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A Nonparametric Test of Exogenous Participation in First-
Price Auctions

By Nianqing Liu and Yao Luo

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Paul Nianqing Liu[†]

Yao Luo[‡]

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Abstract

This paper proposes a nonparametric test of exogenous participation in first-price auctions. Exogenous participation means that the valuation distribution does not depend on the number of bidders. Our test is motivated by the fact that two valuation distributions are the same if and only if their generalized Lorenz curves are the same. Our method avoids estimating unobserved valuations and does not require smooth estimation of bid density. We show that our test is consistent against all fixed alternatives and has power against root- n local alternatives. Monte Carlo experiments show that our test performs well in finite samples. We implement our method on data from the U.S. Forest Service timber auctions. We also discuss how our test can be adapted to other testing problems in auctions.

Keywords: Auctions, Exogenous Participation, Nonparametric, Hypothesis Test, Generalized Lorenz Curve

JEL: D44, D82, C12, C14

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[†]Shanghai University of Finance and Economics. e-mail: nliu@mail.shufe.edu.cn

[‡]University of Toronto. e-mail: yao.luo@utoronto.ca

1 Introduction

The distinction of different models is important for choosing empirical approaches and making policy recommendations. From an empirical point of view, models with certain restrictions are usually simpler to work with. Moreover, identification of the models relies on behavioral assumptions and on assumptions about the underlying demand and information structure. However, their relevance has to be decided through formal testing procedures.

This paper proposes a nonparametric test of exogenous participation in first-price auctions. Exogenous participation means that the valuation distribution does not depend on the number of bidders (Athey and Haile, 2002). Assuming exogenous participation has been used to identify various auction models, such as first-price auctions with risk aversion (Guerre, Perrigne, and Vuong, 2009), first-price auctions under ambiguity (Grundl and Zhu, 2013), and ascending auctions (Aradillas-López, Gandhi, and Quint, 2013). Many testing problems in auctions reduce to the standard form of testing exogenous participation, such as detecting collusion (Aryal and Gabrielli, 2012), distinguishing private value and common value auctions (Haile, Hong, and Shum, 2003), testing different models of entry (Marmer, Shneyerov, and Xu, 2013).

Testing equality of distributions is a standard problem in statistics. Classic examples are the Kolmogorov-Smirnov test, the Cramer-von Mises test, and the Anderson-Darling test. In first-price auctions, complications arise from the fact that valuations are estimated rather than observed directly. The seminal paper by Guerre, Perrigne, and Vuong (2000) transformed the First-Order Conditions (FOC) for optimal bids to express a bidder's value as an explicit function of the submitted bid, the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of bids. Thus, value density function can be estimated using constructed pseudo values. Although their estimator converges to the true value at the optimal rate with an appropriate choice of the bandwidth, the asymptotic distribution of this estimator is as yet unknown. A key difficulty is that both steps of the Guerre, Perrigne, and Vuong (2000) method are nonparametric with estimated values entering the second stage.

This paper proposes a nonparametric test of equality of valuation distributions without constructing pseudo valuations in auctions. It is motivated by a simple idea: two valuation distributions are the same if and only if their generalized Lorenz curves are the same.¹ We show that the bidders' FOC allows us to express the generalized Lorenz curve of a valuation distribution as a simple linear functional of the quantile function of the bids. In light of this observation, we propose a test statistic measuring the L^1 -distance between the sample analogues of this linear functional for two bid samples. Consequently, our test statistic only involves the two empirical quantile functions of the bids.

Our test has two attractive features. First, the test statistic is calculated in one step, which allows us to characterize its asymptotic properties under regularity conditions. In particular, we show that the test statistic converges to the L^1 -norm of a Gaussian process

¹Recently, Barrett, Donald, and Bhattacharya (2014) proposed nonparametric tests for Lorenz dominance.

with mean zero at a parametric rate under the null hypothesis. We also show that our test is consistent against any fixed alternative and can detect local alternatives converging to the null hypothesis at a rate of root- n . Second, our test statistic is easy to calculate, as it involves no density estimation. Moreover, since the empirical quantile function of bids is a step function, the empirical counterpart of the generalized Lorenz curve is piecewise linear. Therefore, the test statistic, i.e., the L^1 -distance between two empirical counterparts, is simply the total area of a finite number of trapezoids and triangles. This feature makes our test easy to implement in practice.

In the auction literature, there has been an increase in attention paid to the development of statistical tests. Examples include tests for affiliation such as [Li and Zhang \(2010\)](#) and [Jun, Pinkse, and Wan \(2010\)](#), for monotonicity of bid function such as [Liu and Vuong \(2013\)](#), for discriminating entry models such as [Marmer, Shneyerov, and Xu \(2013\)](#) and for risk aversion such as [Fang and Tang \(2014\)](#).

Although our testing approach is new, we are not the first to use quantile-based approaches in auctions. [Marmer and Shneyerov \(2012\)](#) and [Marmer, Shneyerov, and Xu \(2013\)](#) propose a quantile-based estimator in first-price auctions and a quantile-based test for distinguishing different entry models, respectively. [Liu and Vuong \(2013\)](#) provide a quantile-based test of monotonicity of bidding strategy in first-price auctions. For nonlinear pricing models, [Luo, Perrigne, and Vuong \(2014\)](#) propose a quantile-based estimator which achieves root- n consistency.

The remainder of the paper is organized as follows. In Section 2, we describe our testing problem and introduce the test statistic. We then derive the asymptotic properties (i.e., asymptotic distribution under the null hypothesis, size and power) of the test in Section 3. In Section 4, we report the results of a Monte Carlo study for moderate sample sizes. Section 5 discusses applications of our test to auctions with risk aversion, auction-specific heterogeneity and entry. In Section 6, we apply our method to data from the U.S. Forest Service timber auctions. Section 7 summarizes our results and indicates some future lines of research. All the proofs are gathered in the Appendix.

2 Exogenous Participation Test in First-Price Auctions

We briefly present the first-price sealed-bid auction model with independent private values. A single and indivisible object is auctioned. I potential bidders are symmetric and risk neutral. Their private values are i.i.d. drawn from a common distribution $F(\cdot)$, which is absolutely continuous with density $f(\cdot)$ and support $[\underline{v}, \bar{v}]$. The equilibrium bid function takes the form of:

$$b = s(v|F, I) \equiv v - \frac{1}{F(v)^{I-1}} \int_0^v F(x)^{I-1} dx.$$

2.1 Motivation

The seminal paper [Guerre, Perrigne, and Vuong \(2000\)](#) shows that a bidder's value can be expressed as an explicit function of the submitted bid, the PDF and CDF of bids

$$v = \zeta(b) \equiv b + \frac{1}{I-1} \frac{G(b)}{g(b)}, \quad (1)$$

which can be rewritten in terms of the quantile functions of values and bids as

$$v(\alpha) = b(\alpha) + \frac{1}{I-1} \frac{\alpha}{g(b(\alpha))}, \quad (2)$$

where $v(\alpha)$ and $b(\alpha)$ are the α quantiles of the valuations and bids, respectively. [Marmer and Shneyerov \(2012\)](#) use the well-known formula $f(v) = 1/v'(F(v))$ to estimate $f(\cdot)$. [Haile, Hong, and Shum \(2003\)](#) and [Marmer, Shneyerov, and Xu \(2013\)](#) exploit this quantile relationship to construct tests for detecting common value and to distinguish entry models, respectively. Note that the bid density $g(\cdot)$ is involved in both Equations (1) and (2). Thus, estimating the valuation distribution (quantile) function requires estimating both the bid distribution (quantile) function and the density function. Recovering the latter could be troublesome, as we need to choose a tuning parameter, namely, some bandwidth h , in density estimation.

To motivate our test, we focus on the integral of the quantile function (generalized Lorenz curve) of valuations. In particular, we rewrite the quantile relationship as follows:

$$\begin{aligned} v(\alpha) &= \frac{I-2}{I-1} b(\alpha) + \frac{1}{I-1} \left[b(\alpha) + \frac{\alpha}{g(b(\alpha))} \right] \\ &= \frac{I-2}{I-1} b(\alpha) + \frac{1}{I-1} \frac{d(b(\alpha) \cdot \alpha)}{d\alpha}. \end{aligned}$$

Taking integration on both sides from 0 to β leads to our basic formula:

$$V(\beta) \equiv \int_0^\beta v(\alpha) d\alpha = \frac{I-2}{I-1} \int_0^\beta b(\alpha) d\alpha + \frac{1}{I-1} b(\beta) \beta. \quad (3)$$

Note that the right-hand side involves only the bid quantile function $b(\cdot)$, which can be nonparametrically estimated at a \sqrt{N} rate.

2.2 Test Statistic

We now introduce our testing problem formally. We observe two independent i.i.d. samples $\{B_{1,1}, \dots, B_{1,N_1}\}$ and $\{B_{2,1}, \dots, B_{2,N_2}\}$, where $N_k = I_k \times L_k$ is the number of bids generated from L_k first-price auctions with I_k bidders and valuation distribution $F(\cdot|I_k)$ for $k = 1, 2$. We want to test whether the two valuation distributions $F(\cdot|I_1)$ and $F(\cdot|I_2)$ are the same.

Specifically, our hypothesis of interest is:

$$H_0 : F(v|I_1) = F(v|I_2), \forall v \in [\underline{v}, \bar{v}] \quad \text{v.s.} \quad H_1 : F(v|I_1) \neq F(v|I_2), \text{ for some } v \in [\underline{v}, \bar{v}].$$

Denote $v(\cdot|I_k)$ as the corresponding quantile function of $F(\cdot|I_k)$, where $k = 1, 2$. Define $V_k(\cdot) \equiv \int_0^\cdot v(\alpha|I_k)d\alpha$. Our testing problem is equivalent to

$$H_0 : V_1(\beta) = V_2(\beta), \forall \beta \in [0, 1] \quad \text{v.s.} \quad H_1 : V_1(\beta) \neq V_2(\beta), \text{ for some } \beta \in [0, 1]$$

where functions $V_1(\cdot)$ and $V_2(\cdot)$ can be expressed in terms of bid quantile functions

$$V_k(\beta) \equiv \int_0^\beta v(\alpha|I_k)d\alpha = \frac{I_k - 2}{I_k - 1} \int_0^\beta b(\alpha|I_k)d\alpha + \frac{1}{I_k - 1} b(\beta|I_k)\beta,$$

where $b(\cdot|I_k)$ is the quantile function of bids generated from auctions with I_k bidders and valuation distribution $F(\cdot|I_k)$.

Under the null hypothesis, $V_1(\cdot)$ and $V_2(\cdot)$ are identical on the whole domain $[0, 1]$. In other words, the distance between $V_1(\cdot)$ and $V_2(\cdot)$ is zero under the null and positive under the alternative. Naturally, we propose a test statistic which measures the distance between their sample analogue

$$t \equiv \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \cdot \int_0^1 |\widehat{V}_1(\beta) - \widehat{V}_2(\beta)| d\beta \quad (4)$$

where $\widehat{V}_1(\cdot)$ and $\widehat{V}_2(\cdot)$ are the sample analogues of $V_1(\cdot)$ and $V_2(\cdot)$, respectively. When sample size is large, $\widehat{V}_1(\cdot)$ and $\widehat{V}_2(\cdot)$ will be close to their true functions. Thus, the test statistic has a small value under the null hypothesis. It will, however, diverge to infinite under the alternative. Consequently, our test rejects the null hypothesis when the test statistic is large enough.

$\widehat{V}_k(\cdot)$ is piecewise linear with possible jumps at $\{1/N_k, 2/N_k, \dots, N_k/N_k\}$. To see this, we order the bids in each sample. Denote $B_{1,(1)} \leq \dots \leq B_{1,(N_1)}$ and $B_{2,(1)} \leq \dots \leq B_{2,(N_2)}$ as the order statistics of sample 1 and 2, respectively. Simple algebra yields

$$\widehat{V}_k(\beta) = B_{k,(i_k(\beta))} \times \beta + \frac{I_k - 2}{N_k(I_k - 1)} \left[\sum_{j=1}^{i_k(\beta)} B_{k,(j)} - i_k(\beta) \times B_{k,(i_k(\beta))} \right],$$

where $i_k(\beta)$ is chosen such that $\frac{i_k(\beta)-1}{N_k} < \beta \leq \frac{i_k(\beta)}{N_k}$.²

Thanks to this property, our test statistic is easy to implement in practice. The knots

²For a given sample X_1, \dots, X_n , we define the empirical quantile function F_n^{-1} as the inverse mapping of the empirical distribution function F_n , i.e.

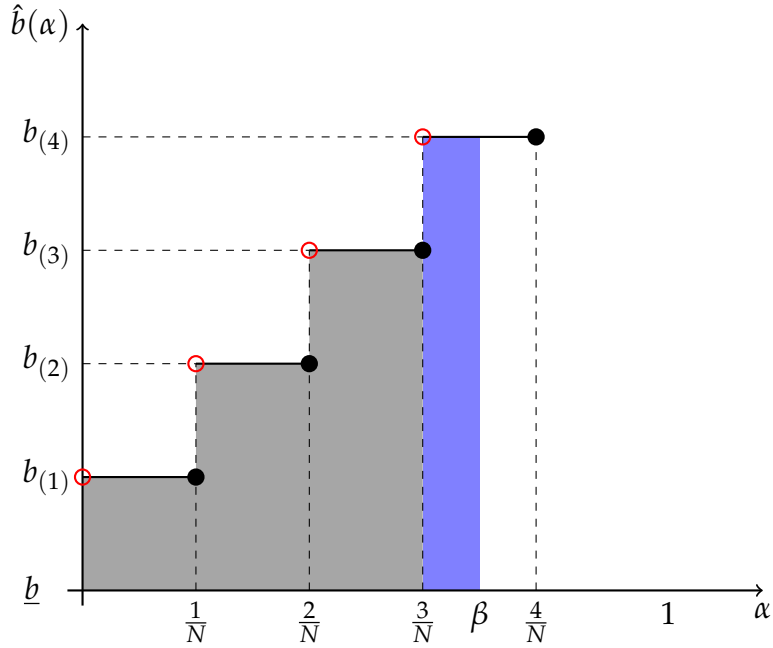
$$F_n^{-1}(\alpha) = \inf\{x : F_n(x) \geq \alpha\} = X_{(i)}$$

where i is chosen such that $\frac{i-1}{n} < \alpha \leq \frac{i}{n}$, and $X_{(1)}, \dots, X_{(n)}$ are the order statistics of the sample, that is, $X_{(1)} \leq$

$\{1/N_1, 2/N_1, \dots, N_1/N_1\}$ and $\{1/N_2, 2/N_2, \dots, N_2/N_2\}$ divide the interval $(0, 1]$ into subintervals. Denote these subintervals as $\{(\tau_\ell, \tau_{\ell+1}]\}_{\ell=0}^{N_0-1}$, where $\tau_\ell < \tau_{\ell+1}$, $\tau_0 = 0$ and $\tau_{N_0} = 1$. Since $\widehat{V}_1(\cdot)$ and $\widehat{V}_2(\cdot)$ are both linear in $(\tau_\ell, \tau_{\ell+1}]$, the integral $\int_{\tau_\ell}^{\tau_{\ell+1}} |\widehat{V}_1(\beta) - \widehat{V}_2(\beta)| d\beta$ is essentially the area of a trapezoid if the curves of $\widehat{V}_1(\cdot)$ and $\widehat{V}_2(\cdot)$ do not cross in $(\tau_\ell, \tau_{\ell+1}]$ or two similar triangles otherwise. The area is bounded by the curves of $\widehat{V}_1(\cdot)$ and $\widehat{V}_2(\cdot)$ and the two lines $\beta = \tau_\ell$ and $\beta = \tau_{\ell+1}$. Denote $d_\ell^- \equiv \widehat{V}_1(\tau_{\ell-}) - \widehat{V}_2(\tau_{\ell-})$ and $d_\ell^+ \equiv \widehat{V}_1(\tau_{\ell+}) - \widehat{V}_2(\tau_{\ell+})$, where $\widehat{V}_k(\beta_-) \equiv \lim_{u \uparrow \beta} \widehat{V}_k(u)$ and $\widehat{V}_k(\beta_+) \equiv \lim_{u \downarrow \beta} \widehat{V}_k(u)$. Our test statistic has an explicit formula:

$$t = \sqrt{\frac{N_1 N_2}{N_1 + N_2}} \cdot \sum_{\ell=0}^{N_0-1} \left(\frac{\tau_{\ell+1} - \tau_\ell}{2} \right) \cdot \left(|d_\ell^+| + |d_{\ell+1}^-| - \mathbf{1}(d_\ell^+ \cdot d_{\ell+1}^- < 0) \cdot \frac{2|d_\ell^+| \cdot |d_{\ell+1}^-|}{|d_\ell^+| + |d_{\ell+1}^-|} \right).$$

Figure 1: Integration of Empirical Bid Quantile Function



3 Asymptotic Properties

Following the literature, we make the following regularity assumptions on how the bids are generated:

$\dots \leq X_{(n)}$ and $(X_{(1)}, \dots, X_{(n)})$ is a permutation of the sample X_1, \dots, X_n . As Figure 1 shows,

$$\int_0^\beta F_n^{-1}(\alpha) d\alpha = X_{(i)} \left(\beta - \frac{i-1}{n} \right) + \frac{1}{n} \sum_{k=1}^{i-1} X_{(k)}.$$

Assumption 1. For any $I \in \{I_1, I_2\}$, the random variables B_1, \dots, B_I are independent and identically distributed (i.i.d.) with common true distribution $G(\cdot | I)$.

Assumption 1 is a regularity assumption on the bid data generation process. In addition, we impose some specific properties on the bid distribution in Assumption 2.

Assumption 2. For any $I \in \{I_1, I_2\}$, the bid distribution $G(\cdot | I)$ is absolutely continuous with a positive density $g(\cdot | I)$ on a support of $[\underline{b}, \bar{b}]$.

Assumption 2 requires that the bid distribution has a density function bounded away from zero on a compact support. It guarantees that the inverse bid function $\xi(b) = b + \frac{1}{I-1} \cdot \frac{G(b|I)}{g(b|I)}$ is well defined on the whole support of bid distribution.

Assumption 3. $\frac{N_1}{N_1+N_2} \rightarrow \lambda \in [0, 1]$ as $\min\{N_1, N_2\} \rightarrow \infty$.

Assumption 3 means that the size of the first bids sample is proportional to the second one in the limit when the parameter λ is in the interior of $[0, 1]$. The second sample tends to have many more (fewer) observations than the first one when the parameter λ is 0 (is 1).

3.1 Asymptotic Distribution under the Null Hypothesis

In this subsection, we provide the asymptotic null distribution of our test statistic. First, we define two mappings $T_k, k = 1, 2$, as

$$T_k(f)(\beta) = \frac{I_k - 2}{I_k - 1} \int_0^\beta f(\alpha) d\alpha + \frac{1}{I_k - 1} \cdot f(\beta) \cdot \beta, \quad \beta \in [0, 1], \quad (5)$$

where $f(\cdot)$ is any integrable function defined on $[0, 1]$. The following lemma show that the mappings $T_k, k = 1, 2$ are linear.

Lemma 1. The mappings $T_k, k = 1, 2$, defined in Equation (5) are linear, i.e., for any $c \in \mathbb{R}$ and any integrable functions $f(\cdot)$ and $h(\cdot)$ defined on $[0, 1]$,

$$T_k(c \cdot f + h)(\beta) = c \cdot T_k(f)(\beta) + T_k(h)(\beta), \quad \forall \beta \in [0, 1].$$

Proof. See Appendix A.1. □

Since a linear mapping of a Gaussian process is still Gaussian process, we now show that the null asymptotic distribution of our test statistic is the L^1 -norm of a Gaussian process. This Gaussian process is the sum of two independent Gaussian processes generated by applying T_1 and T_2 on two quantile processes. Denote $\mathbf{G}_k(\cdot) \equiv \frac{\mathbf{B}(\cdot)}{g(b(\cdot|I_k)|I_k)}$ for $k = 1, 2$.

Theorem 1. Suppose Assumptions 1, 2 and 3 hold. Then under H_0 , the test statistic $t \xrightarrow{d} \int_0^1 |\mathbf{G}(\beta)| d\beta$ as $\min\{N_1, N_2\} \rightarrow \infty$, where \mathbf{G} is a Gaussian process with a mean of zero and a covariance function of

$$\text{cov}(\mathbf{G}(t), \mathbf{G}(s)) = (1 - \lambda) \cdot \text{cov}(T_1(\mathbf{G}_1)(t), T_1(\mathbf{G}_1)(s)) + \lambda \cdot \text{cov}(T_2(\mathbf{G}_2)(t), T_2(\mathbf{G}_2)(s)).$$

Proof. See Appendix A.2. □

Theorem 1 provides the asymptotic distribution of our test statistic under the null hypothesis. It states that our test statistic converges in distribution to the L^1 -norm of a Gaussian process with mean zero when sample sizes are large enough. Theorem 1 is important for two reasons. First, the test statistic converges to the asymptotic distribution at a parametric rate, although our testing procedure is fully nonparametric. This feature makes our test applicable in data sets with moderate sample sizes. Second, the asymptotic distribution of the test statistic under the null hypothesis can be well characterized since it is essentially the integral of the absolute value of a Gaussian process.

3.2 Asymptotic Critical Value

Next, we give asymptotic critical value based on the asymptotic null distribution.

We propose a plug-in type of critical value $c_{1-\alpha}$ as

$$\{c_{1-\alpha} : c_{1-\alpha} \text{ is the } (1 - \alpha)\text{-th quantile of the distribution of } \int_0^1 |\widehat{\mathbf{G}}(\beta)| d\beta\},$$

where $\widehat{\mathbf{G}}(\cdot)$ is obtained by replacing $g(b(\cdot|I_k)|I_k)$ with its nonparametric estimate $\widehat{g}(\widehat{b}(\cdot|I_k)|I_k)$ in $\mathbf{G}(\cdot)$.

We then show its properties in the following theorem:

Theorem 2. *Suppose Assumptions 1, 2 and 3 hold. Then the asymptotic critical value $c_{1-\alpha}$ satisfies:*

1. *For any $(b(\cdot|I_1), b(\cdot|I_2)) \in H_0$, $\lim_{N_1, N_2 \rightarrow \infty} \Pr(t > c_{1-\alpha}) = \alpha$ for any $\alpha \in (0, 1)$;*
2. *For any $(b(\cdot|I_1), b(\cdot|I_2)) \in H_1$, $\lim_{N_1, N_2 \rightarrow \infty} \Pr(t > c_{1-\alpha}) = 1$ for any $\alpha \in (0, 1)$.*

Proof. See Appendix A.3. □

Theorem 2 shows that the asymptotic critical value has two properties. Property 1 says that the critical value has the correct size asymptotically, and Property 2 shows that, as sample sizes are large enough, the test based on the asymptotic critical value will reject almost surely for any size value and any Data Generation Process (DGP) in the alternative. Consequently, our test based on the asymptotic critical value has correct size asymptotically and is also consistent.

3.3 Bootstrap Critical Value

We investigate the properties of critical value for the test statistic from bootstrapping in this subsection. The bootstrap treats the given data as if they were the population. It develops the bootstrap empirical distribution of the test statistic by repeatedly sampling the given data and computing the test statistic from the resulting bootstrap samples.

Specifically, we can bootstrap by independently drawing samples of size N_k with replacement from each of the two original samples for $k = 1, 2$. Let $\{B_{1,1}^m, \dots, B_{1,N_1}^m\}$ and $\{B_{2,1}^m, \dots, B_{2,N_2}^m\}$ be the pair of bootstrap samples, where $m = 1, \dots, M$ and M is the number of bootstrap sample pairs. For each bootstrap sample pair m , we can compute a value of the bootstrap test statistic as

$$t^m = \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \cdot \int_0^1 \left| \left(\widehat{V}_1^m(\beta) - \widehat{V}_2^m(\beta) \right) - \left(\widehat{V}_1(\beta) - \widehat{V}_2(\beta) \right) \right| d\beta,$$

where $\widehat{V}_k^m(\cdot)$ is computed by the m th bootstrap sample drawn from original sample k , and $\widehat{V}_k(\cdot)$ is computed by the original sample k . We then define the bootstrap critical value as

$$\{c_\alpha^M : \alpha\text{-quantile of bootstrap statistics}\{t^1, \dots, t^M\}\},$$

whose asymptotic properties are given in the following theorem:

Theorem 3. *Suppose Assumptions 1, 2 and 3 hold. Then the bootstrapping critical value c_α^M satisfies:*

1. *For any $(b(\cdot|I_1), b(\cdot|I_2)) \in H_0$, $\lim_{N_1, N_2 \rightarrow \infty} \Pr(t > c_{1-\alpha}^M) = \alpha$ for any $\alpha \in (0, 1)$;*
2. *For any $(b(\cdot|I_1), b(\cdot|I_2)) \in H_1$, $\lim_{N_1, N_2 \rightarrow \infty} \Pr(t > c_{1-\alpha}^M) = 1$ for any $\alpha \in (0, 1)$.*

Proof. See Appendix A.4. □

Theorem 3 establishes that the bootstrap critical value has the correct size asymptotically and is consistent. Consequently, our bootstrap critical value is asymptotically valid.

3.4 Asymptotic Local Power

We now study the asymptotic local power properties of our test. We consider the following class of local alternatives,

$$H_{1n} : V_2(\beta) = V_1(\beta) + n^{-\gamma} \cdot h(\beta), \quad \forall \beta \in [0, 1],$$

where $n = \frac{N_1 \cdot N_2}{N_1 + N_2}$ and $h(\cdot)$ is nonzero for some β and is differentiable on $[0, 1]$.³ These local alternatives are equivalent to $v(\cdot|I_2) = v(\cdot|I_1) + n^{-\gamma} h'(\cdot)$. If $h(\cdot) = 0$, they degenerate to the null hypothesis H_0 . The following theorem describes the local alternatives our test can detect.

Theorem 4. *Suppose that the DGPs satisfy the local alternative hypothesis H_{1n} . Then the following statements hold: Let $n \rightarrow \infty$,*

1. *If $\gamma < \frac{1}{2}$, then the test statistic $t \xrightarrow{p} +\infty$;*

³ Notice that $h(0) = 0$ since $V_1(0) = V_2(0) = 0$. Moreover, the differentiability of $h(\cdot)$ is due to the differentiability of $V_1(\cdot)$ and $V_2(\cdot)$ under Assumption 2.

2. If $\gamma = \frac{1}{2}$, then $t \xrightarrow{d} \int_0^1 |\mathbb{G}(\beta) - h(\beta)| d\beta$, where $\mathbb{G}(\cdot)$ is the process defined by Theorem 1;
3. If $\gamma > \frac{1}{2}$, then the test statistic t converges in distribution to $\int_0^1 |\mathbb{G}(\beta)| d\beta$, which is the asymptotic distribution of t under H_0 .

Proof. See Appendix A.5. □

Theorem 4 shows that, despite our test being fully nonparametric, it has non-trivial power against local alternatives approaching to the null hypothesis at a rate of root-n.

4 Finite Sample Performance

To study the finite sample performance of our testing procedure, we conduct Monte Carlo experiments. We consider two groups of auctions: one group has $I_1 = 3$ bidders, and the other group has $I_2 = 7$ bidders. The true valuation distribution of group k is

$$F(v|I_k) = \begin{cases} 0 & \text{if } v < 0, \\ v^{\gamma_k} & \text{if } 0 \leq v \leq 1, \\ 1 & \text{if } v > 1, \end{cases} \quad (6)$$

where $\gamma_k > 0$ and $k = 1, 2$.⁴ Such a choice of private value distributions is convenient since the distributions correspond to linear bidding strategies as:

$$s(v|I_k) = \left(1 - \frac{1}{\gamma_k(I_k - 1) + 1}\right) \cdot v. \quad (7)$$

The number of Monte Carlo replications is 1000. For each replication, we first generate randomly $N_1 = I_1 \cdot L_1$ and $N_2 = I_2 \cdot L_2$ private values from $F(\cdot|I_1)$ and $F(\cdot|I_2)$, respectively. Second, we calculate the corresponding bids $B_{1,i}$ and $B_{2,i}$ using the linear bidding strategies in Equation (7). Third, we compute the one-step test statistic t using Equation (4). Fourth, we obtain the bootstrap critical value by applying the bootstrapping procedure described in Section 3.3 with 1000 pairs of bootstrap samples. Comparing the test statistic and the bootstrap critical value, we conclude whether the null hypothesis H_0 can be rejected for this Monte Carlo replication. We can then obtain the simulated rejection rate by the rejection rate of these 1000 Monte Carlo replications.

We now conduct several experiments to study the size and local power of our test in finite samples.

4.1 Size

We first study the size of our test, that is, the probability that the test will reject the null hypothesis when it is true. In this experiment, we consider $\gamma_1 = \gamma_2 \in \{0.25, 0.50\}$, $N_1 =$

⁴We adopt the setup of the Monte Carlo simulations from [Marmor and Shneyerov \(2012\)](#).

$N_2 \in \{105, 525, 735\}$, and the size $\alpha \in \{0.10, 0.05, 0.01\}$. The results are summarized in Table 1.

Table 1: Simulated size for $I_1 = 3, I_2 = 7$ and $\gamma_1 = \gamma_2 = \gamma$

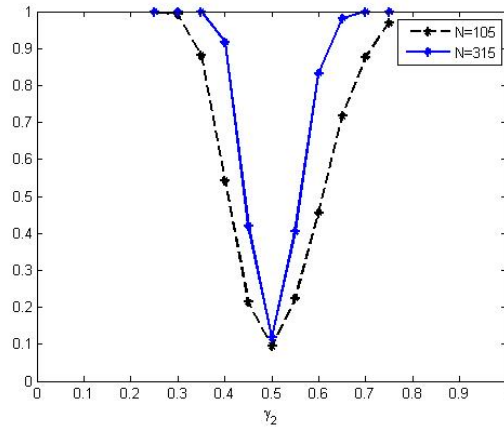
Nominal size	$N = 105$			$N = 525$			$N = 735$		
	0.1	0.05	0.01	0.1	0.05	0.01	0.1	0.05	0.01
$\gamma = 0.25$	0.0980	0.0460	0.0110	0.1200	0.0630	0.0160	0.1030	0.0550	0.0120
$\gamma = 0.5$	0.1070	0.0510	0.0090	0.0950	0.0510	0.0140	0.1070	0.0580	0.0160

Table 1 shows that our test has good performance in terms of size with moderate sample sizes. Consider $\gamma_1 = \gamma_2 = 0.5$. When $N_1 = N_2 = 105$ (i.e., the number of auctions $L_1 = 35$ and $L_2 = 15$), the simulated rejection rates are 0.1070, 0.0510 and 0.0090, respectively. They are close to their nominal values.

4.2 Power

We now study the power of our test, namely, the probability that the test will reject the null hypothesis when it is false. First, we show the power of our tests against fixed alternatives. Figure 2 displays the simulated rejection rate for a nominal size of $\alpha = 0.10$. We fix $\gamma_1 = 0.5$, and let γ_2 vary between 0.25 and 0.75 with a step size of 0.05 and sample size $N_1 = N_2 \in \{105, 315\}$.

Figure 2: Simulated Rejection Rate ($\gamma_1 = 0.5, \gamma_2 \in [0.25, 0.75], \alpha = 0.10$)



As shown in Figure 2, for a given sample size, the rejection rate converges to 100% when

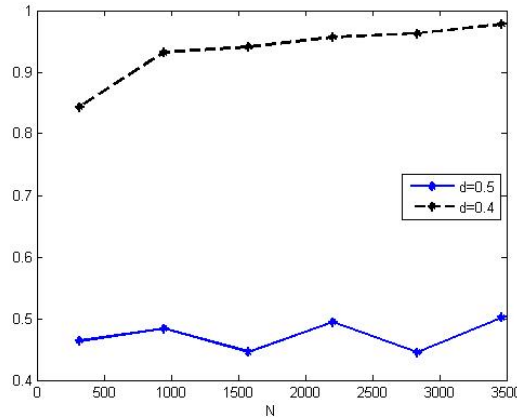
γ_2 moves further away from $\gamma_1 = 0.5$, and for a given value of γ_2 , the rejection rate is higher when the sample size is larger.

We then show the results for the local power, that is, the power of the test when the alternatives are approaching to the null at some rate when the sample size increases. In particular, we fix $\gamma_1 = 0.5$ and let γ_2 approach γ_1 as the sample size increases

$$\gamma_2 = 0.5 + \frac{1}{N^d},$$

where $N_1 = N_2 = N$. Note that $v_2(\beta) = \beta^{1/\gamma_2}$. Consider a large N . Taylor expansion gives $v_2(\beta) \approx v_1(\beta) - \frac{\beta^{1/\gamma_1} \log \beta}{\gamma_1^2} \frac{1}{N^d}$. Thus, Theorem 4 applies. We let the significance level $\alpha = 0.10$, and the sample size N increase from 315 to 3465 with an increment of 630. Figure 3 displays

Figure 3: Simulated Local Power ($\gamma_1 = 0.5$, $\gamma_2 = 0.5 + 1/N^d$, $\alpha = 0.10$)



the simulated local power of our test for $d = 0.5$ in solid line and $d = 0.4$ in dashed line. The former shows the probabilities of detecting the root-n local alternatives. The simulated rejection rate is higher than 40%, which is significantly higher than the nominal size $\alpha = 0.10$. When $d = 0.4$, the rejection rate increases towards 100% as the sample size N increases. In sum, Figure 3 shows some evidence that our test can detect the local alternatives converging to the null hypothesis at a rate of root-n.

5 Extensions

In this section, we discuss how to adapt our test to a variety of auction settings. First, we consider testing exogenous participation, allowing for risk aversion and observed auction heterogeneity. Second, we consider adapting our method to other testing problems, such as discriminating entry models of auctions and detecting risk aversion.

5.1 Auction Models with Risk Averse Bidders

We can generalize our test to allow for risk averse bidders.⁵ We consider the bidders having a Constant Relative Risk Aversion (CRRA) vNM utility function $U(\omega) = \omega^{1-c}$ where $0 \leq c < 1$ and $\omega \in [0, +\infty)$. [Guerre, Perrigne, and Vuong \(2009\)](#) obtain the inverse bidding function as

$$\zeta(b) = b + (1 - c) \cdot \frac{1}{I - 1} \cdot \frac{G(b)}{g(b)},$$

which yields the following integration of private value quantile functions:

$$V_k(\beta) = \frac{I_k - 2 + c}{I_k - 1} \cdot \int_0^\beta b(\alpha|I_k) d\alpha + \frac{1 - c}{I_k - 1} \cdot b(\beta|I_k) \cdot \beta.$$

Therefore, our test applies with a slight modification.

5.2 Auction Models with Auction-Specific Heterogeneity

In this subsection, we discuss how to generalize our testing procedure to allow for auction-specific heterogeneity. Let $X \in \mathbb{R}^d$ be a random vector that describes the heterogeneity of auctions. The bidders' inverse bidding function in an auction with characteristic x is⁶

$$v = \zeta(b|x) \equiv b + \frac{1}{I - 1} \frac{G(b|x)}{g(b|x)},$$

where I is the number of bidders. After some algebra, we obtain:

$$V_k(\beta|x) \equiv \int_0^\beta v(\alpha|I_k, x) d\alpha = \frac{I_k - 2}{I_k - 1} \int_0^\beta b(\alpha|I_k, x) d\alpha + \frac{1}{I_k - 1} b(\beta|I_k, x) \beta, \quad k = 1, 2.$$

It is then fairly straightforward to give a testing procedure of exogenous bidders' participation in auctions with heterogeneity. Notice that our testing procedure involves the curse of dimensionality due to the estimation of conditional quantile function $b(\cdot|I_k, x)$. However, its convergence rate is still faster than estimating $g(\cdot|I_k, x)$ if one were to compare the valuation distribution (quantile) functions.

Alternatively, if we impose an additively separable structure on the valuation, we can "homogenize" the bids and then implement our test in a two-stage procedure following [Haile, Hong, and Shum \(2003\)](#).⁷ In particular, assume that

$$v(w, I, x) = \delta(x) + v(w, I),$$

⁵Risk aversion has been shown to be an important component of bidders' behavior in auctions by both the empirical and experimental literature (see, for example, [Athey and Levin \(2001\)](#), [Lu and Perrigne \(2008\)](#) and [Campo, Guerre, Perrigne, and Vuong \(2011\)](#) for empirical literature, and [Cox, Smith, and Walker \(1988\)](#) and [Bajari and Hortacsu \(2005\)](#) for experimental literature).

⁶We denote a random variable/vector by an uppercase letter and its realization by a lowercase letter.

⁷A similar approach applies to a multiplicatively separable structure on the valuation.

where w is bidder's private value, and bidder's private value W is independent of auction heterogeneity X . Haile, Hong, and Shum (2003) show that bidders' bidding strategy satisfies

$$b(w|I_k, x) = \delta(x) + b(w|I_k),$$

which we rewrite as

$$b(w|I_k, x) = \delta_0 + b_0(I_k) + \tilde{\delta}(x) + \tilde{b}(w|I_k),$$

where $\delta_0 = E[\delta(X)]$, $b_0(I_k) = E[b(W|I_k)|I_k]$, $\tilde{\delta}(x) = \delta(x) - \delta_0$ and $\tilde{b}(w|I_k) = b(w|I_k) - b_0(I_k)$. $b^*(w|I_k) \equiv \delta_0 + b_0(I_k) + \tilde{b}(w|I_k)$ is the bid that a bidder would have submitted in equilibrium if she were in an "average" auction (i.e., $\delta(x) = E[\delta(X)]$) with I_k bidders. Moreover, the term $\tilde{b}(W|I_k)$ has mean zero conditional on (I_k, X) by independence of X and W . In the first stage, we regress bids on X and calculate the homogenized bids as $\hat{b}^*(w|I_k) = b(w|I_k, x) - \hat{\tilde{\delta}}(x)$, where $\hat{\tilde{\delta}}(x)$ is the estimate of $\tilde{\delta}(x)$ from the regression. In the second stage, we apply our testing procedure to the "homogenized" bids \hat{b}^* as if they were from a sample of auctions of identical goods. With such a two-stage procedure, we can avoid smoothing over X when estimating quantiles of bids in the test statistic.

5.3 Auction Models with Entry

We can adapt our test to discriminate the auction models with entry studied in Marmer, Shneyerov, and Xu (2013). They consider the selective entry model (SEM), which nests the Levin and Smith (1994) model of entry (LS) and the Samuelson (1985) model (S). In a selective entry model, a potential bidder observes a private signal correlated with his valuation of the good at the entry stage, which can be learned upon incurring entry cost. The inverse bidding strategy is identifiable as

$$\xi(b|I) = b + \frac{1}{I-1} \left(\frac{G^*(b|I)}{g^*(b|I)} + \frac{1-p(I)}{p(I)} \frac{1}{g^*(b|I)} \right),$$

where $p(I)$ is the equilibrium probability of bidding and $G^*(\cdot|I)$ is the conditional distribution of active bidder's bids.

Marmer, Shneyerov, and Xu (2013) show that as the number of potential bidders increases, those who enter tend to have larger valuations. From this selection effect, the restriction of the SEM and LS model can be stated as: if $I_1 < I_2$,

$$\begin{aligned} H_{LS} : & \quad v(\beta|I_1) = v(\beta|I_2), \\ H_{SEM} : & \quad v(\beta|I_1) \leq v(\beta|I_2). \end{aligned}$$

They propose a test statistic based on pairwise differences between the sample quantiles corresponding to different numbers of potential bidders, which are estimated by plugging in nonparametric estimators of the inverse bidding strategy and conditional bid quantile func-

tion. Thus, the test statistic requires multiple steps and involves kernel density estimators.

After some algebra, we obtain: for $\beta \in [0, 1]$ and $k = 1, 2$,

$$V_k(\beta) \equiv \int_0^\beta v(\alpha|I_k)d\alpha = \frac{I_k - 2}{I_k - 1} \int_0^\beta b(\alpha|I_k)d\alpha + \frac{1}{I_k - 1} b(\beta|I_k) \left[\beta + \frac{1 - p(I_k)}{p(I_k)} \right]$$

Therefore, our method can be an alternative to distinguish auction models with entry.

5.4 A Nonparametric Test of Risk Aversion

We can construct a nonparametric test of risk aversion in the framework of [Guerre, Perrigne, and Vuong \(2009\)](#). They show how to identify risk aversion nonparametrically in first-price auctions under exogenous participation and characterize all the theoretical restrictions. The key idea is that the invariance of the quantile $v(\alpha)$ for two different numbers of bidders I_1 and I_2 leads to the compatibility conditions:

$$b_1(\alpha) + \lambda^{-1} \left(\frac{1}{I_1 - 1} \frac{\alpha}{g_1(b_1(\alpha))} \right) = b_2(\alpha) + \lambda^{-1} \left(\frac{1}{I_2 - 1} \frac{\alpha}{g_2(b_2(\alpha))} \right) \quad (8)$$

where $\alpha \in [0, 1]$ and $\lambda(\cdot) = U(\cdot)/U'(\cdot)$. Note that the risk neutral case is obtained when $U(\cdot)$ is the identity function, in which case both $\lambda(\cdot)$ and $\lambda^{-1}(\cdot)$ are also the identity function.

Under exogenous participation, the auction model implies stochastic dominance between any two observed bid distributions regardless of bidders' risk attitude. Maintaining both of the exogenous participation and stochastic dominance assumptions, our test applies to detect risk aversion nonparametrically: (i) if bidders are risk neutral, we obtain our hypothesis of neutrality H_0 by replacing $\lambda^{-1}(\cdot)$ with the identity function in Equation (8); and (ii) if bidders are risk averse, the compatibility conditions are not satisfied when we replace $\lambda^{-1}(\cdot)$ with the identity function in Equation (8); otherwise, bidders are risk neutral for that the compatibility conditions lead to identification of $\lambda^{-1}(\cdot)$ as shown in [Guerre, Perrigne, and Vuong \(2009\)](#). Consequently, the detection of risk aversion is equivalent to discriminating the following two hypotheses:

$$H_0 : b_1(\alpha) + \frac{1}{I_1 - 1} \frac{\alpha}{g_1(b_1(\alpha))} = b_2(\alpha) + \frac{1}{I_2 - 1} \frac{\alpha}{g_2(b_2(\alpha))}, \forall \alpha \in [0, 1] \quad \text{v.s.} \quad H_1 : \text{not } H_0.$$

Our testing approach can be applied after integrating H_0 with respect to α .

6 Application

In this section, we implement our test to some real life data. We study the U.S. Forest Service timber auctions, which sell the timber harvesting rights from publicly owned forests. In particular, we analyze the sealed-bid auction data used in [Lu and Perrigne \(2008\)](#). It covers the western half of the U.S. in 1979.

There are $N_1 = 107$ auctions with 2 bidders and $N_2 = 108$ auctions with 3 bidders. For each auction, the data contain the estimated total volume of the timber, the appraisal value per unit of timber, the identity of the bidders and their bids per unit of timber. We calculate the FS estimate by multiplying the total volume by the appraisal value. Similarly, we obtain the total bid amount by multiplying the bid per unit by the total volume. Table 2 gives some basic statistics on these variables.

Table 2: Summary Statistics

	Variable	Mean	Std. Dev.	Min.	Max.
$I = 2$	total bid	77417.7	191996.5	2234	1700522
	FS estimate	50489.76	145024.4	425.0198	1333586
	$\log b$	9.9823	1.4933	7.7115	14.3465
	$\log X$	9.5596	1.4474	6.0521	14.1034
	bid_{new}	27231.99	31988.8	13190.1	284547.7
$I = 3$	total bid	96747.53	159255.8	1066.502	1170227
	FS estimate	48665.83	80497.11	1020	415506.8
	$\log b$	10.4073	1.5157	6.9721	13.9727
	$\log X$	9.7369	1.4552	6.9276	12.9373
	bid_{new}	54180.64	98493.89	12592.48	968014.3

Following Haile, Hong, and Shum (2003), we homogenize the total bids before implementing our method to control for observable heterogeneity. In particular, for each sample (number of bidders = 2 or 3), we regress the logarithm of the total bids ($\log b$) on the logarithms of the FS estimate ($\log X$). Table 3 displays the results. The FS estimate explains a large amount of the variation in the total bids.

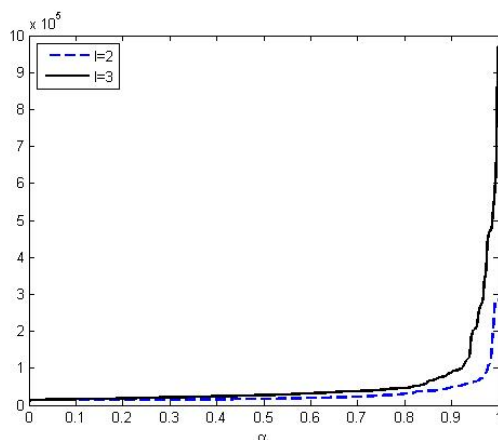
Table 3: Homogenization of Bids

	I=2	I=3
$\log X$	0.9584*** (0.0277)	0.8929*** (0.0333)
Adjusted R2	0.8622	0.7341

The homogenized bids (bid_{new}) are calculated as the exponential of the differences between the logarithm of the original total bids and the demeaned fitted values of the regression. Table 2 presents some summary statistics of the homogenized bids. Under exogenous participation, the auction model implies stochastic dominance between the observed bid distributions, i.e., $G_3(\cdot) > G_2(\cdot)$. Figure 4 displays the two empirical distributions of bid_{new} . Eyeballing does not reject that $\hat{G}_3(\cdot)$ stochastically dominates $\hat{G}_2(\cdot)$.

We now implement our test on the “homogenized” bid samples. The test statistic is calculated to be 91479.3318 with a p-value of 0.025. Therefore, our test rejects the null hypothesis of exogenous participation at the 5% significance level.

Figure 4: $\widehat{G}_2(\cdot)$ and $\widehat{G}_3(\cdot)$



7 Conclusion

This paper develops a generalized Lorenz curve-based nonparametric test for exogenous participation in first-price auctions. The exogenous participation assumes that the bidders' private value distributions are the same across auctions with different numbers of bidders. Our test is convenient in practice since it only involves one step estimation of quantile functions of bids. We show that, at a parametric rate, the test statistic converges to the L^1 -norm of a Gaussian process with mean zero under the null hypothesis. We propose both asymptotic and bootstrap critical values, and we show that our test has the correct size and is consistent against all fixed alternatives. Moreover, our test detects local alternatives approaching the null at a parametric rate, despite the nonparametric nature of our test. Our simulation results show that our test behaves well in finite samples. Additionally, we extend our test to allow for CRRA utility and observable auction-specific heterogeneity. We also adapt our test to distinguish auction models with entry and to detect risk aversion in auctions under exogenous participation.

There are several directions in which this work may be extended. First, we have shown that our test can be generalized to auctions with risk averse bidders for CRRA utility. Further study is needed to extend it to auctions with risk averse bidders for nonparametric utility. Second, our testing procedure assumes that we observe all auction-specific heterogeneity. It is interesting but challenging to allow for unobserved heterogeneity, that is, something bidders observe but econometricians do not. We leave these extensions for future research.

A Proofs

A.1 Proof of Lemma 1

For any $\beta \in [0, 1]$, any $c \in \mathbb{R}$, and any integrable functions $f(\cdot)$ and $h(\cdot)$ defined on $[0, 1]$, by definition, we have

$$\begin{aligned} T_k(c \cdot f + h)(\beta) &= \frac{I_k - 2}{I_k - 1} \int_0^\beta [c \cdot f(\alpha) + h(\alpha)] d\alpha + \frac{1}{I_k - 1} \cdot [c \cdot f(\beta) + h(\beta)] \beta \\ &= c \cdot \left[\frac{I_k - 2}{I_k - 1} \int_0^\beta f(\alpha) d\alpha + \frac{1}{I_k - 1} \cdot f(\beta) \beta \right] + \frac{I_k - 2}{I_k - 1} \int_0^\beta h(\alpha) d\alpha + \frac{1}{I_k - 1} \cdot h(\beta) \beta \\ &= c \cdot T_k(f)(\beta) + T_k(h)(\beta), \end{aligned}$$

which says that T_k is linear. □

A.2 Proof of Theorem 1

Proof. Under the null hypothesis H_0 , we have $V_1(\beta) = V_2(\beta)$ for any $\beta \in [0, 1]$. Consequently, we can rewrite the test statistic as

$$\begin{aligned} t &= \int_0^1 \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \cdot \left| \widehat{V}_1(\beta) - \widehat{V}_2(\beta) \right| d\beta \\ &= \int_0^1 \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \cdot \left| \widehat{V}_1(\beta) - V_1(\beta) - \left(\widehat{V}_2(\beta) - V_2(\beta) \right) \right| d\beta \\ &= \int_0^1 \left| \frac{I_1 - 2}{I_1 - 1} \int_0^\beta \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \left(\widehat{b}(\alpha|I_1) - b(\alpha|I_1) \right) d\alpha + \frac{1}{I_1 - 1} \cdot \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \left(\widehat{b}(\beta|I_1) - b(\beta|I_1) \right) \beta \right. \\ &\quad \left. - \frac{I_2 - 2}{I_2 - 1} \int_0^\beta \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \left(\widehat{b}(\alpha|I_2) - b(\alpha|I_2) \right) d\alpha - \frac{1}{I_2 - 1} \cdot \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \left(\widehat{b}(\beta|I_2) - b(\beta|I_2) \right) \beta \right| d\beta, \end{aligned}$$

where the second equality holds due to $V_1(\beta) = V_2(\beta)$ for any $\beta \in [0, 1]$ under the null H_0 , and the last equality comes from the definitions of $\widehat{V}_k(\cdot)$ and $V_k(\cdot)$ for $k = 1, 2$.

The test statistic can then be rewritten as

$$t = \int_0^1 \left| \sqrt{\frac{N_2}{N_1 + N_2}} \cdot T_1 \left(\widehat{\mathcal{G}}_1 \right) (\beta) - \sqrt{\frac{N_1}{N_1 + N_2}} \cdot T_2 \left(\widehat{\mathcal{G}}_2 \right) (\beta) \right| d\beta,$$

where $\widehat{\mathcal{G}}_k(\cdot) \equiv \sqrt{N_k} \left(\widehat{b}(\cdot | I_k) - b(\cdot | I_k) \right)$, $k = 1, 2$, are the empirical quantile processes.

Notice that the empirical quantile processes $\widehat{\mathcal{G}}_k(\cdot) \Rightarrow \mathbb{B}(\cdot) / g(b(\cdot | I_k) | I_k)$ on $(0, 1)$ as $N_k \rightarrow \infty$. In addition, both mappings T_1 and T_2 are linear by Lemma 1, and the empirical quantile processes $\widehat{\mathcal{G}}_1(\cdot)$ and $\widehat{\mathcal{G}}_2(\cdot)$ are independent of each other because the bids under $I = I_1$ are independent of bids under $I = I_2$. Consequently,

$$\sqrt{\frac{N_2}{N_1 + N_2}} \cdot T_1 \left(\widehat{\mathcal{G}}_1 \right) (\cdot) - \sqrt{\frac{N_1}{N_1 + N_2}} \cdot T_2 \left(\widehat{\mathcal{G}}_2 \right) (\cdot) \Rightarrow \mathbf{G}(\cdot) \quad \text{on } (0, 1),$$

where $\mathbf{G}(\cdot)$ is a Gaussian process with mean zero and covariance function of

$$\text{cov}(\mathbf{G}(t), \mathbf{G}(s)) = (1 - \lambda) \cdot \text{cov}(T_1(\mathbf{G}_1)(t), T_1(\mathbf{G}_1)(s)) + \lambda \cdot \text{cov}(T_2(\mathbf{G}_2)(t), T_2(\mathbf{G}_2)(s));$$

and $\mathbf{G}_k(\cdot) \equiv \frac{\mathbb{B}(\cdot)}{g(b(\cdot|I_k)|I_k)}$ for $k = 1, 2$.

By continuous mapping theorem, our test statistic $t \xrightarrow{d} \int_0^1 |\mathbf{G}(\beta)| d\beta$. \square

A.3 Proof of Theorem 2

Proof. Let n denote the minimum of N_1 and N_2 . We first show part 1. Note that, for any fixed $\alpha \in (0, 1)$, the critical value $c_{1-\alpha}$ is continuous in $\hat{g}(\hat{q}(\cdot|I_1)|I_1)$ and $\hat{g}(\hat{q}(\cdot|I_2)|I_2)$. Consequently, under the null hypothesis H_0 ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \Pr(t \geq c_{1-\alpha}) \\ &= 1 - \lim_{n \rightarrow \infty} [\Pr(t \leq c_{1-\alpha}) - \Pr(t_\infty \leq c_{1-\alpha})] - \lim_{n \rightarrow \infty} \Pr(t_\infty \leq c_{1-\alpha}) \\ &= 1 - \lim_{n \rightarrow \infty} \Pr(t_\infty \leq c_{1-\alpha}) \\ &= \alpha \end{aligned}$$

where t_∞ has a distribution the same as the asymptotic distribution of t ; the second-to-last equality holds by Polya's theorem since the asymptotic distribution of t is continuous; and the last equality holds due to the continuity of $c_{1-\alpha}$ in $\hat{g}(\hat{b}(\cdot|I_1)|I_1)$ and $\hat{g}(\hat{b}(\cdot|I_2)|I_2)$. The conclusion of part 1 therefore follows.

We then show part 2. Under the alternative hypothesis H_1 , there exists a β in $[0, 1]$, denoted as β^* , such that $V_1(\beta^*) \neq V_2(\beta^*)$. Consequently, there must exist an interval \mathcal{C}^* with positive measure around β^* such that $V_1(\beta) \neq V_2(\beta)$ for $\beta \in \mathcal{C}^*$, since V_1 and V_2 are continuous by Assumption 2. We therefore have

$$\begin{aligned} t &= \int_0^1 \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \cdot \left| \widehat{V}_1(\beta) - \widehat{V}_2(\beta) \right| d\beta \\ &\geq \int_{\mathcal{C}^*} \sqrt{\frac{N_1 \cdot N_2}{N_1 + N_2}} \cdot \left| \widehat{V}_1(\beta) - \widehat{V}_2(\beta) \right| d\beta \\ &= \int_{\mathcal{C}^*} \left| \sqrt{\frac{N_1 N_2}{N_1 + N_2}} [\widehat{V}_1(\beta) - V_1(\beta)] - \sqrt{\frac{N_1 N_2}{N_1 + N_2}} [\widehat{V}_2(\beta) - V_2(\beta)] + \sqrt{\frac{N_1 N_2}{N_1 + N_2}} [V_1(\beta) - V_2(\beta)] \right| d\beta \end{aligned} \tag{9}$$

where the first two terms on the right-hand side of last equality in Equation (9) are stochastically bounded, but its third term diverges to infinity as $n \rightarrow \infty$ since $V_1(\beta) \neq V_2(\beta)$ for any β in the positively measured interval \mathcal{C}^* . Thus, the right-hand side of last equality in Equation (9) diverges to $+\infty$ as n goes to infinity, which implies that the statistic t goes to $+\infty$ as $n \rightarrow \infty$. We therefore have that, for any $\alpha \in (0, 1)$,

$$\lim_{N_1, N_2 \rightarrow \infty} \Pr(t \geq c_{1-\alpha}) = 1$$

under the alternative hypothesis H_1 . Part 2 follows immediately. \square

A.4 Proof of Theorem 3

Proof. Lemma 1 shows that the mapping T_k is linear for $k = 1, 2$. In addition, the quantile function is a Hadamard differentiable functional of the cumulative distribution function. Consequently, $V_k(\cdot)$ is a Hadamard differentiable functional of the cumulative distribution function. By functional delta method, it suffices to show that the bootstrap applied to the empirical distributions yields processes with the same asymptotic covariance properties as those for the empirical distributions of the original sample.

Notice that the bootstrap empirical processes are (for details, see Chapter 3.6 of [van der Vaart and Wellner \(1996\)](#)),

$$\begin{aligned}\mathbf{G}_1^*(b_1) &= \frac{1}{\sqrt{N_1}} \sum_{j=1}^{N_1} \mathbf{1}\{B_{1,j}^m \leq b_1\} - \widehat{G}_1(b_1) = \frac{1}{\sqrt{N_1}} \sum_{j=1}^{N_1} (M_{1j} - 1) \cdot \mathbf{1}\{B_{1,j} \leq b_1\}, \\ \mathbf{G}_2^*(b_2) &= \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} \mathbf{1}\{B_{2,j}^m \leq b_2\} - \widehat{G}_2(b_2) = \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} (M_{2j} - 1) \cdot \mathbf{1}\{B_{2,j} \leq b_2\},\end{aligned}$$

where M_{1j} and M_{2j} are independent multinomial random variables with N_1 and N_2 cells and success probabilities of $1/N_1$ and $1/N_2$, respectively. Notice that M_{1j} and M_{2j} are also independent of the original sample. It is easy to show that, given the original sample, $\mathbf{G}_1^*(\cdot)$ and $\mathbf{G}_2^*(\cdot)$ are independent mean zero processes with covariance kernels of:

$$E(\mathbf{G}_k^*(b_k) \mathbf{G}_k^*(b'_k) | B_{k,1}, \dots, B_{k,N_k}) = \widehat{G}_k(b_k) - \widehat{G}_k(b_k) \widehat{G}_k(b'_k),$$

for $b_k \leq b'_k$ and $k = 1, 2$. Such covariance kernels converge to the ones of the limiting processes of the corresponding empirical processes. Consequently, the bootstrap empirical processes $\mathbf{G}_1^*(\cdot)$ and $\mathbf{G}_2^*(\cdot)$ have the same limiting processes as the empirical processes based on the corresponding empirical distributions of the original sample.

Following the delta method for bootstrap as shown by Theorem 3.9.11 of [van der Vaart and Wellner \(1996\)](#) and the continuous mapping theorem, we can show that, given the original sample, the bootstrap test statistic

$$t^m \xrightarrow{d} \int_0^1 |\mathbf{G}(\beta)| d\beta.$$

Consequently, the bootstrap test statistic has the same limiting distribution as the original test statistic. The desired conclusion therefore follows by Theorems 1 and 2. \square

A.5 Proof of Theorem 4

Proof. Notice that there must exist an interval \mathcal{C}^* with positive measure such that $h(\beta) \neq 0$ for any $\beta \in \mathcal{C}^*$, because that $h(\cdot)$ is nonzero and is differentiable on $[0, 1]$. Under the local alternative hypothesis H_{1n} , we can rewrite the test statistic t as follows:

$$\begin{aligned}t &= \int_0^1 \sqrt{n} \cdot \left| \widehat{V}_1(\beta) - \widehat{V}_2(\beta) \right| d\beta \\ &= \int_0^1 \left| \sqrt{n} \left(\widehat{V}_1(\beta) - V_1(\beta) \right) - \sqrt{n} \left(\widehat{V}_2(\beta) - V_2(\beta) \right) - n^{\frac{1}{2}-\gamma} \cdot h(\beta) \right| d\beta\end{aligned}$$

As shown by the proof of Theorem 1 in Appendix A.2,

$$\sqrt{n} \left(\widehat{V}_1(\cdot) - V_1(\cdot) \right) - \sqrt{n} \left(\widehat{V}_2(\cdot) - V_2(\cdot) \right) \Rightarrow \mathbf{G}(\cdot) \quad \text{on } (0, 1).$$

Consequently, if $\gamma < \frac{1}{2}$, then the test statistic $t \xrightarrow{p} +\infty$, since $n^{\frac{1}{2}-\gamma} \cdot h(\beta) \rightarrow \infty$ for all $\beta \in \mathcal{C}^*$; and by the continuous mapping theorem, if $\gamma = \frac{1}{2}$, then $t_n \xrightarrow{d} \int_0^1 |\mathbf{G}(\beta) - h(\beta)| d\beta$, due to the fact that $n^{\frac{1}{2}-\gamma} \cdot h(\beta) = h(\beta)$ for all $\beta \in [0, 1]$; if $\gamma > \frac{1}{2}$, then $t_n \xrightarrow{d} \int_0^1 |\mathbf{G}(\beta)| d\beta$ for that $n^{\frac{1}{2}-\gamma} \cdot h(\cdot) \rightarrow 0$ uniformly on $[0, 1]$. The desired result therefore follows. \square

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