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Measuring Polarization and Convergence as Transitional Processes in the Absence of a Cardinal Ordering.

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Abstract.

Conceptually Polarization and Convergence, objects of study in a variety of fields, are dynamic processes relating to specific types of transition between departure and arrival state distributions. Indeed the axiomatic development of polarization indices has been couched in terms of the impact on the shape of a consequent “final” distribution of cardinally measurable changes in locations and spreads of components of an initial distribution. The resultant indices end up as “distance weighted” summary statistics of the anatomy of the “final” distribution. However Polarization and Convergence concepts often pertain to situations where measurement is not cardinal. For example in many applications in the social sciences the departure and arrival states, which may be quite different in nature, frequently have just an ordinal ranking (e.g. social class departure state – economic or educational outcome arrival state). Such states are defined over one or more groups of agents and the dynamic processes are usually concerned with realignments of said agents within and between groupings. Here it is argued that in such situations polarization/convergence issues are more conveniently analyzed in the context of the anatomy of transitions between states which do not of necessity depend upon a between or within group cardinal ordering. Accordingly indices are proposed which are based upon the structure of an underlying transition process rather than the structure of the final state distribution. The measures do not depend upon the existence of a cardinal ordering but can be augmented to incorporate cardinality if such a metric is available. They do not depend upon the “square-ness” of the transition matrix, that is to say they can deal with disappearing and emerging groups. 3 examples from Canadian Generational Education Data, the world size distribution of Gross National Product per capita and Chinese Class Structures illustrate their use.

Introduction.

The disappearance or emergence of classes in a society and the analytical connection with polarization or convergence concepts has been a recent subject of interest (Heisz 2007, US Department of Commerce 2010, Kharas 2010, Beach and Slotsve1996, Foster and Wolfson 1992, Zhang and Kanbur 2001). There has also been a developing theoretical and empirical literature on polarizing and converging groupings of countries in economic growth and other literatures (Barro, Sala-i-Martin 1992, Quah 1993, 1996, Galor 1996, Anderson 2004, 2004a, Anderson, Linton and Leo 2012, Pittau, Zelli and Johnson 2010, Keefer Knack 2002). Conceptually similar themes have also found expression in contemporary conflict and social polarization literatures which have shifted emphasis toward emerging diversity is a factor for conflict[1]. In the equality of opportunity literature, transitions from circumstances to achievements are evaluated in terms of the extent to which the outcome distributions of different circumstance classes have converged (Lefranc et al 2008, 2009) in the sense that there is some commonality (i.e. absence of dominance) in the outcome distributions of different circumstance classes.

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[1] The Herfindahl-Hirschman fractionalization index (Herfindal (1950)) as a measure of diversity has frequently been used as a "diversity index" in several empirical studies of conflict (see, e.g., Lichbach 1986, Esteban and Schneider 2008, Collier and Hoffler (2004), Fearon and Laitin (2003) Miguel, Satyanath and Sergenti (2004), Easterly and Levine (1997) Rodriguez, and Salas 2002,). D'Ambrosio and Permanyer (2013) examine and characterize axiomatically the measurement of social polarization with categorical and ordinal data which is particularly useful in many contexts where cardinal data are not available. Reynal-Querol 2002 produce an index which is maximal when mass is evenly located at the extremes. Ray and Esteban (2011) study a behavioral model of conflict that provides a basis for choosing certain indices of dispersion as indicators for conflict and show that equilibrium levels of conflict can be expressed as an (approximate) linear function of the Gini coefficient, the Herfindahl-Hirschman fractionalization index, and a specific measure of polarization due to Esteban and Ray (1994).

All of the above analyze states which are the evolutionary consequence of the transition from one distribution into another. The polarized or converged state is a result of a combination of changes in the relative sizes of classes caused by people transiting from one class to another and, when they are definable i.e. cardinally ordered, changes in the location of those classes. However by definition, Polarization: “the production of polarity, a sharp division, as of a population or group, into opposing factions” or “The action of concentration about opposing extremes of groups or interests formerly ranged on a continuum” and convergence: “an act or instance of converging, the contraction of a vector field”[2] describe particular processes of change in environments which are not necessarily cardinally ordered. They also speak to the division and/or amalgamation of subgroups (i.e. vector field expansion or contraction) implying changes in group sizes without reference to their cardinal ordering. This is particularly pertinent when the concepts are applied to situations of conflict, reconciliation and changing class structures where a cardinal ordering over a relevant domain is not available.

Although polarization indices have been axiomatically developed on the basis of dynamic movements (slides and squeezes) over some metric of classes of fixed size, the concepts usually end up being measured in the static context of the anatomy of a final state distribution (Esteban and Ray 1994, Duclos, Esteban and Ray 2004) rather than in the context of the process by which states are evolving. On the other hand early work on convergence (Barro, R.J., Sala-i-Martin, X. 1992) focused on the nature of the dynamic process in a regression to the mean typology, almost without reference to the changing nature of the outcome distribution.

[2] Definitions drawn from Websters and Concise Oxford Dictionary.

Typically the evolution of states has been studied in the statistics literature in the context of Markov Chain Processes (Billingsley 1961, Shorrocks 1976, Geweke et. al. 1986) and Copula theory (Nelsen 1999, Jaworski, Durante, Härdle and Rychlik 2010) characterizing transitions and mobility between classes, frequently without reference to a metric by which those classes can be differentiated. Indeed there is a considerable literature on mobility and transitions between categorical (though usually ordered) classes for which no distance metric is available (for excellent surveys see Fields and Ok 1996, Maassoumi 1999), in such situations movements of agents between the classes is the sole issue. In this literature mobility indices are invariably based upon square matrices, precluding differences in the numbers of classes in departure and arrival states and thus not allowing for the disappearance or emergence of classes. They rarely address movements in the values associated with classes that they are cardinally ranked by, with good reason, because such values do not exist, i.e. classes only have an ordinal ranking[3].

Here Polarization and Convergence indices will be developed in the context of patterns of transition based upon the balance of probabilities of movement between groups which may be augmented by some cardinal distance metric when it is available. Section 1 relates the axiomatic bases for the transition based indices to the axioms employed in Esteban and Ray

[3] As far as the evaluation of societal wellbeing engendered by transition structures is concerned, Dardanoni (1993) considers an expected infinite lifetime welfare ranking of 2 monotone mobility matrices with the same long run stationary solution. His Theorem 1 demonstrates the equivalence of the welfare ranking of the two processes with a dominance ranking between respective rows of the two transition matrices, posing and answering the question as to which transition matrix is offering an agent emerging from each class in period t the best lottery on outcomes in period t+1, if dominance prevails for all classes then the dominating transition matrix will yield the highest welfare (see also Lefranc et al 2008, 2009). The degree of improvement could be evaluated using Anderson and Leo (2015).

(1994) and Duclos, Esteban and Ray (2004). Section 2 outlines a family of transition based polarization / convergence indices which are exemplified in section 3 with 3 examples.

Section 1. Developing Polarization/Convergence Indices for Transition Matrices.

In the seminal works of Esteban and Ray (1994) and Duclos, Esteban and Ray (2004) polarization indices[4] were formulated for destination distributions by positing a collection of axioms whose consequences should be reflected in a Polarization measure. The axioms are founded upon a so-called Identification-Alienation nexus wherein notions of polarization are fostered jointly by an agent's sense of increasing within-group identity and between-group distance or alienation. They were couched in terms of movements of subgroup distributions defined over a metric. The four axioms (D for discrete and C for continuous equivalents) may be summarized as follows:

Axiom 1D: A mean preserving reduction in the spread of a distribution cannot increase polarization.

Axiom 1C: If a distribution is composed of a single basic density, then a squeeze of that density cannot increase polarization.

Axiom 2D: Mean preserving reductions in the spread of sub-distributions at the extremes of a density cannot reduce polarization.

[4] See also Wang and Tsui 2000, Wolfson 1994, Anderson 2004, 2004a, 2012, Anderson Linton Leo 2012.

Axiom 2C: If a symmetric distribution is composed of three basic densities with the same root and mutually disjoint supports, then a symmetric squeeze of the side densities cannot reduce polarization

Axiom 3D: Separation of two sub-densities towards the extremes of the distributions range must increase polarization.

Axiom 3C: Consider a symmetric distribution composed of four basic densities with the same root and mutually disjoint supports. Slide the two middle densities to the side (keeping all supports disjoint). Then polarization must go up.

Axiom 4: Polarization measures should be population-size invariant.

The general polarization indices developed as a consequence of these axioms for discrete and continuous distributions at the arrival state may be written respectively as:

$$P_\alpha \propto \sum_{i=1}^{n}\sum_{j=1}^{n} | x_i - x_j | \pi_i^{1+\alpha} \pi_j \quad [1]$$

$$P_\alpha (F) \propto \int_y f(y)^\alpha \int_x | y - x | dF(x) dF(y) \quad [1a]$$

In the discrete version π_i is the sample weight of the i'th observation value x_i and $\alpha \geq 0$ is a polarization sensitivity factor chosen by the investigator. It may readily be seen that $\alpha = 0$ yields a sample weighted Gini coefficient. In the continuous analogue α is the polarization sensitivity factor which in this case is confined to [0.25,1]. The axiomatic construction of polarization indices was couched in terms of potential changes in location of and squeezes (reductions in the spread) of the basic densities describing sub-populations defined on a particular cardinal metric. The populations of basic densities remain intact, they just change locations and domains, the possibility of the relative size of component populations changing (i.e. people switching classes) is not entertained, neither is the possibility of classes polarizing when they only possess an ordinal ranking. In this regard the development of a measure of transitional polarization, the axioms need to be articulated in the absence of, or at least without reliance upon, a cardinal ranking of sub populations.

Here Axioms 1 and 2 are re - interpreted in terms of the effect of a particular transition matrix on a probability distribution with Axiom 1 addressing transitions that transfer mass to the center of the distribution and Axiom 2 addressing transitions that concentrate mass at the extremities of a distribution. However they could also be contemplated in terms of locational movements of sub distributions when a metric is available. In essence the polarization measures summarize, in an index form, the effects on the anatomy of a distribution that has been the subject of such transformations. Axiom 3 addresses changes in the location of sub-distributions in the form of movements over the metric away from the center of the distribution, which could be reflected in movements of mass away from the center of a distribution. Axiom 4 is just a size invariance property. Fundamentally, in the absence of a metric, the axioms are about the balance of two opposing forces, converging forces that transfer mass from groups on the peripheries to groups at the center and polarizing forces that transfer mass from groups at the center to groups at the peripheries and a transition matrix based polarization/convergence measure should reflect such forces. Consider the following 2 axioms.

A1 Transitions that promote net relocation of mass, either by population transfers and/or relocation of classes, toward the peripheries of a distribution increase (reduce) polarization (convergence).

A2 Transitions that promote net relocation of mass, either by population transfers and/or relocation of classes, toward the center of a distribution reduce (increase) polarization (convergence).

For some intuition, consider some simple 3 x 3 examples of transition matrices in a model of transitions between initial and final state classes in the 3 dimensional model $x_F = Tx_I$ where x_F ,and x_I are final and initial state vectors of relative class sizes respectively. Note that the initial state could be subject to a sequence of K transitions of the form T in which case $x_F = T^K x_I$, and, in thinking in terms of polarization and convergence as processes, it is necessary to think of the properties of such a sequence in terms of the properties of T. Some examples of Polarizing, Converging and Socially Static (Immobile and Mobile) Class Transition matrices together with their matrix properties for the departure state - arrival state transition and their 10 period sequence counterparts are presented in Tables 1. Arrival state outcomes after 1 and 10 transit periods starting with equi-probable classes, a single middle class and equal sized poor and rich classes are reported in Tables 2a, 2b and 2c.

Table 1.

T^1 *Transition Matrices*

$$\text{Polarizing} \quad \begin{bmatrix} 1 & 0.3 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0.3 & 1 \end{bmatrix} \quad \text{Converging} \quad \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \quad \text{Static }(IM) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Static }(MO) \quad \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

T^{10} *Transition Matrices*

$$\text{Polarizing} \quad \begin{bmatrix} 1 & 0.500 & 0 \\ 0 & 0.000 & 0 \\ 0 & 0.500 & 1 \end{bmatrix} \quad \text{Converging} \quad \begin{bmatrix} 0.001 & 0 & 0 \\ 0.999 & 1 & 0.999 \\ 0 & 0 & 0.001 \end{bmatrix} \quad \text{Static }(IM) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Static }(MO) \quad \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Observe that after a 10 period sequence the middle row is vanishing in the non-singular polarizing matrix as is the first and last rows in the non-singular converging matrix, ultimately as k -> ∞ the polarizing matrix will have rank 2 and the converging matrix will have rank 1. Note also that $|T^k|$-> 0 as k increases for non-static transition matrices with the "disappearance" of rows (classes) i.e. (0<|T|<1) whereas $|T^k|$ does not change with k in the static matrices. Thinking of the outcomes after 1 and 10 periods of the same transition structure observe in

Table 2a. Outcomes after 1 and 10 transit periods starting with equi-probable classes

		Polarizing	Converging	Static (Immobile)	Static (Mobile)
1 Transition Period	Poor Class	0.4333	0.1666	0.3333	0.3333
	Middle Class	0.1333	0.6666	0.3333	0.3333
	Rich Class	0.4333	0.1666	0.3333	0.3333
10 Transition Periods.	Poor Class	0.5000	0.0003	0.3333	0.3333
	Middle Class	0.0000	0.9993	0.3333	0.3333
	Rich Class	0.5000	0.0003	0.3333	0.3333

table 2a the polarizing matrix engenders rich and poor classes by dissipating the middle class in the long run (the disappearing middle class scenario) as it does in table 2b, in table 2c it simply reinforces the existence of the 2 classes. The converging matrix engenders a middle class by dissipating the rich and poor classes in the long run as it does in table 2c, in Table 2b it simply reinforces the existence of the middle class.

Table 2b. Outcomes after 1 and 10 transit periods starting with a single middle class.

		Polarizing	Converging	Static (Immobile)	Static (Mobile)
1 Transition Period	Poor Class	0.3000	0.0000	0.0000	0.3333
	Middle Class	0.4000	1.0000	1.0000	0.3333
	Rich Class	0.3000	0.0000	0.0000	0.3333
10 Transition Periods.	Poor Class	0.5000	0.0000	0.0000	0.3333
	Middle Class	0.0000	1.0000	1.0000	0.3333
	Rich Class	0.5000	0.0000	0.0000	0.3333

Table 2b. Outcomes after 1 and 10 transit periods starting with equal sized rich and poor class

		Polarizing	Converging	Static (Immobile)	Static (Mobile)
1 Transition Period	Poor Class	0.5000	0.2500	1.0000	0.3333
	Middle Class	0.0000	0.5000	0.0000	0.3333
	Rich Class	0.5000	0.2500	1.0000	0.3333
10 Transition Periods.	Poor Class	0.5000	0.0000	1.0000	0.3333
	Middle Class	0.0000	1.0000	0.0000	0.3333
	Rich Class	0.5000	0.0000	1.0000	0.3333

In the following consider a general n_F x n_I transition matrix T of the form:

$$T = \begin{bmatrix} t_{11} & t_{12} & . & t_{1n_I} \\ t_{21} & t_{22} & . & t_{2n_I} \\ . & . & t_{ij} & . \\ t_{n_F 1} & t_{n_F 2} & & t_{n_F n_I} \end{bmatrix}$$

Describing the transition from initial n_I x 1 ordered (lowest 1, to highest n_I) vector state probabilities π_I to final n_F x 1 ordered vector state probabilities π_F as the system of equations:

$$\pi_F = T\pi_I$$

Note that t_{ij} is the conditional probability of emerging in Final state "i" given an Initial state "j" as such T is a stochastic, but not necessarily bi-stochastic, matrix with $t_{ij} \geq 0$ and $\Sigma_i\ t_{ij} = 1$ for j=1,..,n_F. For later analysis, when a cardinal ordering is available, define x_F to be the corresponding vector of final state values associated with π_F and x_I to be the corresponding vector of initial state values associated with π_I. The i'th element of x_K can be construed as the average wellbeing measure of agents in class K whose share of the population is the i'th element of π_K for K = F, I.

An index of Mobility for Square and Non-Square Transition matrices.

Since in much of this work non-square transition matrices are common the Shorrocks (1978) suggestion of a simple trace based index of mobility is not viable. The j'th column of T corresponds to the probability distribution over the Final State outcome space for agents emerging from Initial State "j". As such, perfect mobility (where the Final State is uninfluenced by or independent of the initial state) is characterized by T having common columns which all sum to 1 and in the square transition matrix case complete immobility is characterized by T = I, the identity matrix. Writing the i'th row of T as $t_{i.}$ and MAXR() and MINR() as an operators which return the maximum and minimum value in a row vector respectively , an index of mobility, TM, which immediately suggests itself is:

$$TM(T) = 1 - \frac{\sum_{i=1}^{n_F} (MAXR(t_{i.}) - MINR(t_{i.}))}{n_c}$$

TM is one minus an n distribution version of Gini's two distribution dissimilarity "transvariation" index (Gini (1915))[5] rescaled by n (the number of distributions being compared) instead of 2. When each column of T is identical, the Final State Outcome distributions emerging from the n Initial States will overlap perfectly, the sum of maximums will equal the sum of minimums and TM = 1. If on the other hand the Final State Outcome distributions are orthogonal as in the Perfect Immobility case the intersections of the overlaps will be null (the sum of minimums will be 0) and the sum of maximums will equal n_c, the number of conditional distributions so that TM = 0 so that in the square T case TM(T) will be 0 when T=I.

Since it may readily be shown that $0 \leq TM(T) \leq 1$, the index will satisfy the Normalization, Immobility and Perfect Mobility Axioms of Shorrocks (1978), while it satisfies the Strong Perfect Mobility Axiom (TM(T) = 1 if and only if T has common columns) it does not satisfy the Strong Perfect Immobility Axiom (TM(T) =0 if and only if T = I) since TM(T) =0 for any column rearrangement of the identity matrix . The Monotonicity axiom requires TM(T) > TM(T*) when $t_{ij} \geq t_{ij}^*$ for all $i \neq j$ with strict inequality holding somewhere. Period Consistency requires TM(T) $\geq$ TM(T*) implies $TM(T^k) \geq TM(T^{*k})$ for positive integer k > 0. Obviously if the initial and outcome states are ordered and associated with some scale the index would satisfy the usual

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[5] It could be based on the discrete multivariate distribution Overlap measure of Anderson and Leo (2011) the continuous version of which is given in Anderson, Linton and Wang (2012).

scale invariance and scale independence axioms found in the inequality index literature (see for example Kobus and Miło's 2012 and below).

Transition Matrix Based Polarizing – Converging Indices.

In the spirit of Esteban and Ray (1994) and Duclos, Esteban and Ray (2004), the class of transitions from and to the center of a distribution with respect to it peripheries are the focus of attention. Working with the notion is that there are 3 classes Poor, Middle and Rich, two basic alternative models are considered, when there is no distance metric between the groups available (classes only have an ordinal ranking) and when there is (classes possess a cardinal ranking). Here the transition matrix T, where T_{ij} corresponds to the conditional probability of transiting from initial period class j to final period class i given an agent is in class j initially, is given by:

$$T = \begin{matrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{matrix}$$

T is related to the joint density matrix P = $||P_{ij}||$ of basic probabilities that an agent is in class j in the initial state and class i in the final state by the equations:

$$T = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} x \begin{bmatrix} \sum_{i=1}^{3} P_{i1} & 0 & 0 \\ 0 & \sum_{i=1}^{3} P_{i2} & 0 \\ 0 & 0 & \sum_{i=1}^{3} P_{i3} \end{bmatrix}^{-1}$$

To develop polarization – depolarization indices of class transitions, the net transfer of mass to the final state middle class that is brought about by the transition process is contemplated. This depends upon the balance of (i.e. difference in) probabilities that an agent outside the initial state middle class would move into the final state middle class (a converging transition) versus an agent inside the initial state middle class would move out of it in the final state (a polarizing transition). Negative values of this difference correspond to a Polarizing Transition, positive values correspond to a Converging Transition. When a cardinal class ranking is available these probabilities can be scaled by a distance measure.

"Balance of Probabilities" Polarization and Net Advancement Measures (classes only have ordinal ranking).

Processes that promote more transitions from the middle class to the peripheries than transitions from the peripheries to the middle class are polarizing. The probability that an agent would move out of the middle class given that she is inside the middle class initially (the divergent component) and the probability that an agent would move in to the middle class given she is outside of it initially (the convergent component) are respectively given by:

$$P(Agent \notin Final\ State\ Middle\ class \mid Agent \in Initial\ State\ Middle\ class) =$$

$$\frac{P(Agent \notin Final\ State\ Middle\ class \cap Agent \in Initial\ State\ Middle\ class)}{P(Agent \in Initial\ State\ Middle\ class)}$$

$$= \frac{P_{12} + P_{32}}{\sum_{i=1}^{3} P_{i2}} = T_{12} + T_{32}$$

$$(Note\ given\ 2\ initial\ middle\ classes\ 2a\ and\ 2b\ this\ becomes:$$

$$= \frac{P_{12a} + P_{32a} + P_{12b} + P_{32b}}{\sum_{i=1}^{3} P_{i2a} + \sum_{i=1}^{3} P_{i2b}} = w(T_{12a} + T_{32a}) + (1-w)(T_{12b} + T_{32b})$$

$$where\ w = P(Agent \in Initial\ State\ class\ 2a \mid Agent \in Initial\ State\ Middle\ class)$$

and

$$P(Agent \in Final\ State\ Middle\ class \mid Agent \notin Initial\ State\ Middle\ class) =$$

$$\frac{P(Agent \in Final\ State\ Middle\ class \cap Agent \notin Initial\ State\ Middle\ class)}{P(Agent \notin Initial\ State\ Middle\ class)}$$

$$\frac{P_{21} + P_{23}}{\sum_{i=1}^{3} P_{i1} + \sum_{i=1}^{3} P_{i3}} = \frac{\sum_{i=1}^{3} P_{i1}T_{21} + \sum_{i=1}^{3} P_{i3}T_{23}}{\sum_{i=1}^{3} P_{i1} + \sum_{i=1}^{3} P_{i3}} = wT_{21} + (1-w)T_{23}$$

$$where\ w = \frac{P(Agent \in Initial\ State\ Low\ class)}{P(Agent \notin Initial\ State\ Middle\ class)}.$$

When the classes have only an ordinal ranking this yields a Transition Matrix based Polarization/Depolarization index of the form:

$$PT = wT_{21} + (1-w)T_{23} - \left(T_{12} + T_{32}\right)$$

This has the intuition of the net proportion of the population transferring to the peripheries corresponds to polarization when PT < 0 and convergence when PT > 0. Note that if focus on a “conflict” version of a polarization index (Esteban and Ray 2011) is desired where equilibrium levels of conflict are characterized by equal sized polar distributions then the above index could be modified to:

$$PT = wT_{21} + (1-w)T_{23} - \left(T_{12} + T_{32}\right) + | T_{12} - T_{32} | - | wT_{21} - (1-w)T_{23} |$$

This would be maximally negative or positive when the net transfers to the poles are balanced.

For an index bounded between 0 and 1 consider the transformation PTB = 0.5 + PT/2.

The forgoing index focusses on convergence/polarization to and from the center but the transition could be to or from upper or lower classes, asymmetric convergence or polarization as it were. Clearly upwardly mobile societies are preferred to downwardly mobile societies, which is largely what is being captured in Dardanoni (1993), so an index needs to be oriented around that sentiment. Here interest is focused on the balance of probabilities that an agent will end up in a higher category than she started versus the chance that she will end up in a lower category. These probabilities are respectively given by:

$$P(Agent \in Final\ State\ Upper\ class \mid Agent \in Initial\ State\ Lower\ class) =$$

$$\frac{P(Agent \in Final\ State\ Upper\ class \cap Agent \in Initial\ State\ Lower\ class)}{P(Agent \in Initial\ State\ Lower\ class)}$$

$$P(Agent \notin Final\ Low\ State \mid Agent \in Initial\ LowState) * P(Agent \in Initial\ LowState)$$
$$+P(Agent \in Final\ High\ State \mid Agent \in Initial\ Middle\ State) * P(Agent \in Initial\ Middle\ State)$$

$$= (1 - T_{11}) \sum_{i=1}^{3} P_{i1} + T_{32} \sum_{i=1}^{3} P_{i2}$$

and

$$P(Agent \in Final\ State\ Lower\ class \mid Agent \in Initial\ State\ Upper\ class) =$$

$$\frac{P(Agent \in Final\ State\ Lower\ class \cap Agent \in Initial\ State\ Upper\ class)}{P(Agent \in Initial\ State\ Upper\ class)}$$

$$P(Agent \notin Final\ High\ State \mid Agent \in Initial\ High\ State) * P(Agent \in Initial\ High\ State)$$
$$+P(Agent \in Final\ Low\ State \mid Agent \in Initial\ Middle\ State) * P(Agent \in Initial\ Middle\ State)$$

$$= (1 - T_{33}) \sum_{i=1}^{3} P_{i3} + T_{12} \sum_{i=1}^{3} P_{i2}$$

When the classes have only an ordinal ranking this yields a Transition Matrix based Upward Advancement index of the form:

$$PUT = (1 - T_{11}) w_1 - (1 - T_{33}) w_3 - (T_{12} - T_{32}) w_2$$

$$where\ w_j = \sum_{i=1}^{3} P_{ij}$$

This has the intuition of the net proportion of the population transferring upwards when PT > 0 and downwards when PT < 0.

Axiomatic foundations for inequality indices for ordinal data have been explored elsewhere (Kobus and Miło's (2012)) and given an ordering of the departure and destination classes the above indices can be seen to satisfy those same axioms. Let C be the set of scaling functions associated with the classes such that in a typical k x 1 vector c, elements c_i and c_j are such that $c_i < c_j$ for all i < j i,j = 1,..,k. and indices I(T,c) based upon the transition matrix T from the family of transition matrices **T**.

Axioms:

A1. Continuity: I(T,c) T ε **T**, c ε C : is a continuous function.

A2. Scale Invariance ($I(T_1,c_1) < I(T_2,c_1) \Leftrightarrow I(T_1,c_2) < I(T_2,c_2)$ for all $c_1 \neq c_2$, c_1, c_2 ε C is invariant to scaling of the index.

A3. Scale Independence ($I(T, c_1) = I(T, c_2)$) for any $c_1, c_2 \in C$

A4. Normalization $0 \leq I(T,c) \leq 1$

A5. Coherence If $T_1 <_{PDIS} T_2$, $I(T_1,c) \leq I(T_2,c)$ for any $c \in C$ where $<_{PDIS}$ is a Polarization/Mobility ordering over the distributions.

These axioms parallel standard axioms used in inequality measurement with A1 requiring continuity, A2 requiring invariance to scale changes and A3 demanding independence of scale changes (Obviously if A3 holds, then so does A2). A4 requires that the index be normalized, i.e. zero is assigned to the lowest valued distribution and one is assigned to the highest valued distribution.

Polarization and Advancement Measures (When classes are ranked cardinally).

For polarization and advancement indices that incorporate a sense of distance moved, the index needs to be scaled by the distance between the classes which has the interpretation of the net average distance moved in the population toward the peripheries. In the context of the final state, the average distance that would have been moved to the peripheries is given by the probability that an agent would move out of the middle class given that she is in it initially scaled by the final state distance moved and the average distance moved to the center is given by the probability that an agent would move to the middle class given she is out of it initially, each of which is respectively given by:

$$dist * P(Agent \notin Final\ State\ Middle\ class \mid Agent \in Initial\ State\ Middle\ class) =$$

$$dist * \frac{P(Agent \notin Final\ State\ Middle\ class \cap Agent \in Initial\ State\ Middle\ class)}{P(Agent \in Initial\ State\ Middle\ class)}$$

$$= T_{12} \mid y_{f1} - y_{f2} \mid + T_{32} \mid y_{f3} - y_{f2} \mid$$

and

$$dist * P(Agent \in Final\ State\ Middle\ class \mid Agent \notin Initial\ State\ Middle\ class) =$$

$$dist * \frac{P(Agent \in Final\ State\ Middle\ class \cap Agent \notin Initial\ State\ Middle\ class)}{P(Agent \notin Initial\ State\ Middle\ class)}$$

$$= wT_{21} \mid y_{f1} - y_{f2} \mid + (1-w)T_{23} \mid y_{f3} - y_{f2} \mid$$

$$where\ w = \frac{P(Agent \in Initial\ State\ Low\ class)}{P(Agent \notin Initial\ State\ Middle\ class)}.$$

A final state distance scaled Polarization index is given by:

$$PTSDF = T_{12}\,|\,y_{f1} - y_{f2}\,| + T_{32}\,|\,y_{f3} - y_{f2}\,| - \left(wT_{21}\,|\,y_{f1} - y_{f2}\,| + (1-w)T_{23}\,|\,y_{f3} - y_{f2}\,|\right)$$

If there is interest in the distance moved from the initial state to the final state an Initial State to Final State distance scaled Polarization index is given by:

$$PTSDIF = T_{12}\,|\,y_{f1} - y_{i2}\,| + T_{32}\,|\,y_{f3} - y_{fi}\,| - \left(wT_{21}\,|\,y_{f1} - y_{fi}\,| + (1-w)T_{23}\,|\,y_{f3} - y_{i2}\,|\right)$$

An Index that relates to final state distances relative to initial state distances is given by:

$$PTSDR = T_{12}\frac{|\,y_{f1} - y_{f2}\,|}{|\,y_{i1} - y_{i2}\,|} + T_{32}\frac{|\,y_{f3} - y_{f2}\,|}{|\,y_{i3} - y_{i2}\,|} - \left(wT_{21}\frac{|\,y_{f1} - y_{f2}\,|}{|\,y_{i1} - y_{i2}\,|} + (1-w)T_{23}\frac{|\,y_{f3} - y_{f2}\,|}{|\,y_{i3} - y_{i2}\,|}\right)$$

Distance based advancement indices may be written as:

$$\begin{aligned} PUTD &= ((1-T_{11})w_1 - (1-T_{33})w_3 - \left(T_{12} + T_{32}\right)w_2) * Dist \\ &= (T_{12}\,|\,y_2 - y_1\,| + T_{13}\,|\,y_3 - y_1\,|)w_1 - (T_{23}\,|\,y_3 - y_2\,| + T_{13}\,|\,y_3 - y_1\,|)w_1 \\ &\quad - \left(T_{12}\,|\,y_2 - y_1\,| + T_{32}\,|\,y_3 - y_2\,|\right)w_2 \end{aligned}$$

Note that Stationary Transition Matrices (The Identity and EOP transition matrices) can still produce polarization and convergence when the class values are changing. Think about M(T) > 0 and μ_H-μ_L > as sufficient for polarization by class outcomes.

3. 3 Applications.

Example 1. Generational Relationships in Educational Attainment in Canada

Anderson, Leo and Muelhaupt (2013) studied alternative versions of equality of opportunity in the context of generational educational relationships of males and females and their respective parents over 5 age cohorts of children in Canada. The generational relationship, represented by a transition matrix, can be seen as a transition from a parentally endowed initial state to the child's educational attainment final state, for a sample of parent – child pairs. Whilst the states are ordered, they are not cardinally ordered. If the transition matrix is constant over cohorts and applied to successive generations it could deliver a polarized, converged or stationary educational structure dependent upon whether it is a polarizing, converging or stationary transition matrix. Essentially the transition matrix delivers information about the progress of educational classes and the indices for such processes tell us whether the process is polarizing or converging.

The data for the empirical analysis on academic achievements of children and their parents in Canada are drawn from Statistics Canada's General Social Survey Cycle 19 (2005). Educational attainment is indexed from 1 to 5 as follows: 1 for some secondary/elementary/no education; 2 for high school diploma; 3 for some university; 4 for Diploma/Certificate in a Trade/Technical skill, and 5 for a university degree. This categorization is for all individuals above the age of 25, including both parents and their children.

Commonality of Transitions

A test for the overall commonality of transition structure across all cohorts is given by using the $TM(T)/n_{ic}$ transvariation index above where in this case T is formed from the vectorized transition matrices for each cohort being compared and n_{ic} is the number of initial classes. If there are no differences in cohorts the statistic returns the value 1, if there is no commonality at all the statistic returns the value 0. This yields a test of commonalities in transitions over all 5 cohorts of 0.6277 (0.0075) for boys and 0.6057 (0.0066) for girls establishing substantial differences in the cohort transitions. Table E1.1 presents tests of differences in transitions by gender over the 5 cohorts. Clearly there are significant differences in parent – child transitions by cohort.

Table E1.1 Cohort by cohort difference in transitions

	25-34 v 35-44	35-44 v 45-54	45-54 v 55-64	55-64 v Over 64
Boys	0.9126 (0.0064)	0.8581 (0.0077)	0.8856 (0.0078)	0.7639 (0.0121)
Girls	0.9001 (0.0060)	0.8726 (0.0066)	0.8147 (0.0085)	0.7489 (0.0103)

Transition Analysis

Table E1.2 presents Boy - Girl transition comparisons together with mobility analysis and the Polarization – Convergence Statistics for each of the cohorts. Note that mobility has generally improved for younger cohorts for both genders with girls being uniformly more mobile than boys across all cohorts. The transition patterns are polarizing for the 3 oldest cohorts in males (they are marginally convergent for the two youngest male cohorts) and all cohorts in females. The intensity of polarization being much stronger for males as compared to females and diminishes monotonically the younger the cohort for males while only approximately so for females. In all cases the balance of advancement was upward especially so in the case of girls (a phenomena familiar to observers of western societies).

Table E1.2 Mobility and Transition Analysis.

Cohort	25 – 34	35 – 44	45 – 54	55 - 64	Over 64
Boys					
Sample Size	895	1039	995	659	569
Mobility	0.5719	0.5260	0.5527	0.4706	0.3219
Pol Index +	0.3830	0.4733	0.5097	0.5421	0.7029
Pol Index –	0.4466	0.4851	0.4062	0.3682	0.2668
Polarization Index	-0.0636	-0.0118	0.1035	0.1739	0.4361
Upward Advancement Index	0.5334	0.7459	0.7843	0.8664	0.8166

Girls					
Sample Size	1187	1340	1201	884	887
Mobility	0.5910	0.6307	0.5628	0.6047	0.4635
Pol Index +	0.4830	0.5221	0.5012	0.5594	0.3946
Pol Index –	0.4427	0.4995	0.4860	0.4037	0.2835
Polarization Index	0.0403	0.0226	0.0152	0.1557	0.1111
Upward Advancement Index	0.7219	0.7784	0.8357	0.8522	0.7176

Example 2 The Disappearing Middle Class in the World Income Distribution.

Employing a semi-parametric approach to income size distribution analysis Anderson, Pittau and Zelli (2015) studied the progress of cardinally ordered Poor, Middle and Rich income groupings in the world size distribution of PPP-adjusted real GDP per capita from 1970 – 2010 with a view to allowing group sizes to vary so that classes could potentially emerge or disappear. PPP-adjusted real GDP per capita, chained series at constant prices 2005 (2005 International dollar per person), from the Penn World Table (PWT) Version 7.1 (Heston et al., 2012) from 1970 to 2010 were employed in the analysis, the sample consists of data on 155 countries. They compared this to a conventional fixed class size analysis in which of course classes cannot emerge or disappear and found very different outcomes. The year 1993 appeared to be a watershed in the analysis with substantially different structures pre and post that year. Whereas the fixed class size model characterized increasing polarization trends between rich and poor and rich and middle classes in the post 1993 era with relatively little action in the pre-1993 era, the variable class size model characterized a converging world with a disappearing middle class in the post 1993 era with a significant polarizing trend in the pre-1993 era.

The analysis of the polarization / convergence structure of 15 year transitions matrices is reported below in tables E2.1 and E2.2.

Table E2.1. Variable Class Size Transition Matrices.	1970-1993 Poor(t)	Mid(t)	Rich(t)	1993-2010 Poor(t)	Mid(t)	Rich(t)
Poor(t-15)	0.993	0.007	0.000	0.990	0.010	0.000
Middle(t-15)	0.095	0.876	0.029	0.691	0.304	0.005
Rich(t-15)	0.000	0.100	0.900	0.000	0.046	0.954
PT(w=0.788)	-0.0973			-0.7578		
Mobility	0.110			0.254		

Table E2.2. Fixed Class Size Transition Matrices.	1970-1993 Poor(t)	Mid(t)	Rich(t)	1993-2010 Poor(t)	Mid(t)	Rich(t)
Poor(t-15)	0.918	0.083	0.000	0.944	0.034	0.023
Middle(t-15)	0.199	0.554	0.248	0.307	0.629	0.064
Rich(t-15)	0.001	0.038	0.961	0.000	0.165	0.835
PT(w=0.500)	-0.345			-0.272		
Mobility	0.201			0.205		

Note the substantially different behavior patterns that fixed class size and variable class size models reveal. In the variable class size model Mobility is low pre 1993 era and substantially higher post 1993 in the fixed class size model mobility is constant across the era’s. The fixed class size model reveals modest polarizing patterns in both pre and past 1993 era’s whereas the variable class size model reveals modest polarizing behavior pre 1993 with strong polarizing behavior in the post 1993 era. Furthermore the fixed class size model suggests that the polarizing behavior is upward transiting in the pre-1993 era and downward transiting in the post 1993 era in terms of countries leaving the middle class whereas the variable class size model reveals a downward transiting polarizing pattern in both eras with strong downward transitions in the post 1993 era. A feature and a finding of Anderson, Pittau and Zelli (2015) is the disappearing middle class, with countries in the middle class transiting downward and merging with the poor class which is clearly captured in the statistics for the variable class model as polarizing behaviour.

Example 3. Polarizing Grandparent - parent - child generational transition patterns in China.

In the early stage of the Chinese revolution (the late 1940's and early 1950's) the entire urban and rural population (the "grandparents" in this study) was classified into ordered social groups according to family employment status, income sources, and political loyalties. The Cultural Revolution 1966-76 (the educational period of the "parents" in this study) saw mass school closures (Gregory and Meng 2002, Deng and Treiman 1997) and a "class enemy" purge of "elites", a relatively small portion of the population. When schools reopened children from formerly lower class families were given opportunities for education and occupational attainment, while those from formerly bourgeois families were shut out (Clark 2014). However Gregory and Meng (2002) suggest that the largest negative impact was faced by children from lower educational achievement and lower social class families. Hence there would be an elevation of outcomes for the middle classes and diminishing outcomes for poor classes, potentially a polarizing transition from old class structures to subsequent generation's educational class outcomes. Post 1980 (when the "children" in our study would have been educated) saw the Economic Reforms and the effects of the One Child Policy with increased investment in child education (Anderson and Leo (2009)). It would be of interest to see whether the generational transitions were indeed converging (i.e. favouring equality of opportunity) or polarizing.

The Chinese Household Income Project (Li, Luo, Wei, and Yue 2008) is a rich dataset containing information on grandparent's social class designation, parent's educational status and child's (grandchildren's) educational status providing information on the transition from Grandparents Social class to parent's educational status and ultimately a child's educational status. The social class classification (Chengfen) at the time of the land reforms of grandparents was as follows C1: Poor Peasant or landless (53.96%), C2: Lower Middle Peasant (14.14%), C3: Upper Middle Peasant (4.81%), C4 : Rich Peasant (2.01%), C5: Landlord (2.82%), C6: Manual Worker (8.21%), C7: Office Worker (3.30%), C8: Enterprise Owner (0.43%), C9 : Petty Proprietor (3.75%), C10: Revolutionary Cadre (1.38%), C11: Revolutionary Army man (1.03%), C12: Other (4.16%). To simplify this analysis, and because some cells were very small this categorization was condensed to 5 social classes SC1 = {C1}, SC2 = {C2}, SC3 = {C3, C4, C5}, SC4 = {C6, C7}, SC5 = {C8, C9, C10, C11, C12}. The educational categories were 0 no category, 1 if Never Schooled, 2 if classes for eliminating illiteracy, 3 elementary school, 4 if junior middle school, 5 if senior middle school(including professional middle school), 6 if technical secondary school, 7 if junior college, 8 if college/university, 9 if graduate. Educational categories 0 through 9 were condensed to EDC1 = {0,1,2,3}, EDC2 = {4}, EDC3 = {5}, EDC4 = {6}, EDC5 = {7}, EDC6 = {8}, EDC7 = {9}. Information was available on 9020 parent - grandparent pairings and 1514 parent – child pairings (only children over 22 years old were used under the assumption they would have completed their education).

Table E3.1 presents the grandparent social class -> parent education class transition structure. As may be observed the columns of the transition matrix are very similar indicating a considerable amount of equality of opportunity, indeed the mobility index is 0.9248 suggesting almost complete equality of opportunity much as one would expect from an equalizing society. However closer inspection reveals a somewhat different story.

Table E3.1 Grand Parent Social Class -> Parent Education Class Transition Structure.

		Grandparents Social Classification				
		Social Class 1	Social Class 2	Social Class3	Social Class 4	Social Class5
Inheritors Class	Marginals	0.4911	0.1399	0.1202	0.1177	0.1310
EDUCATION CLASS 1	0.0520	0.0585	0.0460	0.0295	0.0508	0.0558
EDUCATION CLASS 2	0.2901	0.3140	0.2583	0.2343	0.2825	0.2927
EDUCATION CLASS 3	0.2723	0.2774	0.2742	0.2362	0.3315	0.2310
EDUCATION CLASS 4	0.1134	0.1088	0.1244	0.1347	0.1008	0.1108
EDUCATION CLASS 5	0.1879	0.1713	0.2060	0.2297	0.1676	0.2107
EDUCATION CLASS 6	0.0772	0.0632	0.0864	0.1273	0.0603	0.0905
EDUCATION CLASS 7	0.0069	0.0068	0.0048	0.0083	0.0066	0.0085

The middle class SC2 and SC3 inheritors seem to do somewhat better than their counterparts. Indeed, as Table E3.2 reveals, the outcome distribution of SC3 first order dominates that of all other Social Class outcome distributions (the one negative term in the SC5 – SC3 comparison is not significantly different from 0). Since SC3 dominates all at the first order it will also dominate all at the second order contradicting the equality of opportunity hypothesis (Lefranc, Pistolesi and Trannoy 2008, 2009). This is in essence brought about by middle social class inheritors transiting to high class educational outcomes, indeed if the SC3 is excluded from the analysis the mobility index increases to 0.9327. Similarly SC2 first order dominates the outcome distribution of SC1, and SC4 and second order dominates SC5. These transitions are seen more clearly in the context of the Polarization – Convergence analysis where social classes are

Table E3.2 1st order dominance comparisons of middle social class outcomes over other classes.

	SC(1v3)	SC(2v3)	SC(4v3)	SC(5v3)	SC(1v4)	SC(2v4)	SC(5v4)
P(A≤ 1)-P(B≤ 1)	0.0289	0.0164	0.0213	0.0263	0.0125	0.0049	0.0099
P(A≤ 2)-P(B≤ 2)	0.1086	0.0404	0.0695	0.0847	0.0682	0.0291	0.0443
P(A≤ 3)-P(B≤ 3)	0.1499	0.0784	0.1648	0.0795	0.0714	0.0863	0.0011
P(A≤ 4)-P(B≤ 4)	0.1240	0.0682	0.1309	0.0557	0.0558	0.0627	-0.0125
P(A≤ 5)-P(B≤ 5)	0.0656	0.0445	0.0688	0.0366	0.0211	0.0243	-0.0079
P(A≤ 6)-P(B≤ 6)	0.0015	0.0035	0.0017	-0.0002	-0.0020	-0.0018	-0.0037
P(A≤ 7)-P(B≤ 7)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

amalgamated into Poor Social Class {SC1}, Middle Social Class {SC2, SC3}, Upper Class {SC4, SC5} and the education classes were amalgamated into Low Education {ED1, ED2} Middle Education {ED3} High Education {ED4, ED5, ED6, ED7}. The transition structure in the 3 Social Class – 3 Education Class model is reported in Table E3.3. The mobility index for this structure is 0.9278 which is very close to the mobility measure on the 5 Social 7 educational class structure. The polarization/convergence index for this structure is 0.4656 indicating strong polarization in the transition from social class to educational class structure. As may be seen the middle class inheritors outcomes first order dominate both Lower and Upper class inheritors outcomes following the transition of middle class inheritors to high class educational outcomes (the Upward Advancement Index is 0.1988).

Turning to the next generation Parent - Child educational transitions using the educational categories for both departure and arrival states a somewhat different story is observed in Table

E3.4. There is much less mobility (0.7387) implying a greater likelihood of a child being in the same class as their parent and the dominance relations are somewhat less clear in the disaggregated structure (inheritance class 2 dominates 1 and 3 dominates 2 and 7 dominates all other than that all other relations are less clear details from the author on request).

Table E3.3. 3 Grand Parent Social Class -> 3 Parent Education Class Transition Structure.

		Poor SC	Middle SC	Upper SC
Inheritor Class		0.4911	0.2601	0.2488
Low Education Class	0.3421	0.3725	0.2856	0.3415
Middle Education Class	0.2722	0.2774	0.2566	0.2785
High Education Class	0.3856	0.3501	0.4578	0.3801

Table E3.4 Parent Educational Class -> Child Education Class Transition Structures.

		Parent Educational Class						
		1	2	3	4	5	6	7
Child Educational Class	Marginals	0.0581	0.3336	0.1783	0.1810	0.1744	0.0707	0.0040
1	0.0205	0.0568	0.0158	0.0074	0.0255	0.0189	0.0374	0.0000
2	0.1328	0.3636	0.2178	0.0778	0.0876	0.0341	0.0467	0.0000
3	0.2569	0.3409	0.3109	0.2815	0.2555	0.1477	0.1589	0.0000
4	0.1347	0.1136	0.1168	0.1926	0.1314	0.1402	0.0935	0.0000
5	0.2939	0.1023	0.2416	0.3037	0.3175	0.4015	0.3364	0.5000
6	0.1526	0.0227	0.0950	0.1296	0.1752	0.2386	0.3084	0.3333
7	0.0086	0.0000	0.0020	0.0074	0.0073	0.0189	0.0187	0.1667

Mobility 0.7387

In the condensed 3 departure state – 3 arrival state class structure the mobility index is much the same though the dominance relations are now very much clearer with inheritance A dominates inheritance class A-1 for A = 2, 3, reflecting the diminished mobility in the class structure. The polarizing effect of the transitions is weaker (0.1997) again reflecting the greater propensity for a child to stay in its parental class, however upward advances continue to outweigh downward advances in the transition with the Upward Advancement index being 0.3395.

Table E3.5 3 Parent Education Class -> 3 Child Education Class Condensed Transition Structure.

		Parent condensed education class Low	 Middle	 Upper
Child Condensed Ed Class		0.3917	0.3593	0.2490
Low Education Class	0.5601	0.2614	0.0993	0.0610
Middle Education Class	0.1575	0.4317	0.4301	0.2732
High Education Class	0.2824	0.3069	0.4706	0.6658

Polarization index 0.1997, Mobility 0.7608

Finally the 2 generation transition from parents to grandchildren can be constructed and is reported in Tables E3.6 and E3.7. The lack of mobility in the Parent – Child transitions means that the Grand Parent – Grand Child transition is a close re-run of the Grand Parent – Parent relationship. The greater probability of a child following its parental footsteps means that it will follow in the Grandparents footsteps in a fashion similar to how its parents followed their

Table E3.6 Grand Parent Social Class -> Grand Child Education Class Condensed Transition Structure.

		Social Class 1	Social Class 2	Social Class3	Social Class 4	Social Class5
	Marginals	0.4911	0.1399	0.1202	0.1177	0.1310
EDUCATION CLASS 1	0.0205	0.0187	0.0190	0.0197	0.0178	0.0197
EDUCATION CLASS 2	0.1328	0.1296	0.1163	0.1057	0.1232	0.1231
EDUCATION CLASS 3	0.2569	0.2588	0.2491	0.2380	0.2585	0.2489
EDUCATION CLASS 4	0.1347	0.1410	0.1415	0.1380	0.1450	0.1376
EDUCATION CLASS 5	0.2939	0.2941	0.3040	0.3133	0.2970	0.3010
EDUCATION CLASS 6	0.1526	0.1488	0.1603	0.1740	0.1494	0.1595
EDUCATION CLASS 7	0.0086	0.0090	0.0098	0.0113	0.0091	0.0102

Mobility 0.97986492

parents. In so doing the polarization in Chinese societies will continue with middle social class inheritors transiting to Higher and Lower educational classes in greater numbers than are Low and High class inheritors transiting to middle educational classes, essentially engendering a disappearing middle class in Chinese society.

Table E3.7 3 Grand Parent Social Class -> 3 Grand Child Education Class Transition Structure.

		Grand Parent condensed education class Low	Middle	Upper
Grand Child Condensed Ed Class		0.4911	0.2601	0.2488
Low Education Class	0.5601	0.1483	0.1307	0.1420
Middle Education Class	0.1575	0.2588	0.2439	0.2534
High Education Class	0.2824	0.5929	0.6253	0.6046

Polarization index 0.4991 Condensed Mobility 0.9784

3. Conclusions.

Polarization and convergence are inherently dynamic processes which correspond to transitions between departure and arrival states defined over groups which are in some sense ordered. They are can be concerned with the realignment, disappearance or emergence of groups, yet their measurement is conventionally based upon a "distance weighted" analysis of the anatomy of the arrival state distribution at a given point in time. Here it has been argued that polarization and convergence are more appropriately studied and understood in the context of indices reflecting the anatomy of transitions between states. Accordingly indices have been proposed and developed which identify polarization or convergence based upon the nature of an underlying transition process. They do not necessarily depend upon a between or within group cardinal ordering, however, if a metric by which the states are cardinally ordered is available, an appropriate distance weighting of the indices is possible. They do not depend upon the “square-ness” of the transition matrix, that is to say they can deal with disappearing and emerging groups. 3 examples exemplify their use. The first example, a study of generational dependencies in educational attainments in Canada, revealed considerable heterogeneity across successive cohorts and across genders in generational dependence patterns with polarizing transitions that decline in intensity with younger cohorts with the polarizing effect being more substantive for males. Advancement was upward in all cases. The second example, a study of mobility in the size distribution of world GDP per capita in the context of a variable class size model revealed polarizing behavior that resulted in a disappearing middle class with downward transiting behavior. The third example studies the anatomy of transitions from the early revolutionary class structure classification in China to the educational class structure of the modern day. Again in this context some polarizing transitional structures are revealed.

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