## University of Toronto Department of Economics <br> 

Working Paper 515

# TIGHTENING BOUNDS IN TRIANGULAR SYSTEMS 

By Desire Kedagni and Ismael Mourifie

July 07, 2014

# TIGHTENING BOUNDS IN TRIANGULAR SYSTEMS 

DÉSIRÉ KÉDAGNI AND ISMAEL MOURIFIÉ

Université de Montréal and the University of Toronto


#### Abstract

This note discusses partial identification in a nonparametric triangular system with discrete endogenous regressors and nonseparable errors. Recently, [Jun, Pinkse and Xu (2011, JPX). Tighter Bounds in Triangular Systems. Journal of Econometrics 161(2), 122-128] provides bounds on the structural function evaluated at particular values using exclusion, exogeneity and rank conditions. We propose a simple idea that often allows to improve the JPX bounds without invoking a new set of assumptions. Moreover, we show how our idea can be used to tighten existing bounds on the structural function in more general triangular systems.


Keywords: Nonparametric triangular systems; Partial identification; Instrumental variables; Rank conditions.

JEL subject classification: C14, C30, C31.

## 1. Introduction

In this note, we consider the following nonparametric triangular model:

$$
\left\{\begin{array}{l}
Y=g(D, U)  \tag{1.1}\\
D=h(Z, V)
\end{array}\right.
$$

where $Y \in \mathcal{Y} \subset \mathbb{R}, D \in \mathcal{D} \subset \mathbb{R}^{d}, Z \in \mathcal{Z} \subset \mathbb{R}^{d_{z}}$ are observables and $g$ and $h$ are unknown functions with $g$ nondecreasing in $U$ and left-continuous for all values of $D$ and $h_{j}\left(Z, V_{j}\right)$ the $j^{\text {th }}$ component of the vector $h(Z, V)=\left[h_{1}\left(Z, V_{1}\right), \ldots, h_{d}\left(Z, V_{d}\right)\right]$ nondecreasing in $V_{j}$ and left-continuous for all values of $Z$, for $j=1, \ldots, d$. $U \in \mathcal{U}=(0,1], V \in \mathcal{V} \subseteq \mathcal{U}^{d}$ are errors. We refer to $D$ as endogenous regressors and $Z$ as instruments which need not to be continuous. Our objective is identification of the object

$$
\begin{equation*}
\psi^{*}=\psi\left(d^{*}, \tau^{*}, v^{*}\right)=g\left(d^{*}, Q_{U \mid V}\left(\tau^{*} \mid v^{*}\right)\right) \tag{1.2}
\end{equation*}
$$

for given values of $\left(\tau^{*}, d^{*}, v^{*}\right) \in \mathcal{U} \times \mathcal{D} \times \mathcal{V}$, where $Q_{U \mid V}(\tau \mid v)=\inf \{u: \mathbb{P}(U \leq u \mid V=v) \geq \tau\}$. A model similar to (1.1) has been studied in Chesher (2003, 2005) and in Jun, Pinkse, and Xu (2011, JPX) . Chesher (2003) used an assumption of strict monotonicity to identify the partial derivatives of $g$ with respect to

[^0]$D$. However, when $D$ is discrete the strict monotonicity assumption does not hold and then fails to point identify the quantity of interest $\psi^{*}$. Therefore, Chesher (2005) proposed to bound $\psi^{*}$ under a dependence condition on $U$ and $V$, as well as "local exclusion", and "local exogeneity" conditions on the instrument $Z$. JPX proposed the use of "global" rather than "local" conditions in the sense that they imposed a global exclusion restriction ( Z does not enter $g$ ) and assume that $Z$ is independent of $(U, V)$. Although their global conditions are stronger than the Chesher (2005) local ones, they have some interesting advantages. First, the global conditions allow them to replace a rank condition in Chesher (2005) with an alternative weaker rank condition that in some cases permits the construction of tighter bounds on $\psi^{*}$ than those obtained in Chesher (2005). Second, this weaker rank condition allows them to construct meaningful bounds on $\psi^{*}$ when $D$ is binary something that Chesher (2005) cannot do. Therefore, JPX proposed a general method to derive tighter bounds on $\psi^{*}$ under a set of global conditions.

In this note, we propose a simple idea that allows us to tighten the JPX bounds without invoking a new set of assumptions. Indeed, we show that the weak monotonicity and left-continuity assumptions imposed on both $g$ and $h_{j}$ plus the global conditions allow identification of the sign of $\left[\psi\left(d, \tau^{*}, v^{*}\right)-\psi\left(d^{\prime}, \tau^{*}, v^{*}\right)\right]$ for $\left(d, d^{\prime}\right) \in(\mathcal{D} \times \mathcal{D})$ in some cases. We show how this new information can help tighten the bounds proposed by JPX. The JPX method uses variation in the instrument to provide meaningful bounds on $\psi^{*}$. In addition to their strategy, we propose to use variation in $D$ (across treatment) to tighten their bounds and then propose sharper bounds on $\psi^{*}$. For instance, we show that whenever $Y, D$, and $Z$ are binary, the JPX bounding approach may fail to provide meaningful lower or upper bounds for either $\psi\left(1, \tau^{*}, v^{*}\right)$ or $\psi\left(0, \tau^{*}, v^{*}\right)$ while our strategy does.

For the sake of simplicity, we initially consider a simple case where $Y$ and $D$ are both binary and generalize our argument later. We only show the improvement that we can obtain on the JPX bounds when $D$ is binary, but this improvement would become more important when $D$ takes multiple values or/and in the presence of other exogenous covariates that enter both $g$ and $h$.

The rest of the note is organized as follows. In section 2, we consider a simple binary triangular case of model (1.1). This simple case helps us illustrate ideas and demonstrates the improvement obtained on the JPX bounds using our approach. Section 3 discusses the generalization of our argument for the nonbinary triangular system. The last section concludes.

## 2. Simple case: Binary triangular system

We adopt, without loss of generality (w.l.o.g), the framework of the potential outcomes model $Y=$ $Y_{1} D+Y_{0}(1-D)$, where $Y_{d}=g(d, U), d \in\{0,1\}$ are binary unobserved potential outcomes. Since $g(d, U)$ is weakly increasing in its second argument and $U$ is uniform on $[0,1]$, we have $g(d, u)=\inf \left\{y: \mathbb{P}\left(Y_{d} \leq y\right) \geq u\right\}$, which is equal to $1\left\{\mathbb{P}\left(Y_{d}=0\right)<u\right\}$ since $Y_{d}$ is binary. Left-continuity holds, because $\mathbb{P}\left(Y_{d} \leq y\right)$ is a cadlag function of $y$. Therefore, the binary triangular system can be written w.l.o.g as follows:

$$
\left\{\begin{array}{l}
Y=1\{\vartheta(D)<U\}  \tag{2.1}\\
D=1\{p(Z)<V\}
\end{array}\right.
$$

where $U, V \sim \mathcal{U}[0,1]$. Then, we have $\vartheta(d)=\mathbb{P}\left(Y_{d}=0\right)$ and $p(Z)=\mathbb{P}(D=0 \mid Z)$. The formal assumptions we use in this section may be expressed as follows:

Assumption 1. $(U, V)$ are independent of $Z$.
Assumption 2. $U$ is positive regression dependent on $V$, i.e. $Q_{U \mid V}(\tau \mid v)$ is nondecreasing in $v$ for all values of $\tau$.

Assumption 3. $\mathcal{Z}\left(d^{*}, v^{*}\right)=\left\{z \in \mathcal{Z}: 1\left\{p(z)<v^{*}\right\}=d^{*}\right\}$ is nonempty for $d^{*} \in\{0,1\}$.

This latter assumption ensures observation of individuals in both treatment groups i.e $D=0$ and $D=1$ when $V=v^{*}$. The assumptions made above are presumed to hold throughout the rest of the paper. Under assumptions 1-3, Lemma 1 in JPX states that $\psi\left(d^{*}, \tau^{*}, v^{*}\right)=Q_{Y \mid D, V}\left(\tau^{*} \mid d^{*}, v^{*}\right)$. Then $\psi\left(d^{*}, \tau^{*}, v^{*}\right)=\inf \left\{y: \mathbb{P}\left(Y \leq y \mid D=d^{*}, V=v^{*}\right) \geq \tau^{*}\right\}=1\left\{\mathbb{P}\left(Y_{d^{*}}=0 \mid D=d^{*}, V=v^{*}\right)<\tau^{*}\right\}$. The last equality holds since $Y_{d}$ is binary. Note that, under assumption 3 , there exists $z^{*} \in \mathcal{Z}$ such that $\mathbb{P}\left(Y_{d^{*}}=\right.$ $\left.0 \mid D=d^{*}, V=v^{*}\right)=\mathbb{P}\left(U \leq \vartheta\left(d^{*}\right) \mid Z=z^{*}, V=v^{*}\right)$, which is equal to $\mathbb{P}\left(U \leq \vartheta\left(d^{*}\right) \mid V=v^{*}\right)$ under assumption 1. Then, in this special case our function of interest is

$$
\begin{equation*}
\psi^{*}=1\left\{\mathbb{P}\left(U>\vartheta\left(d^{*}\right) \mid V=v^{*}\right)>1-\tau^{*}\right\}, d^{*} \in\{0,1\} . \tag{2.2}
\end{equation*}
$$

Therefore, the main issue is to provide the tightest bounds for $\mathbb{P}\left(U>\vartheta\left(d^{*}\right) \mid V=v^{*}\right)$. For the sake of clarity, we will explain briefly the bounding approach proposed by Chesher (2005) and JPX before presenting our refinement. In the rest of the paper, we shall use the following notation $p(z)=\mathbb{P}(D=0 \mid Z=z)$, $\mathbb{P}(1 \mid 1, z)=\mathbb{P}(Y=1 \mid D=1, Z=z)=\mathbb{P}(U>\vartheta(1) \mid V>p(z))$, and $\mathbb{P}(1 \mid 0, z)=\mathbb{P}(Y=1 \mid D=0, Z=z)=$ $\mathbb{P}(U>\vartheta(0) \mid V \leq p(z))$ for $z \in \mathcal{Z}$. Now, we are going to present two special cases of different rank conditions to illustrate our idea.

First illustrative case: The support of $Z$ contains four distinct values i.e. $\mathcal{Z}=\left\{z_{1}, z_{2}, z_{3}, z_{4}\right\}$ such that $0<p\left(z_{1}\right)<p\left(z_{2}\right)<v^{*}<p\left(z_{3}\right)<p\left(z_{4}\right)<1$.
2.1. Chesher (2005) approach. Assumption 2 implies that $\mathbb{P}(U>\vartheta(1) \mid V=v)$ is nondecreasing in $v$ then we have:

$$
\begin{aligned}
\mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right) & \leq \mathbb{P}(U>\vartheta(1) \mid V=v) \text { for } v \in\left[p\left(z_{i}\right), 1\right], i=3,4, \\
\mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right)\left(1-p\left(z_{i}\right)\right) & \leq \int_{\left[p\left(z_{i}\right), 1\right]} \mathbb{P}(U>\vartheta(1) \mid V=v) d v \text { for } i=3,4 .
\end{aligned}
$$

The second inequality holds by taking the integral over both parts. The last inequality implies that:

$$
\begin{aligned}
\mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right) & \leq \frac{1}{1-p\left(z_{i}\right)} \int_{\left[p\left(z_{i}\right), 1\right]} \mathbb{P}(U>\vartheta(1) \mid V=v) d v \text { for } i=3,4, \\
& =\mathbb{P}\left(1 \mid 1, z_{i}\right) \text { for } i=3,4
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
\mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right) \leq \min \left(\mathbb{P}\left(1 \mid 1, z_{3}\right), \mathbb{P}\left(1 \mid 1, z_{4}\right)\right) \tag{2.3}
\end{equation*}
$$

Notice that, this cannot be done for $z_{1}$ and $z_{2}$ since $v^{*} \in\left[p\left(z_{i}\right), 1\right]$ for $i=1,2$. However, we can similarly derive the following:

$$
\begin{equation*}
\max \left(\mathbb{P}\left(1 \mid 0, z_{1}\right), \mathbb{P}\left(1 \mid 0, z_{2}\right)\right) \leq \mathbb{P}\left(U>\vartheta(0) \mid V=v^{*}\right) \text { for } i=1,2 . \tag{2.4}
\end{equation*}
$$

Note that, using the idea behind Chesher (2005) we cannot provide meaningful lower and upper bounds respectively for $\mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right)$ and $\mathbb{P}\left(U>\vartheta(0) \mid V=v^{*}\right)$. JPX introduced an interesting idea that allows them to refine those bounds and provide meaningful bounds when the latter approach fails.
2.2. JPX approach. The Chesher (2005) idea exploits information from the different intervals $\left[p\left(z_{i}\right), 1\right]$ and $\left[0, p\left(z_{i}\right)\right]$. JPX pointed out that it's also possible to exploit information from $\left[p\left(z_{1}\right), p\left(z_{2}\right)\right]$ and $\left[p\left(z_{3}\right), p\left(z_{4}\right)\right]$. Indeed, for $p\left(z_{i}\right)<p\left(z_{j}\right)$, we can easily show the following equalities:

$$
\begin{aligned}
& \mathbb{P}\left(U>\vartheta(1) \mid p\left(z_{i}\right)<V<p\left(z_{j}\right)\right)=\frac{\left.\left.\mathbb{P}\left(1 \mid 1, z_{i}\right)\right)\left(1-p\left(z_{i}\right)\right)-\mathbb{P}\left(1 \mid 1, z_{j}\right)\right)\left(1-p\left(z_{j}\right)\right)}{p\left(z_{j}\right)-p\left(z_{i}\right)} \\
& \mathbb{P}\left(U>\vartheta(0) \mid p\left(z_{i}\right)<V<p\left(z_{j}\right)\right)=\frac{\left.\left.\mathbb{P}\left(1 \mid 0, z_{j}\right)\right) p\left(z_{j}\right)-\mathbb{P}\left(1 \mid 0, z_{i}\right)\right) p\left(z_{i}\right)}{p\left(z_{j}\right)-p\left(z_{i}\right)}
\end{aligned}
$$

Since, $\mathbb{P}\left(U>\vartheta(1) \mid p\left(z_{i}\right)<V<p\left(z_{j}\right)\right)$ is identified from the data, we can use this quantity to bound our function of interest. Indeed, we have

$$
\begin{aligned}
\mathbb{P}\left(U>\vartheta(1) \mid p\left(z_{1}\right)<V<p\left(z_{2}\right)\right) & =\frac{1}{p\left(z_{2}\right)-p\left(z_{1}\right)} \int_{\left[p\left(z_{1}\right), p\left(z_{2}\right)\right]} \mathbb{P}(U>\vartheta(1) \mid V=v) d v \\
& \leq \mathbb{P}\left(u>\vartheta(1) \mid V=v^{*}\right)
\end{aligned}
$$

By using this approach, the bounds proposed by JPX are:

$$
\begin{align*}
& L B_{1}^{J P X} \leq \mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right) \leq U B_{1}^{J P X}  \tag{2.5}\\
& L B_{0}^{J P X} \leq \mathbb{P}\left(U>\vartheta(0) \mid V=v^{*}\right) \leq U B_{0}^{J P X} \tag{2.6}
\end{align*}
$$

where
$L B_{1}^{J P X}=\mathbb{P}\left(U>\vartheta(1) \mid p\left(z_{1}\right)<V<p\left(z_{2}\right)\right), U B_{1}^{J P X}=\min \left(\mathbb{P}\left(U>\vartheta(1) \mid p\left(z_{3}\right)<V<p\left(z_{4}\right)\right), \mathbb{P}\left(1 \mid 1, z_{3}\right), \mathbb{P}\left(1 \mid 1, z_{4}\right)\right)$
$L B_{0}^{J P X}=\max \left(\mathbb{P}\left(U>\vartheta(0) \mid p\left(z_{1}\right)<V<p\left(z_{2}\right)\right), \mathbb{P}\left(1 \mid 0, z_{1}\right), \mathbb{P}\left(1 \mid 0, z_{2}\right)\right), U B_{0}^{J P X}=\mathbb{P}\left(U>\vartheta(0) \mid p\left(z_{3}\right)<V<\right.$ $p\left(z_{4}\right)$ ).
2.3. Tightening the JPX bounds. By adequately adapting Lemma 2.1 of Shaikh and Vytlacil (2011), we can identify the sign of $[\vartheta(0)-\vartheta(1)]$. Indeed, for $p(z)<p\left(z^{\prime}\right)$ we can easily show that

$$
\begin{aligned}
\operatorname{sign}([\vartheta(0)-\vartheta(1)]) & =\operatorname{sign}\left(\mathbb{P}\left(U>\vartheta(1), p(z)<V<p\left(z^{\prime}\right)\right)-\mathbb{P}\left(U>\vartheta(0), p(z)<V<p\left(z^{\prime}\right)\right)\right) \\
& =\operatorname{sign}\left(H\left(z, z^{\prime}\right)\right)
\end{aligned}
$$

where $\operatorname{sign}(a) \equiv 1\{a>0\}-1\{a<0\}, H\left(z, z^{\prime}\right) \equiv[\mathbb{P}(Y=1, D=1 \mid Z=z)-\mathbb{P}(Y=1, D=1 \mid Z=$ $\left.\left.z^{\prime}\right)\right]-\left[\mathbb{P}\left(Y=1, D=0 \mid Z=z^{\prime}\right)-\mathbb{P}(Y=1, D=0 \mid Z=z)\right]$, and $1\{$.$\} denotes the indicator function.$

In the case where $\operatorname{sign}\left(H\left(z, z^{\prime}\right)\right)=1$ i.e. $\vartheta(1)<\vartheta(0)$, we have $\mathbb{P}\left(U>\vartheta(0) \mid V=v^{*}\right) \leq \mathbb{P}(U>$ $\left.\vartheta(1) \mid V=v^{*}\right)$, therefore $\mathbb{P}\left(U>\vartheta(0) \mid V=v^{*}\right) \leq \min \left(U B_{0}^{J P X}, U B_{1}^{J P X}\right)$ and $\mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right) \geq$ $\max \left(L B_{0}^{J P X}, L B_{1}^{J P X}\right)$. After some simplification, we obtain

$$
\begin{aligned}
\min \left(U B_{0}^{J P X}, U B_{1}^{J P X}\right) & =\min \left(\mathbb{P}\left(1 \mid 1, z_{3}\right), \mathbb{P}\left(1 \mid 1, z_{4}\right), \mathbb{P}\left(U>\vartheta(0) \mid p\left(z_{3}\right)<V<p\left(z_{4}\right)\right)\right) \\
& \leq \mathbb{P}\left(U>\vartheta(0) \mid p\left(z_{3}\right)<V<p\left(z_{4}\right)\right)=U B_{0}^{J P X}
\end{aligned}
$$

and

$$
\begin{aligned}
\max \left(L B_{0}^{J P X}, L B_{1}^{J P X}\right) & =\max \left(\mathbb{P}\left(1 \mid 0, z_{1}\right), \mathbb{P}\left(1 \mid 0, z_{2}\right), \mathbb{P}\left(U>\vartheta(1) \mid p\left(z_{1}\right)<V<p\left(z_{2}\right)\right)\right) \\
& \geq \mathbb{P}\left(U>\vartheta(1) \mid p\left(z_{1}\right)<V<p\left(z_{2}\right)\right)=L B_{1}^{J P X}
\end{aligned}
$$

It appears immediately that the JPX bounds can be refined using the information on the sign of $[\vartheta(0)-\vartheta(1)]$. However, when $\operatorname{sign}\left(H\left(z, z^{\prime}\right)\right)=-1$, the new information obtained using the sign of $[\vartheta(0)-\vartheta(1)]$ is redundant. The following proposition summarizes our result. For sake of simplicity, we use the shorthand notation $H \equiv H\left(z, z^{\prime}\right)$.

Proposition 1. Under assumptions 1-3, the bounds on $\psi\left(\cdot, \tau^{*}, v^{*}\right)$ in the model (2.1) are the following:

$$
\begin{aligned}
1\left\{L B_{1}^{*}>1-\tau^{*}\right\} & \leq \psi\left(1, \tau^{*}, v^{*}\right) \leq 1\left\{U B_{1}^{J P X}>1-\tau^{*}\right\}, \\
1\left\{L B_{0}^{J P X}>1-\tau^{*}\right\} & \leq \psi\left(0, \tau^{*}, v^{*}\right) \leq 1\left\{U B_{0}^{*}>1-\tau^{*}\right\},
\end{aligned}
$$

where $L B_{1}^{*}=\max \left(\mathbb{P}\left(1 \mid 0, z_{1}\right) 1\{H>0\}, \mathbb{P}\left(1 \mid 0, z_{2}\right) 1\{H>0\}, \mathbb{P}\left(U>\vartheta(1) \mid p\left(z_{1}\right)<V<p\left(z_{2}\right)\right)\right)$,
$U B_{1}^{J P X}=\min \left(\mathbb{P}\left(U>\vartheta(1) \mid p\left(z_{3}\right)<V<p\left(z_{4}\right)\right), \mathbb{P}\left(1 \mid 1, z_{3}\right), \mathbb{P}\left(1 \mid 1, z_{4}\right)\right)$,
$L B_{0}^{J P X}=\max \left(\mathbb{P}\left(U>\vartheta(0) \mid p\left(z_{1}\right)<V<p\left(z_{2}\right)\right), \mathbb{P}\left(1 \mid 0, z_{1}\right), \mathbb{P}\left(1 \mid 0, z_{2}\right)\right)$,
$U B_{0}^{*}=\min \left(\mathbb{P}\left(1 \mid 1, z_{3}\right)^{1\{H>0\}}, \mathbb{P}\left(1 \mid 1, z_{4}\right)^{1\{H>0\}}, \mathbb{P}\left(U>\vartheta(0) \mid p\left(z_{3}\right)<V<p\left(z_{4}\right)\right)\right)$.

Second illustrative case: The support of $Z$ contains two distinct values i.e. $\mathcal{Z}=\left\{z_{1}, z_{3}\right\}$ such that $0<p\left(z_{1}\right)<v^{*}<p\left(z_{3}\right)<1$. Following similar derivations used in the previous case, we can show that the JPX approach will not provide any refinement over the bounds obtained using the Chesher (2005) approach. In fact the bounds are as follows:

$$
\begin{align*}
& \mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right) \leq \mathbb{P}\left(1 \mid 1, z_{3}\right),  \tag{2.7}\\
& \mathbb{P}\left(1 \mid 0, z_{1}\right) \leq \mathbb{P}\left(U>\vartheta(0) \mid V=v^{*}\right) . \tag{2.8}
\end{align*}
$$

It is immediately apparent that the two previous approaches are not able to provide meaningful lower and upper bounds respectively on $\mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right)$ and $\mathbb{P}\left(U>\vartheta(0) \mid V=v^{*}\right)$ when the instrument is binary. However, using our idea we can provide meaningful lower and upper bounds for the latter quantities. Indeed, in the case where $\operatorname{sign}\left(H\left(z_{1}, z_{3}\right)\right)=1$, we have

$$
\begin{align*}
& \mathbb{P}\left(1 \mid 0, z_{1}\right) \leq \mathbb{P}\left(U>\vartheta(1) \mid V=v^{*}\right) \leq \mathbb{P}\left(1 \mid 1, z_{3}\right),  \tag{2.9}\\
& \mathbb{P}\left(1 \mid 0, z_{1}\right) \leq \mathbb{P}\left(U>\vartheta(0) \mid V=v^{*}\right) \leq \mathbb{P}\left(1 \mid 1, z_{3}\right) . \tag{2.10}
\end{align*}
$$

The improvement obtained using our approach is applicable beyond the binary triangular system case. In the following section, we will show how this approach can be generalized to the case when the outcome is nonbinary.

## 3. Generalization

3.1. Triangular systems with nonbinary outcome. In this section, we consider the case where $Y$ is nonbinary $(Y=g(D, U))$. However, for the sake of simplicity, we maintain the treatment to be binary i.e. $D=1\{p(Z)<V\}$. We shall use the notation $\underline{\psi}_{J P X}\left(d^{*}, \tau^{*}, v^{*}\right)$ and $\bar{\psi}_{J P X}\left(d^{*}, \tau^{*}, v^{*}\right)$ to denote, respectively, the JPX lower and upper bounds of the function of interest $\psi\left(d^{*}, \tau^{*}, v^{*}\right)$ proposed in their Theorem 1, i.e. $\underline{\psi}_{J P X}\left(d^{*}, \tau^{*}, v^{*}\right) \leq \psi\left(d^{*}, \tau^{*}, v^{*}\right) \leq \bar{\psi}_{J P X}\left(d^{*}, \tau^{*}, v^{*}\right), d^{*} \in\{0,1\}$. Our goal in this section, is to show that, without additional assumptions, it is possible to identify in some cases the sign of $\left[\psi\left(1, \tau^{*}, v^{*}\right)-\psi\left(0, \tau^{*}, v^{*}\right)\right]$
and then tighten the JPX bounds as follows:

$$
\begin{aligned}
& \max \left(\underline{\psi}_{J P X}\left(d^{*}, \tau^{*}, v^{*}\right), \underline{\psi}_{J P X}\left(1-d^{*}, \tau^{*}, v^{*}\right)\right) \leq \psi\left(d^{*}, \tau^{*}, v^{*}\right) \leq \bar{\psi}_{J P X}\left(d^{*}, \tau^{*}, v^{*}\right) \\
& \underline{\psi}_{J P X}\left(1-d^{*}, \tau^{*}, v^{*}\right) \leq \psi\left(1-d^{*}, \tau^{*}, v^{*}\right) \leq \min \left(\bar{\psi}_{J P X}\left(d^{*}, \tau^{*}, v^{*}\right), \bar{\psi}_{J P X}\left(1-d^{*}, \tau^{*}, v^{*}\right)\right)
\end{aligned}
$$

where $d^{*}=1$ if $\psi\left(0, \tau^{*}, v^{*}\right) \leq \psi\left(1, \tau^{*}, v^{*}\right)$ and $d^{*}=0$ if $\psi\left(0, \tau^{*}, v^{*}\right) \geq \psi\left(1, \tau^{*}, v^{*}\right)$. To show this result, we proceed in two steps.
First step: Whenever there are $z$ and $z^{\prime}$ such that $p(z)<p\left(z^{\prime}\right)$ we have the following equalities:

$$
\begin{aligned}
H\left(z, z^{\prime}, y\right) \equiv & {\left[\left(\mathbb{P}(Y \leq y, D=1 \mid Z=z)-\mathbb{P}\left(Y \leq y, D=1 \mid Z=z^{\prime}\right)\right)-\right.} \\
& \left.\quad\left(\mathbb{P}\left(Y \leq y, D=0 \mid Z=z^{\prime}\right)-\mathbb{P}(Y \leq y, D=0 \mid Z=z)\right)\right] \\
= & \mathbb{P}\left(U \leq g^{-1}(1, y), p(z)<V<p\left(z^{\prime}\right)\right)-\mathbb{P}\left(U \leq g^{-1}(0, y), p(z)<V<p\left(z^{\prime}\right)\right) \\
= & \operatorname{sign}\left(\left[g^{-1}(1, y)-g^{-1}(0, y)\right]\right) \mathbb{P}\left(\min \left(g^{-1}(1, y), g^{-1}(0, y)\right)<U \leq \max \left(g^{-1}(1, y),\right.\right. \\
& \left.\left.\left.g^{-1}(0, y)\right), p(z)<V<p\left(z^{\prime}\right)\right)\right),
\end{aligned}
$$

where $g^{-1}(d, y)$ denotes the generalized inverse i.e. $g^{-1}(d, y)=\sup \{u: g(d, u) \leq y\}$. Then, the sign of $\left[g^{-1}(1, y)-g^{-1}(0, y)\right]$ is identified from the observable function $H\left(z, z^{\prime}, y\right)$. The first equality holds since $g$ is nondecreasing in $U$ and left-continuous. Indeed, under monotonicity and left-continuity, we have $\left\{U \leq g^{-1}(d, y)\right\} \Leftrightarrow\{g(d, U) \leq y\}$. This latter equivalence can be proved by adequately adapting Proposition 1(4) and 1(5) in Embrechts and Hofert (2013).
Second step: If $\forall y \in \mathcal{Y}, H\left(z, z^{\prime}, y\right) \geq 0$, then $\mathbb{P}\left(U \leq g^{-1}(0, y) \mid V=v^{*}\right) \leq \mathbb{P}\left(U \leq g^{-1}(1, y) \mid V=v^{*}\right)$. Under assumptions 1 and 3 , we can find two appropriate $z_{0}^{*}$ and $z_{1}^{*}$ such that

$$
\begin{aligned}
& \mathbb{P}\left(U \leq g^{-1}(0, y) \mid V=v^{*}\right) \leq \mathbb{P}\left(U \leq g^{-1}(1, y) \mid V=v^{*}\right) \\
& \Leftrightarrow \mathbb{P}\left(g(0, U) \leq y \mid Z=z_{0}^{*}, V=v^{*}\right) \leq \mathbb{P}\left(g(1, U) \leq y \mid Z=z_{1}^{*}, V=v^{*}\right) \\
& \Leftrightarrow \mathbb{P}\left(Y \leq y \mid D=0, V=v^{*}\right) \leq \mathbb{P}\left(Y \leq y \mid D=1, V=v^{*}\right)
\end{aligned}
$$

The first equivalence holds under assumption 1, and the second under assumption 3. Hence, from Lemma 1 in JPX, $\psi\left(0, \tau^{*}, v^{*}\right) \geq \psi\left(1, \tau^{*}, v^{*}\right)$. Similarly, if $\forall y \in \mathcal{Y}, H\left(z, z^{\prime}, y\right) \leq 0$ we have $\psi\left(0, \tau^{*}, v^{*}\right) \leq \psi\left(1, \tau^{*}, v^{*}\right)$.

Remark 1. It is important to note that our set of assumptions is slightly different from the one used in $J P X$. Indeed, we assume that $g$ is left-continuous in $U$, while JPX assumed that only for $h_{j}$; however Chesher (2005) maintained this assumption for both $g$ and $h_{j}$ and related subsequent works of Jun, Pinkse and Xu maintained also the left-continuity assumption for both $g$ and $h_{j}$. Basically, we think that the left-continuity assumption is a very mild technical assumption that turns out to be verified in most of the applications.
3.2. Triangular systems with exogenous covariates. In more general models, there exists another exogenous covariate $X \in \mathcal{X} \subset \mathbb{R}$ which enters both equations. In such a case the triangular system takes the following form

$$
\left\{\begin{array}{l}
Y=g(D, X, U) \\
D=h(Z, X, V)
\end{array}\right.
$$

and the object of interest is the conditional structural function

$$
\psi^{*}=\psi\left(d, x, \tau^{*}, v^{*}\right)=g\left(d, x, Q_{U \mid V}\left(\tau^{*} \mid v^{*}\right)\right)
$$

for given values of $\left(\tau^{*}, d, x, v^{*}\right) \in \mathcal{U} \times \mathcal{D} \times \mathcal{X} \times \mathcal{V}$. To construct bounds on $\psi\left(d, x, \tau^{*}, v^{*}\right)$ in such a model, we can firstly use the JPX approach to construct bounds for any given values of $(d, x)$. Secondly, we can use our approach to identify the sign of $\left[\psi\left(d, x, \tau^{*}, v^{*}\right)-\psi\left(d^{\prime}, x^{\prime}, \tau^{*}, v^{*}\right)\right]$ for $\left(d, d^{\prime}\right) \in(\mathcal{D} \times \mathcal{D})$ and for $\left(x, x^{\prime}\right) \in(\mathcal{X} \times \mathcal{X})$. It is important to note that, in addition to use variation in $D$, we can also use variation in the covariate X. For instance, let us consider a binary $D$ i.e. $D=1\{p(X, Z)<V\}$. I shall use the shorthand notation $p(x, z)=P(D=0 \mid X=x, Z=z)$. Whenever there are $p(x, z)<p\left(x^{\prime}, z^{\prime}\right)$ we have the following equalities:

$$
\begin{aligned}
& H\left(z, z^{\prime}, x, x^{\prime}, y\right) \equiv\left[\left(\mathbb{P}(Y \leq y, D=1 \mid X=x, Z=z)-\mathbb{P}\left(Y \leq y, D=1 \mid X=x^{\prime}, Z=z^{\prime}\right)\right)-\right. \\
&\left.\left(\mathbb{P}\left(Y \leq y, D=0 \mid X=x^{\prime}, Z=z^{\prime}\right)-\mathbb{P}(Y \leq y, D=0 \mid X=x, Z=z)\right)\right] \\
&=\mathbb{P}\left(U \leq g^{-1}(1, x, y), p(x, z)<V<p\left(x^{\prime}, z^{\prime}\right)\right)-\mathbb{P}\left(U \leq g^{-1}\left(0, x^{\prime}, y\right), p(x, z)<V<p\left(x^{\prime}, z^{\prime}\right)\right) \\
&= \operatorname{sign}\left(\left[g^{-1}(1, x, y)-g^{-1}\left(0, x^{\prime}, y\right)\right]\right) \mathbb{P}\left(\min \left(g^{-1}(1, x, y), g^{-1}\left(0, x^{\prime}, y\right)\right)<U \leq \max \left(g^{-1}(1, x, y),\right.\right. \\
&\left.\left.\left.g^{-1}\left(0, x^{\prime}, y\right)\right), p(x, z)<V<p\left(x^{\prime}, z^{\prime}\right)\right)\right) .
\end{aligned}
$$

Therefore, by adequately adapting the previous derivation, we can show that if $\forall y \in \mathcal{Y}, H\left(z, z^{\prime}, x, x^{\prime}, y\right) \leq 0$, we have $\psi\left(0, x, \tau^{*}, v^{*}\right) \leq \psi\left(1, x^{\prime}, \tau^{*}, v^{*}\right)$.

This section shows that our approach can be used in a wide range of circumstances to refine existing bounds in the literature without invoking a new set of restrictive assumptions.

## 4. Conclusion

In a recent paper, Jun, Pinkse and Xu (2011) studied a nonparametric triangular system. They provided bounds on the structural function evaluated at particular values under various conditions. Considering a special case of their model, we show that their bounds can be significantly tightened in some cases without invoking a new set of assumptions. Further, we show how our idea can be used to tighten existing bounds on the structural function in more general triangular systems.

## References

Chesher, A. (2003): "Identification in nonseparable models," Econometrica, 71(5), 1405-1441.
Chesher, A. (2005): "Nonparametric identification under discrete variation," Econometrica, 73, 1525-1550. Embrechts, P., and M. Hofert (2013): "A note on generalized inverses," Mathematical Methods of Operations Research, 77, 423-432.
Jun, S. J., J. Pinkse, and H. Xu (2011): "Tighter bounds in triangular systems," Journal of Econometrics, 161(2), 122-128.
Mourifie, I. (2012): "Sharp bounds on treatment effects in a binary triangular system," Unpublished manuscript.
Shaikh, A. M., and E. J. Vytlacil (2011): "Partial Identification in Triangular Systems of Equations with Binary Dependent Variables," Econometrica, 79(3), 949-955.


[^0]:    Date: The present version is of June 21, 2014. Correspondence address: Department of Economics, University of Toronto, 150 St. George Street, Toronto ON M5S 3G7, Canada, ismael.mourifie@utoronto.ca.

