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Identification of Auction Models Using Order Statistics

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Abstract

Auction data often fail to record all bids or all relevant factors that shift bidder values. In this paper, we study the identification of auction models with unobserved heterogeneity (UH) using multiple order statistics of bids. Classical measurement error approaches require multiple independent measurements. Order statistics, by definition, are dependent, rendering classical approaches inapplicable. First, we show that models with nonseparable finite UH is identifiable using three *consecutive* order statistics or two *consecutive* ones with an instrument. Second, two arbitrary order statistics identify the models if UH provides support variations. Third, models with separable continuous UH are identifiable using two *consecutive* order statistics under a weak restrictive stochastic dominance condition. Lastly, we apply our methods to U.S. Forest Service timber auctions and find evidence of UH.

JEL: C14, D44

Key Words: Unobserved Heterogeneity, Measurement Error, Finite Mixture, Multiplicative Separability, Support Variations, Deconvolution

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1 Introduction

Recent years have seen a growth of the literature combing auction theory with econometric analysis to understand auctions markets and inform policy. While the key elements in auction theory normally match well the practice, it is less so when there is auction-level heterogeneity. Not only auctions differ in observable characteristics, there are factors affecting bidder values that are common knowledge among bidders but unobserved by the analyst. Ignoring such factors, known as unobserved heterogeneity(UH), leads to erroneous estimates and wrong policy conclusions. See, e.g., the recent survey by Haile and Kitamura (2018).

The earliest approach to allow for UH in auctions exploits an auxiliary variable that is monotone in UH. See, e.g., Campo, Perrigne, and Vuong (2003) and Haile, Hong, and Shum (2003). Two streams of measurement error approaches exploit multiplicity of bids in each auction to deal with discrete and continuous UH, respectively. Both treams require that the observed bids are independent conditioning on UH. Such independence, however, may not hold due to auction format or data truncation. Often, we may observe multiple order statistics of all the bids rather than all or multiple independent bids. Naturally, these order statistics are dependent, rendering the classical measurement error approaches inapplicable. For instance, Kim and Lee (2014) observe the second to the fourth highest bids in ascending used-car auctions. U.S. Forest Service timber auctions only record at most the top 12 bids regardless the number of bidders. Freyberger and Larsen (2017) observe the second and third order statistics of bids in eBay auctions for used iPhones.

This paper provides a set of new results on identification of auction models with UH using multiple order statistics of bids. We propose new identification strategies that rely on commonly available *consecutive* order statistics or support variations. In particular, we derive our identification results with the same number of bids as the common measurement error approaches.

Our first result shows that three *consecutive* order statistics of bids identify independent private value (IPV) auction models with nonseparable finite UH. In usual misclassification models, independence among multiple measurements implies that the joint distribution has a multiplicatively separable structure. This structure leads to an eigendecomposition argument that identifies the type-specific distributions. Despite their dependence, we show that three *consecutive* order statistics have a multiplicative separable joint distribution in a portion of the support. This structure leads to a multi-step eigendecomposition argument that identifies the type-specific distributions. Similar identification argument can be extended to the scenario of only two *consecutive* order statistics with a binary instrumental variable.

Our second result shows that the auction models with finite UH can be identified using any two order statistics of bids with strict support variations. This result extends Hu and Sasaki (2017), which identify the model with nonseparable finite UH exploiting monotonicity of support boundaries in UH with two independent measurements. It is also related to D'Haultfœuille and Février (2015) which identify nonseparable measurement error models using support variations and three measurements. While they exploit the joint distribution of the measurements being multiplicatively separable, we can not do the same with order statistics.

Support variations lead to density discontinuities at the upper boundaries of component distributions. These discontinuity points divide bid support into intervals, and order statistics from lower type distributions have positive densities only in lower intervals. Identification then follows from an induction argument. First, the highest type solely determines the conditional distribution of the lower order statistic conditioning on the larger one being in the most upper interval. As a result, we can back out the distribution of the highest type. Second, we can subtract the contribution of the highest type from the same conditional distribution in the second most upper interval to obtain the contribution of the second highest type and then identify its distribution. Iterating this argument identifies all component distributions. Lastly, we extend these results for first price auctions to allow for multiple sources of UH: unobserved competition and unobserved auction type. Identification requires an ordering condition which can be satisfied with first oder stochastic dominance ordering type-specific value distribution.

Our last result concerns identification of auctions with continuous UH. Following the existing literature, we assume UH is additively separable in bidder value, which constitutes an error-in-variable model. While identification with multiple independent measurements is well-known for this model (Li and Vuong (1998), Li, Perrigne, and Vuong (2000), and Krasnokutskaya (2011)), the same problem with multiple order statistics is unsolved (Athey and Haile (2002)). Specifically, the dependence among order statistics precludes applications of the classical approach, which relies on both the "within" independence between value and UH, and the "between" independence of values. However, the latter no longer applies to order statistics.

Instead, we propose a new identification strategy that relies only on the "within" independence. Moreover, it exploits properties of consecutive order statistics. In particular, we show that any two consecutive order statistics are sufficient to identify the model in two steps. First, we show that the top two order statistics are sufficient for identification. The "within" independence identifies the ratio of the characteristic functions of top two order statistics. This ratio determines the parent distribution under a weak assumption. This result extends to the case with a maximum order statistic and any other order statistic. Second, any two consecutive order statistics identify the distributions of a maximum order statistics and a minimum order statistic, which again achieves identification following similar arguments in the first step.

Literature Review

For discrete UH with nonseparable structure by nature, the existing literature uses the eigen-decomposition approach, as in Hu, McAdams, and Shum (2013). They achieve identification using bid information of at least three bidders per auction following the results of Hu (2008). Recently, Hu and Sasaki (2017) show that two independent proxies of the latent factor are sufficient to identify nonseparable measurement error model if the support boundaries are increasing with respect to UH, which can also be applied to auction models. For continuous UH, the existing literature uses the deconvolution approach, as in Li and Vuong (1998), Li, Perrigne, and Vuong (2000), and Krasnokutskaya (2011), among others. They require two random bids in each auction and restrict that the UH has a separable structure on bidders' values.

There have been increasing interest in using order statistics for identification in the auction literature. Many markets can be treated as auctions where only the transaction price is observed. Athey and Haile (2002) show that the auction model with symmetric independent private values (IPV) is identified with the transaction price and the number of bidders. This exploits a one-to-one mapping between a parent distribution and the distribution of an order statistic. Guerre and Luo (2018) show that the auction model with symmetric IPV and unknown competition is identified using the transaction price only. They exploit discontinuities in the density function of the winning bid due to changing competition.

Some data contain multiple order statistics of bids. Song (2004) considers eBay auctions where the number of bidders is random and unobservable. He shows that any two order statistics identifies the symmetric IPV model. Freyberger and Larsen (2017) study eBay auctions for used iPhones with auction-specific unobserved heterogeneity and an unknown number of bidders. They circumvents the issue of correlated order statistics using observed reserve prices, rendering the identification problem classical. Kim and Lee (2014) apply the identification results of Song (2004) to wholesale used-car auctions in Korea. Due to the ascending auction format, they observe the second, third and fourth highest bids but not the first highest bids.

Related to our first result, Mbakop (2017) takes a different approach that tries to restore the conditional independence structure needed in the usual approaches. In particular, he exploits the Markov property of order statistics and shows that the bidder's UH-specific distribution and the distribution of UH is point identified from either (any) five or three order statistics along with an instrument using the joint distribution of three order statistics conditional on the two middle order statistics. In contrast, we take advantage of the separability structure provided by consecutive order statistics and achieve identification using three consecutive order statistics. Moreover, our results also extends to (any) four order statistics.

This paper is organized as follows. In Section 2 we describe the auction models. Section 3 and 4 presents identification results of nonseparable finite UH and separable continuous UH, respectively. Section 5 presents an application of the identification result into USFS timber auctions. Section 6 concludes.

2 IPV Auction Models with UH

In this section, we introduce standard IPV auction with UH. Suppose $n \geq 2$ symmetric bidders participate in an auction with zero reserve price. Bidders are risk neutral. Conditioning on an auction-level UH k, where k can be discrete finite or continuous, bidders' valuations are *i.i.d.* draws from the same distribution $\Phi^k(v)$ with support $[\underline{v}, \overline{v}_k]$. All bidders simultaneously submit bids. We denote the optimal bid distribution for the auction state k as $F^k(x)$, and the optimal bid x is with support $[\underline{x}, \overline{x}_k]$.

2.1 The first-price auction model

In a first-price auction, the bidder with the highest bid wins and pays the price he/she submitted. In a state-k auction, a bidder with a valuation v chooses his bid b to maximize its expected payoff, represented in the following.

$$\max_{b} \quad (v-b)\cdot \Phi^k\big(s_k^{-1}(b)\big)^{n-1},$$

where $s_k^{-1}(\cdot)$ is the inverse of his/her optimal bidding strategy in state-k auctions, and $\Phi^k(s_k^{-1}(b))^{I-1}$ is the chance of winning.

Guerre, Perrigne, and Vuong (2000) study the identification of the type-specific value distribution $\Phi^k(\cdot)$ when the competition level n and the type-specific bid distribution $F^k(\cdot)$ is observed. In particular, the identification comes from the type-specific one-to-one mapping between the unknown value and the observed type-specific bid distribution function:

$$s_k^{-1}(b) = b + \frac{1}{n-1}\frac{G_k(b)}{g_k(b)}, \tag{1}$$

where b is any arbitrary bid in its support, and $G_k(\cdot)$ and $g_k(\cdot)$ are the type-specific bid distribution and density functions, respectively.

2.2 The second-price auction model

In a second-price auction, the bidder with the highest bid wins and pays the second highest price submitted. In the private value framework, second-price auctions are equivalent to ascending auctions. Optimal bidding behaviors in these auctions are straightforward: a weakly dominant strategy is to stay in the bidding until the standing bid reaches your value. In other words, once the next-to-last bidder drops out, the bidder with the highest value wins at a price equal to the second-highest value. Therefore, the highest bid is never observed. Moreover, the observed bids are the rth order statistics from an auction of number of bidders n from the value distributions $\Phi^k(\cdot)$, where $r \leq n-1$.

If the number of bidders n and the distribution for the winning bid, which is the distribution of the $n-1$th order statistics out of a n ordered sample denoted as $F^k_{n-1:n}(\cdot)$, is known, the unknown type-specific bid function $F^k(\cdot)$ can be identified because there is a one-to-one mapping between the distribution of the $(n-1)$th order statistics and the underlying distribution itself. That is,

$$F^k_{n-1:n}(x) = n(n-1)\int_0^{F^k(x)} t^{n-2}(1-t)dt.^1$$

Once we recover the type-specific bid distribution from the distribution of the type-specific order statistics, we identify the type-specific value distribution as

$$\Phi^k(x) = F^k(x). \tag{2}$$

In summary, we can identify the type-specific value distributions $\Phi^k(\cdot)$ from the type-specific bid distributions $F^k(\cdot)$ for the first price auction using equation (1) and for the second price auction using equation (2). Consequently, we focus on identification of the type-specific bid distribution $F^k(\cdot)$ from the joint distribution of order statistics of the bids. We then can follow the above argument to recover the type-specific value distribution.

[1] In general, the distribution function of the rth order statistic $X_{r:n}$ is

$$F_{r:n}(x) = \frac{n!}{(n-r)!(r-1)!}\int_0^{F(x)} t^{r-1}(1-t)^{n-r}dt,$$

which is strictly increasing in $F(x)$ and thus invertable. Therefore, the distribution of any order statistic identifies the parent distribution $F(x)$. This property has been used in several papers including Athey and Haile (2002) and Song (2004).

3 Identification with Nonseparable UH

In this section, we consider the generic identification of nonseparable UH in auction models. First, we derive some properties of order statistics which is essential for our identification. Then we provide sufficient conditions to identify the type-specific bid distributions using three consecutive order statistics. We then extend the identification result to the case with two consecutive order statistics and an instrument variable. Second, we provide identification using only two consecutive order statistics with a first order stochastic dominance condition (FOSD). This result can be extended to the scenarios of multi-dimensional UH such as allowing for the number of bidders to be unknown for first price auction.

We first introduce some notation. There are K unobserved value types, assumed to be finite and discrete, which maps to K unobserved bid distributions. We assume that the cardinality of the UH K is known.[2] The probability of type k is p_k, so $\sum_k p_k = 1$. Let $X_{1:n} \leq X_{2:n} \leq, ..., \leq X_{n:n}$ represent the n order statistics out of an n ordered sample, and x_1 ,..., x_n represent their realizations, respectively. We use $f_{r,s,j:n}(\cdot)$ to denote the joint pdf of the three order statistics $X_{r:n}, X_{s:n}, X_{j:n}$, we use $f_{r|s:n}(\cdot,\cdot)$ to denote the conditional pdf of order statistics $X_{r:n}|X_{s:n}$, and we use $f_{r:n}(\cdot)$ to denote the marginal distribution of order statistics $X_{r:n}$. The cdfs $F(\cdot)$ are defined analogously.

3.1 Preliminaries

We now derive two useful properties of order statistics that we exploits in our identification arguments. We first illustrate that the joint distribution of consecutive order statistics persists the multiplicatively separable structure in a restricted region by the definition of order statistics. We then illustrate how the conditional distribution of order statistics determines the distribution of another order statistic, which leads to the

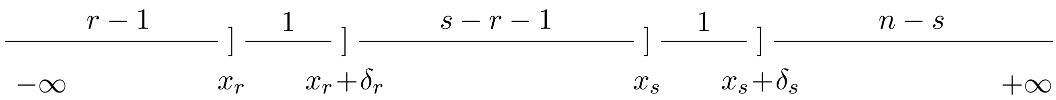
[2]When K is unknown, some rank condition would be sufficient to identify the cardinality of the unobserved types K as in An (2017)

identification of the parent distribution. We omit UH in this subsection.

Separability by consecutiveness

Suppose we observe two order statistics out of an n ordered sample, $X_{r:n}, X_{s:n}$, where $r < s$. To understand the separability generated by two consecutive order statistics, we visualize the event $(x_r < X_{r:n} \leq x_r+\delta_r, x_s < X_{s:n} \leq x_s+\delta_s)$ with $x_r+\delta_r \leq x_s$ in Figure 1. When δ_r and δ_s are both small, the likelihood of this event is approximately proportional to the multiplication of the following three components: (1) the probability that $r-1$ draws are smaller than x_r, $F(x_r)^{r-1}$; (2) the probability that 1 draw is in $(x_r, x_r+\delta_r)$, $f(x_r)\delta_r$; (3) the probability that $n-s$ draws are larger than x_s, $[1-F(x_s)]^{n-s}$; (4) the probability that 1 draw is in $(x_s, x_s+\delta_s)$, $f(x_s)\delta_s$; (5) the probability that $s-r-1$ draws are in between $x_r+\delta_r$ and x_s, $[F(x_s)-F(x_r)]^{s-r-1}$. Thus, the likelihood of this event can be represented in a multiplicatively separable structure in x_r and x_s if and only if $s-r-1=0$, i.e., the two order statistics are consecutive.

Figure 1: Joint Distribution of Two Consecutive Order Statistics



Similar logic can be applied to the joint distribution of three order statistics. Suppose we observe three consecutive order statistics, $X_{r-2:n}, X_{r-1:n}, X_{r:n}$. The well-known Markovian property of order statistics implies that

$$f_{r-2,r-1,r:n}(x_{r-2}, x_{r-1}, x_r) = f_{r-1:n}(x_{r-1}) f_{r-2|r-1:n}(x_{r-2}|x_{r-1}) f_{r|r-1:n}(x_r|x_{r-1}). \quad (3)$$

The two conditional density functions of two consecutive order statistics $f_{r-2|r-1:n}(x_{r-2}|x_{r-1})$ and $f_{r|r-1:n}(x_r|x_{r-1})$ are multiplicatively separable because their joint density functions are multiplicatively separable. Furthermore, we can represent the above joint distribution as the following multiplicatively separable structure:

$$
\begin{aligned}
& f_{r-2,r-1,r:n}(x_{r-2}, x_{r-1}, x_r) \\
= \; & \frac{n!}{(r-3)!(n-r)!}[F(x_{r-2})]^{r-3} f(x_{r-2}) f(x_{r-1}) [1 - F(x_r)]^{n-r} f(x_r) I(x_{r-2} \leq x_{r-1} \leq x_r) \\
= \; & \frac{n!}{(r-2)!(n-r+1)!} f_{r-2:r-2}(x_{r-2}) f(x_{r-1}) f_{1:n-r+1}(x_r) I(x_{r-2} \leq x_{r-1} \leq x_r), \qquad (4)
\end{aligned}
$$

where $I(\cdot)$ is the indicator function. Intuitively, we can visualize the event $(x_{r-2} < X_{r-2:n} \leq x_{r-2} + \delta_{r-2}, x_{r-1} < X_{r-1:n} \leq x_{r-1} + \delta_{r-1}, x_r < X_{r:n} \leq x_r + \delta_r)$ in Figure 2. The likelihood of this event is proportional to the multiplication of the following three components: (1) the probability that one draw is in $(x_{r-1}, x_{r-1} + \delta_{r-1})$, $f(x_{r-1})\delta_{r-1}$; (2) the probability that the maximum of $r-2$ draws are in $(x_{r-2}, x_{r-2} + \delta_{r-2})$, $f_{r-2:r-2}(x_{r-2})\delta_{r-2}$; (3) the probability that the minimum of $n-r+1$ draws are in $(x_r, x_r + \delta_r)$, $f_{1:n-r+1}(x_r)\delta_r$.

Figure 2: Joint Distribution of Three Consecutive Order Statistics

$$
\begin{array}{ccccccccc}
 & r-3 & & 1 & & 1 & & 1 & & n-r & \\
-\infty & \text{———} &] & \text{——} &] & \text{——} &] & \text{——} &] & \text{———} & +\infty \\
 & & x_{r-2} & & x_{r-1} & & x_r & & & &
\end{array}
$$

Even though order statistics are correlated by their nature, the joint distribution of three consecutive order statistics can be separated as the products of three marginal distributions $f_{r-2:r-2}(x_{r-2})$, $f(x_{r-1})$, and $f_{1:n-r+1}(x_r)$ in the area of $x_{r-2} \leq x_{x-1} \leq x_r$. Note that this separability does not indicate that the three order statistics are independent. The separability only holds in the region $x_{r-2} \leq x_{x-1} \leq x_r$, but not for the full support $[\underline{x}, \bar{x}] \times [\underline{x}, \bar{x}] \times [\underline{x}, \bar{x}]$. The joint probability distribution is zero outside the region $x_{r-2} \leq x_{x-1} \leq x_r$. This is due to the definition of the order statistics, i.e., $X_{r-2:n} \leq X_{x-1:n} \leq X_{r:n}$.

Conditional Distribution of Order Statistics

Consider again any two order statistics $X_{r:n}$ and $X_{s:n}$, where $r < s$. The conditional density of $X_{r:n}$ given that $X_{s:n} = x_s$ can be written as

$$f_{r|s:n}(x_r|x_s) = \frac{f_{r,s:n}(x_r, x_s)}{f_{s:n}(x_s)},$$

Figure 1 visualizes the event $(x_r < X_{r:n} \leq x_r + \delta_r, x_s < X_{s:n} \leq x_s + \delta_s)$ and the event $(x_s < X_{s:n} \leq x_s + \delta_s)$. While the likelihood of the former is proportional to $f(x_r)f(x_s) \cdot F(x_r)^{r-1} \cdot [1 - F(x_s)]^{n-s} \cdot [F(x_s) - F(x_r)]^{s-r-1}$ as shown above, the likelihood of the latter is proportional to the multiplication of: (1) the probability that $s-1$ draws are smaller than x_s, $F(x_s)^{s-1}$; (2) the probability that 1 draw is in $(x_s, x_s + \delta_s)$, $f(x_s)\delta_s$; (3) the probability that $n-s$ draws are larger than x_s, $[1 - F(x_s)]^{n-s}$.

Therefore, the ratio that defines the conditional density is proportional to $\frac{f(x_r)}{F(x_s)} \cdot [\frac{F(x_r)}{F(x_s)}]^{r-1} \cdot [\frac{F(x_s)-F(x_r)}{F(x_s)}]^{s-r-1}$. In fact, it is the same as the distribution of the rth order statistic in a sample of size $s-1$ from a distribution that is the parent distribution truncated at x_s. Note that the truncation is no longer effective if $x_s = \overline{x}$. As a result, the conditional distribution is the same as the distribution of the rth order statistic in a sample of size $s-1$ from the parent distribution $F(\cdot)$.

3.2 Identification using three consecutive order statistics

This subsection uses the separability properties of consecutive order statistics in a certain region to provide sufficient condition to identify auction models with nonseparable UH with a known competition level. The identification proceeds in several steps. First, we apply a discretization to the bid support which accounts for the ordering of the observed order statistics so that the separable structure persists. Second, an eigenvalue decomposition argument identifies a key matrix that governs the finite mixture structure in our order statistic setting. Third, we apply this matrix to identify the component distributions in the lower portion of the support and then in the upper part.

The unconditional joint distribution of any three consecutive order statistics, which can be estimated from the data directly, by total probability can be represented as

$$\begin{aligned} & f_{r-2,r-1,r:n}(x_{r-2}, x_{r-1}, x_r) \\ = & \sum_k p_k f^k_{r-2,r-1,r:n}(x_{r-2}, x_{r-1}, x_r) \\ = & \frac{n! \cdot I(x_{r-2} \leq x_{r-1} \leq x_r)}{(r-2)! \cdot (n-r+1)!} \cdot \sum_k p_k f^k_{r-2:r-2}(x_{r-2}) f^k(x_{r-1}) f^k_{1:n-r+1}(x_r). \end{aligned} \tag{5}$$

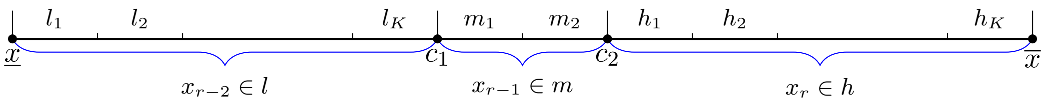
This joint distribution of the consecutive order statistics has a semi-separable structure, in the sense that we can separate the observed joint density function into three density functions, which is similar to that in the finite mixture literature, but it has an extra restriction by the nature of order statistics $I(x_{r-2} \leq x_{r-1} \leq x_r)$, which cannot be separated. This semi-separable structure precludes us from following exactly the identification procedure in the existing literature to identify the type-specific bid distribution. However, the restriction by the indicator function can be safely ignored if we divide the original support by two cutoff points $\underline{x} < c_1 < c_2 < \bar{x}$ to separate the support into three parts, referred to as "low", "middle", and "high" and denoted as $l \equiv [\underline{x}, c_1], m \equiv [c_1, c_2]$, and $h \equiv [c_2, \bar{x}]$, respectively. The separable structure of the joint distribution $f_{r-2,r-1,r:n}(x_{r-2}, x_{r-1}, x_r)$ reappears if we always restrict $x_{r-2} \in l$, $x_{r-1} \in m$, and $x_r \in h$. Specifically, if $x_{r-2} \in l$, $x_{r-1} \in m$, and $x_r \in h$, the joint distribution can be expressed as

$$f_{r-2,r-1,r:n}(x_{r-2}, x_{r-1}, x_r) = \frac{n!}{(r-2)!(n-r+1)!} \cdot \sum_k p_k f^k_{r-2:r-2}(x_{r-2}) f^k(x_{r-1}) f^k_{1:n-r+1}(x_r), \tag{6}$$

which has a familiar structure of finite mixture models but each component has different meanings.

Following the existing literature, we first select K exclusive intervals from the "low" support l and the "high" support h, denoted as l_i and h_i, $i = 1, ..., K$, respectively. We also select 2 exclusive intervals from the "middle" support m, denoted as m_i, $i = 1, 2$. Note that these intervals do not have to be fully exhaustive, and they can be simply points. Figure 3 provides a visualization of the support.

Figure 3: Support Visualization



We then rewrite the above joint distribution with respect to the selection of the support:

$$
\begin{aligned}
& \Pr(X_{r-2:n} \in l_i, X_{r-1:n} \in m_{i'}, X_{r:n} \in h_j) \\
= \; & \frac{n!}{(r-2)!(n-r+1)!} \sum_k \int_{x_r \in h_j} \int_{x_{r-1} \in m_{i'}} \int_{x_{r-2} \in l_i} p_k \\
& \qquad f^k_{r-2:r-2}(x_{r-2}) f^k(x_{r-1}) f^k_{1:n-r+1}(x_r) dx_{r-2} dx_{r-1} dx_r \\
= \; & \frac{n!}{(r-2)!(n-r+1)!} \sum_k \int_{x_{r-2} \in l_i} f^k_{r-2:r-2}(x_{r-2}) dx_{r-2} \quad p_k \int_{x_{r-1} \in m_{i'}} f^k(x_{r-1}) dx_{r-1} \\
& \int_{x_r \in h_j} f^k_{1:n-r+1}(x_r) dx_r. \qquad (7)
\end{aligned}
$$

Similar to the identification in finite mixture and measurement error literature, the identification uses matrix algebra, so we first define the following matrix notation.

$$
\begin{aligned}
\boldsymbol{J}_{l,m_{i'},h} &\equiv \{\Pr(X_{r-2:n} \in l_i, X_{r-1:n} \in m_{i'}, X_{r:n} \in h_j)\}_{i,j} \\
\boldsymbol{L} &\equiv \{\int_{x_{r-2} \in l_i} f^k_{r-2:r-2}(x_{r-2}) dx_{r-2}\}_{i,k}, \\
\boldsymbol{D}_{m_{i'}} &\equiv diag\{\int_{x_{r-1} \in B^2_{i'}} f^k(x_{r-1}) dx_{r-1}\}_k \\
\boldsymbol{D}_p &\equiv diag\{p_k\}_k \\
\boldsymbol{H} &\equiv \{\int_{x_r \in h_i} f^k_{1:n-r+1}(x_r) dx_r\}_{i,k},
\end{aligned}
$$

where $\boldsymbol{J}_{l,m_{i'},h}$ denotes the joint probability matrix with the i,jth element being the probability of the event $\{X_{r-2:n} \in l_i, X_{r-1:n} \in m_{i'}, X_{r:n} \in h_j\}$; $\boldsymbol{L}$ is is the probability matrix for top order statistics $X_{r-2:r-2}$ with the i,kth element being the probability of the event $\{X^k_{r-2:r-2} \in l_i\}$, where $i,k = 1,...,K$; $\boldsymbol{D}_{m_{i'}}$ is the diagonal matrix with the kth diagonal element being the probability of the event $\{X^k \in m_{i'}\}$;$\boldsymbol{D}_p$ is the diagonal type weight matrix with the kth diagonal element being the weight of type k; $\boldsymbol{H}$ is the probability matrix for the bottom statistics $X_{1:n-r+1}$ with the i,kth element being the probability of the event $\{X^k_{1:n-r+2} \in h_i\}$. With the matrix notation, we have the following matrix representation connecting the observed joint probability with the unknown matrix constructed using type-specific probability:

$$\boldsymbol{J}_{l,m_{i'},h} = \frac{n!}{(r-2)!(n-r+1)!}\boldsymbol{L}\boldsymbol{D}_{m_{i'}}\boldsymbol{D}_p\boldsymbol{H}^T, \quad i' = 1,2. \tag{8}$$

Following the existing literature, identification of models with UH using the mixture features usually requires some rank condition, which is stated in the follows.

Assumption 1. *(Full Rank) There exists a mapping from x_{r-2} and x_r into K intervals of the support $[\underline{x}, \bar{x}]$ so that the probability matrix in the "low" support $\boldsymbol{L}$ and the probability matrices in the "high" support $\boldsymbol{H}$ are full rank.*

Note that empirically the above matrix is testable since the joint probability matrix can be estimated directly from the data. This full rank condition can be guaranteed that the type-specific bid distributions are linearly independent. The full rank assumption leads to the following main equation for identification:

$$\boldsymbol{J}_{l,m_1,h}\boldsymbol{J}^{-1}_{l,m_2,h} = \boldsymbol{L}\boldsymbol{D}_{m_1/m_2}\boldsymbol{L}^{-1}, \tag{9}$$

where $\boldsymbol{D}_{m_1/m_2}$ is a diagonal matrix with the kth diagonal element as the ratio of the probability that type k occurs in two middle intervals m_1 and m_2, i.e., $\frac{\int_{x \in m_1} f^k(x)dx}{\int_{x \in m_2} f^k(x)dx}$.

Equation (9) indicates that the observed matrix on the left-hand side and the unknown matrices on the right-hand side are similar. Specifically, the probability matrix involved the "low" support $\boldsymbol{L}$ and the probability ratio matrix involved the "'middle" support $\boldsymbol{D}_{m_1/m_2}$ can be identified as the eigenvector and eigenvalue matrices of the observed joint probability matrix. Unique decomposition requires a condition that the diagonal elements differ from each other, as in the following assumption.

Assumption 2. *(Distinctive Eigenvalues) There exist a mapping from x_{r-1} to two intervals of the "middle" support m so that the ratio of the probability that the type-specific underlying random variable falls into the two intervals differ for any two types. That is,*

$$\frac{\int_{x \in m_1} f^k(x)dx}{\int_{x \in m_2} f^k(x)dx} \neq \frac{\int_{x \in m_1} f^{k'}(x)dx}{\int_{x \in m_2} f^{k'}(x)dx}, \quad \forall \quad k \neq k'. \tag{10}$$

This assumption is empirically testable since we can estimate the eigenvalues directly from the decomposition and can be achieved by choosing the cutoff point of the support. With the assumptions of full rank and distinctive eigenvalues, we can identify the probability matrix involved the "low" support $\boldsymbol{L}$, whose kth column is the probability that the order statistics $X^k_{r-2:r-2}$ falling in the K intervals $\{l_1, ..., l_K\}$ up to relabeling and scales. Note that the probability matrix involved the "low" support ($\boldsymbol{L}$) is identified as the eigenvector matrix through an eigenvalue and eigenvector decomposition. As a result, $\boldsymbol{L}$ is identified up to scales and relabeling.

Identification up to scale is a prevalent feature of identification using the mixture feature. Pinning down the scale requires a normalization condition. In the existing literature of independent measurements, a normalization condition could be that the column sum equals 1 because it represents a total probability. This normalization condition is not applicable here because the column sum is not a total probability anymore after we divide the whole support into three portions. We will keep this identification up to scale for now and provide restrictions to pin down the scales later.

With this identified probability matrix involved the "low" support, we can further identify the density for the order statistics $X^k_{r-2:r-2}$ for any value in this interval. Then we can use the connection between the density of an order statistics and the density of the original distribution to recover the type-specific density distribution for the "low support. In particular, to identify the density for the order statistics $X^k_{r-2:r-2}$, we again use the joint distribution. Note that, the matrix representation in equation (8) also holds for a particular value of $x_{r-2} = x \in l$. Thus, we have

$$\boldsymbol{J}_{x,m_1,h} = \frac{n!}{(r-2)!(n-r+1)!} \boldsymbol{L}_x \boldsymbol{D}_{m_1} \boldsymbol{D}_{\boldsymbol{p}} \boldsymbol{H}^T, \tag{11}$$

where $\boldsymbol{J}_{x,m_1,h}$ and $\boldsymbol{L}_x$ are the counterparts of matrix $\boldsymbol{J}_{l,m_1,h}$ and $\boldsymbol{L}$ with replacing the interval to a particular value of $x_{r-2} = x \in l$, respectively, so that $\boldsymbol{L}_x$ represents density. Note that both equation (8) and the above equation have common components $\boldsymbol{D}_{m_1} \boldsymbol{D}_{\boldsymbol{p}} \boldsymbol{H}^T$, which is invertible. Consequently, we can identify the type-specific density for order statistics $X^k_{r-2:r-2}$ in the "low" support, i.e., $f^k_{r-2:r-2}(x), \forall x \in l$, as in the following closed-form expression:

$$\boldsymbol{L}_x = \boldsymbol{J}_{x,m_1,h} \boldsymbol{J}^{-1}_{l,m_1,h} \boldsymbol{L}. \tag{12}$$

Note that the density vector $\boldsymbol{L}_x$ is identified up to the same scales and ordering of the probability matrix $\boldsymbol{L}$.

Now we proceed to identify the type-specific density using the identified type-specific distribution of the order statistics $f^k_{r-2:r-2}(x)$ in the "low" support through the following closed-form expression.

$$f^k(x) \quad = \quad \frac{1}{r-2} \Big[\int_{\underline{x}}^{x} f^k_{r-2:r-2}(v) dv \Big]^{\frac{1}{r-2}-1} f^k_{r-2:r-2}(x), \quad \forall x \in l. \tag{13}$$

Again the type-specific density is identified up to scales since the type-specific density of order statistics $X^k_{r-2:r-2}$ is identified up to scale. Note that the two scales might differ. We summarize the above results in the following Lemma and leave the proof details in the Appendix.

Lemma 1 (low support)**.** *If Assumptions (1) and (2) are satisfied, the type-specific density in the "low" support, i.e., $f^k(x), \forall x \in l$, is identified up to scales.*

In what follows, we identify the type-specific density function $f^k(x)$ in the "high" support. Specifically, we first identify the probability matrix $\boldsymbol{H}$ involved the "high" support up to scales using the joint distribution in equation (8):

$$\boldsymbol{H}^T = \left[\frac{n!}{(r-2)!(n-r+1)!}\boldsymbol{L}\boldsymbol{D}_{m_1}\boldsymbol{D}_p\right]^{-1}\boldsymbol{J}_{l,m_1,h}. \tag{14}$$

Since $\boldsymbol{L}$ is identified up to scales due to the decomposition, and both $\boldsymbol{D}_{m_1}$ and $\boldsymbol{D}_p$ are diagonal matrices, $\boldsymbol{H}$ can be identified up to scales, but the scales are different from the scales in $\boldsymbol{L}$.

We then can follow the same identification argument in lemma 1 to identify the type-specific probability density for order statistics $f^k_{1:n-r+1}(x)$ by replacing the bins in h by a particular value $x \in h$ and use invertibility of the matrices to cancel out common components. We then use the connection of the type-specific density for order statistics and the type-specific density function to recover the type-specific density. In particular,

$$f^k(x) \quad = \quad \frac{1}{n-r+1}\Big[\int_x^{\bar{x}} f^k_{1:n-r+1}(v)dv\Big]^{\frac{1}{n-r+1}-1} f^k_{1:n-r+1}(x), \quad \forall x \in h \tag{15}$$

Note that $f^k_{1:n-r+1}(x)$ is identified up to scales, so the type-specific density $f^k(x) \forall x \in h$ is also identified up to scales. We summarize this identification result in the following Lemma.

Lemma 2 (high support)**.** *If Assumptions (1) and (2) are satisfied, the type-specific* *density in the "high" support, i.e., $f^k(x), \forall x \in h$, is identified up to scales.*

With the type-specific density function being identified for the "low" and "high" support, we now move to identify the density in the "middle" suport. The identification again is similar as in lemma 2. We again rely on the joint distribution relation similar in in equation (8), leading to the following matrix representation:

$$\boldsymbol{J}_{l,x,h} = \frac{n!}{(r-2)!(n-r+1)!} \boldsymbol{L}\boldsymbol{D}_x\boldsymbol{D}_p\boldsymbol{H}^T, \tag{16}$$

where $\boldsymbol{J}_{l,x,h}$ and $\boldsymbol{D}_x$ are the counterparts of matrix $\boldsymbol{J}_{l,m_1,h}$ and $\boldsymbol{D}$ with replacing the interval to a particular value of $x_{r-1} = x \in m$, respectively. Note that $\boldsymbol{L}$ and $\boldsymbol{H}$ are both identified up to scales and $\boldsymbol{D}_p$ is a diagonal matrix. As a result, $\boldsymbol{D}_x$ can be identified up to scales, with the kth diagonal element being the type-specific density $f^k(x)$ where $x \in m$.

Lemma 3 (middle support). *If Assumptions (1) and (2) are satisfied, the type-specific density in the "middle" support, i.e., $f^k(x), \forall x \in m$, is identified up to scale.*

Note that we identify the type-specific density $f^k(x)$ for the "lower", "middle", and "high" portion up to different scales, which requires three conditions to pin down the scales exactly. First, the three density functions identified separately should be the same at the cutoff points c_1 and c_2 for the two densities overlapping in the cutoff points. Moreover, the fact that the cumulative sum across all three intervals should equal to 1, which provides the third restrictions on the scales. Once the scales being pinned down, we can identify the type-specific weight p_k. To sum up, we can identify the type-specific bid distribution using only three consecutive order statistics of the bids. With the type-specific bid distribution being identified, we then can identify the type-specific value function in both first price or second price IPV auctions when the number of potential bidders is known. We summarize all results in the following theorem.

Theorem 1. *If the competition n is known and Assumptions (1) and (2) are satisfied in IPV auctions, the type weight p_k, the type-specific bid distribution $f^k(x)$ for $x \in [\underline{x}, \bar{x}^k]$, and the type-specific value distribution $\Phi^k(v)$ for $v \in [\underline{v}, \bar{v}^k]$ are identified for all k using three consecutive order statistics.*

Identification using two consecutive order statistics with an instrument

In some scenarios, we might only observe only two consecutive order statistics, such as the top 2 bids, in which case the above identification results cannot be applied. In this scenario, an extra instrument for the UH would work as well as the third order statistics and enables identification of the type-specific bid distributions in a similar manner using the joint distribution of the two consecutive order statistics and the instrument. An instrument is a variable that is independent with the order statistics conditional on the type. Furthermore, the requirement for the instrument is mild, in the sense that as long as there is variation in the instrument, such as binary, the identification argument below can go through.[3]

Suppose we observe two consecutive order statistics of the n ordered sample $X_{r-1:n}, X_{r:n}$ and an instrument Z, where $Z \in \{0, 1\}$. The joint distribution of these three variables are

$$
\begin{aligned}
f_{r-1,r:n,Z}(x_{r-1}, x_r, z) &= \sum_k p_k \frac{n! I(x_{r-1} \leq x_r)}{(r-1)!(n-r+1)!} f^k_{r-1:r-1}(x_{r-1}) f^k_{1:n-r+1}(x_r) \Pr(z|k) \\
f_{r-1,r:n,Z}(x_{r-1}, x_r) &= \sum_k p_k \frac{n! I(x_{r-1} \leq x_r)}{(r-1)!(n-r+1)!} f^k_{r-1:r-1}(x_{r-1}) f^k_{1:n-r+1}(x_r). \qquad (17)
\end{aligned}
$$

Similarly, we divide the space into two exclusive intervals and further discretize both intervals into K exclusive intervals. Fix $z = 0$, we can rewrite the above equations connecting the unknown type distribution with the observed joint probability into a matrix representation. We also can rewrite the joint distribution of the two order statistics in

[3] This is similar to the Hu (2017) 2.1 measurement model.

a similar matrix manner. We thus can follow the previous identification argument to identify the type-specific distributions.

3.3 Identification using two order statistics with support variations

In auctions with UH, sometime the UH not only are drawn from different distributions, but also shifts the support of bids, which provides useful variations for identification. The variation in supports essentially reduce the mixture components in some regions of their supports. This subsection shows that with support variations, two arbitrary order statistics, instead of three consecutive ones, are sufficient to identify the unobserved distribution. Specifically, suppose we observe two order statistics $X_{r:n}$ and $X_{s:n}$ of bids, where $r < s$, for a n ordered sample. Note that $f_{r,s:n}(x_r, x_s) = \sum_k p_k f^k_{r,s:n}(x_r, x_s)$ and $f_{s:n}(x_s) = \sum_k p_k f^k_{s:n}(x_s)$.

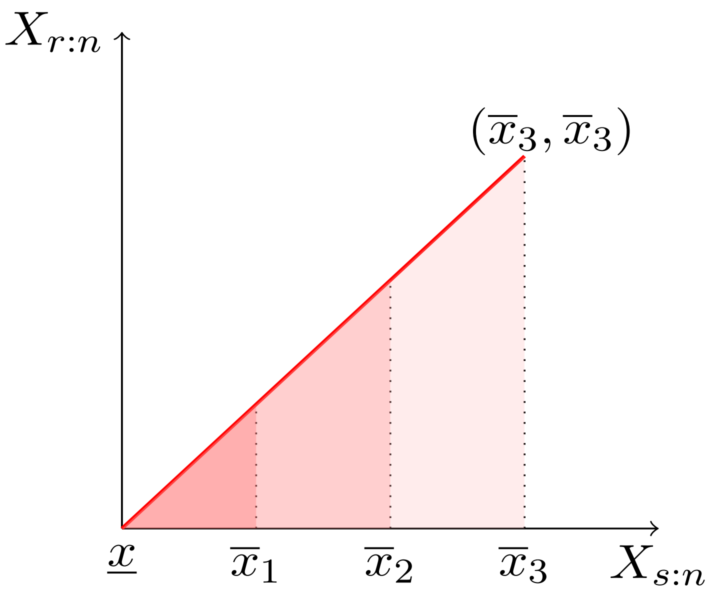
We first formally introduce the support variations in the following assumption.

Assumption 3. *(Support variations)* $\overline{x}_1 < \overline{x}_2 <, \ldots, < \overline{x}_K$.

To illustrate the role of the support variations play in the identification, we display the support of the joint PDF for the order statistics $\{X_{s:n}, X_{r:n}\}$ in Figure 4. We assume that $K = 3$ for illustration purpose. First, the support is below the 45 degree line because $X_{s:n} \geq X_{r:n}$ by definition. Second, there is a jump in density at every upper bound of type $\overline{x}_k$ because the density for lower types vanishes. Specifically, the light red areas between $\overline{x}_2$ and $\overline{x}_3$ involves only the highest type $k = 3$, i.e., the mixture feature degenerates; the middle area involves only the top two types, $k = 2, 3$, i.e., the mixture only have two components instead of three; only the dark red area between $\underline{x}$ and $\overline{x}_1$ involves all three types.

We show the identification of type-specific distribution using this support variation sequentially. First, we identify the distribution of the highest type using the fact that the observed distribution in the area of $\overline{x}_{K-1}$ and $\overline{x}_K$ provides information only about

Figure 4: Support of the Joint PDF for $(X_{s:n}, X_{r:n})$



type K. Then we recover type k's distribution by excluding the identified higher type $k+1, ...K$ from the observed distribution in the area of $[\overline{x}_{k-1}, \overline{x}_k]$, which only involves distributions of type $k = 1, ..., K$. We summarize the whole identification in the following two Lemmas.

Lemma 4. *If Assumption 3 is satisfied, the bid distribution of the highest type $F^K(\cdot)$ and weight p_K are identified.*

Consider $x_{s:n} = x_s \in (\overline{x}_{K-1}, \overline{x}_K]$, the conditional density function of x_r given x_s is

$$\begin{aligned} f_{r|s:n}(x_r|x_s) &= \frac{f_{r,s:n}(x_r, x_s)}{f_{s:n}(x_s)} \\ &= \frac{\sum_k p_k f^k_{r,s:n}(x_r, x_s)}{\sum_k p_k f^k_{s:n}(x_s)} \\ &= \frac{p_K f^K_{r,s:n}(x_r, x_s)}{p_K f^K_{s:n}(x_s)} \\ &= f^K_{r|s:n}(x_r|x_s) \\ &= \frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{f^K(x_r)}{F^K(x_s)} [\frac{F^K(x_r)}{F^K(x_s)}]^{r-1} [1 - \frac{F^K(x_r)}{F^K(x_s)}]^{s-r-1}. \quad (18) \end{aligned}$$

The third equality holds because the area of $(\overline{x}_{K-1}, \overline{x}_K]$ only concerns type K. This conditional density is the density function of rth order statistics from a sample of size $s-1$ from a distribution $F^K(\cdot)$ truncated on the right at x_s.[4] Let $x_s = \overline{x}_K$. The truncation is no longer effective. We then further simplify the conditional density function as

$$f^K_{r|s:n}(x_r|x_s = \overline{x}_K) = \frac{(s-1)!}{(r-1)!(s-r-1)!} f^K(x_r) F^K(x_r)^{r-1}[1 - F^K(x_r)]^{s-r-1} \equiv f^K_{r:s-1}(x_r), \quad (19)$$

where $f^K_{r:s-1}(\cdot)$ is the density function for rth order statistics from a sample of size $s-1$ with a distribution $F^K(\cdot)$. This implies that $F^K_{r:s-1}(\cdot)$ is identified. Thus, $F^K(\cdot)$ is identified following footnote 1.

Moreover, the weight for type K can be identified through a marginal distribution in the area of any $x_s \in (\overline{x}_{K-1}, \overline{x}_K]$. That is,

$$p_K = f_{s:n}(x_s)/f^K_{s:n}(x_s). \quad (20)$$

With the distribution of the highest type being identified, we next identify the distribution of the remainning types using iteration induction, stated in the following Lemma.

Lemma 5. *If $F^k(\cdot)$ and p_k are identified for all $k = \check{k}+1, \dots, K$, $F^{\check{k}}(\cdot)$ and $p_{\check{k}}$ are identified.*

We again use the joint distribution and consider $x_{s:n} = x_s \in (\overline{x}_{\check{k}-1}, \overline{x}_{\check{k}}]$. Note that this areas only involves higher types $k = \check{k}, \dots, K$. In this specific area, we can represent

[4] Song (2004) identifies a model for eBay auctions using a different property that $f_{s|r:n}(\cdot)$ is the same as the density of $(s-r)$th order statistics from a sample of size $(n-r)$ from $F(x)$ truncated on the left at x_r. However, his considers homogeneous auctions where only the number of bidders varies. Moreover, his focus was the identification of the value distribution.

the joint distribution of the two order statistics and the marginal distribution as

$$
\begin{aligned}
f_{r,s:n}(x_r, x_s) &= \sum_k p_k f^k_{r,s:n}(x_r, x_s) = \sum_{k=\check{k}}^{K} p_k f^k_{r,s:n}(x_r, x_s) \\
f_{s:n}(x_s) &= \sum_k p_k f^k_{s:n}(x_s) = \sum_{k=\check{k}}^{K} p_k f^k_{s:n}(x_s).
\end{aligned}
$$

Since the distribution and weight of the higher types are identified, we can identify the distribution of type $\check{k}$ through excluding the identified components from the observed counterparts.

$$
p_{\check{k}} f^{\check{k}}_{r,s:n}(x_r, x_s) = f_{r,s:n}(x_r, x_s) - \sum_{\check{k}+1}^{K} p_k f^k_{r,s:n}(x_r, x_s) \tag{21}
$$

$$
p_{\check{k}} f^{\check{k}}_{s:n}(x_s) = f_{s:n}(x_s) - \sum_{\check{k}+1}^{K} p_k f^k_{s:n}(x_s). \tag{22}
$$

Consequently, following the identification argument in Lemma 4, we can identify the conditional density function of x_r given x_s of type $\check{k}$ through the following equations

$$
f^{\check{k}}_{r|s:n}(x_r|x_s) = \frac{f^{\check{k}}_{r,s:n}(x_r,x_s)}{f^{\check{k}}_{s:n}(x_s)} = \frac{p_{\check{k}} f^{\check{k}}_{r,s:n}(x_r,x_s)}{p_{\check{k}} f^{\check{k}}_{s:n}(x_s)}. \tag{23}
$$

Similar to above, we obtain

$$
f^{\check{k}}_{r|s:n}(x_r|x_s = \overline{x}_{\check{k}}) = f^{\check{k}}_{r:s-1}(x_r),
$$

which identifies $F^{\check{k}}_{r:s-1}(\cdot)$ and thus $F^{\check{k}}(\cdot)$. Moreover, $p_{\check{k}} = p_{\check{k}} f^{\check{k}}_{s:n}(x_s)/f^{\check{k}}_{s:n}(x_s)$. In summary, the distribution function and weight for type $\check{k}$ can be identified.

Lemmas 4 and 5 imply that the distributions $F^k(\cdot)$ and weight p_k are nonparametrically identified. In auction models, we can identify the type-specific bid distributions using the support variation. Consequently, we can identify the type-specific value distributions in IPV auctions.

Theorem 2. *If the competition n is known and Assumptions 3 is satisfied, the weight of type p_k, the type-specific bid distribution $F^k(x)$, and the type-specific value distribution $\Phi^k(v)$ in IPV auctions are identified for all k using any two order statistics.*

Identification with multidimensional UH in first price auction

We extended the above identification argument to first price auctions with two sources of UH: unobserved auction type k and unobserved competition n. We assume that the number of bidders n is exogenous and the maximal number of potential bidders N is known. We first relabel the type as $\omega = (k, n)$. In a first-price auction, each type (k, n) maps with a type-specific value distribution $\Phi^k(v)$ and corresponds to a bid distribution $F^{k,n}(x)$that has support $[\underline{x}, \overline{x}_{(k,n)}]$. We now study how to identify the value distributions $\Phi^k(\cdot)$ and the distribution of each type $p_{(k,n)}$.

Similar to the previous section, in first price auctions, a sufficient condition for the support variation (Assumption 3) is that the type-specific value distribution has a first-order stochastic dominance relation (FOSD).

Assumption 4. *(FOSD) $\Phi^1(v) \leq \Phi^2(v) \leq \ldots \leq \Phi^K(v)$ for all v, with strict inequality at some v for each pair of the value distributions.*

The FOSD condition not only generate support variation for the unobserved type k with a given n, it also provides support variation for the competition n with a given k. Given that Assumption 4 is satisfied, Guerre, Perrigne, and Vuong (2009) and Hu, McAdams, and Shum (2013) discuss how the upper boundary varies with respect to n and k, respectively.

Lemma 6. *For a given k (n), the upper boundary $\overline{x}_{(k,n)}$ is strictly increasing with respect to n (k) if Assumption 4 is satisfied.*

Lemma 6 shows that there is a jump at each upper boundary $\overline{x}_\omega$. Without loss of generality, let $0 < \omega_1 < \omega_2 < \ldots < \omega_{2K}$. These points divide the support of $x_{(w)}$ into $K \times (N-2)$ intervals. The top portion $(\overline{x}_{\omega_{2K-1}}, \overline{x}_{\omega_{2K}}]$ corresponds to highest UH and the biggest number of bidders $\omega = (K, N)$. Following the same lines in Section 3.3, we can identify the weight and bid distribution associated with the highest types, i.e., $p_{(K,N)}$ and $F^{K,N}(\cdot)$, respectively. Consequently, we can identify the value distribution associated with the highest type, $\Phi^K(\cdot)$. Moreover, we can use $\Phi^K(\cdot)$ to identify the upper boundary $\overline{x}_{(K,n)}$ and the corresponding bid distribution function $F_{(K,n)}(\cdot)$ for all $n < N$.

We then proceed to identify the type-specific bid distribution $F^{k,n}(\cdot)$, weight $p_{(k,n)}$, and type-specific value distribution $\Phi^k(\cdot)$, for all $k < K$ and all n, using iteration induction. Specifically, we show that $p_{(\check{k},n)}$, $F^{\check{k},n}(\cdot)$, and $\Phi^{\check{k}}(\cdot)$ are identified if $p_{(k,n)}$, $F^{k,n}(\cdot)$, and $\Phi^k(\cdot)$ are identified for all $k = \check{k}+1, \ldots, K$. Excluding $\overline{x}_{(k,n)}$ for all $k > \check{k}$ and all n, we find the maximum of the other jump points. Lemma 6 implies that this point corresponds to $(k, n) = (\check{k}, N)$. Again, we follow Guerre, Perrigne, and Vuong (2000) to identify the associated type-specific value distribution $\Phi^{\check{k}}(\cdot)$. We then use the identified type value distribution $\Phi^{\check{k}}(\cdot)$ to identify the related bid distribution function with different competition levels $F^{\check{k},n}(\cdot)$, for all competition n.

Theorem 3. *If Assumption 4 is satisfied, the weight $p_{k,n}$, the bid distribution $F^{k,n}(\cdot)$, and the value distribution $\Phi^k(\cdot)$ are identified for first price IPV auctions with unobserved type k and unknown competition n using any two order statistics.*

4 Identification with Separable UH

This section considers the generic identification of separable UH in auction models. Following the existing literature, we assume that the auction-level UH enters in individual bidders' value in an additively separable fashion. Specifically, we consider auctions where individual bidder's value comes from two parts:

$$V_i = V^* + u_i, \tag{24}$$

where V^* is the auction-level UH and u_i are the idiosyncratic part. See, e.g., Li, Perrigne, and Vuong (2000).[5] The distribution of UH and the private part are denoted as $\Phi_{V^*}(\cdot)$ with support $[\underline{v}, \bar{v}]$ and $\Phi_u(\cdot)$ with support $[\underline{u}, \bar{u}]$, respectively. We first impose the mutually independent assumption as in the existing literature.

Assumption 5. *(Independence) The common value V^* and the idiosyncratic value shocks u_i are mutually independent. Moreover, u_i are i.i.d. conditional on the auction heterogeneity V^*.*

With the independence condition, we can further represent the observed bid X_i in the same additively structure in the following, as shown in Haile, Hong, and Shum (2003).

$$X_i = X^* + \epsilon_i, \tag{25}$$

where X^* and ϵ_i map to the value V^* and u_i, and their distributions denoted as $F_{X^*}(\cdot)$ with support $[\underline{x}, \bar{x}]$ and $F_\epsilon(\cdot)$ with support $[\underline{\epsilon}, \bar{\epsilon}]$, respectively. Moreover $X^* = V^*$.

We now provide sufficient conditions to identify both distribution functions $F_{X^*}(\cdot)$ and $F_\epsilon(\cdot)$ using the joint distribution of the top two order statistics of the bids when the competition n is known, i.e., $X_{n:n}$ and $X_{n-1:n}$. We then extend the identification argument to the scenario of any two consecutive order statistics. Note that the usual approach that exploits the joint characteristic function (Li, Perrigne, and Vuong (2000)), no longer applies because order statistics are dependent by definition and thus the joint characteristic function is no longer multiplicatively separable in individual ones. Despite this, Athey and Haile (2002) conjecture that there is enough structure to identify the

[5] In the case of multiplicative separability, $V_i = V^* u_i$, where $V^*, u_i > 0$. We can apply logarithm on both side to obtain an additive separable form $\log V_i = \log V^* + \log u_i$.

model from two order statistics. However, there has been no such result in the literature.[6] One natural approach is to take the difference between two order statistics and study identification of the parent distribution from the distribution of this spacing. However, this approach does not necessarily work. For instance, Athey and Haile (2002) point out a counterexample in Rossberg (1972): if X_1, X_2 are *i.i.d.* random variables with $F(x) = 1 - e^{-x}[1 + \pi^{-2}(1 - \cos 2\pi x)]$ on support R_+, the spacing $X_{2:2} - X_{1:2}$ has a standard exponential distribution.

Our identification procedure also relies on characteristics function of the order statistics but exploit the "within" instead of the "between" independence. In particular, although $\epsilon_{n-1:n}$ and $\epsilon_{n:n}$ are dependent, we can exploit the fact that $X^* \perp \epsilon_{n-1:n}$ and $X^* \perp \epsilon_{n:n}$. We now introduce some notation. We use $\psi_X(t)$ to denote characteristic function for a random variable X. For ease of notation, let $\epsilon_{\check{n}:\check{n}} \equiv \max\{\epsilon_1, \ldots, \epsilon_{\check{n}}\}$, where $\check{n} = n, n-1, \ldots, 1$, which is the top order statistics of i.i.d. random draws from a parental distribution $F_\epsilon(\cdot)$; Let $\psi_{\check{n}}(t)$ denote $\epsilon_{\check{n}:\check{n}}$'s characteristics function, so $\psi_{\check{n}}(t) = \int_{-\infty}^{+\infty} \exp[i(ty)] \cdot [\check{n} F_\epsilon^{\check{n}-1}(y) f_\epsilon(y)] dy$.[7]

We first identify the ratio of the characteristics function of the top order statistics from a $n-1$ and n ordered sample using the characteristics function of the observed order statistics, i.e., $\psi_{n-1}(\cdot)/\psi_n(\cdot)$.

Lemma 7. *If Assumption 5 is satisfied, the ratio of the characteristics function of the top order statistics from a n and $n-1$ ordered sample, i.e., $\psi_{n-1}(t)/\psi_n(t)$, is identified.*

Proof. Under Assumption 5, UH and the order statistics of the idiosyncratic parts are

[6] Freyberger and Larsen (2017) have a similar problem with eBay auctions but they circumvent the issue of correlated order statistics using observed reserve prices, rendering the identification problem classical.

[7] Note that $\psi_1(t) = \psi_\epsilon(t)$.

also mutually independent. That is,

$$\begin{aligned}\psi_{X_{n:n}}(t) &= \psi_{X^*}(t) \cdot \psi_{\epsilon_{n:n}}(t) = \psi_{X^*}(t) \cdot \psi_n(t), &(26)\\ \psi_{X_{n-1:n}}(t) &= \psi_{X^*}(t) \cdot \psi_{\epsilon_{n-1:n}}(t) = \psi_{X^*}(t) \cdot \big(n\psi_{n-1}(t) - (n-1)\psi_n(t)\big). &(27)\end{aligned}$$

Cancelling out the characteristics function of the common UH leads to the following equation

$$\frac{n\psi_{n-1}(t) - (n-1)\psi_n(t)}{\psi_n(t)} = \frac{\psi_{X_{n-1:n}}(t)}{\psi_{X_{n:n}}(t)}.$$

Consequently, we can identify the ratio of the characteristics function of the two top order statistics as

$$\frac{\psi_{n-1}(t)}{\psi_n(t)} = \frac{1}{n}\frac{\psi_{X_{n-1:n}}(t)}{\psi_{X_{n:n}}(t)} + \frac{n-1}{n}. \tag{28}$$

□

Now we proceed to identify the distribution $F_\epsilon(\cdot)$ using the identified ratio of the characteristics functions of the two top order statistics $\psi_{n-1}(t)/\psi_n(t)$. Note that both support for X^* and ϵ is unknown, i.e., $[\underline{x}, \bar{x}]$ and $[\underline{\epsilon}, \bar{\epsilon}]$ are yet to identify together with their corresponding distributions. Consequently, it is observational equivalent if we add a constant to X^* and subtract the same amount from ϵ_i. We make the following location normalization for ease of exposition.[8]

Assumption 6. *(Normalization)* $\underline{\epsilon} = 0$.

We first define identification formally in this context.

Definition 1. *(Identification) The distribution of the idiosyncratic part* $F_\epsilon(\cdot)$ *is identified if any two distribution functions* $F(\cdot)$ *and* $G(\cdot)$ *are the same once they satisfy the*

[8]The existing literature with independent measurements assumes that it is mean zero. See, e.g., Li, Perrigne, and Vuong (2000). Our normalization is without loss of generality.

following two conditions:

1. *their supports are of the form* $[0, \widetilde{\epsilon}]$[9]*;*

2. *they have the same ratio of the characteristic functions of the two top order statistics, i.e.,* $\frac{\psi_{n-1}(t)}{\psi_n(t)} = \frac{\psi'_{n-1}(t)}{\psi'_n(t)}$*, where* $\frac{\psi_{n-1}(t)}{\psi_n(t)}$ *and* $\frac{\psi'_{n-1}(t)}{\psi'_n(t)}$ *are associated with* $F(\cdot)$ *and* $G(\cdot)$*, respectively.*

Note that $\psi_{n-1}(t)/\psi_n(t) = \psi'_{n-1}(t)/\psi'_n(t)$ is equivalent to $\psi_n(t)\psi'_{n-1}(t) = \psi_{n-1}(t)\psi'_n(t)$. Consequently, the identification problem is equivalent to the following problem.

Problem 1. *X_i's and Y_i's are i.i.d. random variables with distribution function $F(\cdot)$ on support $[0, \bar{\epsilon}_F]$ and $G(\cdot)$ on support $[0, \bar{\epsilon}_G]$, respectively. Let $Z_1 = X_{n:n} + Y_{n-1:n-1}$ and $Z_2 = X_{n-1:n-1} + Y_{n:n}$.*[10] *Identification in our context is equivalent to the situation that Z_1 and Z_2 have the same distribution implies that $F(\cdot) = G(\cdot)$ and $\bar{\epsilon}_G = \bar{\epsilon}_F$.*

We now proceed to prove that Z_1 and Z_2 have the same distribution implies that $\bar{\epsilon}_G = \bar{\epsilon}_F$ and $F(\cdot) = G(\cdot)$, so that our original auction structure is identified. Specifically, by the convolutions of probability, Z_1 has a density function

$$\begin{aligned} h_1(z) &= \int_{-\infty}^{+\infty} [nF^{n-1}(z-y)f(z-y) \cdot (n-1)G^{n-2}(y)g(y)]dy \\ &= \int_0^z [nF^{n-1}(z-y)f(z-y) \cdot (n-1)G^{n-2}(y)g(y)]dy, \qquad \text{if } 0 \le z \le \bar{\epsilon} \end{aligned}$$

where $\bar{\epsilon} = \min\{\bar{\epsilon}_F, \bar{\epsilon}_G\}$, and $nF^{n-1}(\cdot)f(\cdot)$ and $(n-1)G^{n-2}(\cdot)g(\cdot)$ represent the density functions of $X_{n:n}$ and $Y_{n-1:n-1}$, respectively.

[9]Their supports are unknown to the analyst and might be different.

[10]Note that Z_1 and Z_2's have support $[0, \bar{\epsilon}_F + \bar{\epsilon}_G]$ and their characteristics functions can be represented as $\psi_n(t)\psi'_{n-1}(t)$ and $\psi_{n-1}(t)\psi'_n(t)$, respectively, because X's and Y's are independent.

Similarly, Z_2 has a density function

$$
\begin{aligned}
h_2(z) &= \int_{-\infty}^{+\infty} [(n-1)F^{n-2}(z-y)f(z-y) \cdot nG^{n-1}(y)g(y)]dy \\
&= \int_{0}^{z} [(n-1)F^{n-2}(z-y)f(z-y) \cdot nG^{n-1}(y)g(y)]dy, \qquad \text{if } 0 \leq z \leq \overline{\epsilon}.
\end{aligned}
$$

Lemma 8. *For any $z \in [0, \overline{\epsilon}]$, if $F(t) = G(t)$ for all $t \leq z$, $h_1(z) = h_2(z)$.*

To show that the distribution $F_\epsilon(\cdot)$ is identified, we rely on an Restricted Stochastic Dominance ordering (RSD) assumption introduced in Luo (2018). That is, $f_\epsilon(\cdot)$ belongs to the set of density functions in which any two functions can be ranked using RSD ordering (Assumption 7).[11]

Assumption 7. *(RSD) $g(\cdot)$ dominates $f(\cdot)$ in the restricted sense if there exists an $y > 0$ such that: (a) $f(\epsilon) \geq g(\epsilon)$ for all $\epsilon \leq y$, and (b) $\exists y_* \leq y, f(y_*) > g(y_*)$.*

Lemma 9. *Under Assumption 7, $h_1(\cdot) = h_2(\cdot)$ implies that $F(\cdot) = G(\cdot)$ and $\overline{\epsilon}_G = \overline{\epsilon}_F$.*

Lemma 9 implies that $F_\epsilon(\cdot)$ is identified once the ratio $\psi_{n-1}(\cdot)/\psi_n(\cdot)$ is identified. Consequently, the distribution of X^*, $F_{X^*}(\cdot)$ is identified. With the identified distribution $F_{X^*}(\cdot)$ and $F_\epsilon(\cdot)$, following the arguments in Section 2, we can identify the distributions of UH and private value.[12]

Identification with any two consecutive order statistics

We further extend the above argument to the scenario of any two consecutive order statistics, $X_{r-1:n}$ and $X_{r:n}$. Denote the distribution of X as $\widetilde{F}(\cdot)$.

[11] It is worthnoting that the counterexample of Rossberg (1972) satisfies this condition and thus our characteristic function approach identifies the parent distribution. In fact, $f(x) - e^{-x} = e^{-x}\pi^{-2}[1 - \cos 2\pi x - 2\pi \sin 2\pi x]$, which approaches 0 when $x \downarrow 0$. Moreover, the slope of the difference is negative at $x = 0$. Therefore, there exists an $x_\dagger$ such that $f(x) < e^{-x}, \forall x \in (0, x_\dagger)$.

[12] Lemma 8 and Lemma 9 extend to the case with a maximum order statistic $X_{m:m}$ and any other order statistic $X_{r:n}$, where $m > r$ or $m = r < n$.

First, note that for any (n, r), we can observe the conditional density functions of one order statistic given a consecutive one and they satisfy the following conditions:

$$\tilde{f}_{r-1|r:n}(x_{r-1}|x_r = \overline{x}) = \tilde{f}_{r-1:r-1}(x_{r-1}),$$
$$\tilde{f}_{r|r-1:n}(x_r|x_{r-1} = \underline{x}) = \tilde{f}_{1:n-r+1}(x_r),$$

which identify $\widetilde{f}_{r-1:r-1}(\cdot)$ and $\widetilde{f}_{1:n-r+1}(\cdot)$, respectively.

Second, following Lemma 7, the corresponding characteristic function are $\widetilde{\psi}_{r-1:r-1}(t) = \psi_{X^*}(t)\psi_{r-1:r-1}(t)$ and $\widetilde{\psi}_{1:n-r+1}(t) = \psi_{X^*}(t)\psi_{1:n-r+1}(t)$, respectively. $\psi_{r-1:r-1}(\cdot)$ and $\psi_{1:n-r+1}(\cdot)$ represent the characteristic functions of two order statistics $\epsilon_{r-1:r-1}$ and $\epsilon_{1:n-r+1}$, respectively. Therefore, we can identify the ratio

$$\frac{\psi_{r-1:r-1}(t)}{\psi_{1:n-r+1}(t)} = \frac{\widetilde{\psi}_{r-1:r-1}(t)}{\widetilde{\psi}_{1:n-r+1}(t)}. \tag{29}$$

Similar to above, the identification problem can be restated as

Problem 2. *X_i's and Y_i's are i.i.d. random variables with distribution function $F(\cdot)$ on support $[0, \bar{\epsilon}_F]$ and $G(\cdot)$ on support $[0, \bar{\epsilon}_G]$, respectively. Let $Z_1 = X_{r-1:r-1} + Y_{1:n-r+1}$ and $Z_2 = X_{1:n-r+1} + Y_{r-1:r-1}$. Identification in our context is equivalent to the situation that Z_1 and Z_2 have the same distribution implies that $F(\cdot) = G(\cdot)$ and $\bar{\epsilon}_G = \bar{\epsilon}_F$.*

The density functions of the minimum $X_{1:n-r+1}$ and the maximum $X_{r-1:r-1}$ are

$$f_{1:n-r+1}(x) = (n - r + 1)[1 - F(x)]^{n-r} f(x),$$
$$f_{r-1:r-1}(x) = (r - 1)F^{r-2}(x)f(x),$$

respectively. Therefore, the density functions of Z_1 and Z_2 are

$$h_1(z) = \int_{-\infty}^{+\infty} [(r-1)F^{r-2}(z-y)f(z-y) \cdot (n-r+1)[1-G(y)]^{n-r} g(y)dy,$$
$$h_2(z) = \int_{-\infty}^{+\infty} [(r-1)G^{r-2}(y)g(y) \cdot (n-r+1)[1-F(z-y)]^{n-r} f(z-y)dy,$$

respectively. This implies that

$$\begin{aligned} &(h_1(z) - h_2(z))/[(r-1)(n-r+1)] \\ = &\int_{-\infty}^{+\infty} f(z-y)g(y)\Big\{F^{r-2}(z-y)[1-G(y)]^{n-r} - G^{r-2}(y)[1-F(z-y)]^{n-r}\Big\}dy, \end{aligned}$$

which is increasing with respect to f and F when the quantity in the curly brackets is positive. The rest follows similar arguments as Lemma 8 and Lemma 9.

Theorem 4. *If Assumptions 5 - 7 are satisfied and the competition is known, the distribution of the auction-level UH and the idiosyncratic part $\Phi_{V^*}(\cdot)$ and $\Phi_u(\cdot)$ in IPV auctions are identified using two consecutive order statistics of the bids.*

5 Empirical Application: USFS timber second-price auctions

In this section, we apply our identification results to an empirical analysis of the United States Forest Service (USFS) timber auctions. Other studies of these auctions include Baldwin, Marshall, and Richard (1997), Haile (2001), and Haile and Tamer (2003). Related to our setting, Aradillas-López, Gandhi, and Quint (2013) consider a more general model and apply a bound approach to the English auctions, which is the first to study these auctions allowing for correlated values. This paper focuses on independent private values with UH and achieves point identification and estimation.

5.1 Institution Background and Data

Timber is one of the most important outputs from National Forests managed by USFS. Auctions are used to allocate the right to harvest from tracts. These tracts are highly heterogeneous. USFS publishes a cruise report that provides information about the tract being auctioned in narrative form. It includes one or more maps showing features of the tract, a discussion of all harvesting cost expected (such as transportation), and conditions of sales (such as planned cutting methods, protective measures for controlling erosion, and contract provisions) et al.

The government uses three methods of sale: English auction, sealed-bid first price auction, and noncompetitive method. We will focus on Enligh auctions, which are conducted in two rounds: a sealed bid auction is followed by oral bidding unless only one sealed bid was received.

Data Sample

In particular, we analyze the ascending English auction data from 1982 to 1990, which are constructed from the data available on Professor Philip A. Haile's website.[13] Following the seminal paper Haile, Hong, and Shum (2003), we consider only scaled sales in Forest Service Regions 1 and 5 to minimize the significance of subcontracting/resale and thus common value. Moreover, we drop salvage sales and sales that are set aside for small businesses. In summary, we focus on auctions that are most likely to satisfy the independent private value assumption. In total, we have 1207 ascending English auctions.

Table 1 provides some summary statistics on auction-specific covariates: winning bid, size of tract (in acres), estimated volume of timber (in MBF), appraisal value (per MBF), estimated selling value (per MBF), estimated harvesting cost (per MBF), estimated manufacturing cost (per MBF), and species concentration index (HHI). We

[13] http://www.econ.yale.edu/ pah29/timber/timber.htm

construct these covariates following Liu and Luo (2017). All dollar values are nominal and all volume values are in thousand board feet (MBF) of timber.

Table 1: Summary Statistics

Variable	Mean	Std. Dev.	Min	Max
winning bid	758182.6	1943825	3615.36	6.15e+07
acres	1079.69	1176.91	14.00	6400.00
vol_sum	6181.44	4870.72	128.00	22100.00
AppValue_avg	34.04	32.51	0.50	192.62
SellValue_avg	347.47	58.97	172.29	523.51
LogCost_avg	132.52	28.06	50.84	276.69
MfgCost_avg	163.96	22.88	87.90	233.54
HHI	0.56	0.25	0.11	1.00

Before applying our methodology, we implement the Haile, Hong, and Shum (2003) method to homogenize the bids. Table 2 displays the regression results. Regressions (1)-(3) use all bids but different control variables. All estimated coefficients have the expected signs. Regression (3) includes all control variables as well as year dummies. As argued above, it is more suitable to model top bids as order statistics. See also Aradillas-López, Gandhi, and Quint (2013). Moreover, multiple top bids (and their "homogenized" counterparts) from the same auction are correlated by definition, leading to biases in the estimated coefficients. Therefore, we use only the winning bid to control for auction-specific covariates. Regression (4) is the same as (3) but using only winning bids. All estimated coefficients have the expected signs. Moreover, the regression with only winning bids shows a much better fit than the other ones. Lastly, we calculate homogenized bids as the exponential of the differences between the logarithm of the original total bids and the demeaned fitted values of Regression (4). In the following analysis, we only use the top three "homogenized" bids.

Table 2: Regression Results

VARIABLES	(1) OLS	(2) Fixed Effects	(3) Fixed Effects	(4) Fixed Effects
log_acres		0.0115	0.00715	-0.00400
		(0.0110)	(0.0110)	(0.0220)
log_vol_sum	1.060***	1.035***	1.032***	1.090***
	(0.0111)	(0.0148)	(0.0150)	(0.0229)
log_AppValue_avg	0.613***	0.388***	0.376***	0.167***
	(0.0139)	(0.0213)	(0.0213)	(0.0328)
log_SellValue_avg		1.412***	1.516***	1.791***
		(0.0930)	(0.0978)	(0.136)
log_LogCost_avg		-0.666***	-0.743***	-0.787***
		(0.0532)	(0.0540)	(0.0822)
log_MfgCost_avg		-0.0534	-0.254**	-0.189
		(0.107)	(0.112)	(0.152)
log_HHI		-0.0264	-0.00470	0.00336
		(0.0221)	(0.0224)	(0.0379)
Constant	1.447***	-2.719***	-1.885***	-1.891***
	(0.0960)	(0.521)	(0.528)	(0.727)
Observations	6,121	6,121	6,121	1,207
R-squared	0.785	0.801	0.804	0.857
Group FE		YES	YES	YES
Year FE			YES	YES

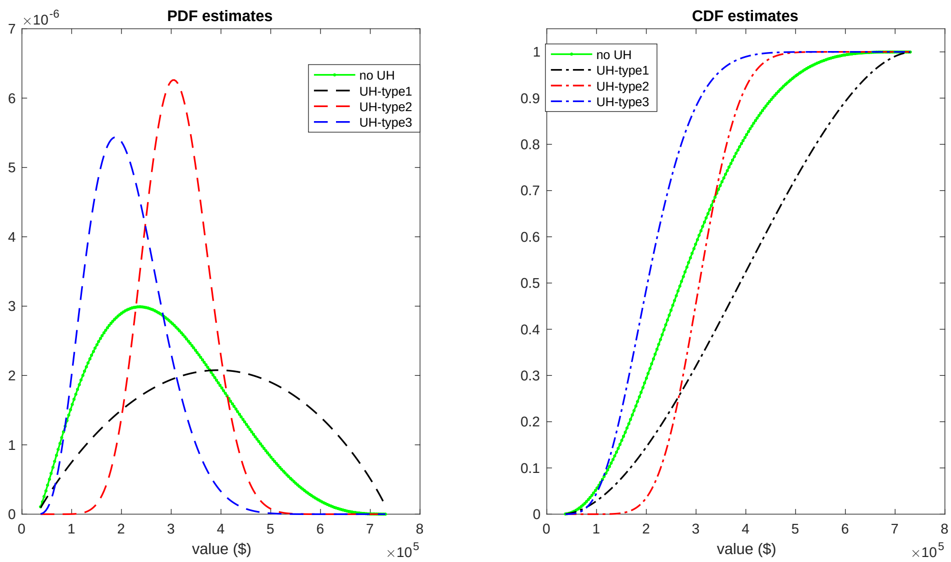
Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

5.2 Empirical Results

To make sure the bids are informative for the value, we exclude the bids that are very closed to the estimated prices by USFS. Furthermore, the highest bid and the second highest bids are quite close to each other, both revealing information on the second highest value among all the bidders. Thus, we use the transaction price as the 2nd highest value among all the bidders and exclude the 2nd highest bid from the data to avoid redundant information for the 2nd highest values. We then use the 3rd and 4rd bids as the proxies to the 3rd and 4th highest values among all the bidders. Note that from the identification, we need to have at least three informative bids, which means at least four bids from the data. We have 496 auctions.

We use beta distributions to approximate the underlying type-specific component distributions and estimate the type-specific beta coefficients and the corresponding type probability using a maximum likelihood estimator.[14] We estimate the underlying component distribution and the type probability for the number of types being three, i.e., 'low', 'middle', and 'high'. The probability for the three types is 0.24, 0.37, and 0.39, respectively. Each type accounts for a nontrivial portion of the sample, confirming the importance of allowing for UH in our analysis. We present the estimated type-specific pdf and cdf for values in figure 5 for the case allowing for UH and the case without UH. As noted in Krasnokutskaya (2011), ignoring UH leads to a biased estimate of the value distribution with an overestimated dispersion.

Figure 5: Estimation of PDF and CDF: no UH vs UH



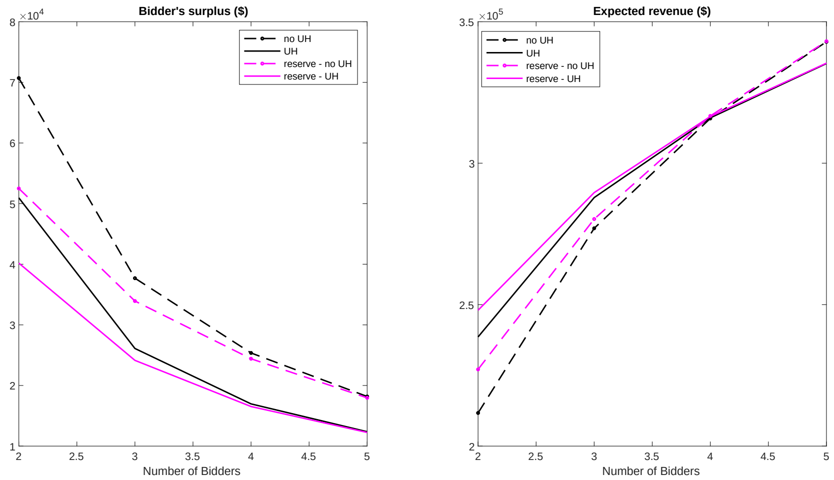
[14] We have also conducted Monte Carlo experiment with a semi-parametric estimator using sieve approximation. The estimator performs quite well with a moderate number of sample size. The results are available upon request.

Bidder Surplus and Expected Revenue

With the estimates of value distributions, we compute the bidder ex ante expected surplus and auctioneer's expected revenue to provide a comparison between the cases of no UH and UH and present the results in figure 6. The solid (dashed) lines represent the results with (without) UH. Black (purple) lines represent the results without (with) the reserve price. The black lines in the left-hand figure suggests that we overestimate bidder expected surplus if UH is ignored. Note that in auctions with less disperse values, a likely winner who has a high value expects the other bidders to have similar high values with a higher probability. Therefore, bidders tend to be more aggressive. When attributing wrongly the variation in UH to bidders' values, we overestimate bidder heterogeneity and underestimate bidders' aggressiveness. As a result, we overestimate bidder expected surplus. The black lines in the right-hand figure suggests that ignoring UH leads to underestimation of expected revenue when the number of bidders is small and overestimation otherwise. Consider the number of bidders $n = 2$. The bidder's surplus is around \$ 70,708 and \$ 50,957 for model estimates without and with UH, respectively; the seller's revenue is around \$ 211,717 and \$ 238,590 for model estimates without and with UH, respectively.

Next, we examine the effects of an optimal reserve price on bidder surplus and seller revenue. Reserve price is the minimum amount that the seller will accept as the winning bid. In a second-price auction, a reserve price does not change bidders' optimal bidding behaviors. Note that the black line and the purple line converge when the number of bidders increase. In other words, an optimal reserve price makes less difference in larger auctions. Nevertheless, most auctions have very few bidders and thus the choice of a reserve price is empirically relevant. Consider again the number of bidders $n = 2$. Using the estimated model with UH, the optimal reserve prices are \$ 321,100, \$ 239,400, and \$ 155,100, respectively, which corresponds to a probability of no bidding of 0.36, 0.13, and 0.24, respectively. The mode of the type-specific value distribution is

Figure 6: Bidder surplus and expected revenue



\$ 394,396, \$ 304,871, and \$ 187,971, respectively. The optimal reserve prices lead to an ex-ante expected bidder surplus of \$ 40,187 and seller revenue of \$ 248,025. Using the estimated model without UH, the optimal reserve price is \$ 219,800, which corresponds to a probability of no bidding of 0.35. The mode of the value distribution is \$ 236,786. This optimal price leads to an ex-ante expected bidder surplus of \$ 52,498 and seller revenue of \$ 227,113. Therefore, ignoring UH leads to substantial bias in the optimal reserve price policy and welfare estimates.

6 Conclusion

Auction data often fail to record all bids or all relevant auction-specific characteristics that shift bidder values. Instead, they contain only order statistics of bids and suffer from unobserved heterogeneity. In this paper, we present a set of new identification results for auction models with UH using order statistics. In particular, we show that, despite being dependent, the same number of order statistics as that of the independent measurements are sufficient to achieve similar identification results for both discrete and continuous UH. Our results rely on consecutive order statistics or support variations.

Note that consecutiveness plays a key role in the identification of nonseparable finite unobserved heterogeneity without support variations. A possible extension to the identification results is to explore the markov property of order statistics and investigate the possible identification results using any four order statistics. The identification strategy is similar to some recent results on identification of dynamic models with unobserved state variables (Hu and Shum (2012) and Luo, Xiao, and Xiao (2018)). Specifically, first, the joint distribution of four order statistics can be represented as a multiplicatively separable mixture structure by the Markov property, with which an eigenvalue decomposition argument identifies a key matrix that governs the finite mixture structure. Second, we apply the matrix identified in the first step to identify the component distributions in the lower portion of the support and then in the upper portion using joint distribution of only three order statistics. Moreover, the same identification argument applies to the scenario of nonseparable continuous unobserved heterogeneity.

7 Appendix

7.1 Proof of Theorem 1

This subsection provides derivation of all the necessary conditions for Theorem 1.

Proof of Lemma 1: identification in the "low" support The full rank assumption leads to the following main equation for identification:

$$\boldsymbol{J}_{l,m_1,h}\boldsymbol{J}_{l,m_2,h}^{-1} = \boldsymbol{L}\boldsymbol{D}_{m_1/m_2}\boldsymbol{L}^{-1},$$

where $\boldsymbol{D}_{m_1/m_2}$ is a diagonal matrix with the kth diagonal element as the ratio of the probability that type k occurs in two middle intervals m_1 and m_2, i.e., $\frac{\int_{x\in m_1} f^k(x)dx}{\int_{x\in m_2} f^k(x)dx}$. With the assumption of distinctive eigenvalues, we can identify the probability matrix involved the "low" support $\boldsymbol{L}$ as the eigenvector matrix through an eigenvalue and eigenvector decomposition.

Note that the components identified from the eigenvalue-eigenvector decomposition are not the probability matrix $\boldsymbol{L}$ itself, but the matrix $\boldsymbol{L}$ with each column multiplied by an unknown constant. That is, denote the eigenvector matrix obtained from the decomposition $\tilde{\boldsymbol{L}}$, we have $\boldsymbol{L} = \tilde{\boldsymbol{L}}\boldsymbol{\lambda}_l$, where $\boldsymbol{\lambda}_l \equiv diag[\lambda_l^1, ..., \lambda_l^K]$ is the scale matrix. Basing on the matrix $\tilde{\boldsymbol{L}}$ from the decomposition, we can compute the corresponding density matrix through equation (12)

$$\begin{aligned}
\tilde{\boldsymbol{L}}_x &\equiv J_{x,m_1,h}J_{l,m_1,h}^{-1}\tilde{\boldsymbol{L}} \\
&= J_{x,m_1,h}J_{l,m_1,h}^{-1}\tilde{\boldsymbol{L}}\boldsymbol{\lambda}_l\boldsymbol{\lambda}_l^{-1} \\
&= J_{x,m_1,h}J_{l,m_1,h}^{-1}\boldsymbol{L}\boldsymbol{\lambda}_l^{-1} \\
&= \boldsymbol{L}_x\boldsymbol{\lambda}_l^{-1} \\
\text{Thus,} \qquad \boldsymbol{L}_x &= \tilde{\boldsymbol{L}}_x\boldsymbol{\lambda}_l. \qquad (30)
\end{aligned}$$

As a result, the type-specific density for order statistics $X^k_{r-2:r-2}$ in the "low" support is identified up to the same scale as the probability matrix $\boldsymbol{L}$, i.e., $f^k_{r-2:r-2}(x) = \lambda^k_l \tilde{f}^k_{r-2:r-2}(x)$, where $\tilde{f}^k_{r-2:r-2}(x)$ is the $k$the element in vector $\tilde{\boldsymbol{L}}_x$.

We then move forward to show that the type-specific density $f^k(x)$ is also identified up to scales in the follows.

$$\begin{aligned}
f^k_{r-2:r-2}(x) &= (r-2)[F^k(x)]^{r-3} f^k(x) \\
\leftrightarrow \quad \int_{\underline{x}}^{x} f^k_{r-2:r-2}(v)dv &= (r-2)\int_{\underline{x}}^{x} [F^k(v)]^{r-2} f^k(v)dv \\
\leftrightarrow \quad \int_{\underline{x}}^{x} f^k_{r-1:r-1}(v)dv &= [F^k(x)]^{r-2} \\
\leftrightarrow F^k(x) &= \Big[\int_{\underline{x}}^{x} f^k_{r-2:r-2}(v)dv\Big]^{\frac{1}{r-2}}
\end{aligned}$$

Thus, we can derive the type-specific density for $x \in l$ as in the follows.

$$\begin{aligned}
f^k(x) &= \frac{1}{r-2}\Big[\int_{\underline{x}}^{x} f^k_{r-2:r-2}(v)dv\Big]^{\frac{1}{r-2}-1} f^k_{r-2:r-2}(x) \\
&= \frac{1}{r-2}\Big[\int_{\underline{x}}^{x} \lambda^k_l \tilde{f}^k_{r-2:r-2}(v)dv\Big]^{\frac{1}{r-2}-1} \lambda^k_l \tilde{f}^k_{r-2:r-2}(x) \\
&= \left(\lambda^k_l\right)^{\frac{1}{r-2}} \frac{1}{r-2}\Big[\int_{\underline{x}}^{x} \tilde{f}^k_{r-2:r-2}(v)dv\Big]^{\frac{1}{r-2}-1} \tilde{f}^k_{r-2:r-2}(v) \\
&\equiv \left(\lambda^k_l\right)^{\frac{1}{r-2}} \tilde{f}^k_l(x),
\end{aligned} \tag{31}$$

where $\tilde{f}^k_l(x) \equiv \frac{1}{r-2}\Big[\int_{\underline{x}}^{x} \tilde{f}^k_{r-2:r-2}(v)dv\Big]^{\frac{1}{r-2}-1} \tilde{f}^k_{r-2:r-2}(x)$ represents the type-specific density computed using the eigenvector matrix directly from the decomposition, which is known. The type-specific density in the "low" support is identified up to scale of $\left(\lambda^k_l\right)^{\frac{1}{r-2}}$.

Proof of Lemma 2: (identification in the "high" support) In what follows, we identify the type-specific density function $f^k(x)$ in the "high" support.

First of all, we can identify the probability matrix $\boldsymbol{H}$ up to scale from the joint distribution equation (8) as the following closed-form expression:

$$\begin{aligned}
\boldsymbol{H}^T &= \left[\frac{n!}{(r-2)!(n-r+1)!}\boldsymbol{L}\boldsymbol{D}_{m_1}\boldsymbol{D}_p\right]^{-1}\boldsymbol{J}_{l,m_1,h} \\
&= \boldsymbol{D}_p^{-1}\boldsymbol{D}_{m_1}^{-1}\left[\frac{n!}{(r-2)!(n-r+1)!}\tilde{\boldsymbol{L}}\boldsymbol{\lambda}_l\right]^{-1}\boldsymbol{J}_{l,m_1,h} \\
&= \boldsymbol{\lambda}_l^{-1}\boldsymbol{D}_p^{-1}\boldsymbol{D}_{m_1}^{-1}\left[\frac{n!}{(r-2)!(n-r+1)!}\tilde{\boldsymbol{L}}\right]^{-1}\boldsymbol{J}_{l,m_1,h} \\
&\equiv \boldsymbol{\lambda}_h\tilde{\boldsymbol{H}}^T, \qquad (32)
\end{aligned}$$

where $\boldsymbol{\lambda}_h \equiv \boldsymbol{\lambda}_l^{-1}\boldsymbol{D}_p^{-1}\boldsymbol{D}_{m_1}^{-1} \equiv diag\{\lambda_h^1, ..., \lambda_h^K\}$ is a diagonal matrix captures the unknown scales, and $\tilde{\boldsymbol{H}}^T$ represent the component that can be computed using result from the decomposition. Since $\boldsymbol{L}$ is identified up to scales due to the decomposition, and both $\boldsymbol{D}_{m_1}$ and $\boldsymbol{D}_p$ are diagonal matrices, $\boldsymbol{H}$ can be identified up to scales, but the scales are different from the scales in $\boldsymbol{L}$.

We then can follow the same logic to identify the type-specific density for the "high" support.

$$\begin{aligned}
\boldsymbol{H}_x^T &= \boldsymbol{\lambda}_h^{-1}\boldsymbol{D}_p^{-1}\boldsymbol{D}_{m_1}^{-1}\left[\frac{n!}{(r-2)!(n-r+1)!}\tilde{\boldsymbol{L}}_x\right]^{-1}\boldsymbol{J}_{l,m_1,h} \\
&\equiv \boldsymbol{\lambda}_h\tilde{\boldsymbol{H}}_x^T, \qquad (33)
\end{aligned}$$

where $\boldsymbol{H}_x^T$ and $\tilde{\boldsymbol{H}}_x^T$ are defined similar to $\boldsymbol{H}^T$ and $\tilde{\boldsymbol{H}}^T$, respectively. This indicates that the type-specific density in the "high" support can be identified up to scale, i.e., $f_{1:n-r+1}^k(x) = \lambda_h^k \tilde{f}_{1:n-r+1}^k(x)$, where $\tilde{f}_{1:n-r+1}^k(x)$ is the kth component in the vector $\tilde{\boldsymbol{H}}_x^T$, which can be computed using results from the decomposition.

We then use the connection of the type-specific density for order statistics and the type-specific density function to recover the type-specific density. In particular, for any value in the "high" support, i.e., $x \in h$, by definition of the order statistics, we have

$$\begin{aligned}
f^k_{1:n-r+1}(x) &= (n-r+1)[1-F^k(x)]^{n-r} f^k(x) \\
\leftrightarrow \quad \int_x^{\bar{x}} f^k_{1:n-r+1}(x)dx &= (n-r+1)\int_x^{\bar{x}} [1-F^k(x)]^{n-r} f^k(x)dx \\
\leftrightarrow \quad \int_x^{\bar{x}} f^k_{1:n-r+1}(x)dx &= [1-F^k(x)]^{n-r+1} \\
\leftrightarrow F^k(x) &= 1-\Big[\int_x^{\bar{x}} f^k_{1:n-r+1}(x)dx\Big]^{\frac{1}{n-r+1}}
\end{aligned}$$

Thus, we can link the type-specific density to the type-specific density of the order statistics in the follows, $\forall x \in h$:

$$\begin{aligned}
f^k(x) &= \frac{1}{n-r+1}\Big[\int_x^{\bar{x}} f^k_{1:n-r+1}(v)dv\Big]^{\frac{1}{n-r+1}-1} f^k_{1:n-r+1}(x) \\
&= \frac{1}{n-r+1}\Big[\int_x^{\bar{x}} \lambda^k_h \tilde{f}^k_{1:n-r+1}(v)dv\Big]^{\frac{1}{n-r+1}-1} \lambda^k_h \tilde{f}^k_{1:n-r+1}(x) \\
&= \left(\lambda^k_h\right)^{\frac{1}{n-r+1}} \Big[\int_x^{\bar{x}} \tilde{f}^k_{1:n-r+1}(v)dv\Big]^{\frac{1}{n-r+1}-1} \tilde{f}^k_{1:n-r+1}(x) \\
&\equiv \left(\lambda^k_h\right)^{\frac{1}{n-r+1}} \tilde{\boldsymbol{f}}^k_h(x),
\end{aligned} \tag{34}$$

where $\tilde{\boldsymbol{f}}^k_h(x) \equiv \Big[\int_x^{\bar{x}} \tilde{f}^k_{1:n-r+1}(v)dv\Big]^{\frac{1}{n-r+1}-1} \tilde{f}^k_{1:n-r+1}(x)$, which can be computed directly. The type-specific density in the "high" support is identified up to scale of $\left(\lambda^k_h\right)^{\frac{1}{n-r+1}}$.

Proof of Lemma 3: identification in the "middle" support We again rely on the joint distribution relation similar in equation (8), leading to the following matrix representation:

$$\boldsymbol{J}_{l,x,h} = \frac{n!}{(r-2)!(n-r+1)!}\boldsymbol{L}\boldsymbol{D}_x\boldsymbol{D}_p\boldsymbol{H}^T, \tag{35}$$

where $\boldsymbol{J}_{l,x,h}$ and $\boldsymbol{D}_x$ are the counterparts of matrix $\boldsymbol{J}_{l,m_1,h}$ and $\boldsymbol{D}$ with replacing the interval to a particular value of $x_{r-1} = x \in m$, respectively. Note that $\boldsymbol{L}$ and $\boldsymbol{H}$ are both identified up to scales and $\boldsymbol{D}_p$ is a diagonal matrix. As a result, $\boldsymbol{D}_x$ can be identified up to scales, with the kth diagonal element being the type-specific density $f^k(x)$ where $x \in m$. Specifically, we can represent the diagonal matrix

$$
\begin{aligned}
\boldsymbol{D}_x &= \left[\frac{n!}{(r-2)!(n-r+1)!}\right]^{-1} \boldsymbol{L}^{-1} \boldsymbol{J}_{l,x,h} \left[\boldsymbol{H}^T\right]^{-1} \boldsymbol{D}_p^{-1} \\
&= \left[\frac{n!}{(r-2)!(n-r+1)!}\right]^{-1} \left[\tilde{\boldsymbol{L}}\boldsymbol{\lambda}_l\right]^{-1} \boldsymbol{J}_{l,x,h} \left[\boldsymbol{\lambda}_h \tilde{\boldsymbol{H}}_x^T\right]^{-1} \boldsymbol{D}_p^{-1} \\
&= \boldsymbol{\lambda}_l^{-1} \left[\frac{n!}{(r-2)!(n-r+1)!}\right]^{-1} \left[\tilde{\boldsymbol{L}}\right]^{-1} \boldsymbol{J}_{l,x,h} \left[\tilde{\boldsymbol{H}}_x^T\right]^{-1} \boldsymbol{\lambda}_h^{-1} \boldsymbol{D}_p^{-1} \\
&\equiv \boldsymbol{\lambda}_l^{-1} \tilde{\boldsymbol{D}}_x \boldsymbol{\lambda}_h^{-1} \boldsymbol{D}_p^{-1} \\
&\equiv \boldsymbol{\lambda}_m \tilde{\boldsymbol{D}}_x, \qquad (36)
\end{aligned}
$$

where $\tilde{\boldsymbol{D}}_x \equiv \left[\frac{n!}{(r-2)!(n-r+1)!}\right]^{-1} \left[\tilde{\boldsymbol{L}}\right]^{-1} \boldsymbol{J}_{l,x,h} \left[\tilde{\boldsymbol{H}}_x^T\right]^{-1}$ is an diagonal matrix, and $\boldsymbol{\lambda}_m \equiv \boldsymbol{\lambda}_l^{-1}\boldsymbol{\lambda}_h^{-1}\boldsymbol{D}_p^{-1} = \boldsymbol{\lambda}_l^{-1}\left[\boldsymbol{\lambda}_l^{-1}\boldsymbol{D}_p^{-1}\boldsymbol{D}_{m_1}^{-1}\right]^{-1}\boldsymbol{D}_p^{-1} = D_{m1}$ is the scale matrix. As a result, we can identify the type-specific density for the "middle" support up to scales, i.e., $f^k(x) = \lambda_m^k \tilde{f}_m^k(x), \forall x \in m$, where $\tilde{f}_m^k(x)$ is the $k$the diagonal element in the matrix $\tilde{\boldsymbol{D}}_x$, which can be directly computed.

Proof of Theorem 1 To pin down the three scale parameters for each type, we first summarize the identified type-specific density in the followings:

$$
f^k(x) = \begin{cases}
\left(\lambda_l^k\right)^{\frac{1}{r-2}} \tilde{f}_l^k(x), & \text{if } x \in l = [\underline{x}, c_1], \\
\lambda_m^k \tilde{f}_m^k(x), & \text{if } x \in m = [c_1, c_2], \\
\left(\lambda_h^k\right)^{\frac{1}{n-r+1}} \tilde{\boldsymbol{f}}_h^k(x), & \text{if } x \in h = [c_2, \bar{x}],
\end{cases}
$$

Note that we identify the type-specific density $f^k(x)$ for the "lower", "middle", and "high" portion up to different scales, which requires three conditions to pin down the scales exactly. First, the three density functions identified separately should be the same at the cutoff points c_1 and c_2 for the two densities overlapping in the cutoff points.

$$\begin{aligned}\left(\lambda_l^k\right)^{\frac{1}{r-2}} \tilde{f}_l^k(c_1) &= \lambda_m^k \tilde{f}_m^k(c_1) \\ \lambda_m^k \tilde{f}_m^k(c_2) &= \left(\lambda_h^k\right)^{\frac{1}{n-r+1}} \boldsymbol{\tilde{f}}_h^k(c_2)\end{aligned}$$

Moreover, the cumulative sum across all three intervals should equal to 1, which provides the third restriction on the scales.

$$\begin{aligned}&\int_{x\in l} \left(\lambda_l^k\right)^{\frac{1}{r-2}} \tilde{f}_l^k(x)dx + \int_{x\in m} \lambda_m^k \tilde{f}_m^k(x)dx + \int_{x\in h} \left(\lambda_h^k\right)^{\frac{1}{n-r+1}} \boldsymbol{\tilde{f}}_h^k(x)dx = 1 \\ &\left(\lambda_l^k\right)^{\frac{1}{r-2}} \int_{x\in l} \tilde{f}_l^k(x)dx + \lambda_m^k \int_{x\in m} \tilde{f}_m^k(x)dx + \left(\lambda_h^k\right)^{\frac{1}{n-r+1}} \int_{x\in h} \boldsymbol{\tilde{f}}_h^k(x)dx = 1\end{aligned}$$

The above three conditions lead to the following linear equation system, which only involves the scales as unknowns.

$$\begin{bmatrix} \tilde{f}_l^k(c_1) & -\tilde{f}_m^k(c_1) & 0 \\ 0 & \tilde{f}_m^k(c_2) & -\boldsymbol{\tilde{f}}_h^k(c_2) \\ \int_{x\in l} \tilde{f}_l^k(x)dx & \int_{x\in m} \tilde{f}_m^k(x)dx & \int_{x\in h} \tilde{f}_h^k(x)dx \end{bmatrix} \begin{bmatrix} \left(\lambda_l^k\right)^{\frac{1}{r-2}} \\ \lambda_m^k \\ \left(\lambda_h^k\right)^{\frac{1}{n-r+1}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Pining down the scales requires that the matrix

$$M \equiv \begin{bmatrix} \tilde{f}_l^k(c_1) & -\tilde{f}_m^k(c_1) & 0 \\ 0 & \tilde{f}_m^k(c_2) & -\boldsymbol{\tilde{f}}_h^k(c_2) \\ \int_{x\in l} \tilde{f}_l^k(x)dx & \int_{x\in m} \tilde{f}_m^k(x)dx & \int_{x\in h} \tilde{f}_h^k(x)dx \end{bmatrix}$$

to be full rank. Note that the determinant of the matrix M can be represented in the following:

$$det(M) = \tilde{f}_l^k(c_1)\tilde{f}_m^k(c_2) \int_{x\in h} \tilde{f}_h^k(x)dx + \tilde{f}_l^k(c_1)\boldsymbol{\tilde{f}}_h^k(c_2) \int_{x\in m} \tilde{f}_m^k(x)dx + \int_{x\in l} \tilde{f}_l^k(x)dx \tilde{f}_m^k(c_1)\boldsymbol{\tilde{f}}_h^k(c_2)$$

since all components are positive, $det(M) > 0$. Thus, matrix M is full rank and the scales can be pined down in the following closed-form expression:

$$\begin{bmatrix} \left(\lambda_l^k\right)^{\frac{1}{r-2}} \\ \lambda_m^k \\ \left(\lambda_h^k\right)^{\frac{1}{n-r+1}} \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Once the scales being pinned down, we can identify the type-specific weight p_k.

Proof of Lemma 8

$$\begin{aligned} h_1(z) &= n(n-1)\int_0^z [G^{n-1}(z-y)g(z-y)G^{n-2}(y)g(y)]dy \\ &= -n(n-1)\int_z^0 [G^{n-1}(x)g(x)G^{n-2}(z-x)g(z-x)]dx \\ &= n(n-1)\int_0^z [G^{n-1}(x)g(x)G^{n-2}(z-x)g(z-x)]dx \\ &= h_2(z), \end{aligned}$$

where the second equality follows from a change of variable $z - y = x$.

Proof of Lemma 9 Proof by contradiction: Suppose that $F(\cdot) \neq G(\cdot)$, we now show that $h_1(z) \neq h_2(z)$ for some $z \in [0, \bar{\epsilon}]$. Since $F(\cdot) \neq G(\cdot)$, without loss of generality, there exists an $y_\dagger \in [0, \bar{\epsilon})$ and $\epsilon > 0$ such that $y_\dagger + \epsilon \in (0, \bar{\epsilon})$, $f(y) = g(y)$ for all $y \leq y_\dagger$ and $f(y) < g(y)$ for $y \in (y_\dagger, y_\dagger + \epsilon]$. Thus, $F(y) < G(y)$ for $y \in (y_\dagger, y_\dagger + \epsilon]$.

First, note that for any $z \in [0, \bar{\epsilon}]$, $F(z-0) > G(0) = 0, F(z-z) = F(0) = 0 < G(z)$, and $F(z-y)$ is decreasing and $G(y)$ is increasing in y. By the intermediate value theorem, there exists $z_\dagger \in (0, z)$ such that $F(z - z_\dagger) = G(z_\dagger)$. Moreover, $F(z - y) - G(y) > 0$ for all $y \in [0, z_\dagger)$ and $F(z - y) - G(y) \leq 0$ for all $y \in [z_\dagger, z]$.

Second, suppose $z = y_\dagger + \epsilon$, we have

$$
\begin{aligned}
&[h_1(z) - h_2(z)]/[n(n-1)] \\
=&\int_0^z \{F^{n-2}(z-y)f(z-y)G^{n-2}(y)g(y) \cdot [F(z-y) - G(y)]\}dy \\
=&\int_{z_\dagger}^z \{F^{n-2}(z-y)f(z-y)G^{n-2}(y)g(y) \cdot [F(z-y) - G(y)]\}dy \\
&\quad + \int_0^{z_\dagger} \{F^{n-2}(z-y)f(z-y)G^{n-2}(y)g(y) \cdot [F(z-y) - G(y)]\}dy \\
<&0 + \int_0^{z_\dagger} \{G^{n-2}(z-y)g(z-y)G^{n-2}(y)g(y) \cdot [G(z-y) - G(y)]\}dy \\
=&0,
\end{aligned}
$$

where the inequality follows from that $F(z-y) - G(y) < 0$ when $y \in (z_\dagger, z)$, and that $f(z-y) \le g(z-y)$ and $F(z-y) \le G(z-y)$ for all $y \in [0, z_\dagger) \subset [0, z]$. The last equality follows from Lemma 8.

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