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Firing Costs, Misallocation, and Aggregate Productivity

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Firing Costs, Misallocation, and Aggregate Productivity*

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Abstract

We assess the quantitative impact of firing costs on aggregate total factor productivity (TFP) in a dynamic general-equilibrium framework where the distribution of establishment-level productivity is not invariant to the policy. We show that firing costs not only generate static misallocation, but also a worsening of the productivity distribution substantially contributing to large aggregate TFP losses. In a calibrated version of the model, firing costs equivalent to 5 years' wages imply a drop in TFP of more than 20 percent. Consistent with the existing literature on firing costs, static misallocation only generates a small drop in TFP, accounting for around 20 percent of the total loss, whereas the remaining 80 percent arises from the endogenous change in the productivity distribution.

Keywords: firing costs, inaction, misallocation, establishments, productivity.
JEL codes: O1, O4, E1, E6.

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1 Introduction

A fundamental issue in economic growth and development is identifying the policies and institutions that account for the large differences in total factor productivity (TFP) and output per capita across countries. A recent literature has emphasized factor misallocation across heterogeneous production units for aggregate TFP differences.[1] While the empirical evidence of factor misallocation across countries is overwhelming, the connection with the specific policies and institutions that create misallocation remain elusive.[2] In this paper, we assess the quantitative impact of a specific policy—firing costs—on aggregate TFP in a framework where the distribution of establishment-level productivity is not invariant to the policy. We focus on firing costs because unlike other specific policies, firing costs are easily measurable in the data, show substantial variation across countries, and have been studied extensively in the misallocation literature. Whereas the literature has attributed a small quantitative impact of firing costs on aggregate productivity, we show that empirically-plausible measures of firing costs generate large aggregate TFP loses arising from changes in the distribution of establishment productivity.

We consider an otherwise standard model of producer heterogeneity building on the work of Hopenhayn (1992) and Hopenhayn and Rogerson (1993). The model is set up in continuous time for analytical tractability. Establishments are heterogeneous in their TFP that follows a stochastic process over time. Crucially, and differently from the related previous literature, policies that distort the size of establishments such as firing costs, have an effect on the evolution of productivity for individual establishments and, hence, on the stationary distribution of productivity across establishments and aggregate TFP. A well-known property of firing costs in the context of dynamic models of producer heterogeneity is that the policy generates

[1] See Banerjee and Duflo (2005), Restuccia and Rogerson (2008), and Hsieh and Klenow (2009).

[2] Some of the specific policies and institutions studied in accounting for factor misallocation and aggregate TFP losses include firing costs (Hopenhayn and Rogerson, 1993), size-dependent policies (Guner et al., 2008), financial frictions (Buera et al., 2011), among many others. See also Restuccia and Rogerson (2013), Restuccia (2013), and Hopenhayn (2014).

an inaction zone in employment decisions whereby small but successful establishments remain small and large but relatively less successful establishments remain large. This type of inaction in employment decisions to changes in productivity generates factor misallocation as the policy weakens the connection between the allocation of employment and productivity across establishments. In addition, in our framework we show that this policy also alters the distribution of productivity across establishments contributing to a substantial reduction (in addition to static misallocation) in aggregate productivity.

We calibrate a benchmark economy with no firing costs to micro and macro data for the United States and consider quantitative experiments that increase the size of the firing cost—the cost for an individual establishment to reduce employment—with a range from one month's wages to 5 years' wages. Relative to the benchmark economy with no firing costs, aggregate TFP in the economy with a firing cost of 6 month's wages is 0.96 and in the economy with 5 years' wages 0.79. These are very large TFP losses when compared to the previous literature in the context of models with exogenous distributions of establishment productivity, e.g. Hopenhayn and Rogerson (1993), Moscoso-Boedo and Mukoyama (2012), and Hopenhayn (2014).[3] Interestingly, when we decompose the total effect of firing costs on aggregate TFP between static misallocation (misallocation of labor across establishments) and dynamic misallocation (changes in the productivity distribution of establishments), we find that dynamic misallocation accounts for around 80 percent of the total effect. This implies that the quantitative effect of static misallocation is of similar magnitude than that in the existing literature.

A desirable property of our framework is that we are able to provide direct analytical results on the main variables of interest. Following the seminal work of Dixit (1989) in the benchmark economy with no firing costs there is only one threshold productivity for which large establishments become small and small establishments become large, in economies with

[3]See also the partial equilibrium analysis in Bentolila and Bertola (1990).

firing costs there is an inaction zone, a range of productivity for which establishments remain either large or small. Additionally, the inaction zone becomes larger with increases in firing costs and general equilibrium effects shift the inaction zone towards lower levels of productivity. This property of establishment decisions entails misallocation: large but less productive establishments remain large and small but more productive establishments remain small. As a result, relative to an undistorted economy, more employment is allocated in less productive establishments and less employment in more productive establishments with firing costs.

Furthermore, we fully characterize the endogenous productivity distribution of establishments and show how changes in firing costs impact the distribution by making the distribution flatter and reducing its average productivity. We solve the model using Laplace transforms techniques, which allows us to fully characterize how changes in firing costs impact the rate at which establishments adjust their size by firing or hiring workers. This characterization allows us to connect the model with well-known moments in the data such as job turnover and the Gini coefficient of the establishment-size distribution. In the model, higher firing costs reduce both the average productivity of the economy and job turnover.

As discussed earlier, our paper relates closely to the literature assessing the aggregate productivity losses of firing costs such as the seminal work of Hopenhayn and Rogerson (1993) and the more recent analysis in Moscoso-Boedo and Mukoyama (2012) and Hopenhayn (2014). A critical distinction between our framework and these previous works is that firing costs affect the distribution of productivity which greatly contribute to amplify the negative aggregate productivity implications of the policy. More broadly, our paper shares with a growing literature emphasizing the dynamic effects of distortionary policies.[4] We differ from this literature in quantifying the effect of a measurable specific policy. A key property of the distortionary effects of firing costs is its connection with the inaction zone in employment

[4] See for instance Hsieh and Klenow (2014), Da-Rocha et al. (2014), among many others.

decisions. In this respect, firing costs shares with many other policies and institutions that tend to produce inaction zones.[5] For example, in our framework, there is an equivalence between firing and hiring costs. We use this equivalence to relate our work to previous analysis of size-dependent policies such as the work by Guner et al. (2008) and more recently the empirical analysis in Gourio and Roys (2014) and Garicano et al. (2013) using micro data from France. Our analysis reveals the importance of considering the dynamic productivity effects of the policy for a more accurate quantitative assessment of the aggregate impacts of these policies.

The remainder of the paper is organized as follows. In the next section and Section 3, we describe the economic environment in detail and characterize its main properties. Section 4 calibrates a benchmark economy with no firing costs to data for the United States and perform a series of quantitative experiments to asses the aggregate implications of firing costs. We conclude in Section 5. Appendix A contains the formal proofs of all the lemmas in the paper.

2 Model

Our framework builds from the work of Hopenhayn (1992), Hopenhayn and Rogerson (1993), and Dixit (1989). Establishments hire labor in a competitive market and their productivity follows a stochastic process. Time is continuous and the horizon is infinite. We focus on a stationary equilibrium of this model and study the impact of firing costs on aggregate measures of TFP and output.

[5] See for instance a trade model of inaction (decision to export or not) in Impullitti et al. (2013).

2.1 General description

The unit of production in the economy is the establishment. Establishments are heterogeneous in their productivity z. They are described by a production function $f(z,n)$ that uses labor to produce output. The function f is assumed to exhibit decreasing returns to scale in labor and to satisfy the usual Inada conditions. The production function is given by:

$$f(z,n) = zn^{\alpha}, \quad \alpha \in (0,1).$$

We assume for simplicity that establishments can only hire two different amounts of labor n_1 and n_2, where n_2 is larger than n_1. Establishment's productivity z follows a Geometric Brownian motion, that depends on the establishment size, the Geometric Brownian motions are given by:

$$dz = \mu_1 z dt + \sigma_z z dw_z \quad \text{and} \quad dz = \mu_2 z dt + \sigma_z z dw_z,$$

where the drift of the Brownian motion μ_i depends on the establishment's size and the standard deviation σ_z is the same for both sizes. There is a mass one of infinitly-lived households with preferences over consumption goods and labor supply described by the utility function,

$$\int_0^{\infty} e^{-\rho t} \left[u(c) - v(n) \right] dt,$$

where c is consumption, n is labor supply, and ρ is the discount rate. Households own equal shares of the establishments. We next introduce firing costs that distort the decision of establishments to adjust their size.

2.2 Policy distortions

In a distorted economy, we assume that establishments have to pay a firing cost τ in units of labor per worker in order to reduce employment from large n_2 to small n_1. The firing costs policy creates inertia in employment decisions because establishments delay their decisions of firing and hiring workers and consequently adjusting their employment size. We assume that the revenue from firing costs paid by establishments is redistributed to households in the form of a lump-sum transfer T.

2.3 Incumbents' problem

Incumbents maximize the present value of profits. At each point in time, establishments observe their current TFP shock z and their employment size n_i, where $i \in \{1, 2\}$, and decide to keep their current size or to adjust by hiring or firing workers. This is a standard optimal switching problem described by Dixit (1989). The problem is characterized by the value function at the current state and by the value matching condition at the switching points. We first describe the dynamic problem of a small incumbent n_1 and then we describe the dynamic problem of a large incumbent n_2.

Small establishments observe their productivity and choose to keep their current size n_1 or to hire workers and become large n_2. They receive revenue from selling output and pay a wage bill at every point in time. Formally, the dynamic problem of a small establishment is defined by:

$$\begin{aligned} \rho V_1(z) \quad &= \quad z n_1^{\alpha} - w n_1 + E_z \frac{dV_1(z)}{dt}, \\ s.t. \quad & dz = \mu_1 z dt + \sigma_z z dw_z, \end{aligned} \tag{1}$$

and by the value matching condition at the switching point z_1 where small establishments hire workers and become larger, $V_1(z_1) = V_2(z_1)$, and the smooth pasting condition at the switching, $V_1'(z_1) = V_2'(z_1)$.

Large establishments observe their current productivity z and choose to keep their current size n_2 or to fire workers and become small n_1, paying firing costs $\tau w(n_2 - n_1)$. Large establishments receive revenue from selling output and pay a wage bill. Formally, the dynamic problem of a large establishment is defined by:

$$\begin{aligned} \rho V_2(z) &= z n_2^{\alpha} - w n_2 + E_z \frac{dV_2(z)}{dt}, \\ s.t. \quad dz &= \mu_2 z dt + \sigma z dw_z, \end{aligned} \tag{2}$$

and by the value matching condition at the switching point z_2, where large establishments are indifferent between paying the firing costs τ and become small or being large, $V_1(z_2) - \tau w(n_2 - n_1) = V_2(z_2)$, and the smooth pasting condition at the switching point, $V_1'(z_2) = V_2'(z_2)$. In Lemma 1 we characterize the value function of small $V_1(z)$ and large $V_2(z)$ establishments and the two policy functions $\{z_1, z_2\}$.

Lemma 1. *Given a wage rate w, interest rate ρ, and firing costs τ, the value function of small establishments $V_1(z)$ and the value function of large establishments $V_2(z)$ that solve the small establishment's problem (1) and the large establishment's problem (2) are given by*

$$V_i(z) = \frac{{n_i}^{\alpha}}{\rho - \mu_i} z - \frac{w n_i}{\rho} + B_i z^{\beta_i},$$

where $\beta_i = -\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}$ for $i \in \{1, 2\}$, and the constants $\{B_1, B_2\}$ and the policy functions $\{z_1, z_2\}$ solve the two value matching conditions and the two smoothing pasting conditions together.

Proof See Appendix A.1.

In order to fully characterize the impact of firing costs on the optimal decision of establishments, we choose the positive root β_1 for small establishments and the negative root β_2 for large establishments. The positive root for small establishment has the desirable property that the option value of becoming larger increases when the productivity increases, while the negative root for large establishment has the desirable property that the option of becoming smaller decreases when the productivity increases. In the next Lemma 2, we show that B_1 and B_2 are positive.

Lemma 2. *If β_1 is the positive root and β_2 is the negative root, then B_1 and B_2 are positive.*

Proof See Appendix A.2.

The value functions of large and small establishments have an intuitive interpretation, where the first two terms are the present value of being a small or a large establishment when switching is not allowed and the the last term is the present value of the switching option. Changes in the firing costs have two effects on the incumbents' problem. It has a direct effect on the present value of being large and small through the constants B_1 and B_2 and a general equilibrium effect through changes in wages. In the next Lemma 3 we characterize these two effects.

Lemma 3. *Given a wage rate w, interest rate ρ, and firing costs τ.*

1) The inaction rate, $\theta = z_2/z_1$, is the solution of the following non-linear equation:

$$\varphi(\theta) = \frac{(\Omega_1(\theta)+1)}{(\Omega_1(\theta)\theta^{\beta_1}+\theta^{\beta_2})} \quad = \quad \frac{1}{1-\rho\tau}, \tag{3}$$

where $\Omega_1(\theta) = \frac{(1-\beta_1)\beta_2}{(1-\beta_2)\beta_1}\frac{(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)}{(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}$.

2) The policy functions z_1 and z_2 are given by:

$$z_1 = \kappa\Omega_2(\theta)w \quad \textit{and} \quad z_2 = \theta z_1, \tag{4}$$

where $\Omega_2(\theta) = \frac{\beta_1\beta_2(\theta^{\beta_2-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}{((1-\beta_1)\beta_2(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)+(1-\beta_2)\beta_1(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1))}$.

Proof See Appendix A.3.

Lemma 3 is key to understand the model dynamics. In the first part of Lemma 3, we characterize the inaction rate θ, which is a function of the productivity process, summarized by β_1 and β_2, the interest rate ρ, and the firing cost τ. Overall, an increase in firing costs generates a decrease in the inaction rate. In an economy without firing costs, the inaction rate is equal to one and establishments do not delay their decision of firing and hiring workers. In this case, there is a unique switching point and no inaction zone.

An important result from Lemma 3 is that the inaction rate is independent of prices, but the policy functions are linear in prices. A reduction in the wage rate moves the policy functions to the left, reducing the average productivity in the economy. Consequently, the final impact of an increase in firing costs on the inaction zone depends on the combination of the impact on the inaction rate, summarized by θ, and on the general equilibrium impact on the wage rate w.

In Figure 1, we illustrate these two mechanisms. In the left panel, we illustrate how a decrease in the inaction rate, increases the inaction zone measured by the area between z_1 and z_2. In the right panel, we illustrate how an increase in firing costs τ reduces the inaction rate θ^*.

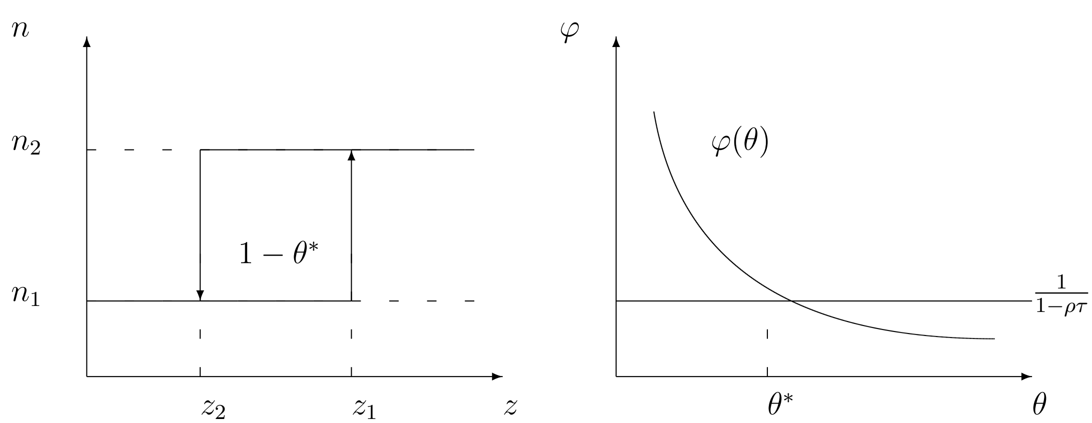


Figure 1: Inaction Zone and Inaction Zone Rate

The results in Lemma 3 are not restricted to firing costs. Instead, these results can be easily extended to the case where the costs are on hiring workers instead of firing and this is relevant empirically as many labor market policies generate costs associated with hiring workers above a certain threshold size. We establish an equivalence between firing costs and hiring costs. The equivalence allows us to relate our results to the broad literature on size-dependent policies in Guner et al. (2008) and Gourio and Roys (2014).[6] We can rewrite the small and large establishment's problem where establishments face hiring costs τ_h instead of firing costs τ. The new value-matching conditions are:

$$
\begin{aligned}
V_1(z_1) &= V_2(z_1) - \tau_h w(n_2 - n_1), \\
V_1(z_2) &= V_2(z_2),
\end{aligned}
$$

and the new smoothing pasting conditions are the same in both the firing costs and hiring costs problem. In Lemma 4 we show that solving the model with hiring costs is equivalent to solving the model with firing costs.

Lemma 4. *Given hiring costs τ_h, there is firing costs τ that generates the same inaction zone rate, given by:*

$$
\frac{1}{1 - \rho\tau} = 1 + \rho\tau_h,
$$

where ρ is the interest rate.

Proof See Appendix A.4.

Lemma 4 demonstrates that there is a simple relationship between firing costs τ and hiring costs τ_h. Given firing costs τ, we can find hiring costs τ_h that generates the same inaction zone, and consequently the same equilibrium in both economies. In the next section, we characterize the stationary distribution.

[6] Gourio and Roys (2014) study a size-dependent regulation in the form of firing costs in France that only apply to establishments with 50 or more employees.

2.4 Stationary distribution

We first characterize the stationary distribution in a distorted economy with firing costs and then solve for the stationary distribution of the undistorted economy.

2.4.1 Distorted economy

We characterize the solution of the stationary distribution of large establishments, leaving the solution of the stationary distribution of small establishments to the Appendix. In order to solve for the stationary distribution, it is easier to work in the logarithm of the establishment productivity z instead of levels. Let x be the logarithm of an establishment with productivity z and size i relative to the switching point z_i, that is $x = \log(z/z_i)$. The variable x is equal to zero at the switching point and has domain in $[0, +\infty)$.

Let $m_2(x,t)$ denote the number density function of large establishments with productivity x at time t. At time t, the total number of large establishment is equal to the integral from zero to plus infinity of the number density function, $M_2(t) = \int_0^{+\infty} m_2(x,t)dx$. The large establishments' productivity process can be characterized by a *modified* Kolmogorov-Fokker-Planck equation of the form:

$$\frac{\partial m_2(x,t)}{\partial t} = -\hat{\mu}_2 \frac{\partial m_2(x,t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 m_2(x,t)}{\partial x^2} + B_2(x,t), \tag{5}$$

where the drift $\hat{\mu}_i$ is equal to $\mu_i - \frac{\sigma_z^2}{2}$ and the function $B_2(x,t)$ is the mass of new large establishments that arrive with productivity x at time t.[7] The *modified* Kolmogorov-Fokker-Planck equation (5) is supplemented by two boundary conditions:

$$\lim_{x \to +\infty} m_2(x,t) = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} \frac{\partial m_2(x,t)}{\partial x} = 0.$$

[7] We can rewrite the Geometric Brownian motion of large and small establishments as a Brownian motion in the logarithm of the establishment productivity z_i as $dx_i = \hat{\mu}_i dt + \hat{\sigma}_z dw_z$.

The two boundary conditions guarantee that there is no establishments' mass at the upper limit and that the function is smooth at the upper limit. We are interested in the stationary distribution of the number density function and consequently we are looking for a solution that is separable in time, that is $m_2(x,t) = M_2(t)f_2(x)$ and $B_2(x,t) = M_2(t)b_2(x)$; where $f_2(\cdot)$ is the large establishments' probability density function and $b_2(x)$ is a delta Dirac function that describes the arrival of new establishments. After making this restriction, we can rewrite Kolmogorov-Fokker-Planck equation (5) as:

$$\frac{M_2'(t)}{M_2(t)} f_2(x) = -\hat{\mu}_2 f_2'(x) + \frac{\sigma^2}{2} f_2''(x) + b_2(x), \tag{6}$$

where $\frac{M_2'(t)}{M_2(t)}$ is the growth rate of establishments over time denoted by η_2 and two boundary conditions:[8]

$$\lim_{x \to +\infty} f_2(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} f_2'(x) = 0. \tag{7}$$

These two conditions are supplemented by the additional conditions that guarantee that $f_2(\cdot)$ is a pdf:

$$f_2(x) \geq 0 \quad \text{and} \quad \int_0^{+\infty} f_2(x)dx = 1. \tag{8}$$

In our model, arrival at the large establishment's distribution occurs at the switching point $x_1 = \log(z_1/z_2)$. In this point, small establishments choose to hire more workers and become large. Mathematically, we can express the arrival by using a Dirac delta function:

$$\delta(x - x_1) = \begin{cases} +\infty & \text{if} \quad x = x_1 \\ 0 & \text{if} \quad x \neq x_1, \end{cases} \tag{9}$$

[8]Note that we can rewrite the separation rate as $M_2(t) = e^{\eta_2 t} M_2(0)$. When η_2 is equal to zero the mass of large establishments does not grow over time.

the function is equal to infinity at the point in which new establishments enter x_1 and zero otherwise. Let $\hat{b}_2$ be the arrival rate at point x, the function $b_2(x)$ can be described as:

$$b_2(x) = \hat{b}_2 \delta(x - x_1). \tag{10}$$

The Dirac delta function has two desirable properties: First, the integral over the domain is the arrival rate $\hat{b}_2$, $\int_0^\infty \hat{b}_2 \delta(x - x_1) dx = \hat{b}_2$ and second, the integral weighted by the density function is the mass of establishments at the switching point $\int_0^\infty \delta(x - x_1) f_2(x) dx = f_2(x_1)$.

The constraints in equations (7) to (10) restrict the separation rate η_2, after integrating the Kolmogorov-Fokker-Planck equation (6), applying the boundary conditions, and using the Dirac delta function properties, we find that growth rate of large establishments is given by:

$$\eta_2 = -\hat{\mu}_2 f_2(0) + \frac{\sigma^2}{2} f_2'(0) + \hat{b}_2. \tag{11}$$

The expression for η_2 has a very intuitive interpretation, it states that the growth rate of large establishments η_2 is equal to the rate at which small establishments decide to hire workers and become large $\hat{b}_2$, minus the rate at which large establishments decide to fire workers and become small, $\hat{\mu}_2 f_2(0) - \frac{\sigma^2}{2} f_2'(0)$. As a result, the large establishments' mass grows when the entry rate is larger than the exit rate, and it is constant when the entry rate is equal to the exit rate. Since we are solving for the stationary equilibrium, we restrict the solution to an economy with a constant mass of establishments, that is η_2 equal to zero. We can now solve for the stationary distribution $f_2(\cdot)$.

After substituting η_2 equal to zero into the Kolmogorov-Fokker-Planck equation (6), we find the following second order differential equation:

$$-\hat{\mu}_2 f_2'(x) + \frac{\sigma^2}{2} f_2''(x) + \hat{b}_2 \delta(x - x_1) = 0, \tag{12}$$

which supplemented by the boundary conditions and by $f_2(\cdot)$ being a pdf completly characterize the problem. The presence of the Dirac delta functions in (12) indicates that we cannot expect classical solutions to the problem in $C^2[0,+\infty)$. We thus look for solutions $f_2(x)$ such that $f_2(x) \in C^2[0,+\infty) \cup L^1[0,\infty)$ and $f_2(x)$ has continuous second derivatives for all $x \in [0,+\infty)$ except perhaps at $x = x_1$.

We solve the second order differential equation using Laplace transforms.[9] In order for the probability density function to be bounded we need to impose a boundary condition at 0. This boundary condition guarantees that the mass of establishments that are indifferent between switching or not is equal to zero, $f_2(0) = 0$. In the next Lemma 5 we characterize the small and large establishments' stationary distributions.[10]

Lemma 5. *Let $x_1 = \log(z_1/z_2)$ be the entry point, where small establishments that become large enter the large establishment distribution, and let $x_2 = \log(z_2/z_1)$ be the entry point, where large establishments that become small enter the small establishment distribution. The stationary distribution of small establishments $f_1(\cdot)$ and the stationary distribution of large*

[9]Laplace transforms are given by

$$\begin{aligned} \mathscr{L}[f'(x)] &= s\mathscr{L}[f(x)] - f(0), \\ \mathscr{L}[f''(x)] &= s^2\mathscr{L}[f(x)] - sf(0) - f'(0), \\ \mathscr{L}[\delta(x - x_*)] &= e^{-sx_*} \end{aligned}$$

After applying Laplace transforms to equation (12), we find the following Laplace transforms equation:

$$(s^2 - \gamma_1 s)Y(s) = f_2'(0) + (s - \gamma_1)f_2(0) - \gamma_2 e^{-sx_1}.$$

After applying the Laplace inverse and some algebraic manipulation, we find the following solution to the boundary value problem:

$$f_2(x) = \frac{f_2'(0)}{\gamma_1}\left(e^{\gamma_1 x} - 1\right) - \gamma_2 \frac{H(x_1)}{\gamma_1}\left[e^{\gamma_1(x - x_1)} - 1\right] + f_2(0),$$

where $H(x_1)$ is Heaviside step function given by:

$$H(x_1) = \begin{cases} 0 & \text{if} \quad x \leq x_1, \\ 1 & \text{if} \quad x > x_1. \end{cases}$$

[10] The characterization of the stationary distribution of small establishments follows the same methodology and we leave it to the Appendix A.5.

establishments $f_2(\cdot)$ *are given by:*

$$f_1(x) = \begin{cases} \frac{-1}{x_2}(e^{\alpha_1(x-x_2)} - e^{\alpha_1 x}) & \forall x \in (-\infty, x_2] \\ \\ \frac{-1}{x_2}(1 - e^{\alpha_1 x}) & \forall x \in (x_2, 0) \end{cases} \quad f_2(x) = \begin{cases} \frac{1}{x_1}(1 - e^{\gamma_1 x}) & \forall x \in [0, x_1) \\ \\ \frac{1}{x_1}(e^{\gamma_1(x-x_1)} - e^{\gamma_1 x}) & \forall x \in (x_1, +\infty) \end{cases}$$

where $\gamma_1 = \dfrac{2\hat{\mu}_2}{\sigma^2}$ *and* $\alpha_1 = \dfrac{2\hat{\mu}_1}{\sigma^2}$.

Proof See Appendix A.6.

In Figure 2 we illustrate the stationary distribution of small and large establishments, and the stationary equilibrium dynamics. A small establishment hire workers at x equal to zero, at this point the establishment leaves the small establishments' distribution and enter the large establishments' distribution at point x_2. This establishment will stay at the large establishments' distribution until its productivity reaches the point x equal to zero, at this point this establishment fires workers, becomes small again, and a new cycle starts. The rectangular area between the entry points and zero is the inaction zone caused by the firing costs policy, in the absent of firing costs large and small establishments switch at the same point and there is no inaction zone. In the next section, we examine the stationary distribution of an undistorted economy.

2.4.2 Undistorted economy

We characterize the stationary distribution of the undistorted economy following the same methodology as in the distorted economy. In the undistorted economy, there is an unique switching point z_*. We solve the economy again in logs and after the renormalization, the switching point is equal to zero, $x_* = \log(z_*/z_*) = 0$. The stationary distribution of large establishments in the undistorted economy is the solution of a *modified* Kolmogorov-Fokker-Planck equation that after following the same steps as in the distorted economy we obtain

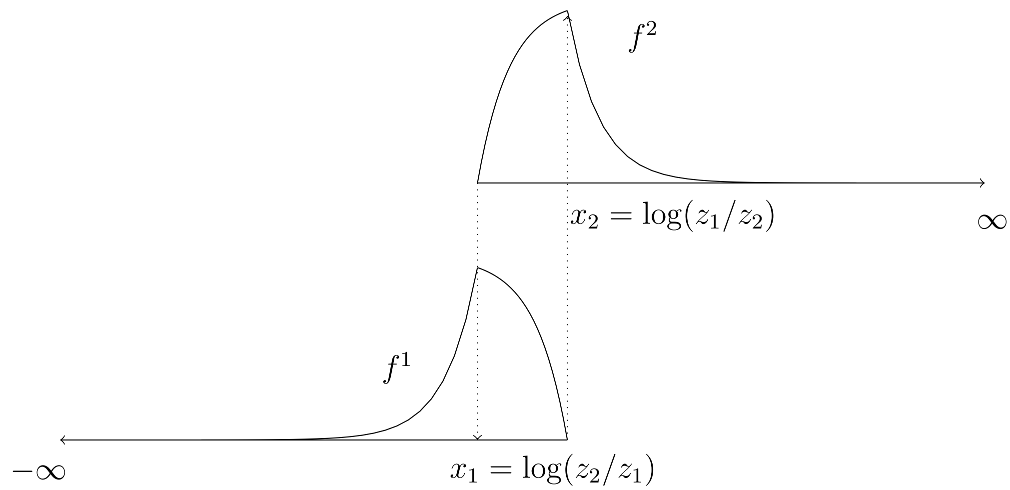


Figure 2: Stationary Distribution Dynamics

the following differential equation:

$$-\hat{\mu}_2 f_2'(x) + \frac{\sigma^2}{2} f_2''(x) + \hat{b}_2 \delta(x) = 0, \tag{13}$$

subject to the respective boundary conditions and $f_2(\cdot)$ being a pdf. The main difference between the distorted and the undistorted economy is that in the undistorted economy there is no inaction zone and establishments hire and fire workers in the same point x_*. We solve the boundary-value problem using Laplace transforms.[11] After applying the Laplace transforms, we find that the large establishments' boundary conditions are satisfied only when μ_2 is negative, and following the same methodology to small establishments, we find

[11] After applying Laplace transforms on the large establishments' second order differential equation (13) we obtain the following differential equation:

$$(s^2 - \gamma_1 s) Y_2(s) \quad = \quad (f_2'(0) - \gamma_1 f_2(0) - \gamma_2) + s f_2(0),$$

because the growth rate of large establishments is zero in equilibrium, the second term between parenthesis must be zero, leading to the following solution to the stationary distribution of large establishments:

$$Y_2(s) = \frac{f_2(0)}{(s - \gamma_1)},$$

and after solving for the Laplace inverse we obtain the following pdf $f_2(x) = f_2(0)e^{\gamma_1 x}$. We solve for the constants $f_2(0)$ to guarantee that the integral over the domain is equal to one.

that the small establishments' boundary conditions are satisfied only when μ_1 is positive. In Lemma 6 we formalize this result.

Lemma 6. *If γ_1 is negative and α_1 is positive, the stationary distribution of small establishments $f_1(\cdot)$ and the stationary distribution of large establishments $f_2(\cdot)$ are given by:*

$$\begin{aligned} f_1(x) &= \alpha_1 e^{\alpha_1 x}, \quad x \leq 0, \\ f_2(x) &= -\gamma_1 e^{\gamma_1 x}, \quad x \geq 0, \end{aligned}$$

where $\gamma_1 = \frac{2\hat{\mu}_2}{\sigma^2}$ and $\alpha_1 = \frac{2\hat{\mu}_1}{\sigma^2}$.

Proof See Appendix A.7.

In the next section, we discuss the necessary equilibrium conditions to find a stationary equilibrium.

2.5 Entry and exit balance condition

Our focus is on a stationary equilibrium in which the distribution of small establishments $f_1(\cdot)$ and large establishments $f_2(\cdot)$ are constant, and the mass of small M_1 and large establishments M_2 are also constant. In order to guarantee that in the stationary equilibrium, the mass of small and the mass of large establishments are constant, it is necessary that the mass of establishments that leaves an establishment's size distribution is the same as the mass of establishments that enter on the other distribution.

The mass of establishments that leaves a distribution is equal to the total mass of establishments multiplied the rate at which establishments reach the switching points, normalized to zero, and the mass of establishments that arrives is equal to the born rate multiplied by the mass. This condition gives rise to the following two equilibrium conditions one to small and another one to large establishments:

$$M_1\left(-\hat{\mu}_1 f_1(0)+\frac{\sigma_z^2}{2}f_1'(0)\right) = -M_2\hat{b}_2, \tag{14}$$

$$-M_2\left(-\hat{\mu}_2 f_2(0)+\frac{\sigma_z^2}{2}f_2'(0)\right) = M_1\hat{b}_1. \tag{15}$$

These two conditions guarantee that the total of mass of establishment is constant in the stationary equilibrium.

2.6 Household's problem

Households solve a static consumption-leisure maximization problem:

$$\max_{c,n}\left[u(c)-v(n)\right],$$

subject to the budget constraint $c = wn + \Pi + T$, where the right-hand side of the budget constraint is given by the wage income wn, the lump-sum transfer given by the government T, and the total profits of operating establishments Π. Now, we are ready to define the equilibrium.

2.7 Equilibrium definition

Definition The stationary equilibrium for this economy is an stationary distribution for small and large establishments $\{f_1(\cdot), f_2(\cdot)\}$, a value function for small and large establishments $\{V_1(\cdot), V_2(\cdot)\}$, a mass of small and large establishments $\{M_1, M_2\}$, a policy function for small and large establishments $\{z_1, z_2\}$, prices $\{w, \rho\}$, profits Π, transfer T, and household allocations $\{c, n\}$, such that:

i) Given prices and profits, the allocations $\{c, n\}$ solve the household's problem.

ii) Given prices, incumbents' policy functions $\{z_1, z_2\}$ and value functions $\{V_1(\cdot), V_2(\cdot)\}$ solve the incumbents' problem.

iii) The stationary distributions $\{f_1(\cdot), f_2(\cdot)\}$ solve the Kolmogorov forward equations and determine aggregate profits.

iv) Labor market clears.

v) The entry and exit balance conditions are satisfied.

vi) The government budget constraint is satisfied, $T = \tau(n_2 - n_1)w$.

vii) Mass condition $M = M_1 + M_2$.

Conditions (i) and (ii) are standard. Condition (iii) is the key condition to find the stationary distribution. Condition (iv) is the labor market clearing and condition (v) guarantees that the total mass of establishments is constant. Last, condition (vi) guarantees that the government budget constraint is satisfied and condition (vii) that the mass of establishments clears. In the next section, we characterize the stationary equilibrium.

3 Solution and Characterization

The model is very tractable and we characterize key stationary equilibrium properties in more detail. From the entry and exit balance condition, we characterize the mass of small and large establishments. After substituting the stationary distributions from Lemma 5 into equations (14) and (15) we obtain the following condition:

$$M_1 \frac{\hat{\mu}_1}{\hat{\sigma}^2} = -M_2 \frac{\hat{\mu}_2}{\hat{\sigma}^2}. \tag{16}$$

From this relation we obtain the mass of establishments as a function of the stochastic process. In the case where the drift of small establishments is larger in absolute value than the drift of large establishments, the above expression implies that in the stationary equilibrium the mass of small establishments is smaller than the mass of large establishments. This guarantees that the flows of large and small establishments is constant in the stationary equilibrium.

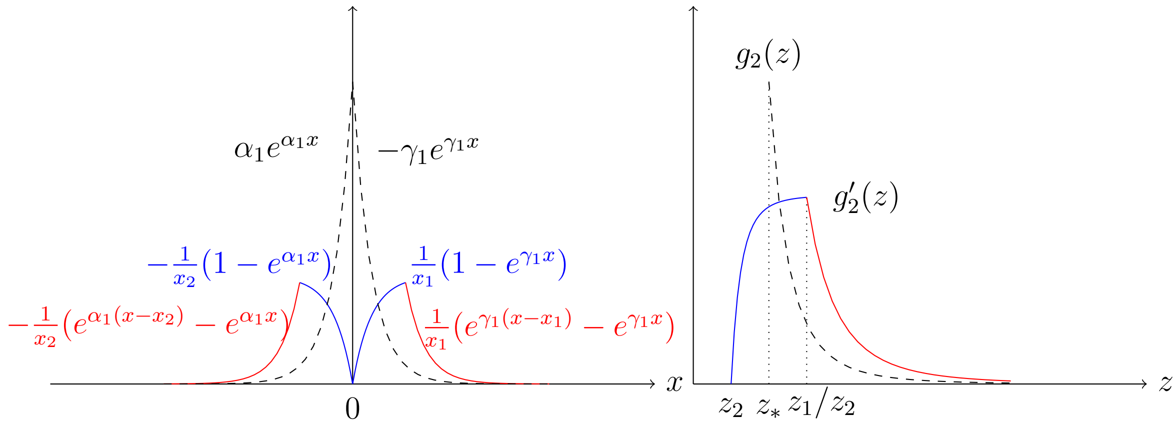
We normalize the total mass of establishments to be equal to one. Finding the share of small and large establishments consist in solving equation (16). Solving the stationary equilibrium is very simple now, for a given firing cost τ, from Lemma 3, we find the inaction rate. Since profits are linear in wages, the equilibrium wage can be found by solving the labor market clearing condition, and then after applying the second part of Lemma 3, we find the policy functions and then the stationary distribution.[12]

In Figure 3, we illustrate the impact of firing costs on the stationary distribution. Panel (a) illustrates the impact of firing costs on the entire distribution, whereas panel (b) focus on the distribution of large establishments.

The distribution of TFP in the benchmark economy denoted by f is a combination of the TFP distribution of large and small establishments. An increase in firing costs makes the distribution flatter because of inaction. In panel (b), the effect of firing costs is decomposed into two effects: (i) static misallocation - caused by large establishments that wait longer to switch reducing the average productivity of large establishments, this is represented by the area between z_2 and z_*, and (ii) dynamic misallocation - caused by the impact firing costs on selection of establishments that enter the large establishments' distribution. When there are firing costs small establishments wait for a larger productivity to hire workers and become large, as result, small establishments enter the large establishment's distribution with higher productivity. This effect is represented by the area after z_1/z_2 between the undistorted economy's density function and the distorted economy's density function.

Since an objective of the paper is to understand the impact of firing costs on aggregate TFP,

[12] We leave the formal solution of the stationary equilibrium to the Appendix A.8.



(a) Distribution of (log) TFP

(b) Misallocation Decomposition

Figure 3: Distribution Dynamics

Notes: The figure reports the stationary distribution of establishment TFP for undistorted and distorted economies. Dashed lines represent the distribution of the undistorted benchmark economy and solid lines for the distorted economy. Panel (a) reports the distributions of log establishment TFP for the entire economy. Panel (b) reports the distribution of TFP levels of large establishments. The TFP level, z_* is the switching point in the benchmark economy, whereas (z_2, z_1) is the inaction zone in the distorted economy.

we now characterize aggregate TFP in the model as:

$$\text{TFP} = (1-s)E_1 z + sE_2 z,$$

which is the weighted average of the productivity of large and small establishments.[13] Changes in firings costs impact aggregate TFP by altering the distribution of large and small establishments.

4 Quantitative analysis

We consider a benchmark economy with no firing costs and calibrate this economy to data for the United States. We then study the quantitative impact of firing costs.

[13]The weight s is given by $s = \frac{M_1 n_1^\alpha}{M_1 n_1^\alpha + M_2 n_2^\alpha}$ and M_1 is given by $M_1 = \frac{-\hat{\mu}_2}{\hat{\mu}_2 - \hat{\mu}_1}$, where we assume that $M_1 + M_2 = 1$.

4.1 Calibration

We calibrate a benchmark economy with no firing costs to data for the United States. Our main objective is to study the quantitative impact of firing costs on the distribution of establishments and on aggregate outcomes relative to the undistorted economy in the same spirit of Restuccia and Rogerson (2008) and Hsieh and Klenow (2009). We start by defining briefly our benchmark undistorted economy.

To calibrate this economy, we start by selecting a set of parameters that are standard in the literature, these parameters have either well-known targets which we match or the parameter values have been well discussed in the literature. Our calibration follows closely Hopenhayn and Rogerson (1993). A model period is set to 5 years. Preferences are given by the following utility function:

$$[u(c) - v(n)] = \frac{c^{1-\eta}}{1-\eta} - An.$$

We select η to be equal to 0.5 and normalize A to be equal to 1. We select M_1 equal to 0.5 and we focus on a symmetric stationary equilibrium where the share of large and small establishments is the same. We normalize the size of small establishments n_1 to be equal 1 and we choose the size of large establishments n_2 to be equal to 124.2 to match the average size of establishment equal to 61.7 from Hopenhayn and Rogerson (1993).

Following the literature we assume decreasing returns in the establishment-level production function and set α equal to 0.64, e.g. Hopenhayn and Rogerson (1993). We select the discount rate ρ to generate an annual real interest rate of 0.04.

We calibrate the remaining two parameters $\{\mu_1, \sigma_z^2\}$ by solving the stationary equilibrium so that two moments in the model match two corresponding targets in the data. We construct the following two statistics in the model and match with the corresponding targets in the data:

(1) Job turnover:

$$\text{Job Turnover} \propto \hat{b}_2 = \hat{\mu}_2 f_2'(0) - \frac{\sigma^2}{2} f_2''(0),$$

where $\hat{b}_2$ is the endogenous flow.[14]

(2) Gini coefficient of large establishment's size distribution:

$$\text{Gini} = \frac{-1}{(\frac{4\hat{\mu}_2}{\sigma^2} + 1)}.$$

The two parameters that define the productivity's stochastic process are selected simultaneously. These parameters are selected to match the job turnover rate in the United States, that according to Hopenhayn and Rogerson (1993) is 0.30, and to match the the Gini coefficient reported by Luttmer (2010) for large firms, which is equal to 0.89. Table 1 summarizes the calibrated parameter values.

Table 1: Benchmark Calibration to U.S. Data

Parameter	Value	Target
A	1	Normalization
α	0.64	Literature
ρ	0.04	Literature
η	0.05	Literature
n_1	1.00	Normalization
n_2	124.20	Average firm size
M_1	0.50	Employment share
μ_1	0.27	Job turnover rate
σ_z	0.52	Gini coefficient

[14] We calculate job turnover as the share of workers that are fired divided by the total number of workers, the formal expression for job turnover is given by:

$$\text{Job Turnover} = \frac{(n_2 - n_1)(1-\psi)}{\psi n_1 + (1-\psi) n_2} \hat{b}_2,$$

where in equilibrium $\hat{b}_2 = \hat{\mu}_2 f_2'(0) - \frac{\sigma^2}{2} f_2''(0)$.

4.2 Firing Costs

We quantify the impact of firing costs on aggregate TFP, aggregate output, and the job turnover rate by comparing these statistics in each distorted economy relative to the benchmark undistorted economy.

We study the impact of different firing cost policies by changing τ and report statistics relative to the benchmark economy. Firing costs τ have a direct interpretation with other values in the model. Since a period in the model is equal to 5 years, a value of τ equal to 0.1 corresponds to firing costs equivalent to 6 months' wages, a value of τ equal to 0.2 corresponds to firing costs of 1 years' wages, and a value of τ equal to 1 corresponds to firing costs of 5 years' wages. We report the results of these experiments in Table 2 for a number of statistics such as aggregate output, aggregate TFP, wages, and the job turnover rate. All statistics, except for job turnover, are reported relative to the benchmark economy in percent.

Table 2: Effects of Changes in Firing Costs τ

	τ			
	0.0	0.1	0.2	1.0
Relative Y	100.0	97.9	95.8	79.4
Relative TFP	100.0	97.9	95.8	79.4
Relative wages	100.0	98.9	97.7	89.1
Job turnover	0.30	0.13	0.10	0.05

Notes: Values for τ of 0.1, 0.2, and 1 represent firing costs equivalent to 6 months, 1 year, and 5 years' wages.

In Table 2 firing costs have a substantial impact on aggregate output and TFP. An economy with 6 months' wages of firing costs has aggregate output and TFP that is 97.9 percent of the benchmark economy. While an economy with 1 year's wage of firing costs has aggregate output and TFP that is 95.8 percent of the benchmark economy, and an economy with 5 years

Table 3: Effects of Changes in Firing Costs τ

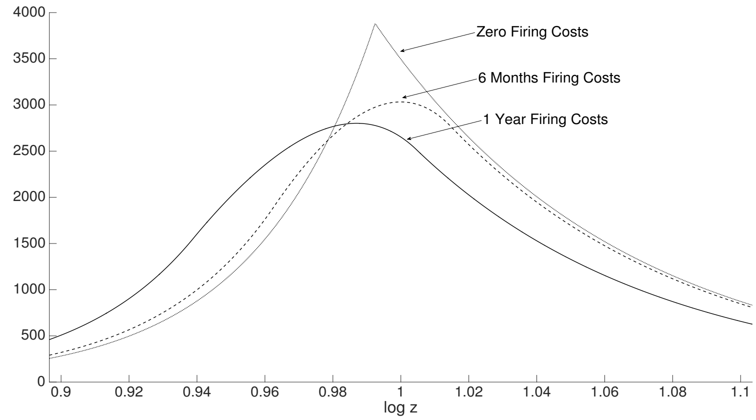
	τ			
	0.0	0.1	0.2	1.0
Inaction rate θ	1.00	0.68	0.61	0.39
Decision rule z_1	1.00	1.17	1.20	1.21
Decision rule z_2	1.00	0.80	0.74	0.48

Notes: Values for τ of 0.1, 0.2, and 1 represent firing costs equivalent to 6 months, 1 year, and 5 years' wages.

of firing costs has aggregate output and TFP of 79.4 percent of the undistorted economy. The impact on TFP is much larger than estimates from the Literature, as an example, Hopenhayn and Rogerson (1993) estimate an impact of 99.2 percent for 6 months' wages and 97.9 percent to 1 year's wage. Overall, differences in output and TFP relative to the benchmark economy are increasing in the amount of firing costs, more firing costs implies lower TFP and consequently output. Regarding job turnover, the impact is also substantial an 6 months' wages of firing costs reduces job turnover by more than half. It is also instructive to analyze how firing costs impact the policy functions of small establishments, large establishments, and the inaction rate of the economy. Table 3 contains all this information relative to the benchmark economy.

The first result from Table 3 is that increases in firing costs decrease the inaction zone rate, increasing the inaction zone, measured by the difference between z_1 and z_2. In Figure 4, we plot the distribution of establishment TFP of the undistorted economy, the distribution of an economy with 6 months of firing costs, and an economy with a year of firing costs. The increase in the inaction zone makes the TFP distribution flatter, increasing its variance. Since wages also decrease, from Table 2, the overall effect on the distribution of TFP is both a move to the left of the distribution, caused by lower wages, and a flatter distribution from the increase in the inaction zone.

Figure 4: Stationary Distribution of Establishment TFP for Different Firing Costs



4.3 Static and Dynamic Misallocation

The impact of firing costs on misallocation can be decomposed in two effects: a static and a dynamic. The static effect is the classic impact on the policy function of small and large establishments, which causes large establishments to switch at lower levels of productivity and small establishments to switch at higher levels productivity. The dynamic effect is due to changes in the average productivity of switching establishments. Since in our model the productivity's distribution is endogenous, when large firms start switching at lower levels of productivity this also impact on the distribution of productivity of small establishments, because entrants have on average lower productivity, and the opposite is true for large firms, since small firms start switching at higher level of productivity, the average productivity of new entrants is higher. In Table 4 we quantify these two effects.

From Table 4 we observe that most of changes in TFP are due to dynamic misallocation. When firing costs are at 6 months's wage dynamic misallocation accounts for 90 percent of changes in TFP, which is around to 0.2 percent, while dynamic misallocation accounts to 90 percent, which is 1.8 percent. As firing costs increase static misallocation becomes more

Table 4: Static and Dynamic Misallocation

τ	0.1	0.2	1.0
Relative TFP	97.8	95.7	79.3
Decomposition:			
Static misallocation (%)	10.0	13.0	23.0
Dynamic misallocation (%)	90.0	87.0	77.0

Notes: Values for τ of 0.1, 0.2, and 1 represent firing costs equivalent to 6 months, 1 year, and 5 years' wages.

relevant, but not as relevant as static misallocation.

5 Conclusions

We developed a tractable framework with heterogeneous production units that builds on the work of Dixit (1989), Hopenhayn (1992), and Hopenhayn and Rogerson (1993). We showed that in this framework firing costs not only impact the employment decision of establishments of firing and hiring workers as emphasized by the existing literature, but also impact the distribution of establishment productivity. We showed that in a calibrated version of the model, firing costs have a substantial negative impact on aggregate TFP, a quantitative effect that is 4 times larger than in the earlier literature. We decomposed the effect of firing costs on aggregate TFP in two channels: a static misallocation effect and a dynamic effect through changes in the distribution of establishment productivity. This decomposition allows us to show that the quantitative impact of the static misallocation channel is in line with previous estimates in the literature and that the bulk of the effect of firing costs on aggregate TFP in our model is due to the dynamic channel affecting the distribution of establishment TFP.

We also established an equivalence between firing costs and hiring costs. This equivalence allows us to connect our findings to the vast literature that studies the impact of specific size-dependent policies on aggregate productivity. In future work, this equivalence could help connect our framework with micro data in specific contexts to obtain more accurate empirical estimates of the impact of the much broader size-dependent policies on aggregate TFP.

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A Appendix

A.1 Proof Lemma 1

The proof is by guessing and verifying. We guess the following functional form for the value function $W_i(z) = a_i + A_i z + B_i z^{\beta_i}$ and solving the Hamilton-Jacobi-Bellman equation we find that $a_i = -\frac{wn_i}{\rho}$, $A_i = \frac{{n_i}^{\alpha}}{\rho - \mu_i}$ and β_i is equal to

$$\beta_i = -\left(\frac{\mu_i}{\sigma_z^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma_z^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_z^2}}. \tag{17}$$

Finally from the boundary and smooth pasting conditions we find B_1, B_2, z_1, and z_2 solving the following system of nonlinear equations:

$$\begin{aligned}
(1-\beta_1)B_1 z_1^{\beta_1} &= (a_2 - a_1) + (1-\beta_2)B_2 z_1^{\beta_2}, & (18)\\
(1-\beta_1)B_1 z_2^{\beta_1} &= (a_2 - a_1) + (1-\beta_2)B_2 z_2^{\beta_2} + \tau(n_2 - n_1)w, & (19)\\
\beta_1 B_1 z_1^{\beta_1} &= (A_2 - A_1)z_1 + \beta_2 B_2 z_1^{\beta_2}, & (20)\\
\beta_1 B_1 z_2^{\beta_1} &= (A_2 - A_1)z_2 + \beta_2 B_2 z_2^{\beta_2}. & (21)
\end{aligned}$$

And this conclude the proof. ■

A.2 Proof Lemma 2

First we need to show that the positive root of (17) is greater than one. First note that this polynomial is convex, since $\Omega''(\beta) = 1$, and at zero $\Omega(0) = -\frac{\rho}{\sigma_z^2}$, which is negative. So, $\Omega(\beta)$ has a positive and a negative root. At one $\Omega(1) = \frac{\mu_1}{\sigma_z^2} - \frac{\rho}{\sigma_z^2}$, since ρ is greater than μ_1, $\Omega(1)$ is also negative. Consequently, the positive root must be greater than one. Now, we can prove that B_1 and B_2 are positive.

From Lemma 1 and the two equations on the smoothing pasting conditions (21) and (21), we can write B_1 and B_2 as a function of parameters, z_1 and z_2, the constant B_2 is equal to:

$$B_2 = \frac{(A_2 - A_1)(z_2^{1-\beta_1} - z_1^{1-\beta_1})}{\beta_2(z_1^{\beta_2-\beta_1} - z_2^{\beta_2-\beta_1})}. \tag{22}$$

Note that the numerator is positive (negative), because $A_2 - A_1$ is positive and $(z_2^{1-\beta_1} - z_1^{1-\beta_1})$ is positive, since $z_1 > z_2$ and $\beta_1 > 1$, as we can see from ($\beta_1 < 1$). The denominator is also positive, because β_2 is negative and $(z_1^{\beta_2-\beta_1} - z_2^{\beta_2-\beta_1})$ is algo negative. As a result B_2 is positive. Substituting the expression for B_2 into equation (21), we find B_1 equal to:

$$B_1 = \frac{(A_2 - A_1)(z_1 z_2)^{-\beta_1}(z_1^{\beta_2} z_2 - z_1 z_2^{\beta_2})}{\beta_1(z_1^{\beta_2-\beta_1} - z_2^{\beta_2-\beta_1})}. \tag{23}$$

Note that the numerator is negative, because $z_1 > z_2$ and $z_1^{\beta_2} < z_2^{\beta_2}$ since $\beta_2 < 0$. In addition, the denominator is also negative, because $(z_1^{\beta_2-\beta_1} - z_1^{\beta_2-\beta_1})$ is negative and β_1 is positive, as result B_1 is positive. ■

A.3 Proof Lemma 3

In order to characterize the inaction zone rate, we first assume that $z_2 = \theta z_1$ for a $\theta \in (0, 1]$, substituting $z_2 = \theta z_1$ into the expression of B_1 in equation (23) and B_2 into equation (22), we find:

$$B_1 = \frac{(A_2 - A_1)z_1^{1-\beta_1}(\theta^{1-\beta_2} - 1)}{\beta_1(\theta^{\beta_1-\beta_2} - 1)}, \tag{24}$$

$$B_2 = \frac{(A_2 - A_1)z_1^{1-\beta_2}(\theta^{1-\beta_1} - 1)}{-\beta_2(\theta^{\beta_2-\beta_1} - 1)}. \tag{25}$$

Second we divide both sides of of the boundary condition (18) and the boundary condition (20) by $(1 - \beta_2)B_2$ and we substitute $z_2 = \theta z_1$ to obtain:

$$\frac{(1-\beta_1)B_1}{(1-\beta_2)B_2} z_1^{\beta_1} - z_1^{\beta_2} = \frac{(a_2 - a_1)}{(1-\beta_2)B_2}, \tag{26}$$

$$\frac{(1-\beta_1)B_1}{(1-\beta_2)B_2} (\theta z_1)^{\beta_1} - (\theta z_1)^{\beta_2} = \frac{(a_2 - a_1) + \tau(n_2 - n_1)w}{(1-\beta_2)B_2}. \tag{27}$$

We can substitute B_1 and B_2 from equation (24) and (25) into equation (26) and (27) and divide the two equations to obtain:

$$\varphi(\theta) = \frac{(\Omega_1(\theta) + 1)}{(\Omega_1(\theta)\theta^{\beta_1} + \theta^{\beta_2})} = \frac{1}{1 - \rho\tau},$$

where $\Omega_1(\theta) = \frac{(1-\beta_1)\beta_2}{(1-\beta_2)\beta_1} \frac{(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)}{(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}$.

To find z_1 we can substitute B_1 and B_2 from equations (26) and (27) into the boundary condition (18), after some algebraic manipulation we find z_1 equal to:

$$z_1 = \frac{-\left[\frac{n_2}{\rho} - \frac{n_1}{\rho}\right]}{\left[\frac{n_2^\alpha}{\rho-\mu_2} - \frac{n_1^\alpha}{\rho-\mu_1}\right]} \Omega_2(\theta) w,$$

where $\Omega_2(\theta) = \frac{\beta_1\beta_2(\theta^{\beta_2-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}{((1-\beta_1)\beta_2(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)+(1-\beta_2)\beta_1(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1))}$. ■

A.4 Proof Lemma 4

Consider the optimal switching problem of small and large establishments facing a hiring cost. This problem is characterized by the two value matching conditions one for small and one for large establishments and by the two smoothing pasting conditions:

$$\begin{aligned} V_1(z_1) &= V_2(z_1) - \tau_h w(n_2 - n_1), \\ V_1(z_2) &= V_2(z_2), \\ V_1'(z_1) &= V_2'(z_1), \\ V_1'(z_2) &= V_2'(z_2). \end{aligned}$$

From Lemma 1 we know that the solution of this problem is given by two roots that solve the polynomial $\beta_i = -\left(\frac{\mu_i}{\sigma_z^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma_z^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_z^2}}$ for $i \in \{1, 2\}$, and by the constants $\{B_1, B_2\}$ and the policy functions $\{z_1, z_2\}$ solve the two value matching conditions and the two smoothing pasting conditions together. Note that for a given stochastic process for small and large establishments the two roots are the same in the case of firing or hiring costs. Following the same methodology as Lemma 3, we find:

$$\varphi(\theta) = \frac{(\Omega_1(\theta) + 1)}{(\Omega_1(\theta)\theta^{\beta_1} + \theta^{\beta_2})} \quad = \quad 1 + \rho\tau_h,$$

where $\Omega_1(\theta) = \frac{(1-\beta_1)\beta_2}{(1-\beta_2)\beta_1} \frac{(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)}{(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}$. Since $\varphi(\theta)$ only depends on β_1 and β_2 that are independent of the costs we find that for a given hiring costs τ_h there exist a firing costsτ that solves

$$\frac{1}{1 - \rho\tau} \quad = \quad 1 + \rho\tau_h$$

and generate the same inaction zone rate θ. ■

A.5 Small establishment's stationary distribution

In order to find the stationary distribution of small establishments we apply the same methodology as in the distribution of large establishments. First, let $m_1(x, t)$ denote the number density function of small establishments. At time t, the small establishments productivity process follows the modified Kolmogorov-Fokker-Planck equation below:

$$\frac{\partial m_1(x,t)}{\partial t} = -\hat{\mu}_1 \frac{\partial m_1(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 m_1(x,t)}{\partial x^2} + B_1(x,t), \tag{28}$$

where the function $B_1(x, t)$ are the new small establishment that arrival with productivity x at time t. The partial differential equation (28) is supplement by the two boundary conditions

$$\lim_{x \to -\infty} m_1(x,t) = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{\partial m_1(x,t)}{\partial x} = 0$$

where in the case of small establishments the boundary conditions are at the bottom of the distribution. We are interested in solving for the steady state productivity distribution, as in the case of large establishments, we look for a solutions with the form $m_1(x, t) = M_1(t)f_1(x)$ and $B_1(x, t) = M_1(t)b_1(x)$, where substituting in the Kolmogorov-Fokker-Planck equation we find:

$$\frac{M_1'(t)}{M_1(t)} f_1(x) = -\hat{\mu}_1 f_1'(x) + \frac{\sigma^2}{2} f_1''(x) + b_1(x - x_2), \tag{29}$$

and the new boundary conditions:

$$\begin{aligned}\lim_{x\to-\infty} f_1(x) &= 0,\\ \lim_{x\to-\infty} f_1'(x) &= 0,\end{aligned}$$

and the additional requirement that $f_1(x)$ is also a pdf leads to the conditions:

$$\begin{aligned}f_1(x) &\geq 0,\\ \int_{-\infty}^{0} f_1(x)dx &= 1,\end{aligned}$$

where now differently from large establishments, small establishment domain is from minus infinity to zero. As in the large establishment problem, we calculate the separation rate of small establishments by integrating equation (29) from minus infinity to zero, where small establishment decide to become large. The growth rate for small establishment η_1, is given by

$$\eta_1 \int_{-\infty}^{0} f_1(x)dx = \left(-\hat{\mu}_1 f_1(x) + \frac{\sigma^2}{2} f_1'(x)\right)\Bigg|_{-\infty}^{x=0} + \int_{0}^{+\infty} \hat{b}_1 \delta(u)du,$$

and applying the boundary conditions and using the delta function, we find that growth rate of small firms is equal to:

$$\eta_1 = -\hat{\mu}_1 f_1(0) + \frac{\sigma^2}{2} f_1'(0) + \hat{b}_1,$$

which has the same interpretation as in the large establishment case. As in the large establishments, we look for the stationary equilibrium in which the number of small establishments does not grow over time, which restricts η_1 to be equal to zero. Now, we can rewrite the Kolmogorov-Fokker-Planck equation by substituting η_1 we obtain:

$$-\hat{\mu}_1 f_1'(x) + \frac{\sigma^2}{2} f_1''(x) + \hat{b}_1 \delta(x - x_2) = 0,$$

subject to the boundary conditions $f_1(0) \geq 0$ and $\int_{-\infty}^{0} f_1(x)dx = 1$. Therefore, the stationary pdf is the solution of the boundary-value problem that consists of solving the equation:

$$f_1''(x) - \alpha_1 f_1'(x) + \alpha_2 \delta(x - x_2) = 0,$$

and the boundary conditions $f_1(0) \geq 0$ and $\int_{-\infty}^{0} f_1(x)dx = 1$, where the constants α_1 and α_2 are given by

$$\alpha_1 = \frac{2\hat{\mu}_1}{\sigma^2} \qquad \text{and} \qquad \alpha_2 = \frac{2\hat{b}_1}{\sigma^2}.$$

A.6 Proof Lemma 5

The stationary pdf is the solution of the boundary-value problem that consists of solving the equation:

$$f_2''(x) - \gamma_1 f_2'(x) = -\gamma_2 \delta(x - x_1),$$

subject to the boundary conditions $f_2(0) \geq 0$ and $\int_0^{+\infty} f_2(x)dx = 1$. As in the undistorted economy case, Lemma 6, we are going to use Laplace transforms. After some algebraic manipulation we obtain the following equation:

$$(s^2 - \gamma_1 s)Y(s) = f_2'(0) + (s - \gamma_1)f_2(0) - \gamma_2 e^{-sx_1},$$

and

$$Y(s) = \frac{f_2'(0) + (s - \gamma_1)f_2(0) - \gamma_2 e^{-sx_1}}{(s - \gamma_1)s}.$$

Note that Laplace inverses of each component is equal to:

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{s(s-\gamma_1)}\right] &= \frac{1}{\gamma_1}\left(e^{\gamma_1 x} - 1\right), \\
\mathscr{L}^{-1}\left[\frac{1}{s}\right] &= 1, \\
\mathscr{L}^{-1}\left[\frac{e^{-sx_1}}{s(s-\gamma_1)}\right] &= \frac{H(x_1)}{\gamma_1}\left[e^{\gamma_1(x-x_1)} - 1\right],
\end{aligned}$$

where $H(x_1)$ is Heaviside step function given by:

$$H(x_1) = \begin{cases} 0 & \text{if} \quad x \leq x_1, \\ 1 & \text{if} \quad x > x_1. \end{cases}$$

Substituting the Laplace inverses in the second oder differential equation give us the stationary distribution for large firms:

$$f_2(x) = \frac{f_2'(0)}{\gamma_1}\left(e^{\gamma_1 x} - 1\right) - \gamma_2 \frac{H(x_1)}{\gamma_1}\left[e^{\gamma_1(x-x_1)} - 1\right] + f_2'(0).$$

Note that $f_2(0)$ must be equal to zero in order to the integral to be bounded. This boundary condition guarantee the mass of firms that are indifferent between switching or not is equal to zero. Now substituting the Heaviside step function $H(x_1)$ we find:

$$f_2(x) = \begin{cases} \frac{f_2'(0)}{\gamma_1}\left(e^{\gamma_1 x} - 1\right) & \text{if} \quad x \leq x_1, \\ \\ \frac{f_2'(0)}{\gamma_1}\left(e^{\gamma_1 x} - 1\right) - \frac{\gamma_2}{\gamma_1}\left[e^{\gamma_1(x-x_1)} - 1\right] & \text{if} \quad x > x_1. \end{cases}$$

We only need to solve for $f_2'(0)$ to have the complete characterization of the large establishments distributions. We find $f_2'(0)$ to guarantee that that the integral of $f_2(\cdot)$ is one.

$$\frac{f_2'(0)}{\gamma_1}\int_0^{+\infty}\left(e^{\gamma_1 x}-1\right)dx-\frac{\gamma_2}{\gamma_1}\int_{x_1}^{+\infty}\left(e^{\gamma_1(x-x_1)}-1\right)dx=1,$$

where we can rewrite the integral above as:

$$\frac{f_2'(0)}{\gamma_1}\int_0^{+\infty}e^{\gamma_1 x}dx-\frac{\gamma_2}{\gamma_1}\int_{x_1}^{+\infty}e^{\gamma_1(x-x_1)}dx-\frac{f_2'(0)}{\gamma_1}\int_0^{x_1}1dx-\left(\frac{f_2'(0)}{\gamma_1}-\frac{\gamma_2}{\gamma_1}\right)\int_{x_1}^{+\infty}1dx=1.$$

The last term of the integral is zero, so the mass of establishments is constant in equilibrium, by integrating all the other terms we find the following expression:

$$-\frac{f_2'(0)}{\gamma_1^2}+\frac{\gamma_2}{\gamma_1^2}-\frac{f_2'(0)}{\gamma_1}x_1=1.$$

Substituting γ_2 gives

$$-\frac{f_2'(0)}{\gamma_1^2}+\frac{f_2'(0)}{\gamma_1^2}-\frac{f_2'(0)}{\gamma_1}x_1=1,$$

as a result, $f_2'(0)=-\frac{\gamma_1}{x_1}=-\frac{2\hat{\mu}_2}{x_1\sigma^2}$.

Now we can solve for the small establishment's distribution, first we are going to change variables. Let $f_1(x)=g(-y)$ and $\delta(x)=\delta(-y)$, we can rewrite the second order differential equation for small establishment as:

$$g''(-y)+\alpha_1 g'(-y) \quad = \quad -\alpha_2\delta(-y-x_2).$$

After some algebraic manipulation we obtain the following equation:

$$(s^2-\alpha_1 s)Y(s)=-g'(0)+(s+\alpha_1)g(0)-\alpha_2 e^{-sx_2}$$

and

$$Y(s)=\frac{-g'(0)+(s+\alpha_1)g(0)-\alpha_2 e^{-sx_2}}{(s+\alpha_1)s}.$$

Applying the Laplace inverse as in the large establishment case we obtain the following equation:

$$g(-y)=\frac{-g'(0)}{\alpha_1}\left(1-e^{-\alpha_1 y}\right)-\frac{\alpha_2}{\alpha_1}H(x_2)\left[1-(y-x_2)e^{-\alpha_1(y+x_2)}\right]+g(0),$$

using that

$$f_1(x)=\frac{f_1'(0)}{\alpha_1}\left(1-e^{\alpha_1 x}\right)+\frac{\alpha_2}{\alpha_1}\left[1-H(x_2)\right]\left[e^{\alpha_1(x-x_2)}-1\right]+f_1(0).$$

Note that again as in the large establishment case we need to impose the boundary condition $f_1(0)=0$ to guarantee that integral is bounded and we use again the symmetric Heavside function

$H(x_2)$ equal to

$$1 - H(x_2) = \begin{cases} 0 & \text{if} \quad x \geq x_2, \\ 1 & \text{if} \quad x < x_2. \end{cases}$$

Last we obtain the following stationary distribution,

$$f_1(x) = \begin{cases} \frac{f_1'(0)}{\alpha_1}\left(1 - e^{\alpha_1 x}\right) + \frac{\alpha_2}{\alpha_1}\left(1 - e^{\alpha_1 (x - x_2)}\right) & \text{if} \quad x < x_2, \\ \\ \frac{f_1'(0)}{\alpha_1}\left(1 - e^{\alpha_1 x}\right) & \text{if} \quad x \geq x_2. \end{cases}$$

We only need to solve for $f_1'(0)$ to have the complete characterization of the small establishments distribution. We find $f_1'(0)$ to guarantee that the integral of $f_1(x)$ is one.

$$\frac{f_1'(0)}{\alpha_1}\int_{-\infty}^{0}\left(1 - e^{\alpha_1 x}\right)dx + \frac{\alpha_2}{\alpha_1}\int_{-\infty}^{x_2}\left(1 - e^{\alpha_1(x - x_2)}\right)dx = 1,$$

where we can rewrite the integral above as:

$$-\frac{f_1'(0)}{\alpha_1}\int_{-\infty}^{0} e^{\alpha_1 x}dx - \frac{\alpha_2}{\alpha_1}\int_{-\infty}^{x_2} e^{\alpha_1(x - x_2)}dx - \frac{f_1'(0)}{\alpha_1}\int_{0}^{x_1} 1dx - \left(\frac{f_1'(0)}{\alpha_1} + \frac{\alpha_2}{\alpha_1}\right)\int_{-\infty}^{x_2} 1dx = 1.$$

From the labor market clearing we find that the last term of the integral is zero, by integrating all the other terms we find the following expression:

$$-\frac{f_1'(0)}{\alpha_1^2} - \frac{\alpha_2}{\alpha_1^2} - \frac{f_1'(0)}{\alpha_1}x_2 = 1.$$

Substituting α_2 we obtain

$$-\frac{f_1'(0)}{\alpha_1^2} + \frac{f_1'(0)}{\alpha_1^2} - \frac{f_1'(0)}{\alpha_1}x_2 = 1,$$

as a result, $f_1'(0) = -\dfrac{\alpha_1}{x_2} = -\dfrac{2\hat{\mu}_1}{x_2\sigma^2}$. ■

A.7 Proof Lemma 6

The stationary large establishment's pdf is the solution of the following second order differential equation

$$f_2''(x) - \gamma_1 f_2'(x) = -\gamma_2\delta(x),$$

where the constants γ_1 and γ_2 are given by $\gamma_1 = \dfrac{2\hat{\mu}_2}{\sigma^2}$ and $\gamma_2 = \dfrac{2\hat{b}_2}{\sigma^2}$, subject to the boundary conditions $f_2(0) \geq 0$ and $\displaystyle\int_0^{+\infty} f_2(x)dx = 1$.

We solve the boundary-value problem using Laplace transforms. By applying Laplace transforms in equation ($A.7$) we obtain:

$$(s^2 - \gamma_1 s)\mathscr{L}[f(x)] - (s - \gamma_1)f_2(0) - f_2'(0) = -\gamma_2 \mathscr{L}[\delta(x)],$$

and after some algebraic manipulation, we find:

$$(s^2 - \gamma_1 s)Y(s) = (f_2'(0) - \gamma_1 f_2(0) - \gamma_2) + s f_2(0).$$

First note that the first term between parenthesis in the right hand side is equal to zero, because the growth rate of large firms η_2 is equal to zero. So, we can simplify the expression and above and obtain:

$$Y(s) = f_2(0)\frac{1}{(s - \gamma_1)},$$

and after solving for the Laplace inverse we obtain $f_2(x) = f_2(0)e^{\gamma_1 x}$. Now we need to find the constant $f_2(0) = -\gamma_1$ to guarantee that the integral over the domain is equal to one. In addition, note that the solution above only satisfy the boundary condition for γ_1 negative.

For the small establishment distribution, we are going to follow the same methodology. The stationary small establishment's pdf is the solution of the following second order differential equation

$$f_1''(x) - \alpha_1 f_1'(x) = -\alpha_2 \delta(x),$$

subject to the boundary conditions $f_1(0) \geq 0$ and $\displaystyle\int_{+\infty}^{0} f_2(x)dx = 1$. We solve the boundary-value problem using Laplace transforms and after some algebraic manipulation we obtain:

$$(s^2 - \alpha_1 s)Y(s) = (f_1'(0) - \alpha_1 f_1(0) - \alpha_2) + s f_1(0),$$

as in the large establishment's distribution, the term between the parenthesis is also zero, because small establishments do not grow $\eta_1 = 0$. We can simplify the expression and above and obtain:

$$Y(s) = f_1(0)\frac{1}{(s - \alpha_1)},$$

and after solving for the Laplace inverse we obtain $f_1(x) = f_1(0)e^{\alpha_1 x}$. Now we need to find the constant $f_1(0) = \alpha_1$ to guarantee that the integral over the domain is equal to one. In addition, note that the solution above only satisfy the boundary condition for α_1 positive. ■

A.8 Stationary Equilibrium

Formally, we find M_1, M_2, z_1, z_2, and w are obtained by solving the following 5 equations:

(1) The inaction rate θ is found by solving the non-linear equation from Lemma 3:

$$\frac{(\Omega_1(\theta) + 1)}{(\Omega_1(\theta)\theta^{\beta_1} + \theta^{\beta_2})} = \frac{1}{1 - \rho\tau},$$

where $\Omega_1(\theta) = \frac{(1-\beta_1)\beta_2}{(1-\beta_2)\beta_1}\frac{(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)}{(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}$ and $\beta_i = -\left(\frac{\mu_i}{\sigma_z^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{\mu_i}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}$ for $i \in \{1, 2\}$.

(2) We find the mass of small establishments M_1 using the enter and exit condition:

$$M_1 \frac{\hat{\mu}_1}{\sigma^2} = -(1 - M_1)\frac{\hat{\mu}_2}{\sigma^2},$$

and

(3) we find M_2 as a residual since $M_2 = 1 - M_1$

(4) The wage w is found using the labor market clearing condition:

$$w = A^{\frac{-1}{(\eta-1)}} \left(M_1 n_1^{\alpha} \frac{\alpha_1}{1+\alpha_1} \kappa \Omega_2(\theta) + M_2 n_2^{\alpha} \frac{\gamma_1}{1+\gamma_1} \kappa \Omega_2(\theta)\theta \right)^{\frac{-\eta}{(\eta-1)}},$$

where $\gamma_1 = \frac{2\hat{\mu}_2}{\sigma^2}$, $\alpha_1 = \frac{2\hat{\mu}_1}{\sigma^2}$, and $\Omega_2(\theta) = \frac{\beta_1\beta_2(\theta^{\beta_2-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}{((1-\beta_1)\beta_2(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)+(1-\beta_2)\beta_1(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1))}$.

(5) The policy functions z_1 and z_2 are now characterized using Lemma 3:

$$z_1 = \kappa \Omega_2(\theta) w \quad \text{and} \quad z_2 = \theta z_1$$

where $\Omega_2(\theta) = \frac{\beta_1\beta_2(\theta^{\beta_2-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1)}{((1-\beta_1)\beta_2(\theta^{1-\beta_2}-1)(\theta^{\beta_2-\beta_1}-1)+(1-\beta_2)\beta_1(\theta^{1-\beta_1}-1)(\theta^{\beta_1-\beta_2}-1))}$.

In order to calculate TFP we defineψ as the share of small establishments and $1-\psi$ the share of large establishments. Then, the share of small establishment are such that $\psi\hat{\mu}_1 = -(1-\psi)\hat{\mu}_2$, as result, $\psi = \frac{-\hat{\mu}_2}{\hat{\mu}_2 - \hat{\mu}_1}$, and aggregate TFP in the model is defined by:

$$TFP = \frac{\psi n_1^{\alpha} E_1 z + (1-\psi) n_1^{\alpha} E_2 z}{\psi n_1^{\alpha} + (1-\psi) n_2^{\alpha}}.$$

In the case where there is no firing costsTFP is given by:

$$TFP_u = \frac{\psi n_1^{\alpha} \frac{\alpha_1}{(1+\alpha 1)} z_* + (1-\psi) n_2^{\alpha} \frac{\gamma_1}{(1+\gamma_1)} z_*}{\psi n_1^{\alpha} + (1-\psi) n_2^{\alpha}},$$

while in the case where there is firing costs, the TFP in the economy is given by:

$$TFP_d = \frac{\psi n_1^{\alpha} \frac{\alpha_1}{(1+\alpha 1)} z_1 \left(\frac{e^{x_2}}{x_2} - \frac{1}{x_2}\right) + (1-\psi) n_2^{\alpha} \frac{\gamma_1}{(1+\gamma_1)} z_2 \left(\frac{e^{x_1}}{x_1} - \frac{1}{x_1}\right)}{\psi n_1^{\alpha} + (1-\psi) n_2^{\alpha}}.$$