University of Toronto
Department of Economics

Working Paper 539

Integrated-quantile-based estimation for first price auction models

By Yao Luo and Yuanyuan Wan

May 06, 2015

INTEGRATED-QUANTILE-BASED ESTIMATION FOR FIRST-PRICE AUCTION MODELS

YAO LUO[†] AND YUANYUAN WAN[‡]

UNIVERSITY OF TORONTO

ABSTRACT. This paper considers nonparametric estimation of first-price auction models under the monotonicity restriction on the bidding strategy. Based on an integrated-quantile representation of the first-order condition, we propose a tuning-parameter-free estimator for the valuation quantile function. We establish its cube-root-n consistency and asymptotic distribution under weaker smoothness assumptions than those typically assumed in the empirical literature. If the latter are true, we also provide a trimming-free smoothed estimator and show that it is asymptotically normal and achieves the optimal rate of Guerre, Perrigne, and Vuong (2000). We illustrate our methods using Monte Carlo simulations and an empirical study of the California highway procurements auctions.

Key words: First Price Auctions, Monotone Bidding Strategy, Nonparametric Estimation, Tuning-Parameter-Free

JEL: D44, D82, C12, C14

First version: Wednesday 21st January, 2015

This version: Sunday 3rd May, 2015

We thank the valuable comments from Victor Aguirregabiria, Tim Armstrong, Christian Gourieroux, Emmanuel Guerre, Ruixuan Liu and Ismael Mourifie. We benefited from discussions with participants at CMES 2014. All errors are ours. Luo gratefully acknowledge financial support from the National Natural Science Foundation of China (NSFC-71373283).

[†] Corresponding author. 150 St. George Street, Toronto ON M5S 3G7, Canada. yao.luo@utoronto.ca.

[‡]yuanyuan.wan@utoronto.ca.

1. INTRODUCTION

Since the seminal work of Guerre, Perrigne, and Vuong (2000, GPV hereafter), the nonparametric estimation of auction models has received enormous attention from both the perspectives of econometric analysis and empirical applications. In this paper, we revisit the first-price auction models and propose a novel estimation procedure for the valuation quantile function. Our approach is appealing both computationally and theoretically. We first construct a quantile estimator that is tuning-parameter-free and robust in the sense that it is consistent under weaker smoothness assumptions than typically imposed in the literature (details later). Whenever the typical smoothness assumptions are satisfied, we can construct a trimming-free and asymptotically normal second step estimator that achieves the optimal rate of GPV. Furthermore, our estimation procedure explicitly incorporates the restriction of the monotone bidding strategy, which is important for empirical work but not ensured by most of the existing approaches.

To better illustrate the features of our estimator, we begin by reviewing existing approaches in the literature. We focus on the baseline case of homogeneous auctions and will discuss possible extensions to incorporate auction specific characteristics in our empirical illustration. We consider the standard GPV setup of independent private value (IPV) first price auction. Their novel approach is to transform the first-order condition for optimal bids and express a bidder's value as an explicit function of the submitted bid, the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of bids:

$$v = s^{-1}(b) \equiv b + \frac{1}{I-1}\frac{G(b)}{g(b)}, \tag{1}$$

where b is the bid, I is the number of bidders, and $G(\cdot)$ and $g(\cdot)$ are the distribution and density of bids, respectively. A two-step estimation method follows from this observation: first construct a pseudo value for each bid and then apply kernel density estimation to the sample of pseudo values. GPV establish the consistency of their estimator and the optimal rate.

Haile, Hong, and Shum (2003) made another important observation. They show that the equilibrium bidding strategy is strictly monotone, then there exists a quantile representation of the first-order condition

$$Q_v(\alpha) = Q_b(\alpha) + \frac{1}{I-1}\frac{\alpha}{g(Q_b(\alpha))}, \quad 0 \le \alpha \le 1, \tag{2}$$

where $Q_b(\cdot)$ and $Q_v(\cdot)$ are the bid and valuation quantile functions, respectively. Note that the right-hand side must be strictly increasing in α, too. Based on this representation, Marmer and Shneyerov (2012, MS hereafter) proposed a novel inference method: first nonparametrically estimate $Q_v(\cdot)$ based on Equation (2) and subsequently estimate the valuation density using $f(v) = 1/Q_v'(Q^{-1}(v))$. MS show that their estimator is asymptotically normal and achieves the optimal rate of GPV.

In both estimators of GPV and MS, the bids density $g(\cdot)$ appears in the denominator of the first step estimation; in MS, the derivative of the bids quantile also appears in the denominator of the second step. In practice, trimming near the boundaries is needed but can be troublesome as it is well known that there is no generic guidance. In addition, there is no guarantee that the empirical analog of the right-hand side of Equation (1) or (2) remains strictly increasing.

In this paper, we propose to consider the integrated quantile function of the valuation as in Liu and Luo (2014):

$$V(\beta) \equiv \int_0^\beta Q_v(\alpha)d\alpha = \frac{I-2}{I-1}\int_0^\beta Q_b(\alpha)d\alpha + \frac{1}{I-1}Q_b(\beta)\beta, \quad 0 \le \beta \le 1. \tag{3}$$

The integrated quantile representation has the following merits. First, the sample analog of $V(\cdot)$, denoted by $V_n(\cdot)$, is easy to compute. It essentially requires little more than sorting the observed bids. Neither bandwidth choice nor trimming is needed. Second, the strict monotonicity of the bidding strategy necessarily implies the strict convexity of the right-hand side. Based on this observation, we can use the greatest convex minorant (g.c.m., call it $\widehat{V}(\cdot)$) of $V_n(\cdot)$ as an estimator for $V(\cdot)$. Since $V_n(\cdot)$ is a piece-wise linear function of β, so is its greatest convex minorant $\widehat{V}(\cdot)$, which can be very easily calculated.[1] Then we can estimate $Q_v(\cdot)$ by taking the piece-wise derivatives of $\widehat{V}(\cdot)$. As a matter of fact, as we will formally prove later, this estimator is cube-root-n consistent and requires weaker smoothness on model primitive, i.e., it only requires that $F(\cdot)$ be continuously differentiable, as opposed to twice continuously differentiable in GPV and MS. We called it as our first step estimator $\widehat{Q}_v(\cdot)$. Note that $\widehat{Q}_v(\cdot)$ is tuning-parameter-free. If indeed the model admits enough smoothness, we can improve the convergence rate by considering a kernel smoothed version $\hat{q}_v(\cdot)$ of $\widehat{Q}_v(\cdot)$. We show that $\hat{q}_v(\cdot)$ is asymptotically normal and achieves GPV's optimal rate. Note that despite that one needs to choose a bandwidth for $\hat{q}_v(\cdot)$ (for which we propose an optimal bandwidth), there is no need for trimming.[2]

Another appealing feature of our estimator is that the monotonicity of bidding strategy is imposed in a simple way through the calculation of g.c.m.. As a result, the estimates $\widehat{Q}_v(\cdot)$ and $\hat{q}_v(\cdot)$ are always increasing by construction. To the best of our knowledge, Henderson, List, Millimet, Parmeter, and Price (2012, HLMPP hereafter) were the first to address the imposition of monotonicity in first price auctions. They argued that nonparametric estimators that naturally impose existing economic restrictions have empirical virtue. Our method, however, is different from theirs, who achieve the desired monotonicity constraint by tilting the empirical distribution of the data by the least amount. Their method requires repeated re-weighting of the sample. Bierens and Song (2012)'s sieve approach implicitly imposes the monotonicity constraint, but it can be computationally expensive. Our estimator imposes the monotonicity by taking the greatest convex minorant of the integrated valuation quantile function. The g.c.m. of $V(\cdot)$ is easy to compute since the empirical counterpart of $V(\cdot)$ is piece-wise linear.

[1] Many statistics softwares, for example, R, provide a command for calculating g.c.m..

[2] See Hickman and Hubbard (2014) for a modified version of the GPV estimator which replaces trimming with boundary correction.

We illustrate our method using the California Highway Procurement auction data set. In practice, it is common that researchers observe auction-specific characteristics.[3] Our method still applies if the observed auction-specific characteristics are discrete-valued (or discretization of continuous variables) by conditioning on each realization. The estimate will then be interpreted as conditional valuation quantiles on observed auction characteristics. When the observed auction-specific characteristics are continuous, we follow the homogenization methods proposed by Haile, Hong, and Shum (2003) and apply our estimation methods to the homogenized bids. The homogenization approach imposes additional additive separability structure on how valuation depends on observed characteristics. There are other ways to handle observed auction heterogeneity, for example in GPV and MS, the conditional valuation density is estimated by Kernel method without making such an additive separability assumption. Nevertheless, the trade-off is that Kernel estimation may suffer the "curse of dimensionality" when the covariates are high dimensional. Recently Gimenes and Guerre (2013) proposed an augmented-quantile regression method to overcome such difficulty by observing that a linear specification of valuation quantile function generates a linear specification for the bid quantile function. In this paper we follow Haile, Hong, and Shum (2003)'s homogenization method since it is convenient to implement under its assumptions and plan to explore other possibilities in future research.

The rest of the paper is laid out as follows. We lay out the model and propose our estimator in Section 2. We examine the performance of our estimator in Section 3. Section 4 is the empirical illustration. We conclude the paper in Section 5.

2. Model and Main Results

We consider the first-price sealed-bid auction model with independent private values. A single and indivisible object is auctioned. We make the following assumptions.

[3] In general, the first price auction model is not identified if there is unobserved heterogeneity across auctions, see Armstrong (2013b).

Assumption 1. *There are $L \to \infty$ identical auctions, and for each auction, there are I symmetric and risk neutral bidders. Their private values are i.i.d. draws from a common distribution $F(\cdot)$.*

We consider large number of auctions with finite bidders. Let the total number of bids be $n = LI$. The asymptotics is on the number of auctions, that is, $L \to \infty$. The assumption that number of bidders I is constant across auctions is just for simplifying notation; our analysis can be easily extended to conditional on I as long as I is independent of valuation. For the benchmark model, we assume that there is no observed heterogeneity across auctions, and we will discuss in our empirical application how to handle the observed heterogeneity based on the method proposed in Haile, Hong, and Shum (2003).

Assumption 2. *$F(\cdot)$ is continuously differentiable over its compact support $[\underline{v}, \overline{v}]$. There exists $\lambda > 0$ such that $\inf_{v\in[\underline{v},\overline{v}]} f(v) \geq \lambda > 0$.*

Assumption 2 only requires that $F(\cdot)$ is continuously differentiable, which is weaker than the twice continuously differentiability, as assumed in the literature, e.g., GPV and MS. It is well known that the equilibrium strategy is

$$b = s(v|F, I) \equiv v - \frac{1}{F(v)^{I-1}} \int_0^v F(x)^{I-1} dx.$$

GPV show that the first-order condition can be written as Equation (1). Haile, Hong, and Shum (2003) represents this equation in terms of quantiles as in Equation (2). In this paper, we consider the integrated quantile function of the valuation as in Equation (3).

2.1. **Estimation.** Now let us first propose a tuning-parameter-free estimator for the valuation quantile function. Let $\lfloor \cdot \rfloor$ denote the integer part and $b_{(i)}$ be the i-th order statistic of a sample of bids $\{b_i\}_{i=1}^n$. Employing Equation (3), we construct a raw estimator $V_n(\cdot)$ for $V(\cdot)$ as follows. Let $V_n(0) = 0$. For $\alpha \in \{\frac{1}{n}, \frac{2}{n}, \cdots, 1\}$,

$$V_n(\alpha) = \frac{I-2}{n(I-1)} \sum_{i=1}^{n\alpha} b_{(i)} + \frac{1}{I-1} \alpha b_{(n\alpha)}.$$

For $\alpha \in \left(\frac{j-1}{n}, \frac{j}{n}\right)$, $j = 1, \cdots, n$, define

$$V_n(\alpha) = (j - \alpha n) V_n\left(\frac{j-1}{n}\right) + (\alpha n - j + 1) V_n\left(\frac{j}{n}\right).$$

One raw estimator $Q_{v,n}(\cdot)$ for $Q_v(\cdot)$ can be constructed as the left-derivative of $V_n(\cdot)$. In particular, let $Q_{v,n}(0) = \underline{v}$, and for $\alpha \in \left(\frac{j-1}{n}, \frac{j}{n}\right]$, $j = 1, 2, \cdots, n$,

$$Q_{v,n}(\alpha) = b_{(j)} + \frac{1}{I-1} \frac{j-1}{n} \frac{b_{(j)} - b_{(j-1)}}{1/n}.$$

However, $Q_{v,n}(\alpha)$ may not be increasing in α. The auction model implies that the higher a bidder bids, the higher his/her valuation is. Thus it is desirable to construct a series of pseudo valuations that increase with the corresponding bids.

We impose the monotonicity constraint by taking the left-derivative of the g.c.m. of $V_n(\cdot)$. Note that $V_n(\cdot)$ is piecewise linear. Let $\widehat{V}(\cdot)$ be the g.c.m. of $V_n(\cdot)$, which is also piecewise linear. Define $\widehat{Q}_v(0) = \underline{v}$ and for $\alpha \in \left(\frac{j-1}{n}, \frac{j}{n}\right]$, $j = 1, \cdots, n$,

$$\widehat{Q}_v(\alpha) = n\left\{\widehat{V}\left(\frac{j}{n}\right) - \widehat{V}\left(\frac{j-1}{n}\right)\right\}.$$

By definition, $\widehat{Q}_v(\cdot)$ is a left-continuous and weakly increasing step function.

Theorem 1. *Suppose Assumptions 1 and 2 are satisfied at $v_0 \in \mathcal{V}$, then*

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \xrightarrow{d} C_{\alpha_0} \operatorname{argmax}_t \left\{\mathbb{B}(t) - t^2\right\},$$

where C_{α_0} is a constant depends on α_0 and $\mathbb{B}$ is a two-sided Brownian motion process.

Proof. See Appendix A.1.

We have a few comments on Theorem 1. First, C_{α_0} depends on α_0, g and Q_b and is estimable (detailed expression in Appendix A.1). To conduct inference on $Q_v(\alpha_0)$, one can obtain the critical values by estimating C_{α_0} and simulating the one-dimensional Brownian motion $\mathbb{B}$, which is easy to compute. An alternative way is subsampling whose validity follows straightforwardly from Theorem 1. Second, Theorem 1 establishes the limiting distribution of the quantile estimator; an estimator of valuation distribution and its limiting distribution can be obtained by inverting $\widehat{Q}$ and the Delta-method, respectively. Thirdly, the cube-root-n consistency of the quantile estimator is obtained under weak assumptions on value distribution $F(\cdot)$ and without choosing any tuning parameters. This is similar to the well-known results in the literature on isotonic estimation: without imposing additional smoothness assumptions on the model primitives and without introducing smoothing, one can at most get cube-root-n rate.[4] Fourthly, Theorem 1 indeed provides us a basis for constructing a simple trimming-free smoothed quantile estimator that converges at a faster rate, provided appropriate smoothness conditions as listed in Assumption 3 below. Specifically, for any $0 < \alpha < 1$, let

$$\hat{q}_v(\alpha) = \int_0^1 \frac{1}{h} K\left(\frac{\alpha - u}{h}\right) \widehat{Q}_v(u) du. \tag{4}$$

Note that by construction, $\hat{q}_v(\cdot)$ is necessarily increasing since $\widehat{Q}_v(\cdot)$ is increasing.

Assumption 3. *Assumption 2 is satisfied. Furthermore, f is continuously differentiable.*

Assumption 4. $nh^5 \to c \in (0, \infty)$.

Assumption 5. *The kernel $K : \mathbb{R} \to R$ is a symmetric function satisfying (1) $\int_{\mathbb{R}} K(u)du = 1$; (2) $\int_{\mathbb{R}} uK(u)du = 0$; (3) $\int_{\mathbb{R}} |u^2 K(u)|du < \infty$; (4) $\sup_u |K(u)| = \overline{K} < \infty$; (5) continuously differentiable.*

Theorem 2. *Suppose Assumptions 1 and 3 to 5 are satisfied, and let $\alpha \in (0, 1)$, then $\sqrt{nh}(\hat{q}_v(\alpha) - Q_v(\alpha)) \xrightarrow{d} N(\mathscr{B}, \mathscr{V})$, where*

$$\mathscr{B} = -\frac{c^2 \alpha}{6(I-1)} Q_b'''(\alpha) \int u^3 K'(u) du \quad \mathscr{V} = \frac{\alpha^2}{c(I-1)^2} (Q_b'(\alpha))^2 \int K^2(u) du.$$

Proof. See Appendix A.2.

[4] Under a similar set of smoothness assumptions to ours, Armstrong (2013a) proposes to estimate the bidding strategy by maximizing the sample analog of the bidder's objective function and subsequently estimates the valuation distribution function at cube-root-n rate. Our estimator is based on the integrated-quantile representation of the first order condition. Both estimators are tuning-parameter-free and robust to the degree of smoothness in the model.

Note that the variance and bias depends on c in an analytic form. One can easily estimate the optimal choice of c that minimizes the asymptotic mean squared error, provided the model has enough smoothness for consistent estimation of $Q_b'''(\cdot)$. We do not further pursue it in this paper.

Sometimes, an analyst might be more interested in the valuation density function than the quantile function. The former can be estimated easily with our first step estimator $\widehat{Q}_v(\cdot)$ as well. First, we construct a sample of pseudo valuations empolying $\widehat{Q}_v(\cdot)$. Let $\hat{v}_j = \widehat{Q}_v(j/n)$, where $j = 1, \ldots, n$. Second, we apply a kernel density estimator on the sample of pseudo values $\{\hat{v}_j\}_{j=1}^N$: for $v \in (\underline{v}, \overline{v})$

$$\hat{f}(v) = \frac{1}{nh} \sum_{j=1}^{n} K\left(\frac{\hat{v}_j - v}{h}\right).$$

Since our first step estimator $\widehat{Q}_v(\cdot)$ is tuning-parameter-free, our estimator of the valuation density function requires only one tuning parameter h.

2.2. **Generalization to procurement auctions.** Our methods can be easily adapted to first price procurement auction settings. Suppose that there are I bidders competing for a contract in a first-price sealed bid auction. For each auction, every bidder i simultaneously draws an $i.i.d.$ cost c_i from a common distribution $F(\cdot)$ and submits a bid to maximize his/her expected profit $E[(b_i - c_i)\mathbb{1}(b_i \le s(\min_{j \neq i} c_j))]$. The lowest bid wins the contract, and the bidder is paid the amount he/she bid.

Differentiating the expected profit with respect to b_i gives the following system of first-order differential equations that define the equilibrium strategy $s(\cdot)$

$$(b_i - c_i)(I-1)\frac{f[s^{-1}(b_i)]}{[1 - F(s^{-1}(b_i))]s'[s^{-1}(b_i)]} = 1,$$

which can be rewritten as

$$c_i = b_i - \frac{1}{I-1}\frac{1 - G(b_i)}{g(b_i)}.$$

Therefore, the quantile relationship becomes

$$Q_c(\alpha) = Q_b(\alpha) - (1-\alpha)/[(I-1)g(Q_b(\alpha))],$$

where $Q_c(\cdot)$ represents the cost quantile function. The integrated quantile function becomes

$$C(\beta) \equiv \int_0^\beta Q_c(\alpha)d\alpha = \frac{I-2}{I-1}\int_0^\beta Q_b(\alpha)d\alpha - \frac{1}{I-1}Q_b(\beta)(1-\beta) + \frac{1}{I-1}Q_b(0). \quad (3\text{'})$$

To impose the monotonicity constraint, we consider the g.c.m. of the empirical counterpart of the following function:

$$\widetilde{C}(\beta) \equiv C(1-\beta),$$

which is the reflection of the integrated quantile function over the line $\beta = 1/2$. The idea is to utilize the prior information that the maximum possible bid equals the maximum cost in procurement auctions, i.e. $Q_b(1) = Q_c(1)$. As the pseudo values are constructed sequentially, consider the g.c.m. of $\widetilde{C}(\cdot)$ is preferable to $C(\cdot)$. To see this, note that $[\widehat{C}(1) - \widehat{C}(\frac{n-j}{n})]/(j/n) = \frac{I-2}{I-1}\sum_{k=n-j+1}^{N} b_{(k)}/j + \frac{1}{I-1}b_{(N-j)}$ and $[\widehat{C}(1/n) - \widehat{C}(0)]/(1/n) = b_{(1)}$. By definition, the preferred method starts with the largest pseudo valuation $\widehat{c}_{(n)} = \frac{I-2}{I-1}b_{(n)} + \frac{1}{I-1}b_{(n-1)}$. Note that the right-hand side converges to $Q_b(1) = Q_c(1)$ at a fast rate. On the other hand, considering the g.c.m. of $C(\cdot)$, we would start with an estimate of the smallest pseudo valuation $\widehat{c}_{(1)} \leq b_{(1)}$. Although $b_{(1)}$ converges to $Q_b(0)$ at a fast rate, it does not guarantee that $\widehat{c}_{(1)}$ converges to $Q_c(0)$ at a fast rate.

For estimation, we construct a raw estimator $\widetilde{C}_n(\cdot)$ for $\widetilde{C}(\cdot)$ by plugging in the bid quantile estimator. We then take the g.c.m. of $\widetilde{C}_n(\cdot)$. The pseudo cost of the bidder whose bid is the jth highest is constructed as the negative of the right-derivative of the g.c.m. at $\beta = (j-1)/n$, where $j = 1, \dots, n$. A smooth estimator for the cost quantile function follows naturally: $\hat{q}_c(\alpha) = \int_0^1 \frac{1}{h}K\left(\frac{\alpha-u}{h}\right)\widehat{Q}_c(u)du$. Moreover, we can also apply a kernel density estimator on the sample of pseudo costs: $\hat{f}(c) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{\hat{c}_j - c}{h}\right)$.

3. Simulation

To study the finite sample performance of our estimation method, we conduct Monte Carlo experiments. We adopt the setup of the Monte Carlo simulations from MS. The true valuation distribution is

$$F(v) = \begin{cases} 0 & \text{if } v < 0, \\ v^\gamma & \text{if } 0 \le v \le 1, \\ 1 & \text{if } v > 1, \end{cases}$$

where $\gamma > 0$. Such a choice of private value distributions is convenient since the distributions correspond to linear bidding strategies as:

$$s(v) = \big(1 - \frac{1}{\gamma(I-1)+1}\big) \cdot v. \tag{5}$$

We consider $I = 7$ bidders, $n = 4200$ and $\gamma \in \{0.5, 1, 2\}$. The number of Monte Carlo replications is 1000. For each replication, we first generate randomly n private values from $F(\cdot)$. Second, we obtain the corresponding bids b_i employing the linear bidding strategy (5). Third, we construct a raw estimator $V_n(\cdot)$ for $V(\cdot)$. Let $\widehat{V}(\cdot)$ be the g.c.m. of $V_n(\cdot)$. Fourth, we obtain a sample of pseudo values $\widehat{v}_j$ as the left-derivative of $\widehat{V}(\cdot)$ at j/N and estimate the valuation density function using a kernel estimator.

We compare our method with MS and GPV. For the MS and GPV methods, we use the same setups as in MS: the tri-weight kernel function for the kernel estimators and the normal rule-of-thumb bandwidths in estimation of densities. For our method, we also use the tri-weight kernel function for the kernel estimators and the normal rule-of-thumb bandwidth in estimation of f: $h = 1.06\widehat{\sigma}_v n^{-1/7}$, where $\widehat{\sigma}_v$ is the estimated standard deviation of the constructed pseudo valuations $\{\widehat{v}_j\}_{j=1}^N$.

Table 1 shows the simulation results for density estimation. When the distribution is skewed to the left ($\gamma = 0.5$), our method improves MSE and MAD but seems to produce larger biases near the boundaries. While the MS and GPV methods behave similarly in terms of MSE and MAD, the former seems to produce larger biases. When the distribution is uniform or skewed to the right ($\gamma = 1$ or 2), our method performs similarly to the GPV method, both of which seem perform slightly better than the MS method.

TABLE 1. Simulation Results for Density Estimation

	v		0.2	0.3	0.4	0.5	0.6	0.7	0.8
γ = 0.5	MSE	MS	0.0068	0.0073	0.0103	0.0131	0.0132	0.0171	0.0202
		GPV	0.0056	0.0072	0.0101	0.0132	0.0139	0.0188	0.0218
		Ours	0.0044	0.0057	0.0080	0.0100	0.0109	0.0140	0.0163
	Bias	MS	-0.0041	-0.0019	-0.0086	-0.0029	-0.0159	-0.0156	-0.0185
		GPV	0.0038	0.0018	-0.0034	0.0037	-0.0019	0.0025	0.0072
		Ours	0.0120	0.0043	-0.0016	0.0037	-0.0022	0.0038	0.0056
	MAD	MS	0.0672	0.0689	0.0806	0.0907	0.0908	0.1027	0.1094
		GPV	0.0611	0.0688	0.0800	0.0925	0.0940	0.1106	0.1186
		Ours	0.0543	0.0608	0.0711	0.0806	0.0825	0.0952	0.1030
γ = 1	MSE	MS	0.0036	0.0050	0.0066	0.0076	0.0102	0.0122	0.0148
		GPV	0.0025	0.0035	0.0050	0.0060	0.0082	0.0102	0.0127
		Ours	0.0023	0.0033	0.0049	0.0061	0.0083	0.0102	0.0129
	Bias	MS	0.0003	0.0000	-0.0047	-0.0035	0.0014	-0.0060	-0.0113
		GPV	0.0000	0.0015	-0.0023	-0.0011	0.0053	0.0007	-0.0021
		Ours	0.0000	0.0016	-0.0027	-0.0020	0.0056	0.0007	-0.0026
	MAD	MS	0.0479	0.0557	0.0647	0.0688	0.0800	0.0892	0.0961
		GPV	0.0402	0.0470	0.0563	0.0610	0.0724	0.0806	0.0904
		Ours	0.0389	0.0459	0.0557	0.0615	0.0730	0.0812	0.0901
γ= 2	MSE	MS	0.0016	0.0025	0.0037	0.0063	0.0085	0.0108	0.0154
		GPV	0.0011	0.0016	0.0025	0.0044	0.0060	0.0078	0.0112
		Ours	0.0011	0.0017	0.0028	0.0049	0.0069	0.0091	0.0130
	Bias	MS	-0.0006	-0.0031	-0.0008	-0.0013	-0.0033	-0.0085	-0.0001
		GPV	0.0005	-0.0020	0.0007	0.0002	-0.0004	-0.0044	0.0021
		Ours	0.0006	-0.0019	0.0013	0.0002	-0.0006	-0.0048	0.0020
	MAD	MS	0.0320	0.0394	0.0481	0.0637	0.0739	0.0830	0.1008
		GPV	0.0263	0.0321	0.0396	0.0528	0.0624	0.0707	0.0864
		Ours	0.0266	0.0329	0.0415	0.0555	0.0668	0.0767	0.0929

4. EMPIRICAL ILLUSTRATION

In this section, we implement our methods using the California highway procurement data. In particular, we analyze the data used in Krasnokutskaya and Seim (2011). It covers highway and street maintenance projects auctioned by the California Department of Transportation (Caltrans) between January 2002 and December 2005. We focus on the procurement auctions with 2 to 7 bidders. For each auction, the data contain the engineer's estimate of the project's total cost, the type of work involved, the number of days allocated to complete the project, the identity of the bidders and their bids.

Following Haile, Hong, and Shum (2003), we homogenize the bids before implementing our method to control for observable heterogeneity for each sample (with the same number of bidders). In particular, we regress the logarithm of the bid ($logb$) on the logarithm of the engineer's estimate ($logX$), the logarithm of the number of days ($logDays$) and the project type dummies. Table 2 displays the results. The homogenized bids (bid_new) are calculated as the exponential of the differences between the logarithm of the original bids and the demeaned fitted values of the regression. Table 3 displays the mean and standard deviation of the original and homogenized bids.

TABLE 2. Regression Results

	2	3	4	5	6	7
logX	0.978***	0.966***	1.015***	0.957***	0.932***	0.938***
	(34.11)	(56.68)	(50.59)	(51.81)	(49.91)	(56.58)
logDays	0.00650	0.00473	-0.00271	0.0901***	0.138***	0.00430
	(0.15)	(0.25)	(-0.13)	(4.76)	(6.31)	(0.18)
type	Yes	Yes	Yes	Yes	Yes	Yes
n	206	474	564	470	402	252
adj. R^2	0.871	0.906	0.857	0.929	0.930	0.947

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

We estimate a first price auction model with each sample. Figure 1 displays the estimated inverse bidding strategies, the estimated valuation quantile functions without and with smoothing, respectively. The curves represented are: from the sample with 2 bidders (yellow solid line); 3 bidders (magenta dash-dot line); 4 bidders (cyan solid line); 5 bidders (red

TABLE 3. Summary Statistics

	2	3	4	5	6	7	Total
bid	993.8	967.6	757.7	1136.9	990.9	1769.7	1042.8
	(1644.5)	(1935.9)	(843.7)	(4584.7)	(3350.3)	(7288.0)	(3595.9)
bid_new	652.5	587.7	566.3	508.9	464.4	478.5	540.0
	(208.4)	(190.6)	(178.6)	(129.0)	(135.0)	(137.4)	(174.0)
cost	402.1	468.0	477.8	453.8	423.6	441.7	451.5
	(259.6)	(223.6)	(218.9)	(164.4)	(156.3)	(159.5)	(200.0)
profit	250.4	119.7	88.46	55.09	40.79	36.81	88.59
	(75.81)	(77.65)	(60.62)	(49.56)	(42.83)	(46.68)	(83.51)
profit rate	0.439	0.244	0.197	0.136	0.109	0.0978	0.190
	(0.213)	(0.208)	(0.194)	(0.167)	(0.153)	(0.158)	(0.206)

Std. Dev. in parentheses. $profit = bid\_new - cost$. *Profit rate=profit / bid.*

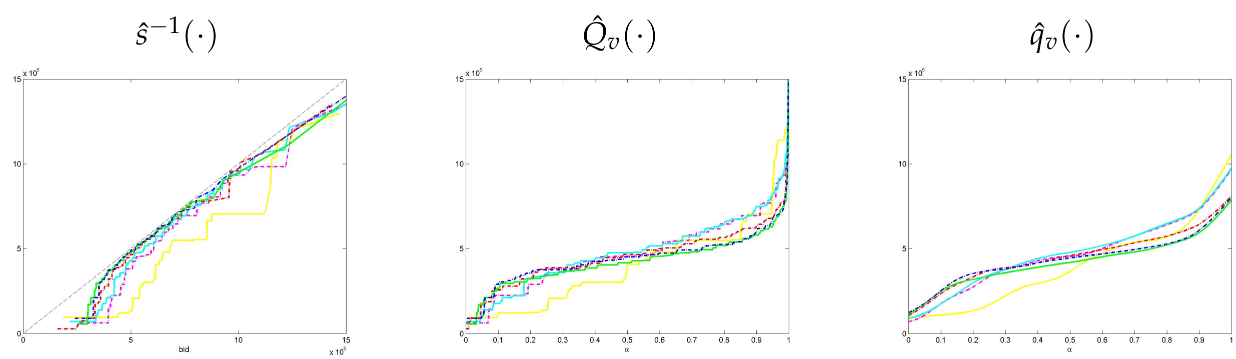

FIGURE 1. Estimation results

dash-dot line); 6 bidders (green solid line); 7 bidders (blue dash-dot line), and the 45-degree line (black dash line).

All inverse bidding strategies are increasing. The valuation quantile functions seem to be close except for $I = 2$. Table 3 displays some summary statistics of the estimated pseudo costs. The auctions with two bidders tend to be less costly to finish in percentage terms. In fact, the generated profit rate is almost twice that of the sample with three bidders. As the auction becomes more competitive when the number of bidders increase from two to seven, the profit rate decreases from 44% to about 10%.

5. Conclusion

This paper considers the nonparametric estimation of first-price auction models based on an integrated-quantile representation of the first-order condition. The monotonicity of bidding strategy is imposed in a natural way. We propose two estimators for the valuation quantile function and derive their asymptotics: a non-smoothed estimator that is tuning-parameter-free and a smoothed one that is trimming-free. We show the former is cube-root consistent under weaker smoothness assumptions and the latter achieves the optimal rate of GPV under standard ones. A series of Monte Carlo simulations shows our methods work well in finite samples. We apply our methods to data from the California highway procurements auctions.

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APPENDIX A. PROOF OF MAIN THEOREMS

A.1. **Proof of Theorem 1.** For $a > 0$, let $Z_n(a) = \text{argmin}_{t\in[0,1]}\{V_n(t) - at\}$. If the argmin is a set, then we take the inf of the set. For any $\tau \in [0,1]$, by van Es, Jongbloed, and Zuijlen (1998, Theorem 2), the two following events are equivalent

$$Z_n(a) \leq \tau \Leftrightarrow \widehat{Q}_v(\tau) \geq a.$$

Therefore, we have for a fixed $\alpha_0 \in [0,1]$,

$$\begin{aligned}
n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) < z \Leftrightarrow \widehat{Q}_v(\alpha_0) < zn^{-\frac{1}{3}} + Q_v(\alpha_0) \Leftrightarrow Z_n(zn^{-\frac{1}{3}} + Q_v(\alpha_0)) > \alpha_0 \\
\Leftrightarrow \underset{s\in[0,1]}{\text{argmin}}\{V_n(s) - (zn^{-\frac{1}{3}} + Q_v(\alpha_0))s\} > \alpha_0 \\
\Leftrightarrow \underset{\{t:\alpha_0+tn^{-\frac{1}{3}}\in[0,1]\}}{\text{argmin}} \{V_n(\alpha_0 + tn^{-\frac{1}{3}}) - (zn^{-\frac{1}{3}} + Q_v(\alpha_0))(\alpha_0 + tn^{-\frac{1}{3}})\} > 0 \\
\Leftrightarrow \underset{t\in[-\alpha_0 n^{\frac{1}{3}}, 1-\alpha_0 n^{\frac{1}{3}}]}{\text{argmin}} \{V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}} - ztn^{-\frac{2}{3}}\} > 0 \\
\Leftrightarrow \underset{t\in[-\alpha_0 n^{\frac{1}{3}}, 1-\alpha_0 n^{\frac{1}{3}}]}{\text{argmin}} \{n^{\frac{2}{3}}V_n(\alpha_0 + tn^{-\frac{1}{3}}) - n^{\frac{2}{3}}V_n(\alpha_0) - Q_v(\alpha_0)tn^{\frac{1}{3}} - zt\} > 0,
\end{aligned}$$

where we conduct changing variable $s = \alpha_0 + tn^{-\frac{1}{3}}$ and use the fact that $s > \alpha_0 \Leftrightarrow t > 0$. Let $W_n(t) = n^{\frac{2}{3}}\left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}}\right]$, the the above expression reduces to

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) < z \Leftrightarrow \underset{t\in[-\alpha_0 n^{\frac{1}{3}}, 1-\alpha_0 n^{\frac{1}{3}}]}{\text{argmin}} \{W_n(t) - zt\} > 0$$

It remains to analyze the asymptotic behavior of $W_n(t)$. Decompose W_n as following

$$\begin{aligned}
W_n(t) = n^{\frac{2}{3}}\left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0)\right] - n^{\frac{2}{3}}\left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0)\right] \\
+ n^{\frac{2}{3}}\left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}}\right].
\end{aligned}$$

The second component equals to $\frac{1}{2}Q_v'(\alpha_0)t^2 + o(1)$ by Assumption 2. By Lemma 3, the first right hand side term converges weakly to $\frac{\alpha_0}{(I-1)\sqrt{g(Q_b(\alpha_0))}}\mathbb{B}(t)$, where $\mathbb{B}$ is a two sided Brownian Motion

Therefore, we have

$$W_n(t) \xrightarrow{w} \frac{\alpha_0}{(I-1)\sqrt{g(Q_b(\alpha_0))}}\mathbb{B}(t) + \frac{1}{2}Q_v'(\alpha_0)t^2.$$

To simplify the notation, let the constants in front of $\mathbb{B}$ and t^2 be a and b, respectively. Note that $a > 0$ and $b > 0$. By Van Der Vaart and Wellner (1996, Theorem 3.2.2) and the property of Brownian motion,

$$\begin{aligned}
\underset{t\in[-\alpha_0 n^{\frac{1}{3}},1-\alpha_0 n^{\frac{1}{3}}]}{\operatorname{argmin}}\{W_n(t) - zt\} &\xrightarrow{d} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{a\mathbb{B}(t) + bt^2 - zt\} \\
&\sim \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{a\mathbb{B}(t) + b(t - \frac{z}{2b})^2 - \frac{z^2}{4b}\} \sim \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{a\mathbb{B}(t) + b(t - \frac{z}{2b})^2\} \\
&\sim \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\frac{a}{b}\mathbb{B}(t) + (t - \frac{z}{2b})^2\} \sim \left(\frac{a}{b}\right)^{2/3} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\mathbb{B}(t) + t^2\} + \frac{z}{2b}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left(n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) < z\right) \to \mathbb{P}\left(\left(\frac{a}{b}\right)^{2/3} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\mathbb{B}(t) + t^2\} + \frac{z}{2b} > 0\right) \\
&= \mathbb{P}\left(\underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\mathbb{B}(t) + t^2\} > -\frac{z}{2b}\left(\frac{b}{a}\right)^{2/3}\right) = \mathbb{P}\left(\operatorname{argmax}_{t\in\mathbb{R}}\{\mathbb{B}(t) - t^2\} < \frac{z}{2b}\left(\frac{b}{a}\right)^{2/3}\right)
\end{aligned}$$

Thus we can conclude that

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \xrightarrow{d} 2a^{2/3}b^{1/3}\operatorname{argmax}_{t\in\mathbb{R}}\{\mathbb{B}(t) - t^2\}.$$

A.2. **Proof of Theorem 2.** For notation simplicity, let $K_h(\cdot) = (1/h)K(\cdot/h)$. Then

$$\begin{aligned}
\hat{q}_v(\alpha) = \int K_h(\alpha - u)\, d\widehat{V}(u) &= \int K_h(\alpha - u)\, dV_n(u) + \int K_h(\alpha - u)\, d(\widehat{V} - V_n)(u) \\
&= \int K_h(\alpha - u)\, dV_n(u) + \frac{1}{h}\int K_h'(\alpha - u)(\widehat{V}(u) - V_n(u))du \\
&= \int K_h(\alpha - u)\, dV_n(u) + \frac{1}{h}\int K'(t)(\widehat{V}(\alpha + ht) - V_n(\alpha + ht))dt \\
&= \int K_h(\alpha - u)\, dV_n(u) + O_p((n/\log n)^{-2/3}/h) \qquad (6)
\end{aligned}$$

where the third inequality holds by integration by parts, and the last equality holds by Lemma 8. It is then sufficient to focus on the first right hand side term. Since $Q_{v,n}$ is piecewise flat and is left-continuous, we have

$$\int K_h(\alpha-u)\,dV_n(u) - Q_v(\alpha) = \int K_h(\alpha-u)\,Q_{v,n}(u)du - Q_v(\alpha)$$
$$= \underbrace{\sum_{i=1}^{n} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha-u)\,du - Q_b(\alpha)}_{A_n(\alpha)} + \frac{1}{I-1}\underbrace{\left(\sum_{i=1}^{n}(i-1)(b_{(i)}-b_{(i-1)})\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha-u)\,du - \frac{\alpha}{g(Q_b(\alpha))}\right)}_{B_n(\alpha)}.$$

$A_n(\alpha)$ and $B_n(\alpha)$ are dealt with by Lemmas 9 and 11, respectively.

Appendix B. Lemmas for Theorem 1

Lemma 1. *Suppose that Assumptions 1 and 2 hold, then for any $\alpha_0 \in (0,1)$ and uniformly over $t \in \mathscr{T}$, where $\mathscr{T}$ is compact,*

$$n^{2/3}\left\{\int_{\alpha_0}^{\alpha_0+t/n^{1/3}} Q_{b,n}(\tau)d\tau - \int_{\alpha_0}^{\alpha_0+t/n^{1/3}} Q_b(\tau)d\tau\right\} \xrightarrow{p} 0.$$

Proof. Under Assumption 2, by the Bahadur representation for quantile functions (see, e.g. Bahadur, 1966; Kiefer, 1967), we know that uniform in $\tau \in [\delta, 1-\delta]$,

$$Q_{b,n}(\tau) - Q_b(\tau) = \frac{\tau - \frac{1}{n}\sum_i \mathbf{1}[b_i \le Q_b(\tau)]}{f_b(Q_b(\tau))} + O_{a.s.}\left(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}\right).$$

Since $\alpha_0 \in (0,1)$, we have

$$n^{2/3}\int_{\alpha_0}^{\alpha_0+t/n^{1/3}} (Q_{b,n}(\tau) - Q_b(\tau))\,d\tau = n^{2/3}\int_{\alpha_0}^{\alpha_0+t/n^{1/3}} \left(\frac{\tau - \frac{1}{n}\sum_i \mathbf{1}[b_i \le Q_b(\tau)]}{f_b(Q_b(\tau))}\right) d\tau + o_p(1)$$
$$= n^{2/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} \left(F(u) - \frac{1}{n}\sum_i \mathbf{1}[b_i \le u]\right) du + o_p(1)$$
$$= \frac{1}{\sqrt{n}}\sum_i n^{1/6}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} (F(u) - \mathbf{1}[b_i \le u])\,du + o_p(1)$$
$$= \frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,t) + o_p(1),$$

where $\xi_n(b_i,t) = n^{1/6}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})}(F(u)-\mathbf{1}[b_i\le u])\,du$. Note that $\mathbb{E}[g_n(b_i,t)]=0$ and the summand are i.i.d.. To derive the covariance function, take t and s from a compact set,

$$\mathbb{E}[\xi_n(b_i,t)\xi_n(b_i,s)] = \mathbb{E}\left[n^{1/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})}\mathbf{1}[b_i\le u]du\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+s/n^{1/3})}\mathbf{1}[b_i\le u]du\right]+o(1)$$

$$= n^{1/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+s/n^{1/3})}\mathbb{E}\left\{\mathbf{1}[\min\{u,v\}\ge b_i]\right\}dudv+o(1)$$

$$= n^{1/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+s/n^{1/3})}G(\min\{u,v\})dudv \to 0,$$

where G is the c.d.f. of the bid distribution. The convergence hold uniformly over compact set of t and s. The conclusion therefore holds.

Lemma 2. *Suppose that Assumptions 1 and 2 hold, then*

$$n^{2/3}\alpha_0\left\{Q_{b,n}(\alpha_0+tn^{-1/3})-Q_{b,n}(\alpha_0)-Q_b(\alpha_0+tn^{-1/3})+Q_b(\alpha_0)\right\}\xrightarrow{w}\frac{\alpha_0}{\sqrt{g(Q_b(\alpha_0))}}\mathbb{B}(t),$$

where $\mathbb{B}$ is a two-sided Brownian motion.

Proof. Let $\mathscr{T}_1\subseteq\mathscr{T}_2\subseteq\cdots\subseteq\mathscr{T}_k\subseteq\cdots$ be a sequence of compact sets. Given Assumption 2, we can apply Bahadur representation again (see Lemma 1) and know that uniform in $\tau\in\mathscr{T}_k$,

$$Q_{b,n}(\tau)-Q_b(\tau)=\frac{\tau-\frac{1}{n}\sum_i\mathbf{1}[b_i\le Q_b(\tau)]}{g(Q_b(\tau))}+O_{a.s.}(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}.$$

Let $r_{1n}=O_{a.s.}(n^{-1/12}(\log n)^{1/2}(\log\log n)^{1/4}$, we have uniformly in $t\in\mathscr{T}_k$,

$$n^{2/3}\left\{Q_{b,n}(\alpha_0+tn^{-1/3})-Q_{b,n}(\alpha_0)-Q_b(\alpha_0+tn^{-1/3})+Q_b(\alpha_0)\right\}$$

$$=\frac{n^{1/6}}{\sqrt{n}}\sum_i\left(\frac{\alpha_0+tn^{-1/3}-\mathbf{1}[b_i\le Q_b(\alpha_0+tn^{-1/3})]}{g(Q_b(\alpha_0+tn^{-1/3}))}-\frac{\alpha_0-\mathbf{1}[b_i\le Q_b(\alpha_0)]}{g(Q_b(\alpha_0))}\right)+r_{1n}$$

$$=\frac{n^{1/6}}{\sqrt{n}}\sum_i\left(\frac{tn^{-1/3}-\mathbf{1}[Q_b(\alpha_0)<b_i\le Q_b(\alpha_0+tn^{-1/3})]}{g(Q_b(\alpha_0))}\right)+r_{1n}+r_{2n},$$

where

$$r_{2n} = \frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \le Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \le Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))} \right)$$

$$= n^{1/6} \left(\frac{1}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{1}{g(Q_b(\alpha_0))} \right) \frac{1}{\sqrt{n}} \sum_i \xi_i = n^{1/6} O(n^{-1/3}) O_p(1) = o_p(1),$$

where $\xi_i = \alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \le Q_b(\alpha_0 + tn^{-1/3})]$. For the leading term, it is can be shown by standard result that over each of the compact sets $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \cdots \subseteq \mathcal{T}_k \subseteq \cdots$, we have

$$\frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_0) < b_i \le Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))} \right) \xrightarrow{w} \frac{1}{\sqrt{g(Q_b(\alpha_0))}} \mathbb{B}(t),$$

where $\mathbb{B}$ is a Brownian motion. The stated result follows.

Lemma 3. *Suppose that Assumptions 1 and 2 hold, then*

$$n^{\frac{2}{3}} \left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) \right] - n^{\frac{2}{3}} \left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0) \right] \xrightarrow{w} \frac{\alpha_0}{(I-1)\sqrt{g(Q_b(\alpha_0))}} \mathbb{B}(t)$$

where $\mathbb{B}$ is a two-sided Brownian motion.

Proof. Recall that for any $\tau \in (0,1)$,

$$V_n(\tau) = \frac{1}{n} \frac{I-2}{I-1} \sum_i b_i \mathbf{1}[b_i \le Q_{b,n}(\tau)] + \frac{1}{I-1} \tau Q_{b,n}(\tau) + O_p(1/n)$$

$$\equiv \frac{I-2}{I-1} V_{1n}(\tau) + \frac{1}{I-1} V_{2n}(\tau) + O_p(1/n).$$

Likewise,

$$V(\tau) = \frac{I-2}{I-1} \int_0^\tau Q_v(t) dt + \frac{1}{I-1} \alpha_0 Q_b(\tau) \equiv \frac{I-2}{I-1} V_1(\tau) + \frac{1}{I-1} V_2(\tau).$$

The part associate with V_{1n}, that is, $n^{\frac{2}{3}}\left[V_{1n}(\alpha_0+tn^{-\frac{1}{3}})-V_{1n}(\alpha_0)\right]-n^{\frac{2}{3}}\left[V_1(\alpha_0+tn^{-\frac{1}{3}})-V_1(\alpha_0)\right]$ converges in probability to zero by Lemma 1. For the part associated with V_{2n}, note that

$$
\begin{aligned}
n^{\frac{2}{3}}(I-1)&\left[V_{2n}(\alpha_0+tn^{-\frac{1}{3}})-V_{2n}(\alpha_0)\right]-n^{\frac{2}{3}}\left[V_2(\alpha_0+tn^{-\frac{1}{3}})-V_2(\alpha_0)\right]\\
&=n^{2/3}Q_{b,n}(\alpha_0+tn^{-1/3})(\alpha_0+tn^{-1/3})-n^{2/3}Q_{b,n}(\alpha_0)\alpha_0-n^{2/3}Q_b(\alpha_0\\
&\qquad+tn^{-1/3})(\alpha_0+tn^{-1/3})+n^{2/3}Q_b(\alpha_0)\alpha_0\\
&=n^{2/3}\alpha_0\left\{Q_{b,n}(\alpha_0+tn^{-1/3})-Q_{b,n}(\alpha_0)-Q_b(\alpha_0+tn^{-1/3})+Q_b(\alpha_0)\right\}\\
&\qquad+n^{1/3}t\left\{Q_{b,n}(\alpha_0+tn^{-1/3})-Q_b(\alpha_0+tn^{-1/3})\right\}
\end{aligned}
$$

The second right hand side term, for $|t|<K$, is uniformly bounded by order $n^{1/3}\times n^{-1/2}\times O_p(1)\xrightarrow{p}0$. The first right hand side term is dealt with by Lemma 2.

Appendix C. Lemmas for Theorem 2

We introduce some notation. Let k_n be a sequence of integers such that $k_n\to\infty$ and $n/k_n\to\infty$. Without loss of generality we assume k_n divides n and let $\ell_n=n/k_n$. We therefore can divide $[0,n]$ into k_n equal size intervals with each interval contains ℓ_n consecutive integers. Let $\{s_i, i=1,2,\cdots,k_n\}$ be the set of upper boundary of those intervals such that $s_i=i\ell_n$.

For $(i-1)\ell_n\le s<i\ell_n$, $i=1,2,\cdots,k_n$, define

$$L(s)=\frac{s-(i-1)\ell_n}{\ell_n}V\left(\frac{i}{n}\right)+\frac{i\ell_n-s}{\ell_n}V\left(\frac{i-1}{n}\right),$$

and

$$L_n(s)=\frac{s-(i-1)\ell_n}{\ell_n}V_n\left(\frac{i}{n}\right)+\frac{i\ell_n-s}{\ell_n}V_n\left(\frac{i-1}{n}\right),$$

That is, L and L_n are the linear interpolation of V and V_n on k_n knots $\{s_1/n,s_2/n,\cdots,s_{k_n}/n\}$, respectively. Note that since V is convex under H_0, L is necessarily convex. However L_n may not be convex since V_n is not necessarily convex. Let A_n be the event such that L_n is convex. Since L_n is convex if and only if each segment is convex, the complement of A_n can be written as

$$A_n^c = \bigcup_{i=2}^{k_n-1} \left\{ V_n\left(\frac{(i-1)\ell_n}{n}\right) + V_n\left(\frac{(i+1)\ell_n}{n}\right) < 2V_n\left(\frac{i\ell_n}{N}\right) \right\}$$
$$= \bigcup_{i=2}^{k_n} \Bigg\{ V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right)$$
$$+\Delta_n\left(\frac{(i-1)\ell_n}{n}\right) + \Delta_n\left(\frac{(i+1)\ell_n}{n}\right) - 2\Delta_n\left(\frac{i\ell_n}{n}\right) < 0 \Bigg\},$$

where $\Delta_n \equiv V_n - V$.

Lemma 4. *Suppose that Assumption 3 is satisfied, then there exists a positive c_1 such that $\min_{i=2,\cdots,k_n-1} |V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right)| \geq \frac{c_1}{k_n^2}$.*

Proof. By Assumption 3, there exists $c_1 > 0$ such that $Q_v'(\alpha) \geq c_1 > 0$ for all $\alpha \in [0,1]$. Then we have

$$V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right)$$
$$= \int_{\frac{i\ell_n}{n}}^{\frac{(i+1)\ell_n}{n}} Q_v(\alpha)d\alpha - \int_{\frac{(i-1)\ell_n}{n}}^{\frac{i\ell_n}{n}} Q_v(\alpha)d\alpha \geq \int_{\frac{i\ell_n}{n}}^{\frac{(i+1)\ell_n}{n}} \left[Q_v(\alpha) - Q_v\left(\frac{i\ell_n}{n}\right)\right] d\alpha$$
$$= \frac{\ell_n}{n}\left[Q_v(\alpha_n^*) - Q_v\left(\frac{i\ell_n}{n}\right)\right] \geq c_1\frac{\ell_n^2}{n^2} = \frac{c_1}{k_n^2}. \quad \square$$

Lemma 5. *Let $\|\cdot\|$ denote the sup norm. Conditional on A_n, there is*

$$\|V_n - \widehat{V}\| \leq 2\|(V_n - L_n) - (V - L)\| + 2\|V - L\|.$$

Proof. By Kiefer and van Wolfowitz (1976), for any convex function m, $\|\widehat{V} - m\| \leq \|V_n - m\|$. Therefore,

$$\|V_n - \widehat{V}\| \leq \|V_n - L_n\| + \|L_n - \widehat{V}\| \leq 2\|V_n - L_n\| \leq 2\|(V_n - L_n) - (V - L)\| + 2\|V - L\|. \quad \square$$

Lemma 6. *Suppose that Assumption 3 is satisfied, then there exists $c_3 > 0$ such that for all $s \in [0,n]$,*

$$0 \leq L(s) - V(s) \leq \frac{c_3}{k_n^2}.$$

Proof. $L(s) > V(s)$ follows immediately by the convexity of V. The other inequality holds follows from a similar argument as in Lemma 4 and the fact that $Q_v'(\alpha)$ is bounded from above uniformly.

Lemma 7. *Suppose that Assumptions 1 and 3 is satisfied, then*

$$\|V_n - L_n - V + L\| = O_p\left(\sqrt{\frac{\log k_n}{nk_n}}\right) + O_p\left(\frac{\log n}{n}\right).$$

Proof. Define function V_P such that $V_P(j/N) = V(j/N)$ for each j/N and otherwise equals to its own interpolation. It is obvious that $\|V_P - V\| = O(1/n)$. It is then sufficient to focus on $V_n - L_n - V_P + L$. Note that all four functions are piece-wise linear, and so does there linear combinations. Therefore, the sup must be achieved at some knot(s). Based on this observations, we can write

$$\begin{aligned}&\|V_n - L_n - V_P + L\| \\ &\quad = \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \Delta_n(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\Delta_n(i/n) - \frac{i\ell_n - j}{\ell_n}\Delta_n((i-1)/n) \right|,\end{aligned}$$

where for $t \in [0,1]$,

$$\begin{aligned}\Delta_n(t) &= V_n(t) - V_P(t) = V_n(t) - V(t) + O(1/n) \\ &= \frac{I-2}{I-1} \underbrace{\left\{ \sum_{i=1}^{[tn]} \frac{b_{(i)}}{n} - \int_0^t Q_b(\alpha)d\alpha \right\}}_{\Delta_A(t)} + \frac{1}{I-1} \underbrace{\left\{ \frac{[tn]}{n} b_{(j)} - tQ_b(t) \right\}}_{\Delta_B(t)} + O(1/n)\end{aligned}$$

where $[x]$ denotes the integer part of x. Note that Δ_A is an integrated quantile process. By Tse (2009, Theorem 2.1), there exists a Gaussian process $\mathbb{G}_n$ and Brownian bridge $\mathbb{B}_n^A$ defined on proper measurable space such that for any $\tau < 1/6$,

$$\|\sqrt{n}\Delta_A - \psi_n\| \overset{a.s.}{=} O(n^{-\tau}),$$

where $\psi_n(t) = \mathbb{G}_n(t) + \int_0^t \mathbb{B}_n^A(u)dQ_b(u)$. On the other hand, by Csorgo and Revesz (1978, Theorem 6), there exists a sequence of Brownian bridge B_n such that $\sup_{\delta_n \le t \le 1-\delta_n} |g(Q_b(t))\sqrt{n}\Delta_B(t) -$

$B_n(t)| \overset{a.s.}{=} O_p(n^{-1/2}\log n)$. We can then conclude

$$
\begin{aligned}
&\|V_n - L_n - V_P + L\| \\
&\quad \le \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \Delta_A(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\Delta_A(i/n) - \frac{i\ell_n - j}{\ell_n}\Delta_A((i-1)/n) \right| \\
&+ \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \Delta_B(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\Delta_B(i/n) - \frac{i\ell_n - j}{\ell_n}\Delta_B((i-1)/n) \right| + O_p(1/n) \\
&\overset{d}{=} \frac{1}{\sqrt{n}} \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \psi_n(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\psi_n(i/n) - \frac{i\ell_n - j}{\ell_n}\psi_n((i-1)/n) \right| + O_p(n^{-\tau-1/2}) \\
&+ \frac{1}{\sqrt{n}} \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| B_n(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}B_n(i/n) - \frac{i\ell_n - j}{\ell_n}B_n((i-1)/n) \right| + O_p(\log n/n) \\
&\le \frac{1}{\sqrt{n}} \sup_{0 \le t-s \le \frac{1}{k_n}} |\psi_n(t) - \psi_n(s)| + \frac{1}{\sqrt{n}} \sup_{0 \le t-s \le \frac{1}{k_n}} |B_n(t) - B_n(s)| + O_p(\log n/n) + O_p(n^{-\tau-1/2}) \\
&\qquad \le \frac{\sqrt{2\log\log n}}{\sqrt{n}} \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{n}} \frac{\sqrt{\log\log K_n}}{\sqrt{k_n}} + O_p(\log n/n) + O_p(n^{-\tau-1/2})
\end{aligned}
$$

where the last two inequalities result from the continuity module of Gaussian processes and the fact that $g(b) \ge \underline{b} > 0$ for all b (GPV Proposition 1). Recall that $k_n \propto \frac{n}{\log n}$, we con conclude that the right hand side is of order $O_p((n/\log n)^{-2/3})$.

Lemma 8. *Suppose Assumptions 3 to 5 are satisfied, the* $\|\widehat{V} - V_n\| = O_p((n/\log n)^{-2/3})$.

Proof. The conclusion holds by Lemmas 5 to 7. □

Lemma 9. *Suppose Assumptions 3 to 5 are satisfied, then for* $0 < \alpha < 1$,

$$
A_n(\alpha) \equiv \sum_{i=1}^{n} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u)\, du - Q_b(\alpha) = -\sqrt{n}(F_n(Q_b(\alpha)) - \alpha)/g(Q_b(\alpha)) + o_p(1).
$$

Proof. We just need to verify that conditions of Yang (1985, Theorem 1) are satisfied. Then the results follows. □

Lemma 10. *Let* $z_{(i)} = n(b_{(i)} - b_{(i-1)})$*. Suppose Assumption 3 is satisfied, then for any* $r > 0$, $n^{-r}\max_i |z_{(i)}| \xrightarrow{p} 0$.

Proof. This directly follows from Parzen et al. (1962, Theorem 2.1) since both $\mathbb{E}[z_{(i)}]$ and $V(z_{(i)})$ converge to zero.

Lemma 11. *For* $0 < \alpha < 1$, *let*

$$B_n(\alpha) \equiv \sum_{i=1}^{n} (i-1)(b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u)\, du - \frac{\alpha}{g(Q_b(\alpha))}.$$

If Assumptions 3 to 5 are satisfied, then $\sqrt{nh} B_n(\alpha) \xrightarrow{d} N(\mathscr{B}, \mathscr{V})$, *where constant* $\mathscr{B}$ *and* $\mathscr{V}$ *are defined below in the proof.*

Proof. Define $\tilde{B}_n(\alpha)$ as

$$\tilde{B}_n(\alpha) = \sum_{i=1}^{n} \alpha n (b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u)\, du - \frac{\alpha}{g(Q_b(\alpha))}$$

Note first when n is large,

$$\begin{aligned} n \sum_{i=1}^{n} (b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u)\, du &= n \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u)\, du \\ - n \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i}{n}}^{\frac{i+1}{n}} K_h(\alpha - u)\, du + n b_{(n)} \int_{\frac{n-1}{n}}^{1} K_h(\alpha - u)\, du - n b_{(0)} \int_{0}^{1/n} K_h(\alpha - u)\, du \\ \approx n \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u)\, du - n \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i}{n}}^{\frac{i+1}{n}} K_h(\alpha - u)\, du. \end{aligned}$$

The last equality holds because under Assumption 5, when n is large, $K_h(t) = 0$ for any $t \neq 0$. Then we know that

$$\begin{aligned} \tilde{B}_n(\alpha) - \frac{\alpha}{g(Q_b(\alpha))} &= \alpha n \sum_{i=1}^{n-1} b_{(i)} \left\{ \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u)\, du - \int_{\frac{i}{n}}^{\frac{i+1}{n}} K_h(\alpha - u)\, du \right\} \\ &= \frac{\alpha}{h^2} \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K'\left(\frac{u-\alpha}{h}\right) du = \alpha \frac{\partial \{\sum_{i=1}^{n} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\tau - u)\, du\}}{\partial \tau} |_{\tau=\alpha}. \end{aligned}$$

By Welsh (1988, main theorem), under Assumptions 3 to 5, $\sqrt{nh}(\tilde{B}_n(\alpha) - \frac{\alpha}{g(Q_b(\alpha))}) \xrightarrow{d} N(\mathscr{B}, \mathscr{V})$, where

$$\mathscr{B} = -\frac{\alpha}{6} Q_b'''(\alpha) \int u^3 K'(u) du, \quad \mathscr{V} = \alpha^2 Q_b'(\alpha) \int K^2(u) du.$$

Next we show that $B_n - \tilde{B}_n = o_p(1/\sqrt{nh})$ uniformly over $\alpha \in [\delta, 1-\delta]$ for any $\delta > 0$. Let $z_{(i)} = n(b_{(i)} - b_{(i-1)})$ and r be a small positive constant,

$$
\begin{aligned}
|B_n(\alpha) - \tilde{B}_n(\alpha)| &= |\frac{1}{h}\sum_{i=1}^{n}\left(\frac{i-1}{n} - \alpha\right) z_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K\left(\frac{u-\alpha}{h}\right) du| \\
&\leq n^{-r} \max_i |z_{(i)}| n^r |\frac{1}{h}\sum_{i=1}^{n}\left(\frac{i-1}{n} - \alpha\right) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K\left(\frac{u-\alpha}{h}\right) du| \\
&= o_p(1) n^r |\frac{1}{h}\sum_{i=1}^{n}\left(\frac{i-1}{n} - \alpha\right) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K\left(\frac{u-\alpha}{h}\right) du| = o_p\left(\frac{1}{n^{1-r}h}\right) = o_p(1/\sqrt{nh}),
\end{aligned}
$$

where the last equality holds because $\int uk(u) = 0$ and

$$
\sum_{i=1}^{n}\left(\frac{i-1}{n} - \alpha\right) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K\left(\frac{u-\alpha}{h}\right) du = \int_0^1 (u-\alpha) K\left(\frac{u-\alpha}{h}\right) du + O(1/n) = O(1/n).
$$

Therefore we can conclude that $\sqrt{nh} B_n(\alpha) \xrightarrow{d} N(\mathscr{B}, \mathscr{V})$. □