposition

Year	2006	2009	2012	2015
Gini Total	30.48	31.30	31.74	33.52
WGini (Gini Within)	4.45	4.76	4.79	4.81
BGini (Gini Between)	9.62	9.65	10.90	16.07
NSF (Non Segmentation Factor)	16.41	16.05	15.84	12.64

Table 2 reports the unweighted and weighted Distributional Gini Coefficients (DisGini and DisGiniW) and Multilateral Gini Transvariation (MGT and MGTW). The income densities $f_k(x)$ are kernel estimated including the sample weights and using the Sheather and Jones (1991) bandwidth. Looking at the patterns of the indices quite different stories emerge under population weighted and unweighted versions of the statistics. While the nation weighted versions of the indices, after a slight dip in 2009, show a significant increase, the unweighted versions record a decline over the whole period with respect to 2006. Thus a representative agent view of the Eurozone suggests increasing commonality of household disposable income distributions whereas the population weighted version suggests increasing segmentation. Thus nations with larger populations appear to be segmenting whereas nations with small populations are not.

Table 2: Unweighted and weighted Distributional Gini coefficients and Multilateral Gini Transvariation - nation group analysis

Year	DisGini	DisGiniW	MGT	MGTW
2006	0.386	0.237	0.135	0.291
	(0.003)	(0.003)	(0.004)	(0.005)
2009	0.341	0.222	0.111	0.279
	(0.003)	(0.003)	(0.004)	(0.005)
2012	0.326	0.282	0.106	0.323
	(0.003)	(0.003)	(0.003)	(0.005)
2015	0.341	0.361	0.107	0.349
	(0.003)	(0.003)	(0.004)	(0.005)

Note: Approximate standard errors are in brackets.

Following the interpretation of DisGini as the scaled average subgroup-overall distributional transvariation (see equation **??**), a further insight on the extent to which each country is converging or diverging to the Eurozone norm is given by the bilateral transvariations between each country and the whole Eurozone distribution. Remembering the measure GT equal to 1 means complete segmentation (two distribution are far apart), while GT equal to 0 means complete overlapping, Figure 1 reports a radar chart of the bilateral transvariation between each country and the Eurozone as a whole, a decomposed distributional coefficient of variation as it were. The closer is a point on a nations spoke to the periphery, the higher is the transvariation of that nations income distribution with respect to Eurozone distribution. The closer to the center is a point the closer is that nations income distribution to convergence with the Eurozone income distribution. The points have been colour coded by year, so that intuitively nations with green dots nearer the centre than black dots are converging to the Eurozone distribution, whereas nations with green dots outside of the black dots are diverging from the Eurozone distribution over the observation period. The Radar Chart also suggests another convergence index, the value of the area enclosed by connecting the dots of a common colour. For Figure 1 this yields 2006 0.818, 2009 0.532, 2012 0.536 and 2015 0.571, reiterating the notion that the unweighted distributions are converging to the overall Eurozone distribution over the period although the convergence does not appear to be comprehensive since subsequent period polygons are not completely interior to preceding period polygons.

The bilateral nation-overall transvariations range from 0.03 for Italy to 0.68 for Slovakia in the year 2006. The pattern of this index shows a process of convergence

Figure 1: Radar chart of bilateral transvariation of each country with respect to Eurozone. The center of the wheel corresponds to the minimum value of the measure or complete overlapping with respect to the Eurozone distribution. Moving to the periphery reflects divergence to, or decreasing commonality with, the Eurozone distribution. Countries are clockwise ordered starting with the largest positive difference between 2006 and 2015 (indicating convergence) and ending with the largest negative difference (indicating divergence).
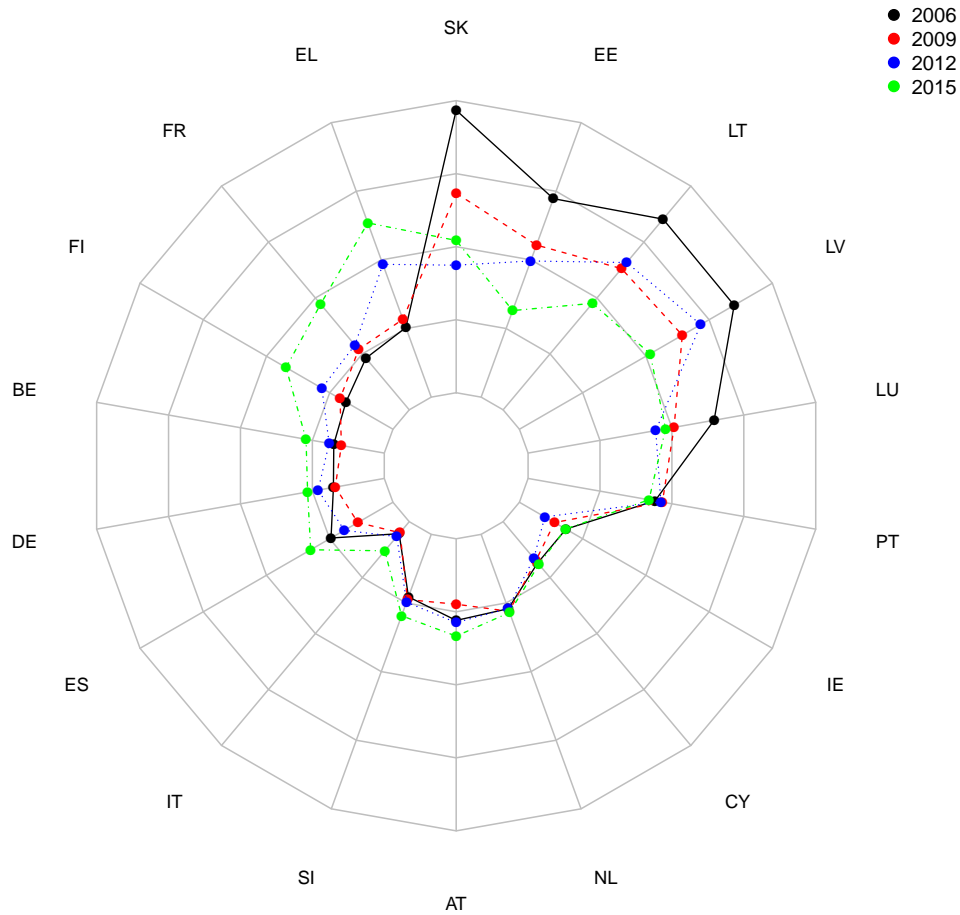
toward the EuroArea distribution for Eastern European countries (notably low population countries) and significant divergence from the Eurozone distribution for Spain, Finland, France and Greece. Figures in appendix show the evolution of the income distributions of constituent nations and their overlapping with respect to the Eurozone distribution in 2006 and in 2015.

Summing up, the cohesiveness of a union of economies has much to do with the ex-

tent to which its respective nation income distributions are segmenting or converging. When constituent nations are equally unequal with relatively similar income levels, the commonality of situation promotes cohesiveness, whereas, when such inequalities are not equally shared, the situation is somewhat more divisive. The new measures are employed to address these distinctions within a substantial subset of nations in the European Union. Population weighted versions of the distributional inequality measures indicate increasing distributional inequality in terms of increasingly segmented nations. However, in a representative agent view of the world, similar to that pursued in the sigma convergence literature wherein nations are equally weighted, a different story is revealed. In this scenario the multilateral results present significant evidence of convergence in nation based household income distributions. Comparison of the weighted and unweighted versions of the statistic reveal that the lesser populated nations of the Eurozone are exhibiting a convergence pattern whereas nations with larger populations appear to be segmenting.

5 Health-income inequalities and the ageing process in China

To exemplify the use of DisGini in situations where only categorical data is available, age related inequalities in health and incomes in China are examined. The world wide prevalence of aging populations has stimulated interest in the aging process and its connection with wellbeing. For elderly populations, health, income and aging is inextricably interlinked in this regard indeed, Anand (2004) argues that health should have primacy over consumption in the wellbeing calculus. Welfare programs, in providing support for the elderly and the poor especially in terms of their health outcomes, are also integral to the process. Given its aging population, its unprecedented economic growth and its recently developed welfare program Dibao, China is of particular interest in this respect. Anderson and Fu (2020) study health and income wellbeing in China's older population groups and the impact that Dibao may have had on them. The categorical nature of self reported health status presents a particular challenge in this regard with respect to quantifying wellbeing levels and inequalities, the Distributional Gini provides a solution.

Gao (2017) presents an extensive analysis of the impact of Dibao on work and welfare, and, along with Kakwani (2019), produces evidence of poor targeting, i.e.

assistance does not always appear to be reaching those for which the program was defined. However, little has been done to examine the health - income inequalities and the impact that Dibao may have had on inequalities in those dimensions, especially with regard to the elderly.

Here, employing survey data drawn from the China Health and Retirement Longitudinal Study (CHARLS) 2013 follow up to a 2011 baseline study, age group based inequalities in health and incomes are examined. Groups based upon gender, urban/rural location, and Dibao recipient were established. Respondents who were at least 45 years of age were asked to categorize their health as poor, fair, good, very good, excellent and placed in income quintiles (adult equivalized incomes based upon the square root rule were used). The sample was partitioned into age groups 45-50, 50-60, 60-70 and over 70 with respective sample sizes of 1823, 5396, 4782 and 2872 yielding an overall sample size of 14873.

Two exercises were performed using the unweighted formulation which treats all groups equally which is as it should be in a representative agent situation which looks at the health and income inequality risks facing a randomly selected member from each group. One formulation separately identified Dibao recipients as a separate group within each category, the other formulation did not separately identify Dibao recipients (see Table 3). What is observed as part of the aging process is significantly increasing inequality in the joint distribution of health and income in post retirement years. When Dibao recipients are separately identified, distributional inequalities increase uniformly across age groups which, from the earlier analysis regarding augmenting of groups, indicates inequalities suffered by those groups are on average even greater than those endured by non-Dibao recipients suggesting that targeting may well not be as bad as has been claimed.

Of particular interest from an aging perspective is the radar chart (Figure 2) showing that the 50-60 year olds polygon is completely inside the 60-70 year olds polygon which in turn is completely inside over 70 year olds polygon reflecting a comprehensive and unequivocal increase in health and income wellbeing inequality over the aging process in later life for every category. This is verified in Table 4 which fails to reject the hypothesis that older age group polygons lay outside younger age group polygons for successive over 50's age groups. Older age groups clearly suffer increasing health and income inequalities with the aging process.

Table 3: Distributional Gini coefficients: Age group analysis when Dibao recipients are separately identified and when they are not.

Age groups	DisGini (Dibao Recipients Separately Identified)	DisGini (Dibao Recipients Not Identified)
45–50	0.3645	0.2170
	(0.0019)	(0.0024)
50–60	0.2625	0.1890
	(0.0010)	(0.0013)
60–70	0.3145	0.2839
	(0.0011)	(0.0016)
>70	0.3943	0.3103
	(0.0015)	(0.0021)

Note: Approximate standard errors are in brackets.

Table 4: Stoline Ury maximum modulus statistics (SUMMS) for spoke changes for successive age groups

Age groups →	45–50 vs 50–60			50–60 vs 60–70			60–70 vs >70		
Subgroups	diff	std err	SUMMS	diff	std err	SUMMS	diff	std err	SUMMS
000	0.0809	0.0115	7.0348	-0.0538	0.008	-6.7102	-0.0274	0.0102	-2.6866
100	0.0884	0.0123	7.1642	-0.0358	0.0086	-4.1649	-0.0129	0.0105	-1.2237
001	0.0548	0.0114	4.8215	-0.0294	0.008	-3.6781	-0.0449	0.0101	-4.4245
101	0.0223	0.0121	1.8381	-0.0154	0.0088	-1.7481	-0.0149	0.0107	-1.397
010	0.1013	0.0129	7.8744	-0.0713	0.0092	-7.7723	-0.0702	0.0115	-6.1136
110	0.2032	0.0129	15.7308	-0.0455	0.0086	-5.2951	-0.2781	0.0113	-24.6222
011	0.0698	0.0124	5.6416	-0.0791	0.009	-8.824	-0.0364	0.0112	-3.2382
111	0.0933	0.0124	7.5327	-0.0336	0.0086	-3.911	-0.074	0.0109	-6.787

Note: Studentisize maximum modulus 5% critical value 2.8.

Figure 2: Radar chart of bilateral transvariation of each group with respect to the overall distribution (GT_{ko}), by age class. The center of the wheel corresponds to the minimum value of GT_{ko}, that is the maximum overlapping. Moving to the periphery reflects more dissimilarity with respect to the overall distribution. Groups are coded from 000 to 111. The first digit indicates Dibao recipient (1) or not recipient (0). The second digit indicates urban(1) or rural (0). The third digit indicates female (1) or male (0).
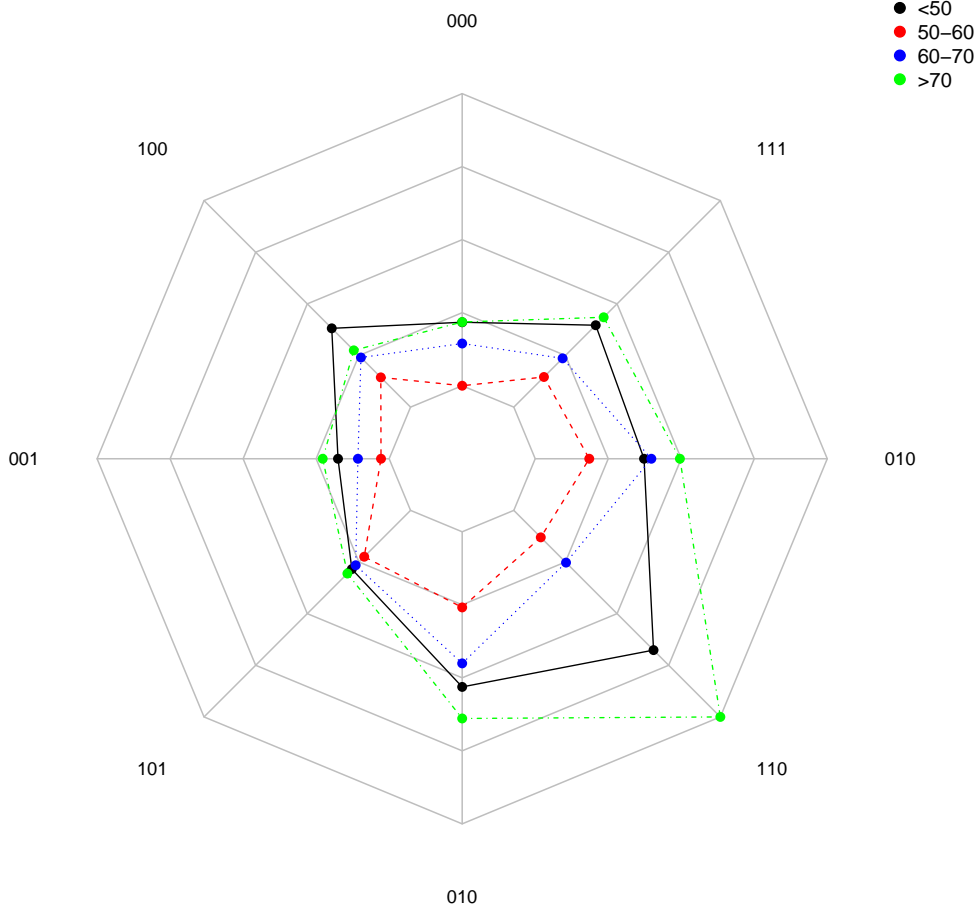
6   Conclusions

When comparing collections of groups, simple first and second order moment multilateral comparisons can overlook substantive differences between groups that a more comprehensive multilateral distributional comparison can reveal. Here, some new tools for the multilateral comparison of many distributions in univariate or multivariate, discrete and continuous, weighted and unweighted environments have been introduced.

Based on extensions of Ginis' Transvariation Measure, new Multilateral Transvariation measures and more comprehensive Gini-like distributional difference measures, together with their asymptotic distributions have been developed. The Distributional Gini measure can be shown to be a scaled weighted sum of subgroup distribution and overall distribution transvariations the magnitude of which can be represented as a polygon within a radar chart which in turn has prompted definition of the notion of comprehensive inequality reduction (increase), the consequence of all subgroups converging to (diverging from) the overall distribution. Assessing distributional differences in categorical - non cardinal environments is particularly challenging and these techniques have been shown to overcome these challenges in these situations. The measures have been exemplified in applications which study national household income distributions in the Eurozone in the 21$^{\text{st}}$ century and income and health inequalities and the aging process.

Acknowledgments

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A   Appendix: DisGini and additional groups

The relationship between DisGini_K and DisGini_{K+1} may be understood as follows. Let the original weights be $w_k, \ \ k = 1, \cdots, K$ where $\sum_{k=1}^{K} w_k = 1$ and the new weights in the extended collection of groups $w_k^{new}, \ \ k = 1, \cdots K+1$ where $\sum_{k=1}^{K+1} w_k^{new} = 1$ are such that:

$$w_k = \frac{w_k^{new}}{\left(1 - w_{K+1}^{new}\right)} = \frac{w_k^{new}}{\theta}, \ \ k = 1, \cdots, K$$

Let $\varphi^{new} = \sum_{k=1}^{K+1} \left(1 - (w_k^{new})^2\right)$, then

$$\text{DisGini}_{K+1} = \frac{1}{\varphi^{new}} \sum_{i=1}^{K+1} \sum_{j=1}^{K+1} w_i^{new} w_j^{new} \text{GT}_{i,j} = \frac{1}{\varphi^{new}} \sum_{i=2}^{K+1} \sum_{j=1}^{i} \int_0^{\infty} w_i^{new} w_j^{new} |f_i(x) - f_j(x)| dx$$

$$= \frac{1}{\varphi^{new}} \sum_{i=2}^{K} \sum_{j=1}^{i} \int_0^{\infty} w_i^{new} w_j^{new} |f_i(x) - f_j(x)| dx + \frac{1}{\varphi^{new}} \sum_{j=1}^{K} \int_0^{\infty} w_{K+1}^{new} w_j^{new} |f_{K+1}(x) - f_j(x)| dx$$

$$= \frac{\theta^2}{\varphi^{new}} \sum_{i=2}^{K} \sum_{j=1}^{i} \int_0^{\infty} w_i w_j |f_i(x) - f_j(x)| dx + \frac{1}{\varphi^{new}} \sum_{j=1}^{K} \int_0^{\infty} w_{K+1}^{new} w_j^{new} |f_{K+1}(x) - f_j(x)| dx$$

$$= \frac{\theta^2 \varphi}{\varphi^{new}} \text{DisGini}_K + \frac{1}{\varphi^n} \sum_{j=1}^{K} \int_0^{\infty} w_{K+1}^{new} w_j^{new} |f_{K+1}(x) - f_j(x)| dx$$

$$= \text{DisGini}_K - \left(1 - \frac{\theta^2 \varphi}{\varphi^{new}}\right) \text{DisGini}_K + \frac{1}{\varphi^{new}} \sum_{j=1}^{K} \int_0^{\infty} w_{K+1}^{new} w_j^{new} |f_{K+1}(x) - f_j(x)| dx$$

$$= \text{DisGini}_K - \left(\frac{\varphi^{new} - \theta^2 \varphi}{\varphi^{new}}\right) \text{DisGini}_K + \frac{w_{K+1}^{new}}{\varphi^n} \sum_{j=1}^{K} \int_0^{\infty} w_j^{new} |f_{K+1}(x) - f_j(x)| dx$$

$$= \text{DisGini}_K - \left(\frac{w_{K+1}^{new}}{\varphi^{new}}\right) 2 \left(1 - w_{k+1}^{new}\right) \text{DisGini}_K + \frac{w_{K+1}^{new}}{\varphi^n} \sum_{j=1}^{K} \int_0^{\infty} w_j^{new} |f_{K+1}(x) - f_j(x)| dx$$

$$= \text{DisGini}_K - \left(\frac{w_{K+1}^{new}}{\varphi^{new}}\right) \left(2\theta \text{DisGini}_K - \sum_{j=1}^{K} \int_0^{\infty} w_j^{new} |f_{K+1}(x) - f_j(x)| dx\right)$$

$$= \text{DisGini}_K - \left(\frac{2\theta w_{k+1}^{new}}{\varphi^{new}}\right) \left(\text{DisGini}_K - \frac{1}{2} \sum_{j=1}^{K} \int_0^{\infty} w_j^{new} |f_{K+1}(x) - f_j(x)| dx\right)$$

B   Appendix: Figures of income distribution overlaps

Figure B.1: Income distribution of Slovakia and Estonia and their overlap with the Eurozone income distribution: years 2006 and 2015.
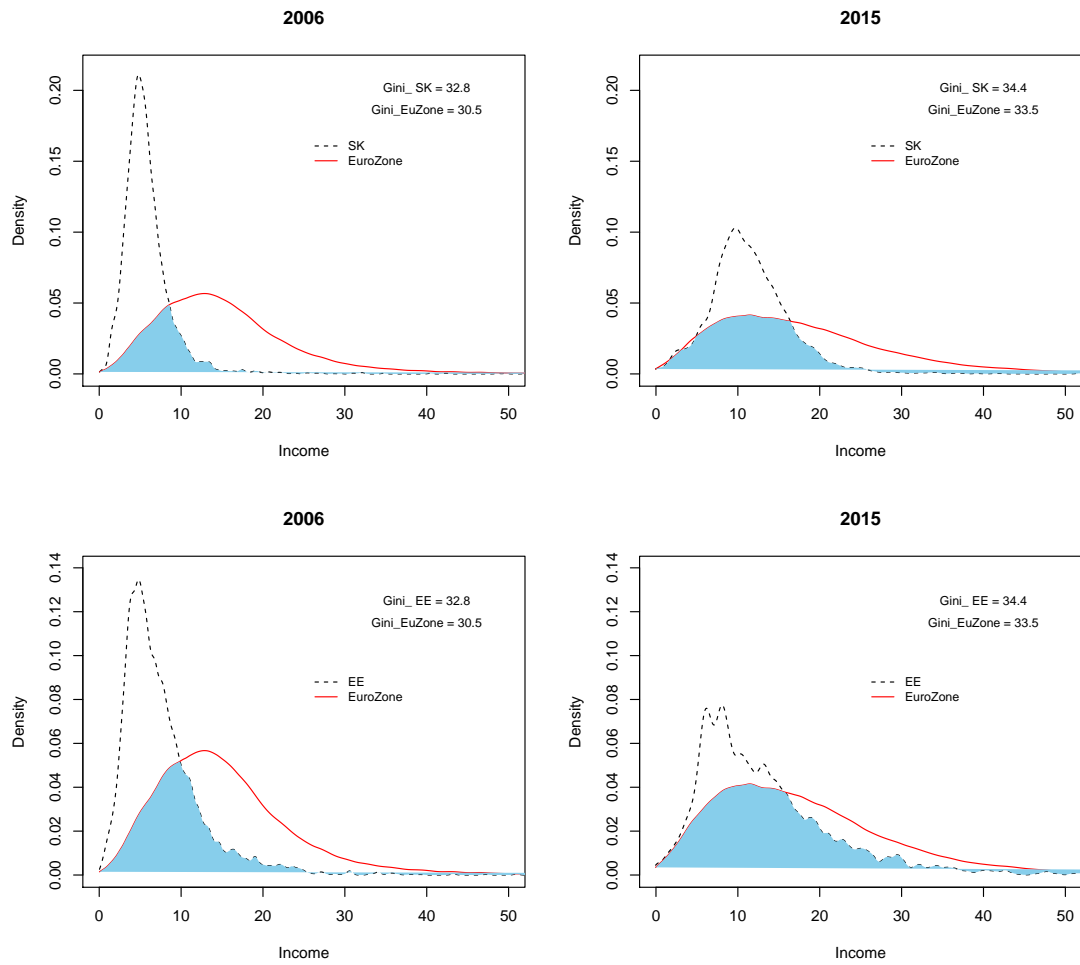
Figure B.2: Income distribution of Latvia and Lithuania and their overlap with the Eurozone income distribution: years 2006 and 2015.
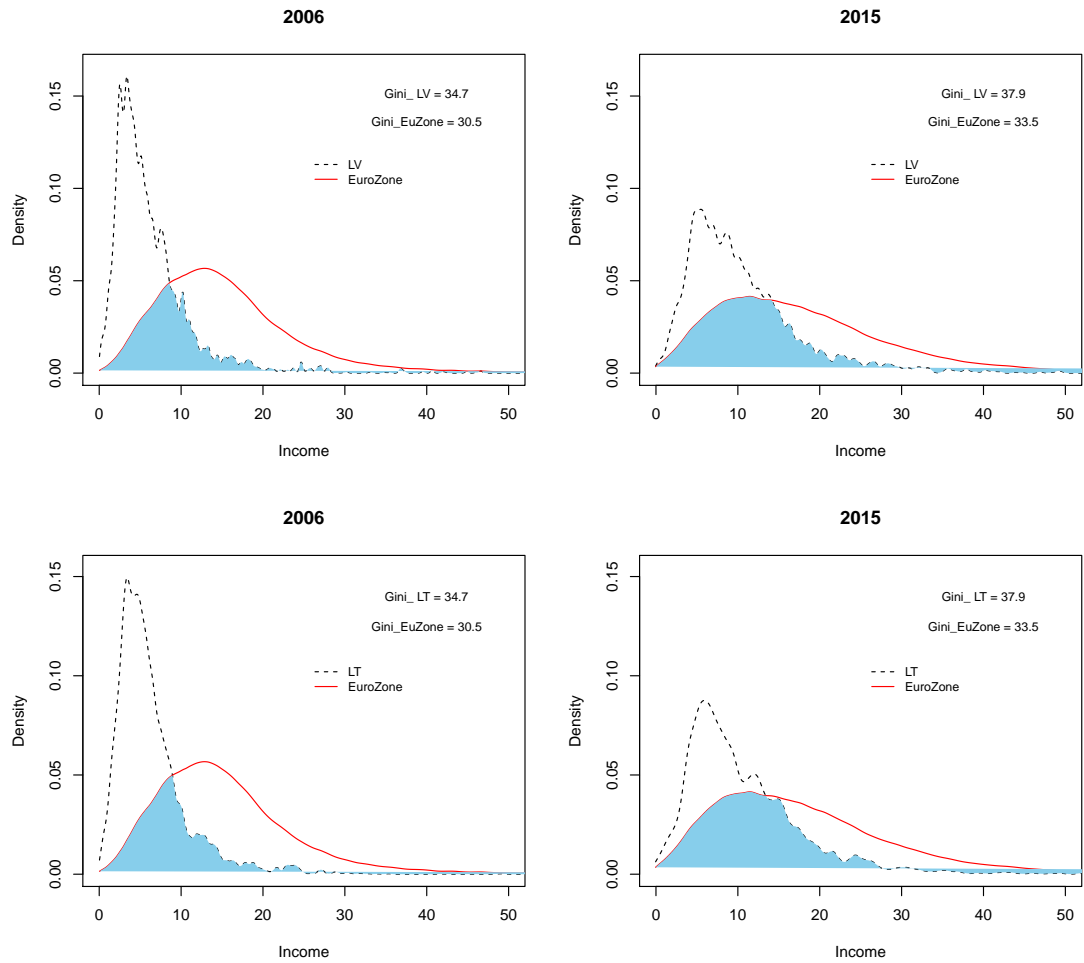
Figure B.3: Income distribution of Greece and Spain and their overlap with the Euro-zone income distribution: years 2006 and 2015.
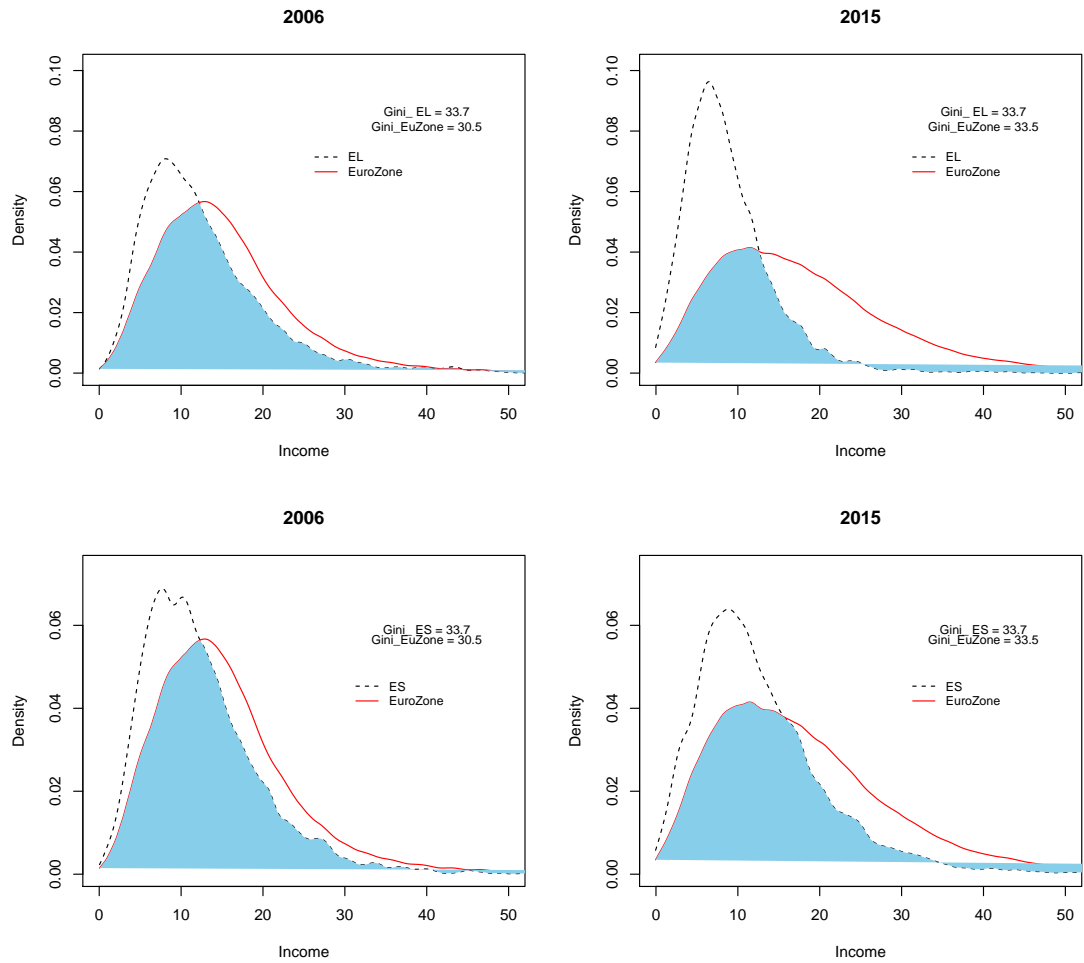
Figure B.4: Income distribution of Finland and France and their overlap with the Eurozone income distribution: years 2006 and 2015.