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Monotonic Norms and Orthogonal Issues in Multidimensional Voting

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Abstract

We study issue-by-issue voting and robust mechanism design in multidimensional frameworks where privately informed agents have preferences induced by general norms. We uncover the deep connections between dominant strategy incentive compatibility (DIC) on the one hand, and several geometric/functional analytic concepts on the other. Our main results are: 1) Marginal medians are DIC if and only if they are calculated with respect to coordinates defined by a basis such that the norm is *orthant-monotonic* in the associated coordinate system. 2) Equivalently, marginal medians are DIC if and only if they are computed with respect to a basis such that, for any vector in the basis, any linear combination of the other vectors is *Birkhoff-James orthogonal* to it. 3) We show how *semi-inner products* and *normality* provide an analytic method that can be used to find all DIC marginal medians. 4) As an application, we derive all DIC marginal medians for l_p spaces of any finite dimension, and show that they do not depend on p (unless $p = 2$).

1 Introduction

We analyze a canonical social choice/mechanism design problem where several privately informed agents take a multidimensional, collective decision. The main results identify the particular issues that can be put to vote in order to obtain robust mechanisms when issue-by-issue voting by (possibly qualified) majority is used to determined the outcome of the

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collective choice. Issue-by-issue voting by majority yields as outcome the issue-by-issue (or marginal, or coordinate-wise) median. Due to the multidimensionality of the decision space, the dimensions on which voting takes place are not uniquely defined. In other words, the issues that are put on the ballot are endogenous, and each feasible set of issues yields a potentially different multidimensional median.

Consider, for example, a legislature that has to decide how much money to allocate to several programs in a given fiscal year. One budgeting procedure, called "bottom-up", is to vote on each program separately in which case the total budget will be the sum of the individual budgets. An alternative, called "top-down", is to vote on the total budget first, and then vote on how to divide the total budget among the individual items.

These two voting-based budgeting procedures yield different outcomes and their relative merit in curbing the budget deficit has been debated among political scientists ever since the U.S. Congress switched from bottom-up to top-down following the 1974 Congressional Budget and Impoundment Control Act.

Of course, there are potentially many other different budgeting procedures with different welfare properties. In order to find the one that is "optimal" according to some criterion (e.g., budget size, or utilitarian welfare), one first has to characterize the set of voting-based budgeting procedures that have good incentive properties.

The goal of this paper is to do just that for an important class of mechanism design problems where there are no monetary transfers among voters, and where the utility of each agent is determined by the distance between a privately known, individual peak (or ideal point) and the taken decision. This distance is derived from a norm on the decision space that is assumed here to be a vector space. The norm can vary across agents, may itself be private information, and it need not be generated by an inner product. In particular, it need not be the Euclidean norm.

Our first main result shows that, with preferences induced by norms, marginal medians are *dominant-strategy incentive compatible* (DIC henceforth) if and only if they are calculated with respect to coordinates determined by an algebraic basis such that the norm is *orthant-monotonic* in the associated coordinate system. Norm *monotonicity* compares all possible pairs of vectors that are ordered with respect to a lattice structure defined on the underlying space, and requires that the norm of a vector with larger coordinates (in absolute values) is larger. Orthant-monotonicity applies the same condition, but requires it to hold only for pairs of comparable vectors in the same orthant. By selecting a coordinate system that aligns the norm structure to the lattice structure, this result allows us to translate to a multidimensional space the one-dimensional insight that a deviation from truthful reporting should move the median peak farther away from one own's true peak.

For any two-dimensional normed space, we prove the existence of at least **two** distinct DIC marginal medians. This is done by invoking an elegant result that goes back to Hermann Auerbach:[1] any convex body that is point-symmetric around a center has at least

[1] The result is attributed to Auerbach by Stefan Banach. But Auerbach was killed by the Nazis, and his

two pairs of *conjugate diameters*. Moreover, the directions of a pair of conjugate diameters define *Birkhoff-James (BJ)-mutually orthogonal* pairs of vectors, and a marginal median calculated with respect to such directions is DIC.

Roughly speaking, BJ-orthogonality translates to any convex set that is point-symmetric around a center (and hence can serve as a unit ball of some norm) the insight that a circle's radius is orthogonal to the tangent through the point where the radius hits the circle's boundary. BJ-orthogonality can be applied to any normed vector space, and it reduces to the usual orthogonality relation in Hilbert spaces, i.e., two vectors are then BJ-orthogonal when their *inner-product* equals zero.

The main "defects" of the general BJ-orthogonality relation, are a lack of symmetry and a lack of additivity. As a consequence, with more than two dimensions, mutual BJ-orthogonality of the vectors in the selected coordinate system is not sufficient in order to induce a DIC marginal median.

Our next main result shows that, for normed spaces of any dimension, marginal medians are DIC if and only if they are computed with respect to a basis such that, for any vector in the basis, any linear combination of the other vectors is BJ-orthogonal to it. To understand the intuition behind this result, consider a three-dimensional normed space, and choose a set of three mutually orthogonal issues (this always exists by Auerbach's construction, which can be performed for any dimension). Assume that a decision has already been taken on two issues (by consecutive majority votes, say): the obtained decision can be any arbitrary vector in the respective two-dimensional subspace. DIC requires the already taken decision to be BJ-orthogonal to the remaining third issue. While this property is automatically satisfied by an orthogonal basis in spaces endowed with an inner product, it needs to be additionally imposed in general normed spaces.

As we argue below, a purely geometric approach cannot yield all BJ-mutually orthogonal vectors (and hence, a priori, not all bases with the above described additivity property). Thus such an approach is not very helpful in mechanism design exercises where a comprehensive class of incentive-compatible mechanisms needs to be identified before maximizing some goal over it.

Our third main result shows how an alternative, analytic approach based on semi-inner products (SIP) can be used to overcome this difficulty. An SIP is a special bivariate form that can be defined for any pair of vectors in any normed space: it resembles an inner product (and is thus related to orthogonality), but is neither symmetric, nor additive. Importantly for our purposes, a *norm-consistent* SIP has an analytic formulation in terms of the underlying norm functional and its directional derivatives. This analytic approach can, in principle, be used to obtain all DIC marginal median mechanisms as the set of solutions to a system of non-linear equations.

As a main application of the SIP approach, we characterize for any finite dimension

dissertation was burnt. Hence there is no trace of the original proof.

$d \geq 2$ and for any $p \geq 1$ the set of coordinate systems yielding DIC marginal medians in the standard $l_p(d)$ space. We also show that, surprisingly, these systems do not depend at all on p (unless $p = 2$, the only inner-product space in this class).[2] Thus, the characterized marginal medians remain DIC even for situations where the norm is allowed to vary across agents within the l_p class, and is their private information.

For example, for $d = 2$ and any $p \geq 1, p \neq 2$, there are exactly **two** distinct DIC marginal median mechanisms for a $l_p(d)$ space:[3] they correspond to marginal medians taken with respect to the standard Cartesian coordinates, or with respect to a 45-degree rotation of these coordinates.[4] Combining this observation with a result of Peters, van der Stel, and Storcken [1993] implies that these are the only DIC, anonymous and Pareto optimal mechanisms in those settings.

Analogous robust results can be obtained for other classes of norms by identifying – via the SIP approach – the set of left-additive bases formed by mutually orthogonal vectors that are shared by all norms in the class. As a further illustration, we consider a setting where agents use individually *weighted* Euclidean norms that are their private information. All these norms are generated by inner-products, and they do not allow for cross-interactions among issues. Then, under a genericity condition on the set of possible weights, there is exactly **one** DIC marginal median mechanism on this class: the standard Cartesian coordinates are the unique jointly orthogonal ones for all the norms in this class. Introducing even the slightest degree of interaction among the issues – by allowing utility functions derived from other, more general, inner-product norms – yields an impossibility result.

1.1 Connections to the Literature and Techniques

Technically, every algebraic basis for a space of decision vectors defines a set of issues (or coordinates) along which issue-by-issue voting by majority yields a combined decision. Since the median is not a linear function of its inputs, the coordinate-wise median varies with the underlying system of coordinates (see Haldane [1948]).

The issue-by-issue median is the prime example for a "structure-induced equilibrium" in the spirit of Shepsle [1979].[5] Besides its ubiquity in practice, this type of voting mechanism (together with its generalization to the so-called "generalized medians" that allow for the presence of additional "phantom" voters with fixed, known peaks) exhaust the set of DIC mechanisms in various settings where the preference domain is sufficiently rich. The first fundamental result in this vein was obtained by Moulin [1980] for the one-dimensional

[2] The case $p < 1$ does not yield a normed space, and it is not considered here.

[3] Recall that Auerbach's theorem yields at least two DIC mechanisms.

[4] It is interesting to note that the welfare analysis in Gershkov, Moldovanu and Shi [2019] for the Euclidean case focused precisely on these coordinates.

[5] Shepsle proposed the structure-induced equilibrium as a response to the lack of equilibria in multidimensional models where voting is not formally constrained by institutional arrangements. See also Feld and Grofman [1988] and Kramer [1972], [1973].

case.[6] Common examples of generalized medians are obtained by issue-by-issue voting with a qualified majority, and the voting thresholds may differ across dimensions (e.g., a bill where one aspect requires a constitutional amendment and hence a higher majority). Our analysis easily generalizes to such mechanisms as well.

With a few notable exceptions, the literature on *incentive compatible* multidimensional voting and its applications to Political Science and Economics has focused on quadratic loss functions. When applied to normed vector spaces, this assumption yields utilities derived from variations on the Euclidean norm (see, for example, the text book by Austen-Smith and Banks [2005]).[7] The reason behind this choice is technical: it allows the use of familiar mathematical methods from Euclidean geometry and/or familiar mean-variance statistical methods associated to the quadratic formulation.

Quadratic loss functions derived from an Euclidean norm are not always suitable for multidimensional applications. Consider, for example, the choice of a budget on two items. Then, under Euclidean distance, the two items are equally weighted, equal deviations from a preferred budget on each item are perceived in the same way, and equal deviations upwards and downwards from the wished spending on one item are also perceived in the same way. Moreover, since utility is separable in the two dimensions, there is no cross-interaction among spending deviations on the two items. While it is possible to extend some of the results based on Euclidean norm to the more general class of quadratic preferences generated by inner-products, and to hereby address some of these concerns,[8] there is an obvious need to understand more general models where preferences do not display such a high degree of spatial symmetry.[9] Moreover, as we show in this paper, inner-product norms constitute, technically, a special, atypical case because the symmetry and additivity of their orthogonality relation. In particular, inner-product norms do not reveal the fundamental difference between the geometries of two-dimensional and higher-dimensional spaces, and its implications for mechanism design.

Even for purely location problems (of a facility, say) the relevant distance function need not be Euclidean: think about the proverbial cab driver in Manhattan who needs to use the "taxicab" norm, driving along the right angles imposed by the array-like city street map. Indeed, distance in US cities is often colloquially measured in "blocks". Such a distance function is not generated by an inner-product norm.

For the Euclidean norm, Kim and Roush [1984] and Peters, van der Stel, and Storcken

[6] See Gershkov, Moldovanu and Shi [2017] and Kleiner and Moldovanu [2017] for implementation of generalized medians by sequentially binary procedures with varying majority requirements. Phantoms are then not required.

[7] The same holds for many other related literatures (e.g., on signaling and cheap talk), and for empirical methodologies (see for example, Clinton, Jackman and Rivers [2004]).

[8] For example, one can consider weighted, quadratic loss functions.

[9] See Eguia [2013] for a critical discussion of these issues, and for references to papers that empirically test this and alternative distance functions.

[1992] connected the DIC property of marginal medians to *orthogonal* coordinate systems.[10] We do indeed show here that orthant-monotonicity of the standard Euclidean norm is equivalent to the requirement that the underlying coordinate system is defined by an orthogonal basis.

Barbera, Gul and Stacchetti [1993] (BGS henceforth) assumed that the decision set is a product of lines. They fixed a system of directions, but did not focus on norm-based preferences. Instead, they studied a richer class of preferences called *multidimensional single-peaked* (m.s.p.) and showed that, on the class of m.s.p. preferences, a mechanism is DIC if and only if it is a generalized marginal median.[11] BGS also showed that their class is maximal in the sense that, if an agent has a preference outside it, there exists a marginal median that is not DIC. In an earlier paper, Border and Jordan [1983] considered a different rich domain of preferences which they called *star-shaped and separable*, and obtained similar results and generalized Moulin's one-dimensional finding.[12]

We show here that a norm-based preference is m.s.p. in the BGS sense if and only if the underlying norm is orthant monotonic, and that it is star-shaped and separable in the Border-Jordan sense if and only if it is monotonic.[13] While our analysis strongly focuses on the dependence on the chosen coordinate system – since norm monotonicity properties crucially depend on the underlying coordinate system – this dependence does not play a role neither in the BGS' nor in Border and Jordan's analysis. We explain in Section 3.2 this discrepancy in terms of the different domains and focuses of the respective studies.

An elegant result due to Peters, van der Stel, and Storcken [1993] shows that marginal medians constitute DIC mechanisms for a general norm on the plane (i.e., when there are two dimensions) if and only if majority voting takes place along two directions that are BJ-mutually orthogonal (see Birkhoff [1935], and James [1947]).[14] As we have already mentioned above, we show here this result holds in this form only for two-dimensional normed spaces.

The existence of BJ-mutually orthogonal vectors (which is necessary but not sufficient for DIC) involves a fixed-point argument, and is therefore not obvious unless the space is Hilbert, where the orthogonality relation is symmetric. For two-dimensional spaces equipped with a strictly convex norm, Peters, van der Stel, and Storcken [1993] constructed a BJ-mutually orthogonal pair of vectors, and therefore proved the existence of at least one DIC marginal median mechanism. In more than two dimensions, an algebraic basis consist-

[10] van der Stel [2000] extends these insights to non-anonymous mechanisms.

[11] Assuming a rich set of preferences, Nehring and Puppe [2007] studied generalized medians on very general abstract domains called "median spaces". These do not necessarily have a vector space structure.

[12] Zhou [1991], and Barbera and Jackson [1994] characterize DIC mechanisms on the larger class of continuous, strictly quasi-concave utility functions with a unique maximizer. The range of such a mechanism must be one-dimensional.

[13] This also shows that the BGS and Border-Jordan domains are different, contrary to some claims in the literature.

[14] See Alonso [**?**] and Martini [2001] for excellent surveys of this, and other related topics.

ing of BJ-mutually orthogonal vectors such that each vector in the basis is orthogonal to any linear combination of the others is called an *Auerbach basis.* Auerbach's construction does generalize to any number of dimensions, and the two constructed bases always possess this *additivity on the right* property.[15,16] What is missing for DIC in more than two dimensions is an *additivity on the left* property.[17]

Gershkov, Moldovanu and Shi [2019] maximize utilitarian welfare over the class of marginal median mechanisms in Hilbert spaces.[18] Technically, they maximize over the continuum, multiplicative group of linear *isometries* (rotations). In inner-product spaces, a linear isometry also preserves angles and hence orthogonality.[19] Thus, applying an isometry to an arbitrary pair of orthogonal issues yields another such pair: because medians are not linear functions, and hence not necessarily isometry-equivariant, this operation may yield a distinct marginal median.[20] Moreover, applying all isometries to any given orthogonal pair exhausts (modulo translations) the set of all relevant orthogonal pairs, and hence the set of all DIC marginal medians (see Kim and Roush [1984] and Peters, van der Stel, and Storcken [1992] for characterizations of DIC mechanisms in terms of orthogonality and isometries for the Euclidean norm).

In non-Hilbert normed spaces, it is not the case that applying all possible isometries to a given pair of BJ-mutually orthogonal directions yields all other possible such pairs. For example, there is no isometry that maps one Auerbach basis into another if the two bases stem from the Auerbach construction mentioned above.[21]

This failure led us to consider an analytical approach, based on semi-inner products, in order to find all sets of coordinates yielding DIC marginal medians. SIP's have been introduced by Lumer [1961]. In an important paper, Giles [1967] has shown that, for the class of smooth (i.e., *Gateaux-differentiable*) norms, BJ-orthogonality coincides with *normality* which means that the SIP equals zero. Moreover, a *norm-consistent SIP* is then unique.

The remainder of the paper is organized as follows: Section 2 presents the social choice model and marginal median mechanisms. Section 3 introduces monotonicity properties of norms and connects DIC to orthant monotonicity. We also relate our insights to those obtained by BGS and Border and Jordan. Section 4 connects orthant-monotonicity (and hence DIC mechanisms) to bases consisting of BJ-mutually orthogonal vectors that satisfy

[15] Additivity on the right of the BJ-orthogonality relation is automatically satisfied if the norm is smooth. This is a consequence of the semi inner-product representation and normality. See Section 5 for details.

[16] Peters et al's two-dimensional construction yields indeed one of the Auerbach bases. These authors generally invoke strict convexity of the norm, but this is not necessary at this step.

[17] If BJ-orthogonality is symmetric or additive and if the dimension is at least three, then the space must be Hilbert (see James [1947], and Marino and Pietramala [1987]).

[18] For two dimensions, this coincides with the class of anonymous, Pareto-optimal and DIC mechanisms. With more dimensions, marginal medians need not be Pareto optimal.

[19] The famous Mazur-Ulam [1932] theorem asserts that any surjective isometry must be linear.

[20] Medians are translation-equivariant, so it is enough to consider isometries that fix at the origin.

[21] In this sense, Euclid's *fourth postulate* does not hold in normed spaces that are not Hilbert.

a left-additivity condition. Section 5 shows how to analytically use semi-inner products in order to find all DIC marginal medians. Section 6 illustrates the various concepts and findings for l_p norms and for inner-product norms, including several robust mechanism design results. Section 7 concludes.

2 The Social Choice Model

An odd number of agents n collectively choose a decision $\mathbf{v} \in V$, where V is a d-dimensional Minkowski (i.e., over the reals) vector space. Since any d-dimensional normed space over the reals is isomorphic to the space $\mathbb{R}^d$, we shall assume here w.l.o.g. that $V = \mathbb{R}^d$, and endow this space with different norms.[22]

Throughout of the paper, the **bold** font is used to denote vectors in $\mathbb{R}^d$. We use $i = 1, ..., n$ to label voters, and j or $k = 1, ..., d$ to label coordinates.

We denote by $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ a generic algebraic basis for $\mathbb{R}^d$, where $\mathbf{x}^1, ..., \mathbf{x}^d$ are linearly independent, and by $\{\mathbf{e}^1, ..., \mathbf{e}^d\}$ the standard Cartesian basis where for vector $\mathbf{e}^j$ only the j-th coordinate is different from zero, and equals one:

$$\mathbf{e}^j = \underbrace{(0, 0, ..., 1}_{j}, 0, ..., 0)$$

Each agent's ideal position is given by a "peak" $\mathbf{t}_i \in \mathbb{R}^d$, $i = 1, 2, ..., n$. The peak $\mathbf{t}_i$ is agent i's private information. The utility of agent i with peak $\mathbf{t}_i$ from decision $\mathbf{v}$ is given by

$$-\|\mathbf{t}_i - \mathbf{v}\|$$

where $\|\cdot\|$ is a *norm* on $\mathbb{R}^d$. Recall that a norm $\|\cdot\|$ is a real-valued function on $\mathbb{R}^d$ that satisfies:

1. $\|\mathbf{x}\| \geq 0$;
2. $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$;
3. $\|a\mathbf{x}\| = |a| \|\mathbf{x}\|, \forall \mathbf{x} \in \mathbb{R}^d, \ a \in \mathbb{R}$;
4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

Since our analysis and results are purely ordinal, they immediately apply to all utility functions of the form

$$-\Delta (\|\mathbf{t}_i - \mathbf{v}\|)$$

where Δ is a strictly monotonic function: all these cardinal utilities represent the same ordinal preferences as the basic norm $\|\cdot\|$.

We do **not** assume that the norm inducing the above utility functions is generated by an inner-product, i.e., the vector space need not be a Hilbert space.

[22] The isomorphism does depend on the assumed basis – we shall make this explicit below.

2.1 Marginal Medians

For the following properties we assume that mechanisms only depend on reported peaks, which holds by definition for the (generalized) marginal medians we consider.

Definition 1 *1. A direct revelation mechanism is a function $\psi : (\mathbb{R}^d)^n \to \mathbb{R}^d$.*

2. A direct revelation mechanism $\psi(\mathbf{t}_i, \mathbf{t}_{-i})$ is dominant-strategy incentive compatible (DIC) if for any voter i and for any realizations $\mathbf{t}_i$ and $\mathbf{t}_{-i}$, it holds that

$$\|\mathbf{t}_i - \psi(\mathbf{t}_i, \mathbf{t}_{-i})\| \leq \left\|\mathbf{t}_i - \psi\left(\hat{\mathbf{t}}_i, \mathbf{t}_{-i}\right)\right\|, \ \forall \hat{\mathbf{t}}_i$$

3. A direct revelation mechanism is anonymous if $\psi(\mathbf{t}_1, .., \mathbf{t}_n) = \psi(\sigma(\mathbf{t}_1, .., \mathbf{t}_n))$ for all $\mathbf{t}_1, ..., \mathbf{t}_n \in \mathbb{R}^d$, and for all permutations σ.

In words, DIC says that any individual manipulation forces the social choice to move weakly away from the true peak, as measured by the distance function used by the manipulating agent. The DIC constraint for agent i only uses the norm considered by agent i, and hence we can easily generalize the above definition to situations where agents use different norms.

Let $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ be an algebraic basis for $\mathbb{R}^d$. Then each $\mathbf{y} \in \mathbb{R}^d$ can be represented as

$$\mathbf{y} = \sum_{j=1}^{d} \alpha^j(\mathbf{y})\mathbf{x}^j,$$

where $\alpha^j(\mathbf{y})$, $j = 1, ..., d$, is the j-th coordinate of $\mathbf{y}$ with respect to this basis.

Definition 2 *The marginal median mechanism (MMM) with respect to basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ is defined as*

$$\psi(\mathbf{t}_1, .., \mathbf{t}_n) = \sum_{j=1}^{d} med(\alpha^j(\mathbf{t}_1), ..., \alpha^j(\mathbf{t}_n))\mathbf{x}^j$$

where $med(\alpha^j(\mathbf{t}_1), ..., \alpha^j(\mathbf{t}_n))$ is the median of the j-th coordinates of the agents' peaks.

All our results below can be extended to *generalized medians* that are obtained by setting a fixed number of "phantom" peaks at some commonly known locations, and then taking the marginal median among the reported peaks of the real agents and the commonly known phantom peaks. Generalized marginal medians are anonymous, but need not be Pareto optimal for three or more dimensions.[23]

[23] The standard marginal median (corresponding to voting by simple majority in each dimension) is Pareto optimal for $d = 2$.

3 Incentive Compatibility and Monotonic Norms

In this section, we first define norm monotonicity and orthant monotonicity with respect to a given algebraic basis in $\mathbb{R}^d$. We then show that a marginal median mechanism computed with respect to a given basis is DIC if and only if the norm is orthant monotonic with respect to that same basis. Finally, we discuss how this result is related to the insights in BGS and Border and Jordan [1983].

3.1 Orthant Monotonicity and DIC Marginal Medians

Fix an algebraic basis in $\mathbb{R}^d$ consisting of d linearly independent vectors $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$. Recall that we can represent each $\mathbf{x} \in \mathbb{R}^d$ as

$$\mathbf{x} = \sum_{j=1}^{d} \alpha^j(\mathbf{x})\mathbf{x}^j,$$

where $\alpha^j(\mathbf{x})$ is the j-th coordinate of $\mathbf{x}$ according to this basis. To simplify notation when confusion cannot arise, we write

$$(x_1, x_2, ..., x_d) = \left(\alpha^1(\mathbf{x}), \alpha^2(\mathbf{x}), ..., \alpha^d(\mathbf{x})\right)$$

and identify $\mathbf{x}$ with the vector of coordinates $(x_1, x_2, ..., x_d)$.

It is important to note that the monotonicity properties described below depend on the underlying coordinate system.

Definition 3 *A norm $\|\cdot\|$ on $\mathbb{R}^d$ is monotonic (Bauer, Stoer and Witzgall [1961]) if*

$$\|(x_1, ..., x_d)\| \leq \|(y_1, ..., y_d)\|$$

whenever

$$|x_j| \leq |y_j| \text{ for all } j = 1, ..., d.$$

A norm $\|\cdot\|$ on $\mathbb{R}^d$ is orthant-monotonic (Gries [1967]) if

$$\|(x_1, ..., x_d)\| \leq \|(y_1, ..., y_d)\|$$

whenever

$$x_j y_j \geq 0 \text{ and } |x_j| \leq |y_j| \text{ for all } j = 1, ..., d.$$

It is clear from the above definition that monotonicity implies orthant-monotonicity.[24] The following lemma provides useful characterizations, repeatedly used below:[25]

[24] A monotonic norm can also be seen as a *lattice norm* since it is consistent with the standard partial order on $\mathbb{R}^d$. Thus, a normed space endowed with a monotonic norm becomes a *Riesz space.*

[25] See Johnson and Nylen [1991] or Horn and Johnson [2013], p. 340, for these, and for other monotonicity properties of norms.

Lemma 1 *1. A norm is monotonic if and only if it is absolute:*

$$\|(x_1,...,x_d)\| = \|(\,|x_1|\,,...,|x_d|)\|$$

for all $\mathbf{x} \in \mathbb{R}^d$*.*

2. A norm is orthant-monotonic if and only if it satisfies

$$\|(x_1,...,x_{j-1},0,...,x_d)\| \leq \|(x_1,...,x_{j-1},x_j,...,x_d)\|$$

for all $\mathbf{x} \in \mathbb{R}^d$ *and all* j*.*

Suppose for visualization that $d = 2$, and consider the unit disc of a smooth norm. If the norm is orthant-monotonic with respect to a basis $\{\mathbf{x}^1, \mathbf{x}^2\}$, then the tangents at the points where the disc intersects the coordinate defined by x^1 must be parallel to the coordinate defined by x^2, and vice versa.

Example 1 *Fix basis* $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$*. For* $\mathbf{x} = (x_1,...,x_d)$ *and* $p \geq 1$*, define*

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{d} |x_j|^p\right)^{1/p}.$$

This is the class of $l_p(d)$ *norms with respect to the given basis. All these norms are absolute, hence monotonic and hence orthant-monotonic. The same holds for the limit norm*

$$\|\mathbf{x}\|_\infty = \max_j |x_j|\,.$$

Example 2 *Let* $d = 2$*, and fix the Cartesian basis.*

1. *Consider the norm with unit ball defined as the hexagon with vertices at* $\pm(1,1), \pm(1,0)$ *and* $\pm(0,1)$*. This norm is orthant-monotonic but **not** monotonic. For example,*

   $$\|(1,-1)\| > 1 = \|(1,1)\|\,,$$

   which contradicts the fact that a monotonic norm can only depend on absolute values (Lemma 1-1).

2. *Consider the norm with unit ball defined as the parallelogram with vertices at* $\pm(2,2)$ *and* $\pm(1,-1)$*. This norm is **not** orthant-monotonic. For example,*

   $$\|(2,0)\| > 1 = \|(2,2)\|\,, \tag{1}$$

which contradicts the characterization in Lemma 1-2.

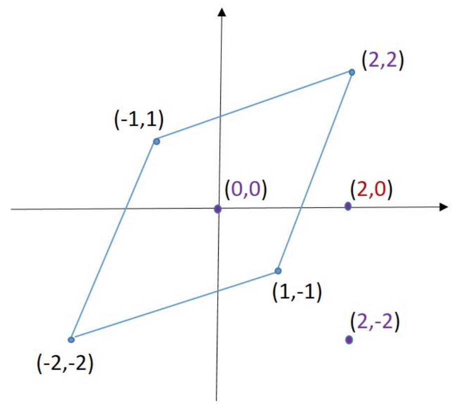


Figure 1: A non-orthant-monotonic norm

To see the implications of non-orthant monotonicity on marginal medians, consider the Cartesian coordinates and consider three agents with peaks at $(0,0)$, $(2,2)$ *and* $(2,-2)$, *respectively. If all agents report truthfully, the marginal median is* $(2,0)$. *If the agent with peak at* $(0,0)$ *deviates and reports instead* $(2,2)$, *the marginal median becomes* $(2,2)$. *By inequality (1), this deviation is profitable for this agent. In contrast, marginal mechanisms are DIC if computed with respect to the coordinates defined by the basis* $\{(-1,1),(1,1)\}$ *or by the basis* $\{(1,3),(1,1/3)\}$.[26]

We can now state our first main result:

Theorem 1 *A marginal median mechanism is DIC if and only if it is computed with respect to a basis* $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ *such that the norm is orthant-monotonic in the associated coordinate system.*

Proof. (If direction). Fix a basis with the required property, $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$. We show that the MMM with respect to this basis is DIC. For any vector $\mathbf{x}$, let $\sum_{j=1}^{d} \alpha^j(\mathbf{x})\mathbf{x}^j$ be its representation in the fixed basis. Let $\mathbf{t}_1$ be agent 1's true peak, and consider a deviation to $\widetilde{\mathbf{t}}_1$. Assume that the other agents $i = 2, ..., n$ make arbitrary reports $\mathbf{t}_2, ..., \mathbf{t}_n$.

Let $\mathbf{m} = \mathbf{m}(\mathbf{t}_1, ..., \mathbf{t}_n)$ and $\widetilde{\mathbf{m}} = \mathbf{m}(\widetilde{\mathbf{t}}_1, ..., \mathbf{t}_n)$ be the marginal medians when agent 1 reports truthfully and when he deviates to $\widetilde{\mathbf{t}}_1$, respectively. The argument for any other agent is analogous. We need to show that

$$\|\mathbf{t}_1 - \mathbf{m}\| \leq \|\mathbf{t}_1 - \widetilde{\mathbf{m}}\|,$$

or equivalently that

$$\left\| \sum_{j=1}^{d} (\alpha^j(\mathbf{t}_1) - \alpha^j(\mathbf{m}))\mathbf{x}^j \right\| \leq \left\| \sum_{j=1}^{d} (\alpha^j(\mathbf{t}_1) - \alpha^j(\widetilde{\mathbf{m}}))\mathbf{x}^j \right\|. \quad (2)$$

[26] We show below that there are always at least two bases with this property.

By the properties of the one-dimensional median we obtain that, for any $j = 1, 2, ..., d$,

$$\text{either} \;\; \alpha^j(\mathbf{t}_1) \leq \alpha^j(\mathbf{m}) \leq \alpha^j(\widetilde{\mathbf{m}}) \;\; \text{or} \;\; \alpha^j(\widetilde{\mathbf{m}}) \leq \alpha^j(\mathbf{m}) \leq \alpha^j(\mathbf{t}_1). \tag{3}$$

This immediately implies that, for all $j = 1, 2, ..., d$,

$$\left(\alpha^j(\mathbf{t}_1) - \alpha^j(\mathbf{m})\right)\left(\alpha^j(\mathbf{t}_1) - \alpha^j(\widetilde{\mathbf{m}})\right) \geq 0, \tag{4}$$

and that

$$\left|\alpha^j(\mathbf{t}_1) - \alpha^j(\mathbf{m})\right| \leq \left|\alpha^j(\mathbf{t}_1) - \alpha^j(\widetilde{\mathbf{m}})\right|. \tag{5}$$

Therefore, (2) follows immediately by applying orthant monotonicity to the two vectors

$$\sum_{j=1}^{d}(\alpha^j(\mathbf{t}_1) - \alpha^j(\mathbf{m}))\mathbf{x}^j \text{ and } \sum_{j=1}^{d}(\alpha^j(\mathbf{t}_1) - \alpha^j(\widetilde{\mathbf{m}}))\mathbf{x}^j.$$

(Only if direction). The proof is a generalization of the insight in Example 2. Assume that the MMM is computed with respect to a basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ whose coordinate system does not yield an orthant-monotonic representation of the norm. By Lemma 1 there exists k, a vector $\mathbf{x} = \sum_{j \neq k} \alpha^j(\mathbf{x})\mathbf{x}^j \neq 0$ and a scalar β such that

$$\|\mathbf{x}\| = \left\|\alpha^1(\mathbf{x}), ..., \alpha^{k-1}(\mathbf{x}), 0, ..., \alpha^d(\mathbf{x})\right\| > \left\|\alpha^1(\mathbf{x}), ..., \alpha^{k-1}(\mathbf{x}), \beta, ..., \alpha^d(\mathbf{x})\right\| = \left\|\mathbf{x} + \beta\mathbf{x}^k\right\|$$

Consider now the following profile of preferences:

$$\begin{aligned} \mathbf{t}_i &= \mathbf{x} - \beta\mathbf{x}^k, \; i = 1, ..., \frac{n-1}{2} \\ \mathbf{t}_i &= \mathbf{x} + \beta\mathbf{x}^k, \; i = \frac{n+1}{2}, ..., n-1 \\ \mathbf{t}_i &= \mathbf{0}, \; i = n \end{aligned}$$

If every agent reports truthfully, we obtain that

$$\mathbf{m}(\mathbf{t}_1, ..., \mathbf{t}_n) = \mathbf{x}.$$

Consider now a deviation of agent n to $\widetilde{\mathbf{t}}_n = \mathbf{x} + \beta\mathbf{x}^k$ which yields

$$\mathbf{m}(\mathbf{t}_1, ..., \widetilde{\mathbf{t}}_n) = \mathbf{x} + \beta\mathbf{x}^k.$$

Since by assumption $\|\mathbf{x}\| > \left\|\mathbf{x} + \beta\mathbf{x}^k\right\|$, this deviation is profitable for agent n who has a true peak at the origin. This is a contradiction to the assumption that the marginal median with respect to this basis is DIC for agent n. ■

The above proof shows that the sufficiency of orthant monotonicity for incentive compatibility is intimately linked to the main properties of the one-dimensional median. In one dimension, a deviation has either no effect on the median, or moves it away from the agent's peak: the old median under truth telling lies between the agent's peak and the new median after the deviation (formally (3)). By applying this observation, dimension by dimension, to a d-dimensional marginal median, we conclude that the two difference vectors, one between the ideal point and the old d-dimensional median and the other between the ideal point and the new median, lie in the same orthant (formally (4)). Moreover, the latter has larger coordinates than the former (formally (5)). Therefore, by orthant monotonicity, the agent's ideal point is nearer to the old median than to the new median[27].

Remark 1 *We can replace the norm $\|\cdot\|$ by individual-specific norms $\{\|\cdot\|_i\}$, and the sufficiency proof (if direction), still goes through. This implies that an MMM remains DIC when agents' preferences are generated by possibly different norms that are all orthant monotonic with respect to a given basis. We provide later alternative characterizations of incentive compatibility via Birkhoff-James orthogonality (Section 4) and via semi-inner products (Section 5). Like the sufficiency part of Theorem 1, these alternative sufficient conditions for DIC (e.g., Theorems 3 and 7) do not require that all agents share the same norm. This observation can be used to construct robust mechanisms for situations where the preference-inducing norms differ from agent to agent, and are their private information.*

3.2 Relations to BGS and to Border and Jordan [1983]

BGS studied a class of preferences called *multidimensional single-peaked (m.s.p.).* BGS showed that, on that class of m.s.p. preferences (that are not necessarily induced by a norm), a mechanism is DIC if and only if it is a generalized marginal median. They also showed that the m.s.p. class is maximal in the sense that, if an agent has a preference outside it, there exists a marginal median which is not DIC.

Border and Jordan [1983] characterized DIC mechanisms on a different domain of preferences that they called *star-shaped and separable.* A norm-based preference is always (weakly) star-shaped in the sense of Border and Jordan: this follows by the convexity of the norm functional (i.e., by the triangle inequality).[28] Here are the respective definitions:[29]

Definition 4 *1. A preference relation with ideal point $\mathbf{x}$ is m.s.p. if for every $\mathbf{y}$ and for every $\mathbf{z}$ on a shortest l_1 path from $\mathbf{x}$ to $\mathbf{y}$, $\mathbf{z}$ is (weakly) preferred to $\mathbf{y}$.*[30]

2. A preference induced by a norm $\|\cdot\|$ is separable if, for all $j = 1, ..., d$ and for all x_{-j}, y_{-j}, x_j, and x'_j,

$$\|(x_j, x_{-j})\| \geq \left\|(x'_j, x_{-j})\right\| \Leftrightarrow \|(x_j, y_{-j})\| \geq \left\|(x'_j, y_{-j})\right\|.$$

[27] Since the one-dimensional generalized medians (with possible phantoms) share the above property with the standard one-dimensional median, the above results easily extends to generalized marginal median mechanisms.

[28] Strictly convex norms – see the definition in Section 6 – are (strongly) star-shaped.

[29] We equivalently re-formulate the BGS one to best highlight the connection with our normed spaces.

[30] This definition implicitly assumes a fixed, given basis for calculating the l_1 norm, e.g., the standard Euclidean one.

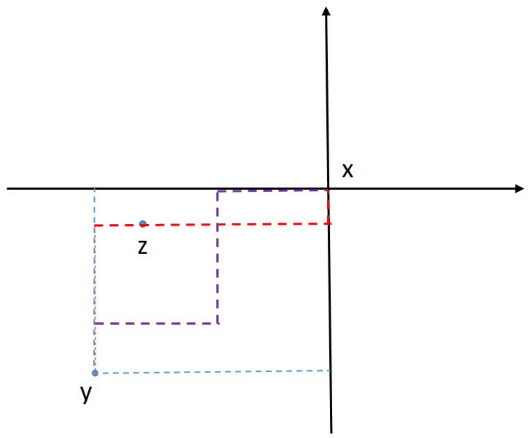


Figure 2: Point $\mathbf{z}$ is on a shortest l_1 path from $\mathbf{x}$ to $\mathbf{y}$

Proposition 1 *Fix a coordinate system determined by a basis* $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$*, and define the* l_1 *norm with respect to **these** coordinates:*

$$\|\mathbf{x}\|_1 = \sum_{j=1}^{d} |x^j| \quad \textit{for any } \mathbf{x} = \sum_{j=1}^{d} x_j \mathbf{x}^j.$$

1. *The preference relation induced by a norm* $\|\cdot\|$ *is m.s.p. if and only if the norm is orthant-monotonic with respect to the chosen system of coordinates.*

2. *The preference relation induced by a norm* $\|\cdot\|$ *is separable if and only if the norm is monotonic with respect to the chosen system of coordinates.*

Proof. 1. Since the $\|\cdot\|$-based preference of an agent with peak at $\mathbf{x}$ is a translation of the preference of an agent with peak at $\mathbf{0}$, we can w.l.o.g. assume below that peaks are at $\mathbf{0}$.

Assume first that $\|\cdot\|$ is orthant-monotonic with respect to the fixed coordinates. Consider any $\mathbf{y}$ and any $\mathbf{z}$ on a shortest l_1 path from $\mathbf{0}$ to $\mathbf{y}$. Then $\mathbf{y}$ and $\mathbf{z}$ must be in the same orthant (see also picture above) and $|z_j| \leq |y_j|$ for all $j = 1, ..., d$.[31] Hence, $\|\mathbf{z}\| \leq \|\mathbf{y}\|$ by orthant-monotonicity, and $\mathbf{z}$ is preferred to $\mathbf{y}$ by an agent with peak at $\mathbf{0}$, yielding m.s.p.

Conversely, assume that the $\|\cdot\|$-based preference is m.s.p. Consider $\mathbf{y}$ and $\mathbf{z}$ such that $z_j y_j \geq 0$ and $|z_j| \leq |y_j|$ for all $j = 1, ..., d$. Then it is easy to see that $\mathbf{z}$ must lie on a shortest l_1 path from $\mathbf{0}$ to $\mathbf{y}$, and hence it must be preferred to $\mathbf{y}$. Since the preference is derived from the norm $\|\cdot\|$, we must have $\|\mathbf{z}\| \leq \|\mathbf{y}\|$ and thus the norm is orthant-monotonic.

2. Assume first that the norm $\|\cdot\|$ is monotonic, and assume that $\|(x_j, x_{-j})\| \geq \left\|(x'_j, x_{-j})\right\|$. We have to show $\|(x_j, y_{-j})\| \geq \left\|(x'_j, y_{-j})\right\|$ for all y_{-j}. By Lemma 1, a

[31] Note that orthants are also defined by the chosen coordinate system.

monotonic norm is absolute. This implies:

$$\begin{aligned}\|(|x_1|,...,|x_j|,...,|x_d|)\| &= \|(x_1,...,x_j,...,x_d)\| \\ &\geq \left\|(x_1,...,x_j',...,x_d)\right\| \\ &= \left\|(|x_1|,...,\left|x_j'\right|,...,|x_d|)\right\|\end{aligned}$$

Monotonicity implies then that $|x_j| \geq \left|x_j'\right|$. Hence, again by monotonicity, we obtain

$$\begin{aligned}\|(y_1,...,x_j,...,y_d)\| &= \|(|y_1|,...,|x_j|,...,|y_d|)\| \\ &\geq \left\|(|y_1|,...,\left|x_j'\right|,...,|y_d|)\right\| \\ &= \left\|(y_1,...,x_j',...,y_d)\right\|\end{aligned}$$

as desired.

For the converse, assume that the preference induced by the norm $\|\cdot\|$ is separable. By Lemma 1, it is enough to show that the norm is absolute: for all $\mathbf{x} = (x_1,...,x_d)$,

$$\|(x_1,...,x_d)\| = \|(|x_1|,...,|x_d|)\|.$$

If for all j, $x_j \geq 0$ or if for all j, $x_j \leq 0$, the implication is clear by the homogeneity of the norm. Assume then that $x_j \geq 0$ for all $j \in S$, and $x_j < 0$ for all $j \in S^C$, and that both S and S^C are not empty. Let $k \in S^C$ be minimal, and consider the vector $(x_1,...,x_k,|x_{k+1}|,...,|x_d|)$. Since k is minimal in S^C, we have

$$\|(x_1,...,x_k,|x_{k+1}|,...,|x_d|)\| = \|(|x_1|,...,-|x_k|,|x_{k+1}|,...,|x_d|)\|.$$

We want to show that

$$\|(|x_1|,...,-|x_k|,|x_{k+1}|,...,|x_d|)\| = \|(|x_1|,...,|x_k|,|x_{k+1}|,...,|x_d|)\|.$$

Assume by contradiction that this is not the case, and let

$$\|(|x_1|,...,-|x_k|,|x_{k+1}|,...,|x_d|)\| < \|(|x_1|,...,|x_k|,|x_{k+1}|,...,|x_d|)\|. \tag{6}$$

The other case is completely analogous. By separability we obtain that

$$\|(-|x_1|,...,-|x_k|,-|x_{k+1}|,...,-|x_d|)\| < \|(-|x_1|,...,|x_k|,-|x_{k+1}|,...,-|x_d|)\|.$$

Multiplying the vectors by -1 and using homogeneity of the norm, we obtain

$$\|(|x_1|,...,|x_k|,|x_{k+1}|,...,|x_d|)\| < \|(|x_1|,...,-|x_k|,|x_{k+1}|,...,|x_d|)\|.$$

which is a contradiction to (6). Hence, we must have

$$\|(x_1,...,x_k,|x_{k+1}|,...,|x_d|)\| = \|(|x_1|,...,|x_k|,|x_{k+1}|,...,|x_d|)\|.$$

Continuing in the same way for all remaining $j \in S^C$ yields the desired result. ■

Remark 2 *We stress the dependence on the chosen coordinate system, but this feature is discussed neither by BGS, nor by Border and Jordan, who implicitly fix a coordinate system. This also fixes the set of l_1 shortest paths BGS consider in the definition of m.s.p.: such paths are then solely composed of segments that are parallel to their fixed coordinates. Both BGS and Border and Jordan consider a rich (even maximal in the BGS analysis) class of preferences and all DIC mechanisms they find are separable in the sense that they can be decomposed into d one-dimensional DIC mechanisms (again, with respect to their fixed coordinate system). Non-separable mechanisms fail DIC because of some preference in their respective rich domains. This also allows BGS, and Border and Jordan to prove converse statements about (separable) generalized medians being the only DIC mechanisms.*

What we do here is different: we fix one particular preference relation generated by a norm (or, for some results below, a relatively "small" set of preferences, such as the preferences generated by all l_p norms) and look instead for ***all*** *possible coordinate systems – each one defines then its own l_1 shortest paths and its own orthants – that yield DIC marginal medians for this particular preference. Thus, in each particular instance, we analyze a small set of preferences, and we uncover a larger set of DIC mechanisms. It is* ***not*** *the case that all these mechanisms are separable with respect to a fixed set of coordinates!*

4 Incentive Compatibility and Orthogonality

In mechanism design exercises one usually seeks an optimal mechanism in a certain class. Thus, one first needs to characterize the relevant incentive compatible mechanisms. How can we construct all systems of issues that induce DIC issue-by-issue voting? In other words, given a norm, what are the coordinates that render this norm orthant-monotonic? We first discuss a geometric approach towards answering this question, and in the next Section we complement the answer via an analytic device.

Let us start with an example showing that the Euclidean norm on the plane has an orthant-monotonic representation if and only if it is computed according to a coordinate system defined by an **orthogonal** basis. This relates to the well-known observation that, under the Euclidean norm, marginal medians on the plane are DIC if and only if they are computed with respect to an orthogonal set of coordinates (see Kim and Roush [1984] and Peters et al [1992]). This result about the Euclidean norm holds for any number of dimensions.

Example 3 *(Orthogonality) Let $\{\mathbf{e}^1, \mathbf{e}^2\}$ be the standard Cartesian basis for $\mathbb{R}^2$, and consider the Euclidean $l_2(2)$ norm with respect to this basis. Consider another algebraic basis $\{\mathbf{f}^1, \mathbf{f}^2\}$ that can be written as $\mathbf{f}^1 = a_1\mathbf{e}^1 + a_2\mathbf{e}^2$ and $\mathbf{f}^2 = b_1\mathbf{e}^1 + b_2\mathbf{e}^2$, where the matrix*

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

is non-singular. Then, any vector $\mathbf{x}$ *can be represented as:*

$$\begin{aligned}\mathbf{x} &= x_1\mathbf{f}^1 + x_2\mathbf{f}^2 \\ &= x_1(a_1\mathbf{e}^1 + a_2\mathbf{e}^2) + x_2(b_1\mathbf{e}^1 + b_2\mathbf{e}^2) \\ &= (x_1a_1 + x_2b_1)\mathbf{e}^1 + (x_1a_2 + x_2b_2)\mathbf{e}^2.\end{aligned}$$

Recall that (x_1, x_2) *denote the coordinates of* $\mathbf{x}$ *according to basis* $\{\mathbf{f}^1, \mathbf{f}^2\}$*. By Lemma 1, the* $l_2(2)$ *norm (defined with respect to Cartesian coordinates) is orthant monotonic in the new coordinate system defined by* $\{\mathbf{f}^1, \mathbf{f}^2\}$ *if and only if*

$$||(x_1, x_2)|| \geq \max\{||(0, x_2)||, ||(x_1, 0)||\}.$$

By the formula of the $l_2(2)$ *norm with respect to the standard Cartesian basis* $\{\mathbf{e}^1, \mathbf{e}^2\}$*, we obtain that*

$$\begin{aligned}||(x_1, x_2)|| &= \sqrt{(x_1a_1 + x_2b_1)^2 + (x_1a_2 + x_2b_2)^2}, \\ ||(0, x_2)|| &= \sqrt{(x_2b_1)^2 + (x_2b_2)^2}, \\ ||(x_1, 0)|| &= \sqrt{(x_1a_1)^2 + (x_1a_2)^2}.\end{aligned}$$

Thus, orthant-monotonicity in the coordinate system defined by $\{\mathbf{f}^1, \mathbf{f}^2\}$ *holds if and only if, for any* x_1 *and* x_2 *we have:*

$$(x_1a_1 + x_2b_1)^2 + (x_1a_2 + x_2b_2)^2 \geq \max\left\{(x_2b_1)^2 + (x_2b_2)^2,\ (x_1a_1)^2 + (x_1a_2)^2\right\}$$

This is equivalent to:

$$x_1^2(a_1^2 + a_2^2) + 2x_1x_2(a_1b_1 + a_2b_2) \geq 0 \quad \textit{and} \quad x_2^2(b_1^2 + b_2^2) + 2x_1x_2(a_1b_1 + a_2b_2) \geq 0.$$

The above inequalities hold for all x_1 *and* x_2 *if and only if* $a_1b_1 + a_2b_2 = 0$*. In other words, the* $l_2(2)$ *norm (with respect to the Cartesian basis) is orthant monotonic with respect to another basis* $\{\mathbf{f}^1, \mathbf{f}^2\}$ *if and only if the inner-product of* $\mathbf{f}^1$ *and* $\mathbf{f}^2$ *is zero, that is,* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *are orthogonal. Every such system yields an orthant-monotonic representation of the* $l_2(2)$ *norm, and thus each marginal median computed with respect to any orthogonal basis is DIC. Since the median is not a linear function (while orthogonal basis transformations are linear), this operation potentially yields new DIC mechanisms.*

4.1 The Birkhoff-James Orthogonality Relation

The standard definition of orthogonality used above (via a zero *inner-product*) can only be applied to *Hilbert spaces* where the norm is generated by an inner product, e.g., the Euclidean norm. In order to characterize the system of coordinates that yield DIC marginal medians in arbitrary normed spaces, we need a more general notion of orthogonality, introduced by Birkhoff [1935] and later masterfully analyzed by James [1947].[32]

[32] There are many definitions of orthogonality. But only the BJ notion is relevant for incentive compatibility.

A vector $\mathbf{x}$ is said to be *Birkhoff-James (BJ) orthogonal* to another vector $\mathbf{y}$ if $\mathbf{x}$ has the smallest norm among all vectors on the line through $\mathbf{x}$ that is parallel to $\mathbf{y}$. Equivalently, the line through $\mathbf{x}$ that is parallel to $\mathbf{y}$ is tangent to the "ball" with radius $\|\mathbf{x}\|$.

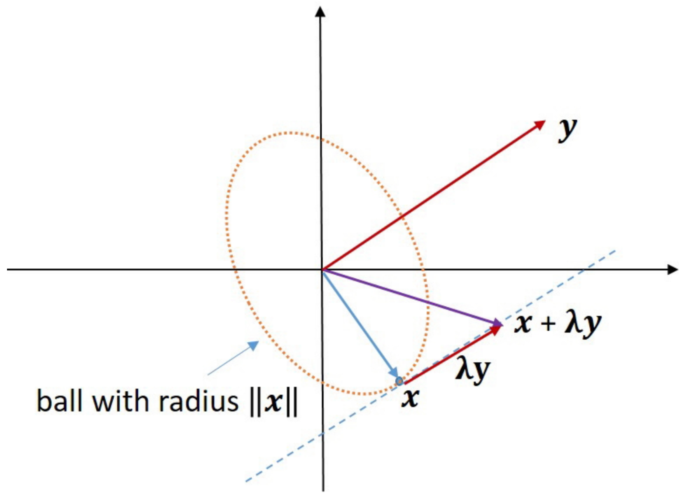


Figure 3: $\mathbf{x}$ is BJ-orthogonal to $\mathbf{y}$

Definition 5 *(Birkhoff-James orthogonality, Birkhoff [1935], James [1947]):*

1. *A vector* $\mathbf{x}$ *is orthogonal to another vector* $\mathbf{y}$, *denoted* $\mathbf{x} \dashv \mathbf{y}$, *if* $\|\mathbf{x} + \lambda\mathbf{y}\| \geq \|\mathbf{x}\|$ *for all real* λ; $\mathbf{y}$ *is orthogonal to* $\mathbf{x}$, *denoted* $\mathbf{y} \dashv \mathbf{x}$, *if* $\|\mathbf{y} + \lambda\mathbf{x}\| \geq \|\mathbf{y}\|$ *for all real* λ.

2. *We call* $\mathbf{x}$ *and* $\mathbf{y}$ *BJ-mutually orthogonal if* $\mathbf{x} \dashv \mathbf{y}$ *and* $\mathbf{y} \dashv \mathbf{x}$.

3. *A vector* $\mathbf{x}$ *is BJ-orthogonal to a subspace* M, *denoted* $\mathbf{x} \dashv M$, *if* $\mathbf{x} \dashv \mathbf{y}$ *for all* $\mathbf{y} \in M$. *A subspace* M *is BJ-orthogonal to a vector* $\mathbf{x}$, *denoted* $M \dashv \mathbf{x}$, *if* $\mathbf{y} \dashv \mathbf{x}$ *for all* $\mathbf{y} \in M$.

The BJ-orthogonality relation is generally **not** symmetric: $\mathbf{x}$ can be orthogonal to $\mathbf{y}$ but not vice-versa. The BJ-orthogonality relation is generally **not** additive, neither on the left, nor on the right: $\mathbf{y} \dashv \mathbf{x}$ and $\mathbf{z} \dashv \mathbf{x}$ need not imply $(\mathbf{y} + \mathbf{z}) \dashv \mathbf{x}$, and also $\mathbf{x} \dashv \mathbf{y}$ and $\mathbf{x} \dashv \mathbf{z}$ need not imply $\mathbf{x} \dashv (\mathbf{y} + \mathbf{z})$. BJ-orthogonality reduces to the standard (symmetric and additive) definition if the space is Hilbert: two vectors are orthogonal if and only if their inner-product is zero.

The next concept corrects for symmetry and for additivity on the right. Given an algebraic basis $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$, denote by X^{-i} the subspace spanned by the vectors $\{\mathbf{x}^1, ..., \mathbf{x}^{i-1}, \mathbf{x}^{i+1}, ..., \mathbf{x}^d\}$.

Definition 6 *An Auerbach basis is an algebraic basis* $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ *such that, for each* $j = 1, ..., d$, $\mathbf{x}^j \dashv X^{-j}$. *Such a basis is orthonormal if* $\left|\left|\mathbf{x}^j\right|\right| = 1,\ j = 1, 2, ..., d$.

Theorem 2 *In any normed space there exist at least **two** distinct Auerbach bases (see Day [1947], Taylor [1947]). There is **no** isometry that transforms one of these bases into another, unless the space is Hilbert (see Plichko [1991]).*

To illustrate the beautiful existence argument, consider a two-dimensional normed space and its unit ball. Let Q be the quadrilateral with largest area inscribed in the unit ball. It exists by compactness. Then its main diagonals are *conjugate diameters* (see Heil and Krautwald [1969]). In particular, the diagonals are in the directions of two BJ-mutually orthogonal vectors. Analogously, let Q' be the quadrilateral with smallest area such that the four sides are all tangents to the unit ball. Again, it exists by compactness. Then the two lines such that each connects two tangency points across each other are also conjugate diameters in the directions of BJ-mutually orthogonal vectors. The two constructions always yield different pairs, unless the space is Hilbert (then there are an infinite number of different pairs).

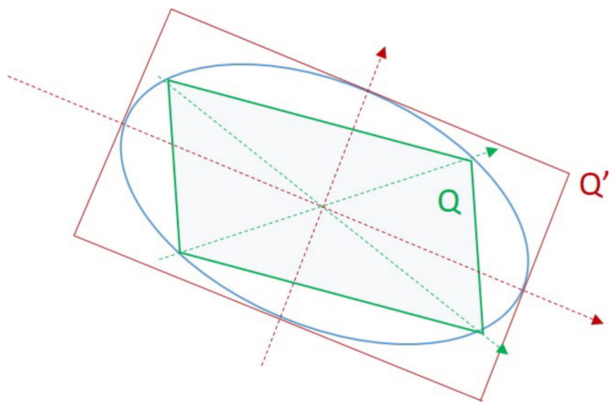


Figure 4: Inscribed parallelogram Q and circumscribing parallelogram Q'

4.2 Orthant-Monotonicity and Orthogonality

The search for DIC mechanisms in higher-dimensional spaces does not reduce to the search for Auerbach bases, and an additional property becomes crucial: in order to induce a DIC marginal median, for any j, any vector in X^{-j} must be orthogonal to $\mathbf{x}^j$. Because of the lack of symmetry and additivity, this property does not automatically follows from the properties of mutually orthogonal vectors unless the space is Hilbert, where orthogonality is always symmetric and additive on both sides.

Theorem 3 *Fix an algebraic basis* $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$. *The norm* $||\cdot||$ *is orthant-monotonic with respect to the associated coordinate system if and only if*

$$X^{-j} \dashv \mathbf{x}^j \quad \textit{for all } j = 1, ..., d. \tag{$*$}$$

Hence, a marginal median mechanism is DIC if and only if it is computed with respect to a basis $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ *that satisfies property* $(*)$ *above.*

Proof. The proof follows by the elegant geometric characterization of orthant-monotonic norms in Gries [1967] and Funderlic [1979]. Although those authors did not observe the relation to the BJ-orthogonality notion, their results are related to it.[33] We adapt here their method of proof, while translating it in terms of BJ-orthogonality.

Suppose first that the norm is orthant monotonic. If $\mathbf{z} \in X^{-j}$, then $\mathbf{z}$ and $\mathbf{z} + \beta \mathbf{x}^j$ lie in the same orthant for any β, since $z_k(z_k + \beta x_k^j) = z_k^2 \geq 0$ for all $k = 1, ..., d$. Moreover, $|z_k| \leq \left| z_k + \beta x_k^j \right|$ for all k. It follows from orthant monotonicity that

$$\|\mathbf{z}\| \leq \left\|\mathbf{z} + \beta \mathbf{x}^j\right\|, \quad \text{for all } \beta,$$

which means that $\mathbf{z} \dashv \mathbf{x}^j$.

Suppose now that $X^{-j} \dashv \mathbf{x}^j$ for all j. Let $\mathbf{y}$ and $\mathbf{z}$ be in the same orthant such that $|z_j| \leq |y_j|$ for all $j = 1, ..., d$. We need to show that

$$\|(z_1, z_2, ..., z_d)\| \leq \|(y_1, y_2, ..., y_d)\|.$$

Let $\mathbf{v}^0 = \mathbf{z}$, $\mathbf{v}^d = \mathbf{y}$ and

$$\mathbf{v}^k = (y_1, y_2, ..., y_k, z_{k+1}, ..., z_d)^T, \ k = 1, 2, ..., d-1.$$

For the norm to be orthant monotonic, it is thus sufficient to show that

$$\left\|\mathbf{v}^{k-1}\right\| \leq \left\|\mathbf{v}^k\right\|, \ k = 1, 2, ..., d.$$

By the construction of $\mathbf{v}^k$ and because $\mathbf{y}$ and $\mathbf{z}$ are in the same orthant with $|z_j| \leq |y_j|$ for all j, there exists $\lambda \in [0, 1]$ such that

$$\mathbf{v}^{k-1} = \mathbf{v}^k - \lambda y_k \mathbf{x}^k = \lambda\left(\mathbf{v}^k - y_k \mathbf{x}^k\right) + (1-\lambda)\,\mathbf{v}^k.$$

By assumption and by the construction of $\mathbf{v}^k$, $\mathbf{v}^k - y_k \mathbf{x}^k \in X^{-k}$, and is thus orthogonal to $\mathbf{x}^k$. Hence, we obtain

$$\left\|\mathbf{v}^k - y_k \mathbf{x}^k\right\| \leq \left\|\left(\mathbf{v}^k - y_k \mathbf{x}^k\right) + y_k \mathbf{x}^k\right\| = \left\|\mathbf{v}^k\right\|.$$

It follows that

$$\begin{aligned}
\left\|\mathbf{v}^{k-1}\right\| &= \left\|\lambda\left(\mathbf{v}^k - y_k \mathbf{x}^k\right) + (1-\lambda)\,\mathbf{v}^k\right\| \\
&\leq \lambda\left\|\mathbf{v}^k - y_k \mathbf{x}^k\right\| + (1-\lambda)\left\|\mathbf{v}^k\right\| \\
&\leq \left\|\mathbf{v}^k\right\|.
\end{aligned}$$

This completes the proof. ■

A necessary condition for property $(*)$ is that the vectors $\mathbf{x}^1, ..., \mathbf{x}^d$ in the basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ be mutually BJ-orthogonal[34]. When $d = 2$, property $(*)$ is equivalent to mutual orthogonality of $\mathbf{x}^1$ and $\mathbf{x}^2$. Therefore, a two-dimensional marginal median computed with respect

[33] Tanaka and Saito [2014] establish similar relations, but are not aware of the earlier papers by Gries and Funderlic.

[34] One cannot require property $(*)$ to generally hold for any basis without excluding all cases of interest here. Marino and Pietramala [1987] proved the following: Let V be smooth, reflexive and strictly convex with dimension $d > 2$, and assume that, for any triple $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, of mutually orthogonal vectors it holds that $(\mathbf{x} + \mathbf{y}) \dashv \mathbf{z}$. Then V is a Hilbert space.

to the basis $\{\mathbf{x}^1, \mathbf{x}^2\}$ is DIC if and only if $\mathbf{x}^1$ and $\mathbf{x}^2$ are BJ-mutually orthogonal – this is shown in Peters et al [1993].

Example 4 *Let $\{\mathbf{e}^1, \mathbf{e}^2\}$ be the standard basis vectors in $\mathbb{R}^2$. Then the $l_p(2)$ norm according to this basis is monotonic. Now consider another basis $\{\mathbf{f}^1, \mathbf{f}^2\}$ with $\mathbf{f}^1 = (1,1)$ and $\mathbf{f}^2 = (-1,1)$. We prove in Section 6 below that for any $l_p(2)$ norm, $\mathbf{e}^1$ and $\mathbf{e}^2$ are BJ-mutually orthogonal, and that $\mathbf{f}^1$ and $\mathbf{f}^2$ are also BJ-mutually orthogonal. For any $\mathbf{x}$ we can write*

$$\mathbf{x} = a\mathbf{f}^1 + b\mathbf{f}^2 = a(\mathbf{e}^1 + \mathbf{e}^2) + b(\mathbf{e}^2 - \mathbf{e}^1) = (a-b)\mathbf{e}^1 + (a+b)\mathbf{e}^2.$$

Hence, for the coordinates according to the basis $\{\mathbf{f}^1, \mathbf{f}^2\}$, the norm's formula is

$$||\mathbf{x}|| = ||a\mathbf{f}_1 + b\mathbf{f}_2|| = (|a-b|^p + |a+b|^p)^{1/p}.$$

For orthant monotonicity we need

$$||a\mathbf{f}_1 + b\mathbf{f}_2|| \geq \max\{||a\mathbf{f}_1 + 0\mathbf{f}_2||\,, \; ||0\mathbf{f}_1 + b\mathbf{f}_2||\}\,,$$

which is equivalent to

$$(|a-b|^p + |a+b|^p)^{1/p} \geq \max\left\{(2\,|a|^p)^{1/p}, \; (2\,|b|^p)^{1/p}\right\}.$$

Thus, it is enough to show that

$$|a-b|^p + |a+b|^p \geq \max\{2\,|a|^p\,, 2\,|b|^p\}\,,$$

which holds by convexity. Hence, a marginal median with respect to the basis $\{\mathbf{f}^1, \mathbf{f}^2\}$ is DIC.

*Consider next a third basis $\{\mathbf{g}^1, \mathbf{g}^2\}$ where $\mathbf{g}^1 = (1,0)$ and $\mathbf{g}^2 = (1,1)$. We show below that $\mathbf{g}^1$ and $\mathbf{g}^2$ are **not** BJ-orthogonal. For any $\mathbf{x}$ we can write*

$$\mathbf{x} = a\mathbf{g}^1 + b\mathbf{g}^2 = (a+b)\mathbf{e}^1 + b\mathbf{e}^2.$$

Hence

$$||\mathbf{x}|| = \left|\left|a\mathbf{g}^1 + b\mathbf{g}^2\right|\right| = (|a+b|^p + |b|^p)^{1/p}.$$

Orthant monotonicity requires that

$$\begin{aligned} \left|\left|a\mathbf{g}^1 + b\mathbf{g}^2\right|\right| &\geq \left|\left|0\mathbf{g}^1 + b\mathbf{g}^2\right|\right| \\ &\Leftrightarrow (|a-b|^p + |b|^p)^{1/p} \geq (2\,|b|^p)^{1/p} \\ &\Leftrightarrow |a-b|^p \geq |b|^p\,. \end{aligned}$$

*The last inequality is false in general. Therefore, a marginal median with respect to $\{\mathbf{g}^1, \mathbf{g}^2\}$ need **not** be DIC.*

5 How to Find All DIC Marginal Medians?

Theorem 3 reduces the quest for all DIC marginal medians to the quest for all bases consisting of BJ-mutually orthogonal vectors that satisfy property $(*)$. We first recapitulate how this is done for the Euclidean norm. We next explain why this geometric procedure does not work for general normed spaces that are not Hilbert. Finally, we introduce an analytic approach based on *semi-inner products*, and show how it can be used to find all bases with property $(*)$.

5.1 Isometries and Orthogonality

Finding orthogonal bases for the standard Euclidean norm is straightforward:

1. Identify one orthogonal basis according to the Euclidean inner product.
2. Identify orientation preserving linear isometries (i.e., rotations) that are known to preserve orthogonality. These isometries yield additional orthogonal bases.
3. Any oriented orthogonal basis can be obtained (modulo translation) from any other via a suitable rotation.

Unfortunately, the property needed at step 3 does **not** hold in general normed spaces, i.e., *Euclid's fourth postulate* about the equivalence of right angles need not hold here. In addition, it is not even clear whether the set of maps that preserve BJ-orthogonality (step 2) is related to the set of linear isometries[35]. Moreover, starkly contrasting inner product spaces, where the group of isometries is always a continuum, general normed spaced may admit only a finite number of these[36].

The above insights imply the following: assuming that we found one BJ-mutually orthogonal pair, we can then identify additional BJ-mutually orthogonal pairs via isometries, but possibly only a finite number of them at a time.[37] Moreover, from Theorem 2 we know that there may exist distinct pairs of mutually orthogonal vectors such that there is no isometry that transforms one pair into another.

5.2 Semi-Inner Products and Normality

Given the above difficulties with a purely geometric approach, we need an analytic way to capture BJ-orthogonality. For each two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ define two real-valued functions

[35] Koldobsky, [1993] proved the following : Let V be a real normed space and let $T : V \to V$ be a linear operator preserving BJ-orthogonality, i.e., $\mathbf{x} \dashv \mathbf{y} \Rightarrow T(\mathbf{x}) \dashv T(\mathbf{y})$. Then $T = \lambda U$ where $\lambda \in \mathbb{R}$ and where U is an isometry.

[36] For example, Garcia-Roig [1997]) considered two-dimensional real normed spaces and showed that the group of isometries is finite if and only if the norm $||\cdot||$ is **not** generated by an inner product.

[37] As we show below for the l_p case, isometries need not yield **any** additional DIC mechanisms because medians are equivariant with respect to all of them. This never happens in Hilbert spaces.

on the real line

$$\begin{aligned} f_{\mathbf{x}}^{\mathbf{y}}(\lambda) &= ||\mathbf{x}+\lambda\mathbf{y}||, \\ f_{\mathbf{y}}^{\mathbf{x}}(\lambda) &= ||\mathbf{y}+\lambda\mathbf{x}||. \end{aligned}$$

By the convexity of norms, these functions are convex. The sub-differential ∂f of a convex function f at λ is the (compact and convex) set of supporting hyperplanes at λ.[38] It contains a unique element, the derivative, whenever the function is differentiable.

By definition, $\mathbf{x} \dashv \mathbf{y}$ if $\|\mathbf{x}+\lambda\mathbf{y}\| \geq \|\mathbf{x}\| = \|\mathbf{x}+0\mathbf{y}\|$ for all real λ. In other words, if $\mathbf{x} \dashv \mathbf{y}$, then $\lambda = 0$ must be a minimum point of $f_{\mathbf{x}}^{\mathbf{y}}$. Analogously, $\mathbf{y} \dashv \mathbf{x}$ if $\|\mathbf{y}+\lambda\mathbf{x}\| \geq \|\mathbf{y}\|$ for all real λ which implies that $\lambda = 0$ must be a minimum point of the function $f_{\mathbf{y}}^{\mathbf{x}}$. A real λ is a minimum of a convex function f if and only if $\mathbf{0} \in \partial f(\lambda)$. These considerations yield:

Lemma 2 *Two vectors* $\mathbf{x}, \mathbf{y}$ *are BJ-mutually orthogonal if and only if*

$$\mathbf{0} \in \partial f_{\mathbf{x}}^{\mathbf{y}}(0) \cap \partial f_{\mathbf{y}}^{\mathbf{x}}(0).$$

We now couple the above observations with the following concept:

Definition 7 *A semi-inner product (SIP) is a real-valued function* $[\cdot,\cdot]$ *defined on* $V \times V$ *with the following properties:*

1. $[\mathbf{x}+\mathbf{z},\mathbf{y}] = [\mathbf{x},\mathbf{y}] + [\mathbf{z},\mathbf{y}],\ \forall \mathbf{x},\mathbf{y},\mathbf{z} \in V$;
2. $[\lambda\mathbf{x},\mathbf{y}] = \lambda[\mathbf{x},\mathbf{y}],\ \forall \mathbf{x},\mathbf{y} \in V,\ \forall\lambda \in \mathbb{R}$;
3. $[\mathbf{x},\lambda\mathbf{y}] = \lambda[\mathbf{x},\mathbf{y}],\ \forall \mathbf{x},\mathbf{y} \in V,\ \forall\lambda \in \mathbb{R}$;
4. $[\mathbf{x},\mathbf{x}] \geq 0,\ \forall \mathbf{x} \in V$, and $[\mathbf{x},\mathbf{x}] = 0 \Rightarrow \mathbf{x} = 0$;
5. $|[\mathbf{x},\mathbf{y}]|^2 \leq [\mathbf{x},\mathbf{x}][\mathbf{y},\mathbf{y}]$, $\forall \mathbf{x},\mathbf{y} \in V$.

An SIP is consistent with the norm of V *if* $[x,x] = ||x||^2$.

A Minkowski space may be simultaneously endowed with many different SIPs. The main differences to an inner-product is that the SIP need not be additive in the second variable, nor commutative.

Definition 8 *(Normality) Let* $[\cdot,\cdot]$ *be an SIP defined on* V. *Then* $\mathbf{x}$ *is normal to* $\mathbf{y}$ *if* $[\mathbf{y},\mathbf{x}] = 0$ *and* $\mathbf{y}$ *is normal to* $\mathbf{x}$ *if* $[\mathbf{x},\mathbf{y}] = 0$.[39]

[38] The simplest example is the absolute value function on the real line: the sub-differential at zero is the entire interval $[-1,1]$, while at all other points it coincides with the derivative, which is either 1 or -1.

[39] The order of the vectors in the above definition is important because the SIP is not commutative.

Theorem 4 *(Giles, [1967]) Assume that the norm $\|\cdot\|$ is smooth, i.e., for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, $\lim_{\lambda \to 0} (||\mathbf{x} + \lambda \mathbf{y}|| - ||\mathbf{x}||) / \lambda$ exists.*[40] *Then there exists a unique SIP, $[\cdot, \cdot]$, that is consistent with the norm $\|\cdot\|$. Moreover, for any $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$,*

$$\frac{df_{\mathbf{x}}^{\mathbf{y}}(\lambda)}{d\lambda}|_{\lambda=0} = \frac{[\mathbf{y}, \mathbf{x}]}{||\mathbf{x}||},$$

and therefore

$$\mathbf{x} \dashv \mathbf{y} \Leftrightarrow [\mathbf{y}, \mathbf{x}] = 0.$$

Remark 3 *If the space is smooth and if $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ are BJ-mutually orthogonal (or Giles-normal in the sense of the SIP), then we have $\mathbf{x}^j \dashv X^{-j}$ for all $j = 1, ..., d$. This property holds because of the left-additivity of the SIP (and hence of normality) in the first input, and because of the equivalence between normality and orthogonality in such spaces ($\mathbf{x} \dashv \mathbf{y} \Leftrightarrow [\mathbf{y}, \mathbf{x}] = 0$). Thus, under smoothness, the additional requirement of an Auerbach basis (see Definition 6) is trivially satisfied, but property $(*)$ may not be.*

We conclude this section with the explicit construction of a SIP that is consistent with a given norm. Fix a basis $\{\mathbf{x}^1, ..., \mathbf{x}^d\}$ in $\mathbb{R}^d$ and let

$$N(\mathbf{x}) = \|\mathbf{x}\|$$

denote the norm functional. Giles [1967] constructed a consistent SIP as follows:

$$[\mathbf{y}, \mathbf{x}] = \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{1}{2} \left(\frac{\partial^2}{\partial x_j \partial x_k} N^2(\mathbf{x}) \right) x_j y_k. \tag{7}$$

We now derive an alternative representation that is more amenable for our analysis below.

Since $N(\mathbf{x})$ is homogeneous of degree 1, we have

$$N(\mathbf{x}) = \sum_{j=1}^{d} N_{x_j}(\mathbf{x}) x_j \quad \text{and} \quad \sum_{j=1}^{d} N_{x_j x_k}(\mathbf{x}) x_j = 0 \text{ for all } k. \tag{8}$$

Note that

$$\frac{1}{2} \frac{\partial^2}{\partial x_j \partial x_k} N^2(\mathbf{x}) = N_{x_j}(\mathbf{x}) N_{x_k}(\mathbf{x}) + N_{x_j x_k}(\mathbf{x}) N(\mathbf{x}).$$

Together with (8), this yields

$$\sum_{j=1}^{d} \frac{1}{2} \left(\frac{\partial^2}{\partial x_j \partial x_k} N^2(\mathbf{x}) \right) x_j = N(\mathbf{x}) N_{x_k}(\mathbf{x}).$$

Therefore, we can re-formulate Giles's construction of the consistent SIP as

$$[\mathbf{y}, \mathbf{x}] = \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{1}{2} \left(\frac{\partial^2}{\partial x_j \partial x_k} N^2(\mathbf{x}) \right) x_j y_k = N(\mathbf{x}) \sum_{k=1}^{d} N_{x_k}(\mathbf{x}) y_k. \tag{9}$$

[40] This property is also called Gateaux differentiability. Norm smoothness requires that, for any vector $\mathbf{x}$ on the unit ball there is a unique tangent to the ball at $\mathbf{x}$.For example, the $l_1(2)$ norm is not smooth at the "corner" $\mathbf{x} = (0, 1)$ and at its signed permutations.

6 Illustrations

In this section we offer several illustrations and applications of the above concepts and insights.

6.1 The two-dimensional case

The two-dimensional case has been previously studied by Peters, van der Stel, and Storcken [1993]. As we observed above, a norm is then orthant monotonic with respect to a given basis if and only if this basis consists of a pair of BJ-mutually orthogonal vectors.

Definition 9 *A norm* $||\cdot||$ *is strictly convex if* $||\lambda\mathbf{x} + (1-\lambda)\mathbf{y}|| < 1$ *for all* $\mathbf{x}, \mathbf{y}$ *with* $||\mathbf{x}|| = ||\mathbf{y}|| = 1$ *and for all* $\lambda \in (0, 1)$.

Theorem 5 *(Peters, van der Stel, and Storcken [1993]) Assume that the norm on* $\mathbb{R}^2$ *is strictly convex, and let the number of agents be odd. A direct revelation mechanism is anonymous, Pareto-optimal and DIC if and only if it is a marginal median with respect to coordinates defined by a basis formed by two BJ mutually orthogonal vectors.*[41]

The following result combines the above insight with ours, and yields an operational method to compute **all** DIC, anonymous and Pareto optimal mechanisms on the plane:

Corollary 1 *Let* $\mathbb{R}^2$ *be endowed with a smooth and strictly convex norm* $N(\cdot)$. *A direct revelation mechanism is anonymous, Pareto-optimal and DIC if and only if it is a marginal median with respect to an Auerbach basis* $\{\mathbf{x}, \mathbf{y}\}$, $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$, *that satisfies the following system of four equations in four unknowns:*

$$\frac{\partial N(\mathbf{x})}{\partial x_1} y_1 + \frac{\partial N(\mathbf{x})}{\partial x_2} y_2 = 0; \tag{10}$$

$$\frac{\partial N(\mathbf{y})}{\partial y_1} x_1 + \frac{\partial N(\mathbf{y})}{\partial y_2} x_2 = 0; \tag{11}$$

$$N(\mathbf{x}) = 1; \tag{12}$$

$$N(\mathbf{y}) = 1. \tag{13}$$

This system has at least ***two*** *pairs of distinct solutions.*

Proof. Since medians are translation equivariant, it is enough by Theorems 3 and 4 to characterize BJ-mutually orthogonal pairs $\mathbf{x}$ and $\mathbf{y}$ with $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. These are equations (12) and (13). It follows from (9) that mutually normal vectors must satisfy the system:

$$[\mathbf{y}, \mathbf{x}] = N(\mathbf{x}) \left(\frac{\partial N(\mathbf{x})}{\partial x_1} y_1 + \frac{\partial N(\mathbf{x})}{\partial x_2} y_2 \right) = 0,$$

$$[\mathbf{x}, \mathbf{y}] = N(\mathbf{y}) \left(\frac{\partial N(\mathbf{y})}{\partial y_1} x_1 + \frac{\partial N(\mathbf{y})}{\partial y_2} x_2 \right) = 0,$$

[41] A direct revelation mechanism ψ is Pareto optimal if $\psi(\mathbf{t}_1, .., \mathbf{t}_n) \in conv(\mathbf{t}_1, .., \mathbf{t}_n)$ for all $\mathbf{t}_1, ..., \mathbf{t}_n \in \mathbb{R}^d$, where $conv(\mathbf{t}_1, ..., \mathbf{t}_n)$ is the convex hull of $(\mathbf{t}_1, .., \mathbf{t}_n)$.

which reduce to equations (10) and (11) since $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$. To conclude, finding all solution to equations (10)-(13) yields all pairs of directions (or the issues) for which marginal medians are DIC. There are at least two distinct solutions by Theorem 2. ■

6.2 The $l_p(d)$ spaces

We now consider the widely used class of $l_p(d)$ norms, and we characterize **all** DIC marginal medians.[42] The respective norm formulae with respect to the standard Cartesian coordinates are

$$N(\mathbf{x}) = \|\mathbf{x}\|_p = \left(\sum_{j=1}^{d} |x_j|^p\right)^{1/p}.$$

6.2.1 The Isometries of l_p spaces

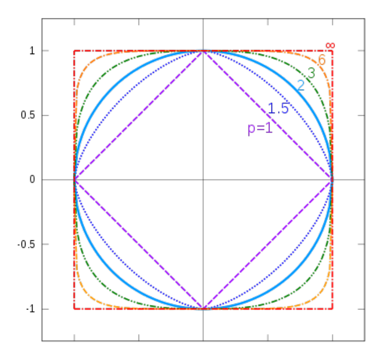
The Hilbert space $l_2(d)$ admits a continuum of different orthogonal bases and, modulo translations, each one can be obtained from another by applying a suitable isometry that preserves orientation (i.e., rotations). For any other $l_p(d)$ space, $p \neq 2$, the set of linear isometries is finite:

Theorem 6 *(Li and Son [1994]) For any $l_p(d)$, $p \geq 1$, $p \neq 2$, the set of linear isometries is independent of p, and is represented by the set of signed permutation matrices, i.e. permutation matrices where some of the 1 entries are replaced by -1 entries.*

For example, for any $l_p(2)$, $p \geq 1$, $p \neq 2$, the set of linear isometries that preserve orientation (and have therefore determinant $+1$) is represented by the four matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This set of matrices represent the easily visualized set of rotations with angles $\{0,\ \pi/2,\ \pi,\ 3\pi/2\}$ that leave invariant the l_1 unit ball in the plane and, as the Theorem shows, also all other l_p unit balls.



[42] See Eguia[2011] for an axiomatic characterization of preferences derived form this class of norms.

Figure 5: The l_p unit balls

Medians are equivariant with respect to signed permutations with determinant +1, and hence, after finding a BJ-mutually orthogonal pair, applying isometries does **not** reveal here new DIC mechanisms (in stark contrast to the Euclidean case). Moreover, we also know from the Theorem 2 that there is no isometry transforming the BJ-mutually orthogonal pair identified via maximal inscribed quadrilateral into the other, identified by minimum circumscribing quadrilateral. Therefore, we apply below the SIP method.

6.2.2 The Semi-Inner Product

The construction of a norm-consistent SIP for the $l_p(d)$ norm is as follows. Suppose $x_j \neq 0$ for all j, and thus $d|x_j|/dx_j = x_j/|x_j|$. If $x_j = 0$, its impact on $[\mathbf{y}, \mathbf{x}]_p$ vanishes in any case. Note that

$$\frac{\partial N(\mathbf{x})}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sum_{j=1}^{d} |x_j|^p\right)^{1/p} = \frac{1}{p}\left(\sum_{j=1}^{d} |x_j|^p\right)^{1/p-1} p\,|x_j|^{p-1}\,\frac{x_j}{|x_j|} = \frac{|x_j|^{p-2}\,x_j}{||\mathbf{x}||_p^{p-1}}$$

It follows from (9) that the SIP is given by

$$[\mathbf{y}, \mathbf{x}]_p = N(\mathbf{x}) \sum_{j=1}^{d} N_{x_j}(\mathbf{x})\, y_j = \sum_{j=1}^{d} \frac{|x_j|^{p-2}\, x_j y_j}{||\mathbf{x}||_p^{p-2}}. \tag{14}$$

For $p \neq 1$ the $l_p(d)$ norm is smooth, and hence this is the unique norm-consistent SIP in those cases.[43]

6.2.3 The set of DIC marginal medians

We next characterize all Auerbach bases with property $(*)$,[44] and hence all DIC marginal medians. We assume below that $p > 1$ because the "taxicab" (or "Manhattan") norm where $p = 1$ is not smooth (and hence the norm-consistent SIP is not unique). By using properties that must be shared by all consistent SIP's, and the fact that the isometries of the spaces l_1 and l_p, $p \neq 2$, coincide (see Theorem 6), this case can be treated separately to yield the same general result. The case $p = 2$, the standard Euclidean norm, is the unique one that is generated by an inner-product space. It is well known that it admits a continuum of orthogonal bases.

Theorem 7 *Fix $d \geq 2$ and assume that $p > 1$ and $p \neq 2$. For any $l_p(d)$ space, a marginal median is DIC if and only if it is computed with respect to coordinates defined by a following modification of the standard Cartesian basis $E = \{\mathbf{e}^1, ..., \mathbf{e}^d\}$:*

[43] Our formula coincides of course with the standard inner product formula for $p = 2$ where $[\mathbf{y}, \mathbf{x}]_2 = \sum_{j=1}^{d} x_j y_j$.

[44] By the smoothness of the norm, the right-additivity property is automatically satisfied here for $p > 1$.

1. *Choose a subset of vectors $E' \subseteq E$ with an even (possibly zero) number of elements, and partition it into distinct pairs $\{\mathbf{e}^j, \mathbf{e}^k\}$.*

2. *Replace each pair $\{\mathbf{e}^j, \mathbf{e}^k\}$ in E' by the pair $\{\mathbf{e}^j + \mathbf{e}^k, \mathbf{e}^j - \mathbf{e}^k\}$.*

3. *Construct a new basis from these new pairs, together with the remaining unit vectors in E/E'.*

Proof. Note first that the construction always yields d linearly independent vectors. Using the SIP, Kinnunen [1984] proved the following: for any vector $\mathbf{x} \in l_p$, there exists a hyperplane H such that $H \dashv \mathbf{x}$ if and only if $\mathbf{x}$ belongs to the union of the one-dimensional subspaces spanned $\mathbf{e}^j, \mathbf{e}^j + \mathbf{e}^k, \mathbf{e}^j - \mathbf{e}^k$, $j, k = 1, ..., d$, $j \neq k$. Kinnunen's result implies that each vector $\mathbf{x}^j$ in a basis with property $(*)$ must be one of the vectors in the statement, because, by property $(*)$, the hyperplane spanned by the other vectors in the basis is necessarily orthogonal to it.[45] Kinnunen's result does not, however, determine the full composition of (Auerbach) bases with property $(*)$, and we do this below.

Let $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ denote a basis as constructed in the statement. We first show that this basis satisfies property $(*)$. There are three cases to consider for vector $\mathbf{x} \in \{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$:

1. $\mathbf{x} \in E \backslash E'$. Then $\mathbf{x} = \mathbf{e}^j$ for some j. Let $\mathbf{y}$ be a vector in the span of the other $d-1$ vectors in the basis. Then we can write $\mathbf{y} = \sum_{k \neq j} \alpha^k \mathbf{e}^k$ for certain coefficients $\{\alpha^k\}$. We obtain from (14) that

$$[\mathbf{y}, \mathbf{x}]_p = \sum_{k=1}^{d} \frac{|x_k|^{p-2} x_k y_k}{||\mathbf{x}||_p^{p-2}} = 0,$$

because $x_k = e_k^j = 1$ for $k = j$, and $x_k = e_k^j = 0$ for $k \neq j$, and because $y_j = 0$.

2. $\mathbf{x} = \mathbf{e}^j + \mathbf{e}^k$. Let $\mathbf{y}$ be a vector in the span of the other $d-1$ vectors in the basis. Then $\mathbf{y} = \sum_{\ell \neq j,k} \alpha^\ell \mathbf{e}^\ell + \beta(\mathbf{e}^j - \mathbf{e}^k)$ for certain coefficients $\{\alpha^\ell\}$ and β. We obtain:

$$\begin{aligned}
[\mathbf{y}, \mathbf{x}]_p &= \left[\sum_{\ell \neq j,k} \alpha^\ell \mathbf{e}^\ell + \beta(\mathbf{e}^j - \mathbf{e}^k), \mathbf{e}^j + \mathbf{e}^k \right]_p \\
&= \sum_{\ell \neq j,k} \alpha^\ell \left[\mathbf{e}^\ell, \mathbf{e}^j + \mathbf{e}^k \right]_p + \left[\mathbf{e}^j - \mathbf{e}^k, \mathbf{e}^j + \mathbf{e}^k \right]_p \\
&= \sum_{\ell \neq j,k} \alpha^\ell \left[\mathbf{e}^\ell, \mathbf{e}^j + \mathbf{e}^k \right]_p + \left[\mathbf{e}^j, \mathbf{e}^j + \mathbf{e}^k \right]_p - \left[\mathbf{e}^k, \mathbf{e}^j + \mathbf{e}^k \right]_p \\
&= 0
\end{aligned}$$

where the second and third equalities follow from the left-additivity of SIP in its first input, and the last equality follows because

$$\left[\mathbf{e}^\ell, \mathbf{e}^j + \mathbf{e}^k \right]_p = 0 \text{ for } \ell \neq j, k, \text{ and } \left[\mathbf{e}^j, \mathbf{e}^j + \mathbf{e}^k \right]_p = \left[\mathbf{e}^k, \mathbf{e}^j + \mathbf{e}^k \right]_p.$$

[45] See also Baronti and Papini [1988] and Lavric [1997] for related results.

3. $\mathbf{x} = \mathbf{e}^j - \mathbf{e}^k$. This case is analogous to case 2 above.

For the converse, we show that an (Auerbach) basis with property $(*)$ must have the form given in the statement of the Theorem. As implied by the result of Kinnunen [1984], each vector $\mathbf{x}^j$ in an Auerbach basis with property $(*)$ must be in the forms of $\mathbf{e}^j$, $\mathbf{e}^k - \mathbf{e}^\ell$ or $\mathbf{e}^m + \mathbf{e}^\ell$. Note that, by the left-additivity property of the SIP,

$$\left[\mathbf{e}^j - \mathbf{e}^k, \mathbf{e}^j\right]_p = \frac{1}{||\mathbf{e}^j - \mathbf{e}^k||_p^{p-2}} \neq 0 \text{ and } \left[\mathbf{e}^j + \mathbf{e}^k, \mathbf{e}^j\right]_p = \frac{1}{||\mathbf{e}^j + \mathbf{e}^k||_p^{p-2}} \neq 0.$$

Hence, if $\mathbf{e}^j$ belongs to an Auerbach basis, vectors of the form, $\mathbf{e}^k - \mathbf{e}^\ell$, $\mathbf{e}^k + \mathbf{e}^\ell$, $\mathbf{e}^\ell - \mathbf{e}^k$, can belong to the basis only if $k, \ell \neq j$.

If all vectors in the basis have their k-th coordinates equal to zero, then they cannot span the entire d dimensional vector space. Assume then that some vector of the form $\mathbf{e}^k + \mathbf{e}^j$ belongs to the basis. We need to show that $\mathbf{e}^k - \mathbf{e}^j$ also belongs to it (the proof of the opposite case is analogous). Assume by contradiction that $\mathbf{e}^k - \mathbf{e}^j$ does not belong to the basis. Since $\mathbf{e}^k$ cannot belong to it by the preceding argument, the only other possible vectors in the basis that have a non-zero k-th coordinate are of the form $\mathbf{e}^k + \mathbf{e}^\ell$ or $\mathbf{e}^k - \mathbf{e}^\ell$ for $\ell \neq j$. But these vectors are not orthogonal to $\mathbf{e}^k + \mathbf{e}^j$ because, for all $\ell \neq j$,

$$\begin{aligned}
\left[\mathbf{e}^k + \mathbf{e}^j, \mathbf{e}^k + \mathbf{e}^\ell\right]_p &= \left[\mathbf{e}^k, \mathbf{e}^k + \mathbf{e}^\ell\right]_p + \left[\mathbf{e}^j, \mathbf{e}^k + \mathbf{e}^\ell\right]_p = \frac{1}{||\mathbf{e}^k + \mathbf{e}^\ell||_p^{p-2}} \neq 0, \\
\left[\mathbf{e}^k + \mathbf{e}^j, \mathbf{e}^k - \mathbf{e}^\ell\right]_p &= \left[\mathbf{e}^k, \mathbf{e}^k - \mathbf{e}^\ell\right]_p + \left[\mathbf{e}^j, \mathbf{e}^k - \mathbf{e}^\ell\right]_p = \frac{1}{||\mathbf{e}^k - \mathbf{e}^\ell||_p^{p-2}} \neq 0.
\end{aligned}$$

Therefore, $\mathbf{e}^k + \mathbf{e}^j$ and $\mathbf{e}^k + \mathbf{e}^\ell$ or $\mathbf{e}^k - \mathbf{e}^\ell$ for $\ell \neq j$ cannot simultaneously be part of an Auerbach basis. This completes the argument. ■

It is clear from Theorem 7 that the set of Auerbach bases with property $(*)$, and thus the set of DIC marginal median mechanisms, is **independent of** p. Theorem 7 tells us exactly how to find these DIC mechanisms for any l_p space. For example, for $d = 5$, the matrix of coordinates of the (column) vectors belonging to an Auerbach basis with property $(*)$ has either one of the forms below, or their signed permutations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The case of $d = 5$

As an application, let us use the above result to identify **all** DIC, anonymous and Pareto-optimal mechanisms for the $l_p(2)$ spaces. The above Theorem shows that, modulo signed permutations, for all these spaces (with the exception of $p = 2$) there are exactly

two pairs of directions on which marginal medians can be taken while preserving DIC: $\{\mathbf{e}^1, \mathbf{e}^2\}$ and $\{\mathbf{e}^1 + \mathbf{e}^2,\, \mathbf{e}^1 - \mathbf{e}^2\}$.[46] In particular, no matter what $l_p(2)$ norm an agent uses, reporting truthfully is DIC if a marginal median mechanism with respect to either one of these two bases is used. It is also clear that the two bases yield two distinct marginal median mechanisms: there is no isometry (i.e., there is no signed permutation) that transforms $\{\mathbf{e}^1, \mathbf{e}^2\}$ into $\{\mathbf{e}^1 + \mathbf{e}^2,\, \mathbf{e}^1 - \mathbf{e}^2\}$. The converse follows by Corollary 1.

A comparison of the two mechanisms from an utilitarian perspective (for the case $p = 2$) is conducted in Gershhkov, Moldovanu and Shi [2019]. There, we also establish the relations between theses two mechanisms and the *bottom-up* vs. *top-down* budgeting procedures used by legislatures (see for example Ferejohn and Krehbiel [1987], Groves [1994], and Poterba and von Hagen [1999])[47].

6.3 Inner-Product Spaces

We briefly illustrate here how our general approach also illuminates the often studied class of quadratic preferences: all such preferences correspond to norm-based preferences, where the norm is generated by an inner-product.

Assume first that agent i has an utility function derived from a weighted Euclidean norm with weights $\boldsymbol{\beta}_i \equiv (\beta_{i1}, \beta_{i2}, ..., \beta_{id}) \neq 0$ where $\beta_{ij} \geq 0$ for all i, j. That is, agent i with ideal point $\mathbf{t}_i = (t_{i1}, t_{i2}, ..., t_{id})$ has a utility from decision $\mathbf{v} = (v_1, ..., v_d)$ given by

$$-\sum_{j=1}^{d} \beta_{ij} \left(v_j - t_{ij}\right)^2,$$

Both $\mathbf{t}_i$ and $\boldsymbol{\beta}_i$ are agent i's private information. Let

$$A_i = \begin{pmatrix} \beta_{i1} & 0 & \cdots & 0 \\ 0 & \beta_{i2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_{id} \end{pmatrix},$$

and define an inner-product and its associated norm by:

$$\begin{aligned} \langle (x_1, ..., x_d), (y_1, ..., y_d) \rangle &\equiv (x_1, ..., x_d) A_i (y_1, ..., y_d)^T, \\ ||(x_1, ..., x_d)|| &\equiv \sqrt{(x_1, ..., x_d) A_i (x_1, ..., x_d)^T} = \sqrt{\sum_{j=1}^{d} \beta_{ij} x_j^2}. \end{aligned}$$

Each such inner-product is symmetric, and its orthogonality relation is also symmetric. The unit ball is an ellipse with axes parallel to the standard Cartesian coordinates, described

[46] These are, precisely, the conjugate diameters identified by Auerbach's theorem (Theorem 2) for these norms.

[47] It is interesting to note that Groves also considered non-Euclidean preferences, but focused then on non-strategic voting.

by

$$\sum_{j=1}^{d} \beta_{ij} x_j^2 = 1,$$

We obtain the following "robust" mechanism design result:

Theorem 8 *Assume that there are at least d agents and that the set of possible weights determining their weighted Euclidean preferences contains d linearly independent vectors $\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_d$. Then, up to translations, the unique DIC marginal median is the one computed with respect to the standard Cartesian coordinates.*

Proof. Consider a realization where there are d agents such that each agent i has a utility function derived from a weighted Euclidean norms with weight vector $\boldsymbol{\beta}_i$. A DIC marginal median for such agents must be computed with respect to coordinates that are orthogonal under all norms generated by these various weights. Let us then look for a basis of the underlying vector space consisting of d vectors that are orthogonal from the joint point of view of all the d agents $i = 1, ..., d$. Property $(*)$ is satisfied here automatically since all norms are generated by inner-products.

Consider then d vectors, $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d$, with $\mathbf{x}^k = (x_1^k, x_2^k, ..., x_d^k)$, $k = 1, ..., d$, all different from zero. Then any two vectors $\mathbf{x}^k, \mathbf{x}^\ell$ such that $\mathbf{x}^k \dashv \mathbf{x}^\ell$ must satisfy the following system of equations[48]:

$$\sum_{j=1}^{d} \beta_{ij} x_j^k x_j^\ell = 0, \text{ for all } i = 1, ..., d$$

Since $\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_d$ are linearly independent, we must have, for all $j, k, \ell \in \{1, ..., d\}$, $k \neq \ell$,

$$x_j^k x_j^\ell = 0.$$

This implies that, for each coordinate j, there is at most one k such that $x_j^k \neq 0$. As a result, there are at most d non-zero numbers in the set $\left\{x_j^k\right\}_{j,k=1,...,d}$. But since $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d$ are all non-zero vectors, there is exactly one non-zero entry for each vector $\mathbf{x}^k$. In other words, the set $\{\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^d\}$ consists of vectors in the direction of the standard Cartesian coordinates, and their signed permutations. The result follows because marginal medians are equivariant with respect to signed permutations of the coordinates. ■

Border and Jordan [1983] studied DIC mechanisms on the entire class of *quadratic separable* preferences - these coincide with our weighted Euclidean norms described above. Combined with Moulin's characterization for one-dimensional domain, their Theorem 1 shows that any unanimous, anonymous and DIC mechanism must be a generalized median. As we showed above, these (generalized) medians must be computed with respect to the standard Cartesian coordinates.

[48] This is, of course, a special case of the SIP approach described above.

Let us now consider what happens when we allow agents to have arbitrary (possibly different) utilities derived from inner-product norms. We recover then the impossibility result Theorem 3 of Border and Jordan [1983].

Theorem 9 *Assume that there are at least $d+1$ agents and that agents have individual preferences derived from arbitrary inner-product norms. Then there is no unanimous, anonymous and DIC mechanisms.*

Proof. Suppose there are d agents whose preference relations are generated from different weighted Euclidean norms. It follows from our result above, and from Theorem 1 in Border and Jordan [1983] that any unanimous, anonymous mechanism that is DIC for these d agents must be a generalized marginal median mechanism with respect to the standard Cartesian basis. Consider then a $d+1$ agent whose preference relation is generated from a norm such that its associated unit ellipse is tilted with respect to the standard Cartesian coordinates. Then the Cartesian coordinates are **not** orthogonal according to this norm. To see this, assume that this third norm is generated by the following inner product

$$\langle (x_1,...,x_d),(y_1,...,y_d)\rangle \equiv (x_1,...,x_d)A_i(y_1,...,y_d)^T$$

where A_i is given by

$$A_i = \begin{pmatrix} 1 & b & \cdots & 0 \\ b & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Since the unit ellipse defined by the norm is assumed to be tilted with respect to the Cartesian coordinates, we must have $b \neq 0$. Consider then the vectors

$$\mathbf{e}^1 = (1,0,...,0) \text{ and } \mathbf{e}^2 = (0,1,...,0).$$

We obtain

$$[\mathbf{e}^1,\mathbf{e}^2] = [\mathbf{e}^2,\mathbf{e}^1] = (1,0,...,0)A_i(0,1,...,0)^T = b \neq 0$$

and hence $\mathbf{e}^1$ and $\mathbf{e}^2$ are not orthogonal here. A marginal median with respect to the standard Cartesian coordinates is thus not DIC for such an agent, and the impossibility result follows. ■

7 Concluding Remarks

We have studied issue-by-issue voting by majority in a multidimensional collective decision situation, and we have identified all special systems of coordinates (the "issues") that render marginal median mechanisms incentive compatible. Our analysis has combined a variety of methods and concepts from geometry/functional analysis, a large part of which are novel to the Economics literature. For fixed, small classes of norm-induced preferences we were able to construct incentive compatible mechanisms that cannot be otherwise identified when the class of feasible preferences is rich. Finally, by going well beyond the Euclidean distance function, our analysis opens a broader scope for applications of spatial voting analysis.

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