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Policy Distortions and Aggregate Productivity with  
Endogenous Establishment-Level Productivity

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# Policy Distortions and Aggregate Productivity with Endogenous Establishment-Level Productivity\*

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## ABSTRACT

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What accounts for differences in output per capita and total factor productivity (TFP) across countries? Empirical evidence points to resource misallocation across heterogeneous production units as an important factor. We study resource misallocation in a model where establishment-level productivity is endogenous and responds to the same policy distortions that create misallocation. In this framework, policy distortions not only misallocate resources across a given set of productive units (static effect), but also create disincentives for productivity improvement (dynamic effect) thereby affecting the productivity distribution and further contributing to lower aggregate output and productivity. The dynamic effect is substantial quantitatively. Reducing the dispersion in revenue productivity in the model by 25 percentage points to the level of the U.S. benchmark implies an increase in aggregate output and TFP by a factor of 2.9-fold. Improved resource allocation accounts for 42 percent of the gain, whereas the change in the productivity distribution accounts for the remaining 58 percent.

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# 1 Introduction

A crucial question in economic growth and development is why some countries are rich and others poor. A consensus has emerged in the literature whereby the large differences in income per capita across countries are mostly accounted for by differences in labor productivity and in particular total factor productivity (TFP).<sup>1</sup> Hence, a key question is what explains differences in TFP across countries. A recent literature has emphasized the (mis)allocation of factors across heterogeneous production units as an important channel.<sup>2</sup> We study factor misallocation in a model where establishment-level productivity is determined endogenously. In this framework, policy distortions not only misallocate resources across a given set of productive units (static effect), but also create disincentives for productivity improvement (dynamic effect) thereby affecting the productivity distribution of establishments in the economy and further contributing to lower aggregate output and productivity.

A recent literature has emphasized the dynamic implications of misallocation by considering variations of the growth model with establishment-level productivity.<sup>3</sup> We build on this literature by endogenizing the entire distribution of productivity as a function of the economic environment which is affected by policy distortions. In our framework, not only there is a tight mapping between abstract policy distortions and the empirical counterparts of dispersion in revenue products and factor misallocation, but also compared to the model with an exogenous distribution of productivity, the quantitative effect of empirically-plausible policy distortions is substantial. For instance, in our framework, the output gain from equalizing the dispersion in log revenue productivity (TFPR) in 1991 India to the dispersion in log

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<sup>1</sup>See, for instance, [Klenow and Rodriguez-Clare \(1997\)](#), [Prescott \(1998\)](#), and [Hall and Jones \(1999\)](#).

<sup>2</sup>See [Banerjee and Dufo \(2005\)](#), [Restuccia and Rogerson \(2008\)](#), [Guner et al. \(2008\)](#), and [Hsieh and Klenow \(2009\)](#). See also surveys of the literature in [Restuccia and Rogerson \(2013\)](#), [Restuccia \(2013a\)](#), and [Hopenhayn \(2014\)](#).

<sup>3</sup>Some of the contributions on the endogenous productivity distribution include [Restuccia \(2013b\)](#), [Bello et al. \(2011\)](#), [Acemoglu et al. \(2013\)](#), [Ranasinghe \(2014\)](#), [Bhattacharya et al. \(2013\)](#), [Gabler and Poschke \(2013\)](#), [Hsieh and Klenow \(2014\)](#), [Bento and Restuccia \(2017\)](#), [Guner et al. \(2015\)](#), [Peters \(2015\)](#), among others. See also [Restuccia and Rogerson \(2017\)](#) for a discussion of this literature.

TFPR in 1997 United States is 120%, which is substantially larger than the 41% static gain from reduced factor misallocation reported in [Hsieh and Klenow \(2009\)](#) and much closer to the estimated gap in manufacturing productivity between these countries of 160%.

We develop a framework with heterogeneous production units that builds on [Hopenhayn \(1992\)](#) and [Restuccia and Rogerson \(2008\)](#). The framework is extended to incorporate the dynamic implications of distortions on the distribution of establishment-level productivity. We use this framework to study the impact of policy distortions on misallocation and aggregate measured productivity and output. There is a large number of homogeneous households with standard preferences over consumption goods. Households accumulate physical capital and supply inelastically their endowment of one unit of time. The key elements of the model are on the production side. A single good is produced in each period. The production unit is the establishment. An establishment has access to a decreasing returns to scale production function with capital and labor as inputs. Establishments are heterogeneous with respect to total factor productivity. Establishments are subject to an exogenous exit rate but differently from the standard framework, the distribution of establishment-level productivity is not exogenous, rather it is determined by establishment's decisions on entry productivity and investment on productivity growth over time. In other words, the level of productivity of entering establishments is determined endogenously in the model by the properties of the economic environment such as policy distortions, as is the evolution of their productivity over time.

Following the literature, the economy faces policy distortions which, for simplicity, take the form of output taxes on individual producers. That is, each producer faces an idiosyncratic tax and it is the properties of policy distortions that generate misallocation in the model. Revenues collected from these taxes are rebated back to the households as a lump-sum transfer. We emphasize that the output distortions we consider are abstract representations (catch all) of the myriad of implicit and explicit distortions faced by individual producers.

While the literature has made substantial progress in identifying the specific policies and institutions that create misallocation—as emphasized for example in [Restuccia and Rogerson \(2017\)](#)—the emphasis in our paper is on the dynamic consequences of misallocation created by all the distortions on the productivity distribution in the economy. As a result, our paper represents a general quantitative assessment of the broader consequences of misallocation.

We characterize the closed-form solution of this model in continuous time. In particular, we solve in closed form for the stationary distribution of establishments which is an endogenous object that varies across economies. We show the equilibrium productivity distribution is a Pareto distribution with tail index that depends on policy distortions and on the investment response of incumbent establishments to distortions. This allows us to characterize the behavior of aggregate output and TFP across distortionary policy configurations as well as the size and productivity growth rate of establishments, the size distribution of establishments, among other statistics of interest.

To assess the quantitative properties of the model relative to the existing literature, we calibrate the model and provide a set of relevant quantitative experiments. We consider a benchmark economy with distortions that is calibrated to data for the United States. The key calibrated parameters in our analysis are the investment cost in establishment-level productivity, the variance in the distribution of productivity, and the size growth rate of establishments, which are targeted to data on the aggregate growth rate of TFP, the average employment growth of establishments, and the right tail index of the share of employment distribution in the U.S. data. We then perform quantitative analysis by exploring the implications of increased distortions for aggregate output and productivity.

Our main result is that the quantitative effect of policy distortions on aggregate output and TFP is substantially larger than in a model with an exogenous distribution of productivity. In particular, in an economy with 50 percent more dispersion in revenue products than the U.S. benchmark economy, reducing the dispersion to that of the benchmark implies a large

increase in aggregate output of 2.9-fold, out of which a 1.56-fold increase is due to improved factor allocation and the remaining 1.89-fold increase is due to an improved productivity distribution. The large effect of policy distortions on the productivity distribution arises from both the effect on entry productivity as well by changes in life-cycle productivity investment by establishments. For instance, the reduction in misallocation in this economy implies a decline in life-cycle productivity growth of one percentage point per year, from 4.7 percent in the benchmark economy to 3.7 percent; and a 25 percent reduction in the relative productivity of entrants. The negative effect of distortions on the life cycle growth of firms is consistent with the evidence in [Hsieh and Klenow \(2014\)](#) where the employment and productivity growth of firms is found to be lower in more distorted economies than in the United States. The negative effect on the productivity of entrants is also consistent with evidence. For example, empirical studies of policy reforms have consistently found that reduced distortions are associated with higher productivity of entrants. More generally, the dynamic effect of distortions is consistent with the evidence in [Bustos \(2011\)](#) showing that a trade reform that increased the profitability of firms encouraged investments in technology upgrading.

Our paper is related to a large literature on misallocation and productivity. By studying the aggregate impact of policy distortions across countries our paper is closely linked to [Restuccia and Rogerson \(2008\)](#) but in addition to the impact of distortions on factor misallocation, we consider the dynamic response of establishment-level productivity to the same distortions creating misallocation. Considering the broader implications of misallocation is of first-order importance in this literature. The literature has emphasized various separate channels such as life-cycle investment of plants, human capital accumulation of managers, experimentation, step-by-step innovation, selection, among many others; and different contexts such as trade and labor policies, financial frictions, and specific sectors.<sup>4</sup> We complement this literature by

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<sup>4</sup>See for instance [Rubini \(2014\)](#) for an analysis of changes in tariffs in the context of a trade model with endogenous establishment-level productivity.

solving for the entire productivity distribution and characterizing this distribution in closed form, which we hope can facilitate empirical applications.

A closely related paper to ours is [Hsieh and Klenow \(2014\)](#) who consider the model of establishment innovation in [Atkeson and Burstein \(2010\)](#) to emphasize the life-cycle growth of establishments and its response to distortions. We emphasize two key distinctions with our work. First, whereas in [Hsieh and Klenow \(2014\)](#) entering establishments draw their productivity from an exogenous and constant distribution across countries, entry size and hence productivity is a key equilibrium object in our framework that responds to policy distortions. We show that this element of the productivity distribution is essential in the quantitative impact of distortions on aggregate output. Second, we differ in the tools used to characterize the economy, in particular, we characterize the model in closed form using continuous time and Brownian motion processes. These tools are increasingly popular in the growth literature.<sup>5</sup> More closely linked, these tools were prominently used by the seminal work of [Luttmer \(2007\)](#) to study the size distribution of establishments in the United States, by [Da-Rocha and Pujolas \(2011\)](#) and [Fattal \(2014\)](#) to study the effect of policy distortions with stochastic productivity and entry/exit decisions, and by [Gourio and Roys \(2014\)](#) to study the productivity effects of size-dependent labor regulations, just to name a few. A key distinction of our work with this existing literature is the emphasis on the amplification effect of policies. We argue that a large propagation effect of policies on aggregate output and productivity is essential in providing a more accurate assessment of the quantitative impact of specific policies such as firing taxes, size-dependent policies, among many other well-studied policies.

The paper proceeds as follows. In the next section and section [3](#), we describe the model and characterize the equilibrium solution. Section [4](#) characterizes aggregate output, measured TFP, and establishment size in the model as a function of distortions. In section [5](#), we

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<sup>5</sup>For instance, [Lucas and Moll \(2014\)](#), [Benhabib et al. \(2014\)](#), [Buera and Oberfield \(2014\)](#), among many others.

calibrate a benchmark economy with distortions to U.S. data. Section 6 performs a series of quantitative experiments to assess the impact of increased policy distortions on aggregate output, TFP, and other relevant statistics. We conclude in section 7.

## 2 Economic Environment

We consider a standard version of the neoclassical growth model with producer heterogeneity as in Restuccia and Rogerson (2008). We extend this framework in order to allow establishments to invest on their productivity. As a result, the extended framework generates an endogenous productivity distribution of establishments associated with the economic environment that may differ across countries. Time is continuous and the horizon is infinite. Establishments have access to a decreasing return to scale technology, pay a one-time fixed cost of entry, and die at an exogenous rate. Establishments hire labor and rent capital services in competitive markets. New entrants enter with a level of productivity  $z_e$  which is endogenous. We study a stationary equilibrium in which the economy grows at an exogenous rate. We then analyze policy distortions that affect the allocation of factors across establishments (static misallocation). In our framework, policy distortions also affect the establishment investment on productivity, the stationary productivity distribution of establishments, and therefore, aggregate measured TFP and output. We contrast the effects of policy distortions in the environment where the distribution of establishment productivity is exogenous. In what follows we describe the economic environment in more detail.



## 2.1 Baseline Model

There is an infinity-lived representative household with preferences over consumption goods described by the utility function,

$$\max \int_0^{+\infty} e^{-\rho t} u(c) dt,$$

where  $c$  is consumption and  $\rho$  is the discount rate. The household is endowed with one unit of productive time at each instant and  $k_0 > 0$  units of the capital stock at date 0.

The unit of production in the economy is the establishment. Each establishment is described by a production function  $f(z, k, n)$  that combines capital services  $k$  and labor services  $n$  to produce output. The function  $f$  is assumed to exhibit decreasing returns to scale in capital and labor jointly and to satisfy the usual Inada conditions. The production function is given by:

$$y = z^{\theta(1-\alpha-\gamma)} k^\alpha n^\gamma, \quad \alpha, \gamma \in (0, 1), \quad 0 < \gamma + \alpha < 1, \quad \theta > 1, \quad (1)$$

where  $\theta$  is a TFP normalization factor that is inconsequential for our results. In our specification, establishment TFP is  $z^{\theta(1-\alpha-\gamma)}$  so  $\theta$  only affects the units in which establishment productivity  $z$  is measured. This normalization is convenient for algebraic manipulations. Establishment productivity  $z$  is stochastic but establishments can invest in upgrading their productivity at a cost. Establishments also face an exogenous probability of death  $\lambda$ .

New establishments can also be created. Entrants must pay an entry cost  $c_e$  measured in units of output and as in the literature the expected value of entry satisfies the zero profit condition in equilibrium. Feasibility in the model requires:

$$C + I + Q = Y - E,$$

where  $C$  is aggregate consumption,  $I$  is aggregate investment in physical capital,  $Q$  is aggregate cost of investing in establishment productivity,  $E$  is the aggregate cost of entry, and  $Y$  is aggregate output.

## 2.2 Policy Distortions

We introduce policies that create idiosyncratic distortions to establishment-level decisions as in [Restuccia and Rogerson \(2008\)](#). We model these distortions as idiosyncratic output taxes but none of our results are critically dependent on the particular source of distortions. While the policies we consider are hypothetical, there is a large empirical literature documenting the extent of idiosyncratic distortions across countries and a nice feature of our framework is that there is a tight mapping between the distortions we consider and empirical observations.<sup>6</sup>

In our framework, distortions not only affect the allocation of resources across existing production units, but also the investment decision in productivity thereby affecting the distribution of productive units in the economy. Specifically, we assume that each establishment faces its own policy distortion (idiosyncratic distortions) reflected as an output tax rate  $\tau_y$ . In what follows, for simplicity in our algebraic expressions we rewrite distortions as  $\tau = (1 - \tau_y)^{\frac{1}{\theta(1-\alpha-\gamma)}}$ . Note that this transformation implies that an establishment with no distortions  $\tau_y = 0$  faces  $\tau = 1$ , whereas a positive output tax  $\tau_y > 0$  implies  $\tau < 1$  and an output subsidy  $\tau_y < 0$  implies  $\tau > 1$ .

In order to generate dispersion in distortions across productive units, we assume that  $\tau$  follows a standard stochastic process, a Geometric Brownian motion,

$$d\tau = \mu_\tau \tau dt + \sigma_\tau \tau dw_\tau,$$

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<sup>6</sup>See for instance [Hsieh and Klenow \(2009\)](#), [Bartelsman et al. \(2013\)](#) and the survey in [Restuccia and Rogerson \(2013\)](#).

where  $\mu_\tau$  is the drift,  $\sigma_\tau$  is the standard deviation and  $dw_\tau$  is the standard Wiener process of the Brownian motion. In this specification  $\sigma_\tau$  controls the dispersion of distortions across producers and hence the dispersion in marginal revenue products that is restricted to data.<sup>7</sup>

Establishment's productivity  $z$  follows a Geometric Brownian motion and establishments can invest in upgrading their productivity by choosing the drift of the Brownian motion  $\mu_z$  that is determined in equilibrium, establishment productivity follows:

$$dz = \mu_z(w, r)zdt + \sigma_z z dw_z.$$

$\sigma_z$  is the standard deviation and  $dw_z$  is the standard Wiener process of the Brownian motion. For tractability, we assume that the output tax and productivity processes are uncorrelated, that is  $E(dw_\tau, dw_z) = 0$ . We note however that much of the quantitative literature has focused on what [Restuccia and Rogerson \(2008\)](#) call correlated distortions, distortions that apply more heavily on more productive establishments and have the potential to generate much larger negative aggregate productivity effects.

At the time of entry, the establishment-entry distortion  $\tau_e$  is known and establishments enter with a productivity  $z_e$  that is determined in equilibrium and implies an expected value of entrants that satisfies a zero profit condition. In this economy, the relevant information for establishment's decisions is the joint distribution over productivity and distortions. We denote this joint distribution by  $g(z, \tau)$ .

A given distribution of establishment-level distortion and productivity may not lead to a balanced budget for the government. As a result, we assume that budget balance is achieved by either lump-sum taxation or redistribution to the representative household. We denote the lump-sum tax by  $T$ .

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<sup>7</sup>As discussed earlier, our specification of distortions is reduced form and abstract, simply standing in for the myriad of policies and institutions that effectively create dispersion in individual producer prices. See for example [Restuccia and Rogerson \(2008\)](#) and [Buera et al. \(2013\)](#).

### 3 Equilibrium

We focus on a stationary equilibrium where the number of establishments grows at an exogenous rate  $\eta$ . The stationary equilibrium is characterized by an invariant distribution of establishments  $g(z, \tau)$  over productivity  $z$  and distortion  $\tau$  and an entry productivity  $z_e$ . In the stationary equilibrium, the mass of establishments grows over time, however, the rank of the establishments' size distribution is constant. In the stationary equilibrium, the rental prices for labor and capital services are constant and we denote them by  $w$  and  $r$ . Before defining the stationary equilibrium formally, it is useful to consider the decision problems faced by incumbents, entrants, and consumers. We describe these problems in turn.

#### 3.1 Incumbent establishments

Incumbent establishments maximize the present value of profits by making static and dynamic decisions. The static problem is to choose the amount of capital and labor services, whereas the dynamic problem involves solving for the amount of investment in establishment productivity. In what follows next, we describe these problems in detail.

**Static problem** At any instant of time an establishment chooses how much capital to rent  $k$  and how much labor to hire  $n$ . These decisions are static and depend on the establishment's productivity  $z$ , the establishment's distortion  $\tau$ , the rental rate of capital  $r$ , and the wage rate  $w$ . Formally, the instant profit function  $\pi(z, \tau)$  is defined by:

$$\pi(z, \tau) = \max_{k, n} (\tau z)^{\theta(1-\alpha-\gamma)} k^\alpha n^\gamma - wn - rk,$$

from which we obtain the optimal demands for capital and labor:

$$n(z, \tau) = \left[ \left( \frac{\alpha}{r} \right)^\alpha \left( \frac{\gamma}{w} \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha-\gamma}} z^\theta \tau^\theta, \quad (2)$$

$$k(z, \tau) = \left[ \left( \frac{\alpha}{r} \right)^{1-\gamma} \left( \frac{\gamma}{w} \right)^\gamma \right]^{\frac{1}{1-\alpha-\gamma}} z^\theta \tau^\theta. \quad (3)$$

For future reference, we redefine instant profits as a function of the optimal demand for factors:

$$\pi(z, \tau) = m(w, r) z^\theta \tau^\theta, \quad (4)$$

where  $m(w, r) = (1 - \alpha - \gamma) \left[ \left( \frac{\alpha}{r} \right)^\alpha \left( \frac{\gamma}{w} \right)^\gamma \right]^{\frac{1}{1-\alpha-\gamma}}$  is a constant across establishments that depends on equilibrium prices.

**Dynamic problem** Incumbent establishments choose how much to invest in upgrading their productivity  $x_z$ . The cost of investing in upgrading productivity is in units of output, described by a cost function  $q(\cdot)$  that is increasing and convex in the productivity parameter  $x_z$ , specifically we assume  $q(x_z) = c_\mu \frac{x_z^\theta}{\theta}$ . The optimal decision of upgrading productivity is characterized by maximizing the present value of profits subject to the Brownian motion governing the evolution of productivity and the Brownian motion governing the evolution of distortions. Formally, incumbent establishments solve the following dynamic problem:

$$W(z, \tau) = \max_{x_z} \left\{ m(w, r) z^\theta \tau^\theta - q(x_z) + \frac{1}{1 + (\lambda + R)dt} E_{z, \tau} W(z + dz, \tau + d\tau) \right\},$$

$$s.t. \quad dz = \frac{x_z}{\tau} dt + \sigma_z z dw_z,$$

$$d\tau = \mu_\tau \tau dt + \sigma_\tau \tau dw_\tau,$$

where  $\lambda$  is the exogenous exit probability of establishments and  $R$  is the stationary equilibrium interest rate. The benefit of upgrading productivity is described by  $\mu_z z = \frac{x_z}{\tau}$ . This specification guarantees Gibrat's law holds as shown in Lemma 1. Next, we define the

Hamilton-Jacobi-Bellman of the stationary solution,

$$(\lambda + R)W(z, \tau) = \max_{x_z} \left\{ m(w, r)\tau^\theta z^\theta - \frac{c_\mu}{\theta} x_z^\theta + \frac{x_z}{\tau} W'_z + \frac{\sigma_z^2}{2} z^2 W''_{zz} + \mu_\tau \tau W'_\tau + \frac{\sigma_\tau^2}{2} \tau^2 W''_{\tau\tau} \right\}.$$

From the first order conditions, we find that the optimal investment rate  $x_z$  is a function of the distortion  $\tau$ , the investment cost  $c_\mu$ , and the marginal present value profits  $W'_z$ ,

$$x_z^{\theta-1} = \frac{W'_z}{c_\mu \tau}.$$

By guessing and verifying, we find that the optimal Hamilton-Jacobi-Bellman equation is given by  $W(z, \tau) = A(w, r)z^\theta \tau^\theta$ , where the constant  $A(w, r)$  is the solution of the polynomial:

$$\left[ \frac{(\lambda + R)}{(\theta - 1)} - \frac{\theta \mu_\tau}{(\theta - 1)} - \frac{\theta}{2} (\sigma_z^2 + \sigma_\tau^2) \right] A(w, r) - \left[ \frac{\theta}{c_\mu} \right] \frac{1}{\theta - 1} A(w, r) \frac{\theta}{\theta - 1} = \frac{m(w, r)}{(\theta - 1)}. \quad (5)$$

Given the solution to this polynomial, the optimal investment rate is linear in  $\tau z$ , i.e.

$$x_z = \left[ \frac{\theta A(w, r)}{c_\mu} \right] \frac{1}{\theta - 1} \tau z. \quad (6)$$

In the following Lemma 1 we characterize formally the impact of distortions on the productivity drift.

**Lemma 1.** *Given a distortion  $\tau$ , a productivity level  $z$ , and operating profits  $m(w, r)$ , the value function that solves the establishment dynamic problem is given by  $W(z, \tau) = A(w, r)\tau^\theta z^\theta$ , and the expected growth rate of establishment's productivity  $z$  follows Gibrat's law:*

$$\frac{dz}{z} = \left[ \frac{\theta A(w, r)}{c_\mu} \right] \frac{1}{\theta - 1} dt + \sigma_z dw_z,$$

where  $A(w, r)$  is the solution of the polynomial in equation (5) and  $\mu_z(w, r) = \left[ \frac{\theta A(w, r)}{c_\mu} \right] \frac{1}{\theta - 1}$ .

The proof of Lemma 1 is straightforward from equation (6). The implication of Lemma 1 is that the growth rate of productivity of individual establishments does not depend on the intrinsic characteristics of the establishment, that is, it does not depend on the establishment productivity  $z$  or the distortion  $\tau$ . As a consequence Gibrat's law holds and productivity growth does not depend on the establishment size. We make this assumption not because we think this is the more plausible case empirically, but instead because this assumption reduces the negative effect on output from distortions and hence, make our results conservative. For example, as emphasized in [Bhattacharya et al. \(2013\)](#) and [Hsieh and Klenow \(2014\)](#), correlated policy distortions can affect the productivity growth of more productive establishments leading to potentially larger negative output effects. We note, however, that the growth rate of productivity can still differ across economies if distortions affect equilibrium wages. What the result in Lemma 1 implies is that the growth rate of productivity would not differ across establishments in the same economy. To provide some intuition regarding the establishment value function we characterize the implicit value function of an incumbent in Lemma 2.

**Lemma 2.** *Given an output tax  $\tau$ , a productivity level  $z$ , and operating profits  $m(w, r)$ , the value function of an establishment that solves the dynamic problem is given by  $W(z, \tau) = A(w, r)\tau^\theta z^\theta$ , where the constant  $A(w, r)$  is implicit given by the following expression:*

$$A(w, r) = \frac{m(w, r)}{\lambda + R - (\theta - 1)\mu_z(w, r) - \theta\mu_\tau - \frac{\theta(\theta - 1)}{2}(\sigma_z^2 + \sigma_\tau^2)}.$$

**Proof** See Appendix [A.1](#).

From Lemma 2 is clear that increases in policy distortions  $\sigma_\tau$  have a direct and an indirect impact on the value of incumbents. The direct impact is positive and the indirect impact through prices  $m(w, r)$  and through establishments' investment in productivity  $\mu_z$  can be

positive or negative. Now, we can characterize the problem of entering establishments.

### 3.2 Entering establishments

Potential entering establishments face an entry cost  $c_e$  in units of output and make their entry decision knowing the output entering tax level  $\tau_e$ . For tractability, we assume that entrants enter with the same level of expected productivity, denoted by  $z_e$ . The initial level of productivity is such that the value of entering establishments satisfies the usual zero profit condition:

$$W_e = W(z_e, \tau_e) - c_e.$$

Note that such a value of productivity  $z_e$  exists and is unique which follows from the fact that the value of entry  $W_e$  inherits the properties of the value of incumbent establishments characterized in Lemma 2. In addition, in the special case where the model is deterministic the value of entering is the same as in Restuccia and Rogerson (2008), which is the establishments' expected profit.

### 3.3 Stationary distribution of establishments

Given the optimal decisions of incumbents and entering establishments, we are now ready to characterize the stationary distribution  $g(z, \tau)$  over productivity  $z$  and distortion  $\tau$ . The first step to characterize this distribution is to rewrite the Brownian motions of productivity  $z$  and distortion  $\tau$  as a function of  $s$ , where  $s \equiv \tau^\theta z^\theta$ .<sup>8</sup> The resulting  $s$  Brownian motion is given by:

$$\frac{ds}{s} = \left[ \theta (\mu_z(w, r) + \mu_\tau) + \frac{\theta(\theta - 1)}{2} (\sigma_z^2 + \sigma_\tau^2) \right] dt + \theta(\sigma_z + \sigma_\tau) dw_s, \quad (7)$$

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<sup>8</sup>Note from the input demands discussed previously, the size of the establishment is proportional to  $s$  so in what follows we generally refer to  $s$  as size.



where the drift  $\mu_s$  is equal to a weighted average of the output tax and the productivity Brownian motion, i.e.  $\mu_s = \theta(\mu_z(w, r) + \mu_\tau) + \frac{\theta(\theta-1)}{2}(\sigma_z^2 + \sigma_\tau^2)$ . It is important to remember that the drift of the productivity Brownian motion  $\mu_z$  is an endogenous object and is given by the solution of the incumbent establishment's dynamic problem. The standard deviation  $\sigma_s$  of the  $s$  Brownian motion is the weighted sum of the standard deviation of the distortion Brownian motion  $\sigma_\tau$  and the standard deviation of the productivity Brownian motion  $\sigma_z$ , i.e.  $\sigma_s = \theta(\sigma_z + \sigma_\tau)$ .

In order to characterize the stationary distribution over size  $s$ , it is useful to rewrite the model in logarithms. Let  $x$  denote the logarithm of  $s$ , that is  $x = \log(s/s_e)$ , where  $s_e$  is the size in which establishments enter. Now we can rewrite the Geometric Brownian motion in equation (7) as a Brownian motion in the logarithm of  $s$ ,

$$dx = \mu_x dt + \sigma_x dw_x,$$

where  $\mu_x = \mu_s - \frac{1}{2}\sigma_s^2$ , and  $\sigma_x = \sigma_s$ . Let  $M(x, t)$  denote the number density function of establishments, i.e. the mass of size  $x$  establishments at time  $t$ . At time  $t$ , the total number of establishments is equal to  $M(t) = \int_{-\infty}^{+\infty} M(x, t) dx$ .

The establishments productivity process can be modeled by a *modified* Kolmogorov-Fokker-Planck equation of the form:

$$\frac{\partial M(x, t)}{\partial t} = -\mu_x \frac{\partial M(x, t)}{\partial x} + \frac{\sigma_x^2}{2} \frac{\partial^2 M(x, t)}{\partial x^2} - \lambda M(x, t) + B(0, t), \quad (8)$$

where  $\lambda$  is the death rate of establishments and the function  $B(0, t)$  are the new establishments that enter at  $t$  and have size 0, after the renormalization. The solution of this problem is discussed in Gabaix (2009), and we solve by applying Laplace Transforms methods.<sup>9</sup>

We are interested in a stationary distribution for the number density function, i.e. solutions

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<sup>9</sup>Da-Rocha et al. (2016) show that the Double Pareto is a particular solution in frameworks with inaction.

that are separable in time  $t$  and are of the form  $M(x, t) = M(t)f(x)$  and  $B(0, t) = M(t)b\delta(x - 0)$ , where  $b$  is the establishment entry rate at point  $x = 0$  and  $\delta(\cdot)$  is a Dirac delta function which is equal to 1 at the entry, normalized to zero, and is equal to zero everywhere else.<sup>10</sup> Therefore, we can rewrite the *modified* Kolmogorov-Fokker-Planck equation equation (8) as:

$$\frac{M'(t)}{M(t)}f(x) = \eta f(x) = -\mu_x f'(x) + \frac{\sigma_x^2}{2}f''(x) - \lambda f(x) + b\delta(x - 0), \quad (9)$$

where  $\frac{M'(t)}{M(t)}$  is the separation rate denoted by  $\eta$  and  $M(t) = e^{\eta t}M(0)$  in the balanced growth path. We normalize  $M(0) = 1$ . We assume four boundary conditions:

$$\lim_{x \rightarrow +\infty} f(x) = 0, \quad \lim_{x \rightarrow +\infty} f'(x) = 0, \quad (10)$$

$$\lim_{x \rightarrow -\infty} f(x) = 0, \quad \lim_{x \rightarrow -\infty} f'(x) = 0, \quad (11)$$

and

$$f(x) \geq 0, \quad \int_{-\infty}^{+\infty} f(x)dx = 1. \quad (12)$$

The first four boundary conditions (10) and (11) guarantee that the stationary distribution is bounded, and equations (12) guarantee that  $f$  is a pdf. The boundary constraints restricts the separation rate  $\eta$ , by integrating (9) we find:

$$\eta \int_{-\infty}^{+\infty} f(x)dx = \left( -\mu_x f(x) + \frac{\sigma_x^2}{2}f'(x) \right) \Big|_{-\infty}^{+\infty} - \lambda \int_{-\infty}^{+\infty} f(x)dx + \int_{-\infty}^{+\infty} b\delta(x - 0)dx,$$

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<sup>10</sup>Mathematically we can express this by using a Dirac delta function that is equal to infinity at the point on which new firms enter and zero otherwise. Let the function  $b(0)$  can be described by:

$$\hat{b}(0) = b\delta(x - 0),$$

where  $\delta$  denotes the Dirac delta function:

$$\delta(x) = \begin{cases} +\infty & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

and applying the boundary conditions and using the Dirac delta function, we find that the growth rate of establishments  $\eta$  is equal to:

$$\eta = b - \lambda.$$

The expression for  $\eta$  has a very intuitive interpretation, it states that growth rate of establishments  $\eta$  is equal to the net entry rate ( $b - \lambda$ ). After some algebraic manipulation from equation (9), we find that the stationary distribution must satisfy the following differential equation:

$$f''(x) - \frac{2\mu_x}{\sigma_x^2} f'(x) - \frac{2(\lambda + \eta)}{\sigma_x^2} f(x) = -\frac{2b}{\sigma_x^2} \delta(x - 0), \quad (13)$$

subject to the boundary conditions and  $f(\cdot)$  being a pdf. We can now characterize the stationary (log) size distribution, which is a double Pareto, with endogenous tail index,  $\xi$ , and endogenous net entry rate,  $b - \lambda$  at  $x = 0$ . Formally, Lemma 3 characterizes the stationary distribution.

**Lemma 3.** *Given wages  $w$  and rental rate of capital  $r$ , the stationary size distribution associated with the distortion Geometric Brownian Motion is a double Pareto:*

$$g(s) = \begin{cases} C \left( \frac{s}{s_e} \right)^{-(\xi_- + 1)} & \text{for } s < s_e. \\ C \left( \frac{s}{s_e} \right)^{-(\xi_+ + 1)} & \text{for } s \geq s_e. \end{cases}$$

where the tail index  $\xi_+$  is the positive root and  $\xi_-$  is the negative root that solves the characteristic equation  $\frac{\sigma_s^2}{2} \xi^2 + \left( \mu_s - \frac{\sigma_s^2}{2} \right) \xi - (\lambda + \eta) = 0$  and  $C = \frac{-\xi_- \xi_+}{s_e(\xi_+ - \xi_-)}$ . Moreover, the average size  $\bar{s}$  of establishments is given by:

$$\bar{s} = s_e \frac{-\xi_- \xi_+}{(\xi_+ - 1)(1 - \xi_-)}.$$

**Proof** See Appendix A.2.

We leave the poof of Lemma 3 to the Appendix. Lemma 3 characterizes the endogenous distribution that is a function of establishments' investment in productivity  $\mu_s$  and entry size  $s_e$ . We can also calculate from Lemma 3, the expected average growth rate of establishments, given by:

$$\frac{\bar{s}}{s_e} = \frac{\eta + \lambda}{\eta + \lambda - \mu_s}.$$

An important implication of the model is with respect to the size of new entrants  $s_e$ . An increase in policy distortions (an increase in the dispersion of distortions  $\sigma_\tau$ ) which increases misallocation, produces an increase in  $\mu_s$  and a decrease in entry size  $s_e$ , generating a larger distance between an incumbent average size and the size of new entrants. An increase in distortions also increases the left tail of the distribution of establishments.

### 3.4 Household's problem

The household problem is standard and essentially help us pin down the stationary interest rate  $R$ . As such, the process for capital accumulation in this model follows the standard neoclassical growth model. The stand-in household seeks to maximize lifetime utility subject to the law of motion of wealth given by:

$$(RK + w + T + \Pi - bc_e - c) dt,$$

where  $w$  is the wage rate,  $R$  is the is the interest rate which in equilibrium is the rental price of capital minus capital depreciation ( $R = r - \delta_k$ ),  $T$  is the lump-sum tax levied by the government,  $\Pi$  is the total profit from the operations of all establishment,  $bc_e$  is the entry cost and  $c$  is consumption.

We assume that households have log utility,  $u(c) = \log(c)$ , and we characterize the equilib-

rium interest rate by solving the household's problem. We define total wealth as:

$$a = K + \frac{w}{R} + \frac{T}{R} + \frac{\Pi}{R} - b \frac{c_e}{R},$$

and we rewrite the law of motion of wealth as  $da = (Ra - c)dt$ . The household solves the following Hamilton-Jacobi-Bellman equation:

$$\rho V(a) = \max_c \{ \log(c) + [Ra - c] V'(a) \}.$$

Lemma 4 establishes that in the stationary equilibrium the interest rate  $R$  is equal to the discount rate  $\rho$ .

**Lemma 4.** *In the stationary equilibrium the interest rate is equal to the discount rate  $R = \rho$ .*

### 3.5 Stationary equilibrium

We assume that population grows at an exogenous rate that is the same as the net growth rate of establishments  $b = \eta + \lambda$ . This assumption guarantees that wages  $w$  are constant in the stationary equilibrium. Since the growth in population is exogenous, we define the stationary equilibrium in per capita terms, i.e.  $N = 1$ . Establishments average size  $\bar{s}$ , establishments entry size  $s_e$ , and establishments investment in productivity,  $\mu_z$  are endogenous in the model.

**Definition** Given the exogenous growth rate of establishments and GDP per capita  $\eta$ , a stationary equilibrium for this economy is a stationary distribution  $g(\cdot)$ , a net entry rate  $b = \eta + \lambda$ , a value function for incumbents  $\{W(\cdot)\}$ , a policy function for new entrants  $\{z_e\}$ , policy functions  $\{k(\cdot), n(\cdot), x_z(\cdot), c(\cdot)\}$ , prices  $\{r, w\}$ , and transfer  $\{T\}$  such that:

- i) Given prices and transfer, the households' policy function  $\{c(\cdot)\}$  solves the household dynamic problem.

- ii) Given prices, the incumbents' policy functions  $\{k(\cdot), n(\cdot)\}$  solve the incumbents' static problem.
- iii) The incumbents' policy function  $\{x_z(\cdot)\}$  together with the value function  $\{W(\cdot)\}$  solve the incumbents' dynamic problem.
- iv) The stationary distribution  $\{g(\cdot)\}$  solve the Kolmogorov forward equation.
- v) The entering establishments' policy function  $\{z_e\}$  solves the free-entry condition.
- vi) Market Clearing:

- a) Capital:  $K = \int_0^{+\infty} k(s, w, r)g(s)ds,$
- b) Labor:  $N = \int_0^{+\infty} n(s, w, r)g(s)ds.$

- vii) The government budget constraint is satisfied,  $T = \int_0^{+\infty} \tau_y y(s)g(s)ds.$

The stationary equilibrium is a fixed point in measure and it is very simple to compute. From the household's problem, we solve for the stationary interest rate  $R$  and hence pin down the rental rate of capital  $r$ . From the incumbents' static problem, we solve the labor and capital demand as a function of prices  $\{r, w\}$  and policies  $\{\tau\}$ . Given the solution of the static problem, incumbents solve the dynamic problem of investing in productivity. The solution of this problem is a policy function  $\{x_z(\cdot)\}$  that determines the Geometric Brownian motion for productivity of the entire economy. Given the Geometric Brownian motion for productivity, we solve for the stationary distribution  $g(\cdot)$  that solves the Kolmogorov forward equation. After solving for the stationary distribution  $g(\cdot)$ , the entry rate at the minimum entry size,  $s_e$  must solve the free-entry condition, and the markets for capital and labor must clear.<sup>11</sup>

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<sup>11</sup>Equilibrium conditions can be found in the Appendix.

## 4 Output, TFP, and Establishment Size

We characterize the impact of policy distortions on output and TFP using the well-known concept of revenue total factor productivity TFPR, which was disseminated in the context of the macro development literature by [Hsieh and Klenow \(2009\)](#). First, we note that in our model an establishment's TFPR is given by:

$$\text{TFPR} = \frac{y}{k^{\alpha/(\alpha+\gamma)} n^{\gamma/(\alpha+\gamma)}} \propto \frac{1}{(1 - \tau_y)},$$

where  $\frac{1}{(1 - \tau_y)} = \tau^{-\theta(1 - \alpha - \gamma)}$ . Since policy distortions follow a Geometric Brownian motion, we can use the same methodology as in Lemma 3 to find the stationary distribution of  $\tau_y$  and consequently TFPR.<sup>12</sup> Aggregate output  $Y$  is computed by integrating over the distribution of establishment's output given by:

$$y(z, \tau) = \frac{s}{(1 - \tau_y)} \left[ \left( \frac{\alpha}{r} \right)^\alpha \left( \frac{\gamma}{w} \right)^\gamma \right]^{\frac{1}{1 - \alpha - \gamma}}. \quad (14)$$

It is clear from this expression that misallocation implies that establishments of the same size may produce different amounts of output depending on the establishment's policy distortions  $\tau_y$ . To compute aggregate output  $Y$  we integrate over the endogenous distribution of productivity  $g_z(\cdot)$  and output distortions  $g_{\tau_y}(\cdot)$  following the same methodology as in Lemma 3, the result is a very intuitive expression:

$$Y \propto \left[ \left( \frac{\eta + \lambda}{\eta + \lambda - \mu_{\tau^{\theta(\alpha + \gamma)}}} \right) \frac{1}{(1 - \tau_{y,e})} \right] \left[ \left( \frac{\eta + \lambda}{\eta + \lambda - \mu_{z^\theta}} \right) \frac{s_e}{\bar{s}^{\gamma/(1 - \alpha)}} \right], \quad (15)$$

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<sup>12</sup>The distribution of distortions  $g_\tau(\tau)$  is a Double Pareto. Therefore,  $\log \text{TFPR}$  follows a Double Exponential with roots  $\xi_{TFPR,-}$  and  $\xi_{TFPR,+}$  that solve the characteristic equation  $\frac{\sigma_{TFPR}^2}{2} \xi^2 + \left( \mu_{TFPR} - \frac{\sigma_{TFPR}^2}{2} \right) \xi - (\lambda + \eta) = 0$  where  $\mu_{TFPR} = -\theta(1 - \alpha - \gamma)\mu_\tau - [\theta(1 - \alpha - \gamma) + 1]\frac{\sigma_\tau}{2}$  and  $\sigma_{TFPR}^2 = \theta^2(1 - \alpha - \gamma)^2\sigma_\tau^2$ .

where the expression in the first square brackets is the static output gain from reducing misallocation and equalizing TFPR across establishments, and the expression in the second square brackets is the dynamic output gain from eliminating distortions.<sup>13</sup>

The dynamic gain can be decomposed further: the first element relates to the impact of policy distortions over the life-cycle growth through the impact on productivity investment and the second element relates to the effect on the entry size and hence establishments' productivity. In our quantitative analysis that follows, we emphasize the relative importance of all these terms in accounting for income differences in our model. For completeness, we compute measured aggregate TFP following standard practice as:

$$\text{TFP} = \frac{Y}{K^{\alpha/(\alpha+\gamma)} N^{\gamma/(\alpha+\gamma)}}, \quad (16)$$

where aggregate demand of capital and labor are given by integrating the demand of capital and labor from the the establishments' static problem equations (2) and (3),

$$N = \left[ \left( \frac{\alpha}{r} \right)^\alpha \left( \frac{\gamma}{w} \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha-\gamma}} \bar{s},$$

$$K = \left[ \left( \frac{\alpha}{r} \right)^{1-\gamma} \left( \frac{\gamma}{w} \right)^\gamma \right]^{\frac{1}{1-\alpha-\gamma}} \bar{s},$$

and they are a function of prices and the average size of establishments.

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<sup>13</sup>Aggregate output  $Y$  is given by:

$$Y = \left[ \left( \frac{\alpha}{r} \right)^\alpha \left( \frac{\gamma}{w} \right)^\gamma \right]^{\frac{1}{1-\alpha-\gamma}} \int_0^{+\infty} z^\theta g_z(z) dz \int_0^{+\infty} \tau^{\theta(\alpha+\gamma)} g_\tau(\tau) d\tau,$$

and we use the fact where we use the fact that  $z_e^\theta \tau_e^\theta = s_e$ , and  $(1 - \tau_{y,e}) = \tau_e^{-\theta(1-\alpha-\gamma)}$ .



## 5 Calibration

Our main objective is to study the quantitative impact of policy distortions on aggregate TFP and GDP per capita in an economy that is relatively more distorted than the United States in the same spirit of [Restuccia and Rogerson \(2008\)](#) and [Hsieh and Klenow \(2009\)](#). For this reason, we calibrate a benchmark economy to U.S. data.

We start by selecting a set of parameters that are standard in the literature. These parameters have either well-known targets which we match or the values have been well discussed in the literature. Following the literature, we assume decreasing returns in the establishment-level production function and set  $\alpha + \gamma = 0.85$ , e.g., [Restuccia and Rogerson \(2008\)](#). Then we split it between  $\alpha$  and  $\gamma$  by assigning 1/3 to capital and 2/3 to labor, implying  $\alpha = 0.283$  and  $\gamma = 0.567$ . We set the annual exit rate  $\lambda$  to be 10 percent, which is in line to the estimates in the literature, e.g., [Davis et al. \(1998\)](#). We set the discount rate  $\rho$  to match a real interest rate of 4 percent and the depreciation rate of capital  $\delta$  to 7 percent to match a capital to output ratio of 2.5. To calibrate the exogenous growth rate of the mass of establishments  $\eta$ , we use the equilibrium implication of the model that the aggregate growth rate of TFP over time is proportional to the growth rate of the mass of establishments. Since the growth rate of TFP in the United States in the last 100 years is roughly equal to 2 percent, we set  $\eta$  equal to 0.02. We normalize  $\tau_e = 1$  for the benchmark economy.

We calibrate the remaining parameters by solving the equilibrium of the model and making sure the equilibrium statistics match some targets. The remaining 6 parameters to calibrate are  $(\sigma_z^2, \mu_\tau, \sigma_\tau^2, c_e, c_\mu, \theta)$ . We construct the following 6 statistics in the model and match with the corresponding targets in the data:

- (1) Standard deviation of log revenue total factor productivity (TFPR):

$$\text{SD log TFPR} = \sqrt{\frac{1}{\xi_{TFPR,-}^2} + \frac{1}{\xi_{TFPR,+}^2}}.$$

(2) Employment growth rate:

$$\mu_s = \left[ \theta (\mu_z + \mu_\tau) + \frac{\theta(\theta - 1)}{2} (\sigma_z^2 + \sigma_\tau^2) \right].$$

(3) Standard deviation of log employment:

$$\sigma_s = \theta(\sigma_z + \sigma_\tau).$$

(4) Productivity growth rate:

$$\mu_z = \left[ \frac{\theta}{s_e} \frac{c_e}{c_\mu} \right] \frac{1}{\theta - 1}.$$

(5) Average establishment size:

$$\bar{s} = s_e \frac{\eta + \lambda}{\eta + \lambda - \mu_s}.$$

(6) Cumulative distribution function (CDF) of establishment size:

$$F(s \leq \bar{s}) = 1 - \left( \frac{-\xi_- \xi_+}{\xi_+ - \xi_-} \right) e^{-(\xi_+) \log(\bar{s}/s_e)}.$$

The six parameters are selected simultaneously to match six targets for the above statistics, but since some parameters have a first-order impact on some targets we discuss them in turn. The policy distortions parameters  $\sigma_\tau$  and  $\mu_\tau$  help matching the standard deviation of log of TFPR in the United States from [Hsieh and Klenow \(2009\)](#) and the employment growth over the life cycle of plants in the United States from [Hsieh and Klenow \(2014\)](#).

The elasticity of the productivity investment cost function  $\theta$  helps matching the annual measured productivity growth rate of establishments of 4.7 percent from [Hsieh and Klenow \(2014\)](#) for the United States. The standard deviation of productivity  $\sigma_z$  is calibrated to match the standard deviation of employment across establishments compatible with Zipf's law, so we set  $\xi_+$  equal to 1.059 from [Gabaix \(2009\)](#). The entry cost  $c_e$  and the level parameter

Table 1: Calibration to U.S. Data

Parameters	Values	Targets	
$c_e$	1.7919	Average establishment size	21.85
$\theta$	1.1610	Productivity growth rate	4.7%
$c_\mu$	0.2553	(CDF) % small establishments	92.5%
$\sigma_z^2$	0.4639	Zipf's law, $\xi_+$	1.059
$\sigma_\tau^2$	0.3764	SD log TFP	0.49
$\mu_\tau$	-0.0741	Employment growth rate	4.7%

of the productivity cost function  $c_\mu$  are calibrated to match the average establishment size and the share of small establishments in the United States in 1997 from [Hsieh and Klenow \(2009\)](#). The implied parameters values from this procedure are summarized in Table 1. We next assess the quantitative impact of increased policy distortions in this model.

## 6 Quantitative Experiments

We quantify the impact of policy distortions on establishment-level productivity, aggregate output, aggregate TFP, and other relevant variables by comparing these statistics in distorted economies relative to the benchmark economy. We highlight the quantitative impact of policy distortions in our model with dynamic effects on the distribution of establishment-level productivity relative to a version of the model where the distribution of productivity is exogenous as in [Restuccia and Rogerson \(2008\)](#) and [Hsieh and Klenow \(2009\)](#). We show that empirically-plausible policy distortions generate substantial negative effects on aggregate output and TFP. These effects are larger than the static effect of misallocation estimated in the literature. In our framework, distortions generate dynamic effects on productivity via two channels affecting the invariant distribution of productivity in the economy. First, distortions reduce the growth of establishments consistent with the empirical evidence on the life-cycle growth of plants, e.g. [Hsieh and Klenow \(2014\)](#). Second, distortions reduce

the productivity of entering establishments, which is also consistent with evidence from policy reforms that reduce misallocation and generate at the same time an improvement in the productivity of entrants. Hence, in our framework, policy distortions generate large differences in output per capita across countries—closer in line with evidence—relative to the existing literature with exogenous distributions of productivity.

## 6.1 Changes in policy distortions $\sigma_\tau$

We quantify the impact of changes in policy distortions via changes in the dispersion of distortions  $\sigma_\tau$ . As noted in the calibration section,  $\sigma_\tau$  was chosen in the benchmark economy to generate the dispersion in log TFPR of 0.49 for the United States in 1997 reported by [Hsieh and Klenow \(2009\)](#). In this section, we consider economies that feature higher dispersion in log TFPR than the United States and that are comparable in magnitude to those estimated by [Hsieh and Klenow \(2009\)](#) for 1991 India of 0.67 and for 1998 China of 0.74. Because China and India are arguably not the poorest or most distorted economies in the world, we also compute results for two other economies that are relatively more distorted than China and India, with dispersion of log TFPR of 0.85 and 1. These are plausible magnitudes of distortions as motivated by the evidence in very poor countries such as Ethiopia and Malawi, with dispersion in log TFPR that are 0.92 and 1.05.<sup>14</sup>

Although there is not an exact mapping between the dispersion in log TFPR and the static efficiency gains of factor reallocation, we are interested in comparing economies that have static gains that are in the ballpark of the empirical gains found in the literature, as this will allow us to quantify the contribution to our results of the dynamic effects on the distribution of establishment-level TFP. For this purpose, we increase  $\sigma_\tau$  to generate the dispersion in log TFPR of 0.63 corresponding to 2005 China and adjust  $\tau_e$  (which was normalized to one in the benchmark economy) to reproduce the static output gains of 30.5% reported by [Hsieh](#)

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<sup>14</sup>See, for instance, [Chen et al. \(2017\)](#) and [Restuccia and Santaaulalia-Llopis \(2017\)](#).

and Klenow (2009) for this economy relative to the gains for the United States. We then do a linear interpolation of the  $\tau_e$  adjustment for all the other values of  $\sigma_\tau$ . Since this is an approximation, the static gains do not necessarily match the gains reported in Hsieh and Klenow (2009) for all the economies, but as we discuss below, the resulting gains are fairly close in magnitude.<sup>15</sup>

Table 2 reports the results for economies that differ in the dispersion of distortions across establishments. The table reports aggregate output and measured TFP for each economy relative to that of the benchmark economy. Hence, the results reported are closely linked to the exercise in Hsieh and Klenow (2009) of calculating the aggregate output gains from reducing the dispersion in marginal revenue products in China and India to the level observed in the United States.<sup>16</sup>

Table 2: Effects of Changes in TFPR dispersion  $\sigma_\tau$

	SD(logTFPR)				
	0.49	0.67	0.74	0.85	1.00
Relative output $Y$	1.000	0.457	0.342	0.225	0.129
Relative TFP	1.000	0.450	0.334	0.218	0.122
Entry size $s_e/\bar{s}$	0.611	0.499	0.454	0.389	0.302
Life-cycle investment $\mu_z$	0.047	0.040	0.037	0.033	0.026

Notes: Output  $Y$  and total factor productivity (TFP) are reported relative to the benchmark economy. Entry size  $s_e/\bar{s}$  is defined as the size of entrants relative to the average incumbent. Life-cycle investment  $\mu_z$  is the growth rate of establishment productivity. Hsieh and Klenow (2009) report that the standard deviation of log TFPR is 0.49 for the United States in 1997, 0.67 in India in 1991, and 0.74 in China in 1998.

Our results are quite striking. For instance, the economies with dispersion in distortions of 0.67 and 0.74, as documented in Hsieh and Klenow (2009) for China and India, have

<sup>15</sup>We emphasize that the static gains from reallocation depend not only on the dispersion of log TFPR but also on the dispersion of productivity in the economy. For this reason, for example, Hsieh and Klenow (2009) report the same dispersion in log TFPR for India in 1991 and 1994 of 0.67, yet the resulting static output gains from reallocation in each case are 41.4% in 1991 and 59.2% in 1994 relative to gains from reallocation in 1997 United States. Hence, we also don't expect the model to exactly reproduce the empirical static gains from reallocation.

<sup>16</sup>We note that while Hsieh and Klenow (2009) report TFP gains, the gains reported are calculated as changes in aggregate output, however, in their static setting with constant factors and number of firms, TFP and output gains are identical.

aggregate output that is 45.7% and 34.2% of that in the benchmark economy. Economies with larger dispersion in distortions feature much lower relative output, 22.5% and 12.9% of the benchmark economy. We find similar quantitative effects for aggregate measured TFP. These results represent substantial decreases in output and TFP compared to the effects from static misallocation reported by [Hsieh and Klenow \(2009\)](#) and others. These larger effects on aggregate output and TFP arise in our model because in more distorted economies, establishments enter relatively small (low relative entry size  $s_e/\bar{s}$ ) and their productivity does not grow as much (low  $\mu_z$ ) relative to the benchmark economy. Whereas the size of entrants is 61.1% of the average incumbent in the benchmark economy, entrants are only 45.4% of the incumbent size in the economy with fifty percent more dispersion in distortions than the benchmark economy. Similarly, while the annual growth in establishment productivity is 4.7% in the benchmark economy, in the economy with fifty percent more dispersion the growth in productivity is 3.7%.

We now comment on the empirical plausibility of these results. The negative effect of distortions on the life cycle growth of firms is consistent with the evidence in [Hsieh and Klenow \(2014\)](#) where the employment and productivity growth of firms is found to be lower in more distorted economies than in the United States. We note however that the quantitative magnitude of the life-cycle effect is not nearly as large as in the data. The negative effect on the productivity of entrants is also consistent with evidence. We emphasize two types of evidence. First, lower entry productivity is consistent with a lower productivity of the median establishment in the productivity distributions reported by [Hsieh and Klenow \(2009\)](#) for China and India relative to the United States. Second, empirical studies of policy reforms have consistently found that reduced distortions are associated with higher productivity of entrants such as, for example, the study of [Pavcnik \(2002\)](#) on the trade reform in Chile and the work of [Kirwan et al. \(2012\)](#) assessing the dismantling of the Tobacco quota in the United States. More generally, the dynamic effect of distortions is consistent with the evidence in [Bustos \(2011\)](#) showing that a trade reform that increased the profitability of firms

encouraged investments in technology upgrading.

## 6.2 Amplification

To illustrate the quantitative importance of the dynamic effects in amplifying the negative impact of policy distortions on aggregate output and to relate our results with the gains from reallocation in [Hsieh and Klenow \(2009\)](#), in Table 3 we report the gains in aggregate output that arise in each economy when eliminating the dispersion in TFPR in the distorted economy relative to the gains of eliminating distortions in the benchmark economy. We decompose the total effect in aggregate output between the static gains from factor misallocation and the dynamic gains from changes in the productivity distribution. This decomposition follows our characterization of aggregate output between the static effect of factor misallocation and the dynamic effects in equation (15). Hence, the total output gain in Table 3 is the product of the static gains from factor misallocation and the dynamic gains from changes in the productivity distribution.<sup>17</sup>

In the economy with dispersion of log TFPR of 0.67, the output gains from eliminating distortions relative to the gains from eliminating distortions in the benchmark economy is 2.2-fold, that is, aggregate output in this economy would increase by 120% when eliminating distortions relative to the corresponding increase in the benchmark economy. Alternatively, this is the increase in aggregate output that results from reducing the dispersion in distortions in this economy to the level of the benchmark economy. Notice that the total increase in output is much larger than that reported in [Hsieh and Klenow \(2009\)](#) for the impact of factor misallocation in India. According to [Hsieh and Klenow \(2009, Table VI\)](#), the output gains in 1991 India from equalizing TFPR relative to 1997 U.S. gains is 41.4%. In our model, the corresponding increase in aggregate output of reducing dispersion in India to the level of the

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<sup>17</sup>To facilitate comparison with the literature, Table 3 reports the results as the ratio of aggregate output without distortions to output with distortions relative to the benchmark economy in each case. Hence, the inverse of the total effect on output coincides with the relative output reported in Table 2 for each economy.

Table 3: Changes in TFPR Dispersion  $\sigma_\tau$ 

	SD(logTFPR)				
	0.49	0.67	0.74	0.85	1.00
Relative output gains from reduced distortions:					
Static effect	1.00	1.40	1.56	1.81	2.17
Dynamic effect	1.00	1.57	1.87	2.45	3.68
Total effect	1.00	2.19	2.93	4.44	7.76

Notes: [Hsieh and Klenow \(2009\)](#) report that the standard deviation of log TFPR is 0.49 in 1997 United States, 0.67 in 1991 India, and 0.74 in 1998 China. We report the results for two other economies that are more distorted than China and India. Static gains refer to the output gains from reducing log TFPR dispersion (i.e., factor misallocation) to the level in the benchmark economy, holding aggregate factors and the productivity distribution of establishments constant. Endogenous Distribution refers to the change in the endogenous productivity distribution and Total refers to the overall impact on aggregate output. These terms follow the decomposition of aggregate output in equation (15). The inverse of the “Total effect” line corresponds to the “Relative output  $Y$ ” line in Table 2.

U.S. is 40% (Static gains 1.40 in Table 3). Hence, the increase in output arising from the reduction in factor misallocation is very close to that estimated empirically. But in our model the distribution of productivity changes generating a larger increase in output. As a result, the endogenous distribution generates a substantial amplification effect over and above the gains from eliminating static misallocation. This amplification effect on output is substantial. For the log TFPR 0.67 economy, whereas the static gain from reducing factor misallocation is 40%, the change in the endogenous distribution of productivity increases aggregate output by 57%. To put it differently, the change in the endogenous distribution accounts for 58% ( $\log(1.57)/\log(2.19)$ ) of the gains in aggregate output from reducing misallocation to the levels in the benchmark economy.

In the economy with dispersion in log TFPR of 0.74, the total increase in aggregate output from the reduction in the dispersion of distortions relative to the benchmark economy is 2.9-fold with the endogenous distribution accounting again for 58% of this increase. Economies with more distortions feature larger increases in relative output and larger contributions of the endogenous distribution. Although the purpose of these quantitative experiments



is not to account for any particular country experience, to gauge the relative importance of the amplification mechanism derived from the endogenous distribution of establishment productivity in our model, it is useful to recall that [Hsieh and Klenow \(2009\)](#) crudely estimate the gap in manufacturing productivity between the United States and China and India to be a factor between 2.3 and 2.6-fold to show that static misallocation is an important component of the factor difference. Our results indicate that the endogenous distribution component can potentially account for most of the remaining difference.

### 6.3 Life-cycle vs. entry effects

We can further decompose the dynamic effect on aggregate output in the model into the change in the life cycle growth of establishments through endogenous investment in productivity and into the entry productivity of establishments affecting entering establishment size. Table 4 reports the quantitative effect in relative aggregate output from these two terms as emphasized in equation (15) for all the economies considered.

Table 4: Changes in TFPR Dispersion  $\sigma_\tau$

	SD(logTFPR)				
	0.49	0.67	0.74	0.85	1.00
Relative output $Y$	1.00	0.46	0.34	0.23	0.13
Life-cycle productivity effect $\left(\frac{\eta+\lambda}{\eta+\lambda-\mu_{z\theta}}\right)$	1.00	0.75	0.67	0.58	0.48
Entry productivity effect $\left(\frac{s_e}{\bar{s}^{\gamma/(1-\alpha)}}\right)$	1.00	0.86	0.80	0.71	0.59

Notes: [Hsieh and Klenow \(2009\)](#) report that the standard deviation of log TFPR is 0.49 in 1997 United States, 0.67 in 1991 India, and 0.74 in 1998 China. We report the results for two other economies that are more distorted than China and India. Life-cycle and entry effects are reported relative to the benchmark economy and refer to the component of aggregate output due to the life-cycle component of productivity investment and entry size as described in equation (15), Section 4.

In Table 4, the product of the life-cycle and entry effects corresponds to the dynamic effect of changes in the productivity distribution emphasized earlier. So the remaining effect on output is the impact of static factor misallocation. For instance, in the economy with a

dispersion in log TFPR of 0.67, the effect of policy distortions on establishment investment over the life cycle generates an aggregate output that is 75% of the benchmark economy, while the entry effect generates an output that is 86% that of the benchmark economy. In other words, the effect of life-cycle investment accounts for 38% ( $\log(0.75)/\log(0.46)$ ) of the lower aggregate output, whereas the distortion in entry size accounts for 20% ( $\log(0.86)/\log(0.46)$ ). The remaining 42% is due to the static effect of factor misallocation. The contribution of these factors to aggregate output differences vary with the magnitude of distortions. For the most distorted economy in Table 4, 36% of the lower relative aggregate output is due to life-cycle investment, 26% due to the entry effect, and the remaining 38% due to static factor misallocation. We conclude that policy distortions have a substantial negative impact on aggregate output not only through factor misallocation but also through the dynamic effects of distorted incentives for life-cycle investment and entry size that affect the productivity distribution in the economy.

## 7 Conclusions

We developed a tractable dynamic model of heterogeneous producers to study the effect of distortions on the distribution of establishment-level productivity across economies. The model tractability allows us to find closed-form solutions that are useful in identifying the response of distortions on aggregate output via static factor misallocation and dynamic effects related to entry productivity as well as life cycle productivity growth. In this framework, policy distortions not only generate differences in factor misallocation as emphasized in a large literature, but also on the distribution of establishment-level productivity. We showed that empirically-reasonable policy distortions have substantial negative effects on aggregate output and TFP in this economy, effects that are orders of magnitude larger than in models with a constant exogenous distribution.

We have considered policy distortions that are uncorrelated to establishment-level productivity and nonetheless have found that these distortions have substantial negative effects on aggregate TFP. Since the literature has emphasized substantially larger productivity impacts of correlated idiosyncratic distortions, it would be interesting to explore the implications of correlated distortions in our framework. This requires a non-trivial extension of the theory and for this reason we leave this important exploration for future work. We also think that it would be interesting to explore specific policies and institutions—such as size-dependent policies, firing taxes, financial frictions—in the context of our framework with dynamic effects of distortions. These explorations of specific policies in our framework may help reconcile the empirically large effects found in the literature relative to models with a constant exogenous distribution. Broadly speaking, we think that our dynamic framework is better suited to explore the data. As a result, further progress aimed to broaden the empirical mapping of the model to the data may provide useful insights. This mapping requires reliable and comparable panel data of producers across countries. While these data are increasingly available for some countries, comparability across countries remains an important limitation. We leave these interesting and important explorations for future work.

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# A Appendix

This appendix presents the proofs of Lemma 2 and Lemma 3 and the algorithm to compute the stationary equilibrium.

## A.1 Proof Lemma 2

The constant  $A(w, r)$  solves the following polynomial equation:

$$\left[ \frac{(\lambda + R)}{(\theta - 1)} - \frac{\theta \mu_\tau}{(\theta - 1)} - \frac{\theta}{2}(\sigma_z^2 + \sigma_\tau^2) \right] A(w, r) - \left[ \frac{\theta}{c_\mu} \right]^{\frac{1}{\theta - 1}} A(w, r)^{\frac{\theta}{\theta - 1}} = \frac{m(w, r)}{(\theta - 1)}. \quad (17)$$

To provide intuition regarding the value function of the firm, we rewrite the implicit value function. From Lemma 1 we know that  $A(w, r) = \frac{c_\mu}{\theta} \mu_z^{(\theta - 1)}$ , after substituting and rewriting the non-linear term, we find:

$$\left[ \frac{\theta}{c_\mu} \right]^{\frac{1}{\theta - 1}} A(w, r)^{\frac{\theta}{\theta - 1}} = \mu_z A(w, r).$$

After this simplification, we rewrite equation (17) finding the following expression for  $A(w, r)$ :

$$A(w, r) = \frac{m(w, r)}{\lambda + R - \theta \mu_\tau - (\theta - 1) \mu_z - \frac{\theta(\theta - 1)}{2}(\sigma_z^2 + \sigma_\tau^2)},$$

where  $\mu_z$  depends on  $A(w, r)$ . ■

## A.2 Proof Lemma 3

The stationary pdf is the solution of the boundary-value problem that consists of solving

$$\begin{aligned} f''(x) - \gamma_1 f'(x) - \gamma_2 f(x) &= 0 & \text{if } x \neq 0, \\ f''(x) - \gamma_1 f'(x) - \gamma_2 f(x) &= -\gamma_3 \delta(x - 0) & \text{if } x = 0, \end{aligned}$$



where the constants  $\gamma_1$ ,  $\gamma_2$ , and  $\gamma_3$  are given by

$$\gamma_1 = \frac{2\mu_x}{\sigma^2} < 0, \quad \gamma_2 = \frac{2(\lambda + \eta)}{\sigma_x^2} > 0, \quad \gamma_3 = \frac{2\hat{b}}{\sigma_x^2} > 0.$$

We solve the boundary-value problem using Laplace transforms.<sup>18</sup> By applying Laplace transforms in equation (9), we obtain:

$$(s^2 - \gamma_1 s - \gamma_2)\mathcal{L}[f(x)] - (s - \gamma_1)f(0) - f'(0) = -\gamma_3\mathcal{L}[\delta(x - 0)]. \quad (18)$$

Using the boundary condition  $f(0) \geq 0$  and  $\mathcal{L}[\delta(x - 0)] = 1$  we find:

$$(s^2 - \gamma_1 s - \gamma_2)Y(s) = f'(0) + (s - \gamma_1)f(0) - \gamma_3, \quad (19)$$

where

$$Y(s) = \frac{f'(0) - \gamma_3 + (s - \gamma_1)f(0)}{(s^2 - \gamma_1 s - \gamma_2)}.$$

We obtain the solution by solving the Laplace inverses when  $x \neq 0$  given by:

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{1}{(s - r_1)(s - r_2)} \right] &= \frac{1}{(r_1 - r_2)} (e^{r_1 x} - e^{r_2 x}), \\ \mathcal{L}^{-1} \left[ \frac{(s - \gamma_1)}{(s - r_1)(s - r_2)} \right] &= \frac{1}{(r_1 - r_2)} [(r_1 - \gamma_1)e^{r_1 x} - (r_2 - \gamma_1)e^{r_2 x}], \end{aligned}$$

where the two roots (one positive, and one negative) are given by  $r = \frac{\gamma_1 \pm \sqrt{\gamma_1^2 + 4\gamma_2}}{2}$ . We can rewrite the final solution for this case as:

$$\begin{aligned} f(x) &= \frac{f'(0)}{(r_1 - r_2)} (e^{r_1 x} - e^{r_2 x}) + \frac{f(0)}{(r_1 - r_2)} [(r_1 - \gamma_1)e^{r_1 x} - (r_2 - \gamma_1)e^{r_2 x}] \quad \text{if } x \neq 0, \\ f(x) &= \frac{f'(0) - \gamma_3}{(r_1 - r_2)} (e^{r_1 x} - e^{r_2 x}) + \frac{f(0)}{(r_1 - r_2)} [(r_1 - \gamma_1)e^{r_1 x} - (r_2 - \gamma_1)e^{r_2 x}] \quad \text{if } x = 0. \end{aligned}$$

When  $x \neq 0$  (that is  $\forall x \in (-\infty, 0) \cup (0, \infty)$ ), we have

$$f(x) = \begin{cases} C_1 e^{r_1 x} + C_2 e^{r_2 x}, & \text{if } x < 0, \\ C_1 e^{r_1 x} + C_2 e^{r_2 x}, & \text{if } x > 0, \end{cases}$$

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<sup>18</sup>Laplace transforms are given by

$$\begin{aligned} \mathcal{L}[f'(x)] &= s\mathcal{L}[f(x)] - f(0), \\ \mathcal{L}[f''(x)] &= s^2\mathcal{L}[f(x)] - sf(0) - f'(0). \end{aligned}$$

where

$$\begin{aligned} C_1 &= \frac{1}{(r_1 - r_2)} [f'(0) + f(0)(r_1 - \gamma_1)], \\ C_2 &= \frac{-1}{(r_1 - r_2)} [f'(0) + f(0)(r_2 - \gamma_1)], \end{aligned}$$

and  $r_1 > 0$  and  $r_2 < 0$ . When  $x > 0$  in order to  $f(\cdot)$  be a pdf, it is necessary that  $C_1 = 0$  and

$$f'(0) = -f(0)(r_1 - \gamma_1) \Rightarrow C_2 = \frac{-1}{(r_1 - r_2)} [f(0)(\gamma_1 - r_1) + f(0)(r_2 - \gamma_1)] = f(0).$$

Symmetrically when  $x < 0$  we need  $C_2 = 0$ . Therefore

$$f'(0) = -f(0)(r_2 - \gamma_1) \Rightarrow C_1 = \frac{1}{(r_1 - r_2)} [f(0)(\gamma_1 - r_2) + f(0)(r_1 - \gamma_1)] = f(0),$$

and

$$f(x) = \begin{cases} f(0)e^{r_1 x} & \text{if } x < 0, \\ f(0)e^{r_2 x} & \text{if } x \geq 0, \end{cases}$$

where  $f(0) = \left( \frac{r_1 r_2}{r_2 - r_1} \right)$ . Finally we need to prove that: 1) for  $x > 0$ ,  $f'(0) = -f(0)(r_1 - \gamma_1)$  (i.e.  $C_1 = 0$ ), and 2) for  $x < 0$ ,  $f'(0) = -f(0)(r_2 - \gamma_1)$  (i.e.  $C_2 = 0$ ); Given that when  $x > 0$   $f'(0) = r_2 f(0)$  (and when  $x < 0$   $f'(0) = r_1 f(0)$ ) this is equivalent to show that

$$(r_2 + r_1)f(0) = \left( \frac{\gamma_1 - \sqrt{\gamma_1^2 + 4\gamma_2}}{2} + \frac{\gamma_1 + \sqrt{\gamma_1^2 + 4\gamma_2}}{2} \right) f(0) = f(0)\gamma_1.$$

When  $x = 0$  we have

$$f(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x},$$

where

$$\begin{aligned} C_1 &= \frac{1}{(r_1 - r_2)} [f'(0) - \gamma_3 + f(0)(r_1 - \gamma_1)], \\ C_2 &= \frac{-1}{(r_1 - r_2)} [f'(0) - \gamma_3 + f(0)(r_2 - \gamma_1)]. \end{aligned}$$

Therefore

$$f(0) = C_1 + C_2 = \frac{1}{(r_1 - r_2)} [r_1 f(0) - r_2 f(0)] = f(0).$$

Using  $s = s_e e^x$ , we can recover the size distribution  $g(s)$ . That is

$$g(s) = \frac{1}{s} f(\ln(s/s_e)) = \begin{cases} f(0) \frac{s^{r_1-1}}{s_e^{r_1}} & \text{if } s < s_e, \\ f(0) \frac{s^{r_2-1}}{s_e^{r_2}} & \text{if } s \geq s_e. \end{cases}$$

Note that this solution is equivalent to the guess and verify solution obtained by solving the characteristic equation  $\frac{\sigma_s^2}{2} \xi^2 + \left(\mu_s - \frac{\sigma_s^2}{2}\right) \xi - (\lambda + \eta) = 0$  with  $r_1 = -\xi_-$  and  $r_2 = -\xi_+$ .

Finally, average establishment size  $\bar{s}$  is given by

$$\bar{s} = s_e \frac{-\xi_- \xi_+}{(\xi_+ - 1)(1 - \xi_-)} = s_e \frac{\eta + \lambda}{\eta + \lambda - \mu_s}. \blacksquare$$

### A.3 Stationary Equilibrium

Formally  $\mu_z$ ,  $A$ ,  $s_e$ ,  $\bar{s}$ ,  $\mu_s$  and  $w$  are obtained by solving the following 6 equations.

- (1) Productivity growth rate is endogenous in distortions, that is establishment's investment,  $\mu_z$  satisfies:

$$\mu_z = \left[ \frac{\theta A(w, r)}{c_\mu} \right] \frac{1}{\theta - 1}. \quad (20)$$

- (2) Establishment's value function,  $A$  satisfies:

$$\left[ \frac{(\lambda + R)}{(\theta - 1)} - \frac{\theta \mu_\tau}{(\theta - 1)} - \frac{\theta}{2} \sigma_\tau^2 \right] A(w, r) - \left[ \frac{\theta}{c_\mu} \right] \frac{1}{\theta - 1} A(w, r) \frac{\theta}{\theta - 1} = \frac{m(w, r)}{(\theta - 1)}, \quad (21)$$

where  $m(w, r) = (1 - \alpha - \gamma) \left[ \left( \frac{\alpha}{r} \right)^\alpha \left( \frac{\gamma}{w} \right)^\gamma \right]^{\frac{1}{1 - \alpha - \gamma}}$ .

- (3) The establishment size growth rate is equal to:

$$\mu_s = \theta (\mu_z(w, r) + \mu_\tau) + \frac{\theta(\theta - 1)}{2} (\sigma_z^2 + \sigma_\tau^2). \quad (22)$$

(4) Minimum establishment's size is compatible with free entry:

$$c_e = A(w, r)s_e. \quad (23)$$

(5) The average endogenous size is given by:

$$\frac{\bar{s}}{s_e} = \frac{\eta + \lambda}{\eta + \lambda - \mu_s}. \quad (24)$$

(6) Labour demand,  $N$ , is compatible with labour supply:

$$\left[ \left( \frac{\alpha}{r} \right)^\alpha \left( \frac{\gamma}{w} \right)^{(1-\alpha)} \right]^{\frac{1}{1-\alpha-\gamma}} \bar{s} = 1. \quad (25)$$