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Sharp Bounds in the Binary Roy Model

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SHARP BOUNDS IN THE BINARY ROY MODEL

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Abstract. We derive the empirical content of an instrumental variables model of sectoral choice with discrete outcomes. The partial identification results extend existing work on sharp bounds in binary choice threshold crossing models in allowing sector specific unobserved heterogeneity. Assumptions on selection include the simple, extended and generalized Roy models. The derived bounds are nonparametric intersection bounds and are simple enough to lend themselves to existing inference methods. Identification implications of exclusion restrictions are also derived. The derived bounds are applied to the analysis of the effect of Swan-Ganz catheterization and the robustness of previous findings to the introduction of procedure-specific unobserved heterogeneity is examined.

Keywords: sectorial choice, treatment specific unobservables, partial identification, intersection bounds.

JEL subject classification: C21, C25, C26

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Introduction

A large literature has developed since Heckman and Honoré (1990) on the empirical content of the Roy model of sectorial choice with sector specific unobserved heterogeneity. Most of this literature, however, concerns the case of continuous outcomes and many applications, where outcomes are discrete, fall outside its scope. They include analysis of the effects of different training programs on the probability of renewed employment, of competing medical treatments or surgical procedures on the probability of survival, of higher education on the probability of migration and of competing policies on schooling decisions in developing countries among numerous others. The Roy model is still highly relevant to those applications, but very little is known of its empirical content in such cases. Sharp bounds are derived in binary outcome models with a binary endogenous regressor in Chesher (2010), Shaikh and Vytlacil (2011), Chiburis (2010), Jun, Pinkse, and Xu (2010) and Mourifié (2011) under a variety of assumptions, which all rule out sector specific unobserved heterogeneity. Finally, Heckman and Vytlacil (1999) derive identification conditions in a parametric version of the binary Roy model.

We consider three distinct versions of the binary Roy model: the original model, where selection is based solely on the probability of success; the extended Roy model (in the terminology of Heckman and Vytlacil (1999)), where selection depends on the probability of success and a function of observable variables (sometimes called "nonpecuniary component"); and the generalized Roy model (in the terminology of Heckman and Honoré (1990)), with selection specific unobservable heterogeneity. When considering the generalized Roy model, we further distinguish restrictions on the selection equation and restrictions on the joint distribution of sector specific unobserved heterogeneity. We specifically consider the case, where selection variables are independent of sector specific unobserved heterogeneity and the case, where sector specific unobserved heterogeneity follows a factor structure proposed in Aakvik, Heckman, and Vytlacil (2005).

Following Heckman, Smith, and Clements (1997), we apply results from optimal transportation theory to derive sharp bounds on the structural parameters, from which a range of treatment parameters can be derived. More specifically, we apply Theorem 1 of Galichon and Henry (2011) (equivalently Theorem 3.2 of Beresteanu, Molchanov, and Molinari (2011)) to derive bounds for the generalized binary Roy model. The latter Theorem was recently applied in a similar context by Chesher, Rosen, and Smolinski (2013) to derive sharp bounds for instrumental variable models of discrete choice. We spell out the point identification implications of the bounds under certain exclusion restrictions. The bounds are simple enough to lend themselves to existing inferential methods, specifically Chernozhukov, Lee, and Rosen (2013) and Andrews and Shi (2013) in the instrumental variables case.

We implement the inference method proposed in Chernozhukov, Lee, and Rosen (2013) to bring a new light on the Connors, Speroff, Dawson, and Thomas (1996) Swan Ganz catheterization data set and analyze treatment effect in a framework with treatment specific unobserved effects. We achieve two main objectives in doing so. First, we illustrate how simple inference on the bounds proposed here can be using the STATA routines developed for Chernozhukov, Lee, and Rosen (2013) in Chernozhukov, Kim, Lee, and Rosen (2013). Second, we propose new results and reinterpret previous ones on the effect of Swan-Ganz catheterization on survival outcomes. In particular, when allowing for treatment selection on treatment specific unobservable heterogeneity and assuming the instrument proposed in Bhattacharya, Shaikh, and Vytlacil (2011) is valid, we show that we cannot reject the hypothesis that medical staff select treatment to maximize survival probability, i.e., that the conditional average treatment on the treated (resp. non treated) is non negative (resp. non positive), except in the case of patients admitted for chronic obstructive pulmonary disease. In addition, we find that even in the cases, where this hypothesis of "correct treatment decision" is not rejected, bounds on survival probabilities under treatment are systematically dominated by bounds on survival probabilities without treatment. In several cases, the sign of conditional average treatment effect is identified and negative. This leads to the interpretation that previous findings of negative treatment effects may be driven by restrictive assumptions on treatment selection in previous work and are in fact compatible with "correct treatment decision".

The remainder of the paper is organized as follows. Section 1 clarifies the analytical framework and the objectives. In Section 2, sharp bounds are derived for the binary Roy model, when selection depends only on the probability of success and possibly on observable variables. Identification implications are spelled out under exclusion restrictions. Section 3 considers the generalized binary Roy model, Section 4 presents inference results on survival outcomes following Swan-Ganz catheterization and the last section concludes.

1. Analytical framework

We adopt the framework of the potential outcomes model $Y = Y_1 D + Y_0(1-D)$, where Y is an observed outcome, D is an observed selection indicator and Y_1, Y_0 are unobserved potential outcomes. Heckman and Vytlacil (1999) trace the genealogy of this model and we refer to them for terminology and attribution. Potential outcomes are as follows:

$$Y_d = 1\{Y_d^* > 0\} = 1\{F(d, X_d, u_d) > 0\},\ d = 1, 0, \tag{1.1}$$

where $1\{.\}$ denotes the indicator function and F is an unknown function of the vector of observable random variables X_d and unobserved random variable u_d. We make the following assumptions throughout Sections 2 and 3.

Assumption 1 (Weak separability). *Potential outcomes can then be written $Y_d = 1\{f_d(X_d) > u_d\}$ for some unknown (measurable) functions f_d, $d = 0, 1$. As can be shown by appropriately adapting* *arguments in Vytlacil (2002), the latter is implied by weak separability of the functions* $F(d, X_d, u_d)$, *d=1,0.*

Assumption 2 (Regularity). *The sector specific unobserved variables* u_d, $d = 1, 0$, *are uniformly continuous with respect to Lebesgue measure, so that they may be assumed without loss of generality to be distributed uniformly on* $[0, 1]$.

The normalization of Assumption 2 is very convenient, since it implies $f_d(x_d) = \mathbb{E}(Y_d|x_d)$ and bounds on treatment effects parameters can be derived from bounds on the structural parameters f_1 and f_0.

Assumption 3 (Instruments). *Observable variables* X_d, $d = 1, 0$, *and instruments* Z *are independent of* (u_1, u_0). *Common components of* X_1 *and* X_0 *will be dropped from the notation in the remainder of the paper and by slight abuse of notation,* X_d *will refer only to the variables that are excluded from the equation for* Y_{1-d} *and* Z *to variables that are excluded from both outcome equations (when the case arises).*

In all the paper, we shall use the notation $\mathbb{P}(i, j|X)$ for $\mathbb{P}(Y = i, D = j|X)$ and $W = (Z, X_1, X_0)$, $\omega = (z, x_1, x_0)$.

Our objective is to characterize all the information that can be gathered from the distribution of observed variables (Y, W) about the unknown elements of the model, namely the functions f_1 and f_0 and the joint distribution (or copula, since the marginals are normalized) of the sector specific heterogeneity vector (u_1, u_0). We shall call this characterizing the empirical content of the model. The empirical content of the model, relative to the unknown functions f_1 and f_2 will be of primary interest and will take the form of sharp bounds such as:

$$\underline{\mathrm{G}}(\omega) \leq f_d(x_d) \leq \bar{G}(\omega), \tag{1.2}$$

in which case, exhibiting the functions $\underline{\mathrm{G}}$ and $\bar{G}$ will be the object of the analysis. In the case of a linear specification of the binary Roy model $f_d(x_d) = \tilde{f}_d(\beta_d' x_d)$, where $\tilde{f}_d : \mathbb{R} \to [0, 1]$ is a known invertible function of the single index $\beta_d' x_d$ and the unknown parameter vector β_d is the object of analysis, we can derive sharp bounds on β_d from (1.2) straighforwardly. From $\underline{\mathrm{G}}(\omega) \leq \tilde{f}_d(\beta_d' x_d) \leq \bar{G}(\omega)$, we derive $\tilde{f}_d^{-1}\underline{\mathrm{G}}(\omega) \leq \beta_d' x_d \leq \tilde{f}_d^{-1}\bar{G}(\omega)$. Hence, the bounds on the parameter vector β_d will be given by the projections of $\tilde{f}_d^{-1}\underline{\mathrm{G}}(\omega)$ and $\tilde{f}_d^{-1}\bar{G}(\omega)$ on x_d.

2. Sharp bounds for the binary Roy and extended Roy models

2.1. **Simple binary Roy model.**

As in the original model proposed by Roy (1951), the sector yielding the highest outcome is selected.

Assumption 4 (Binary Roy Model). *Agents select the sector yielding the highest latent outcome, i.e.,* $Y_1^* > Y_0^* \Rightarrow D = 1$ *and* $Y_1^* < Y_0^* \Rightarrow D = 0$.

The empirical content of the model under this selection rule is characterized in Figures 1 and 2.

FIGURE 1. Characterization of the empirical content of the simple binary Roy model in the unit square of the (u_1, u_0) space.

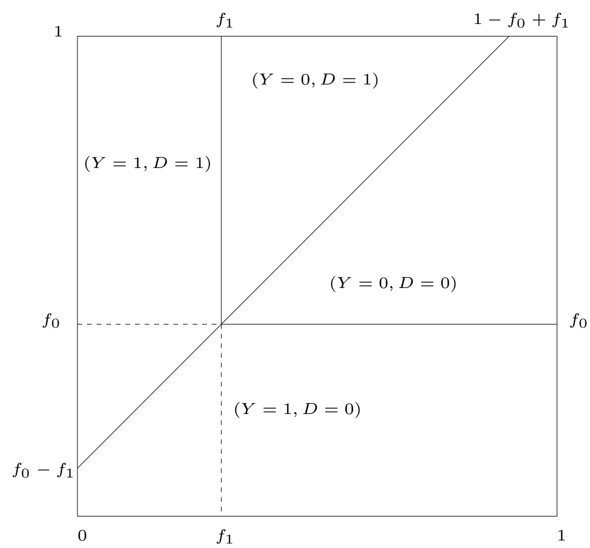
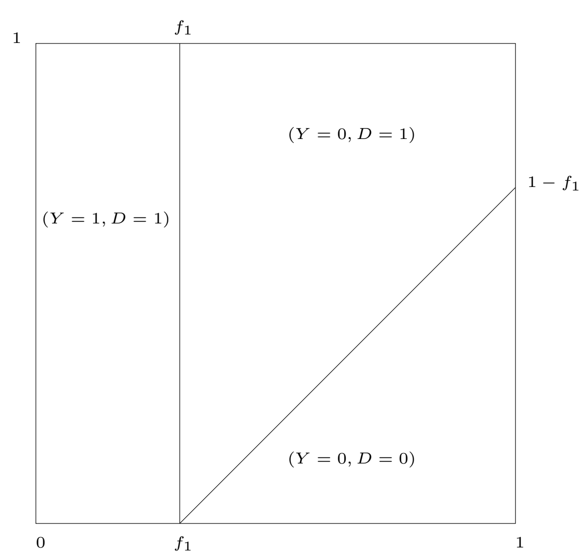


FIGURE 2. Characterization of the empirical content of the simple binary Roy model in the unit square of the (u_1, u_0) space in case $f_0 = 0$.



2.1.1. *Bounds on the structural functions.* For each value of the exogenous observable variables and each value of the pair (u_1, u_0), the outcome is uniquely determined. If the joint distribution were known, the likelihood of each of the potential outcomes $(Y = 1, D = 1)$, $(Y = 1, D = 0)$, $(Y = 0, D = 1)$ and $(Y = 0, D = 0)$ would be determined. However, only the marginal distributions of u_1 and u_0 are fixed, not the copula, so that only the probability of vertical and horizontal bands in Figures 1 and 2 are uniquely determined. Thus we see for instance that $f_1 = \mathbb{P}(Y = 1, D = 1)$ is identified when $f_0 = 0$ (as in Figure 2) and $f_0 = \mathbb{P}(Y = 1, D = 0)$ is identified when $f_1 = 0$ in a way that is akin to identification at infinity, as in Heckman (1990), when $f_d(x)$ follows a single index restriction. But in other cases (as in Figure 1), we only know $\mathbb{P}(Y = 1, D = 1) \leq f_1 \leq \mathbb{P}(Y = 1)$ and $\mathbb{P}(Y = 1, D = 0) \leq f_0 \leq \mathbb{P}(Y = 1)$. The following proposition, proved in the Appendix, shows that these bounds are jointly sharp.

Proposition 1 (Roy model)**.** *Under Assumptions 1-4 in Model (1.1), the following inequalities characterize the identified set for* (f_1, f_0)*.*

$$\sup_{x_0} \mathbb{P}(1,1|x_1,x_0) \leq \quad f_1(x_1) \quad \leq \inf_{x_0} \Big[\mathbb{P}(1,1|x_1,x_0) + \mathbb{P}(1,0|x_1,x_0)\Big] \tag{2.1}$$

$$\sup_{x_1} \mathbb{P}(1,0|x_1,x_0) \leq \quad f_0(x_0) \quad \leq \inf_{x_1} \Big[\mathbb{P}(1,0|x_1,x_0) + \mathbb{P}(1,1|x_1,x_0)\Big] \tag{2.2}$$

where the infima and suprema are taken over the domains of the excluded variables X_1 *or* X_0 *as indicated and when they exist.*

The validity of the bounds was shown above. To prove sharpness, we show in Appendix A that we can construct joint distributions for (u_1, u_0) such that each point in the identified region for $(f_1(x_1), f_0(x_0))$ defined by (2.1) and (2.2) is attained. Since the bounds in Proposition 1 are obtained as intersections over the domains of the excluded variables, they are called "intersection bounds". They are also semiparametric in the non excluded variables. Inference on such bounds can be conducted with existing methods described in Chernozhukov, Lee, and Rosen (2013) or in Andrews and Shi (2013).

A simple implication of Assumption 4 is that actual success is more likely than counterfactual success.

Assumption 5 (Positive ATT)**.** $\mathbb{E}(Y_d|D = d, X_1, X_0) \geq \mathbb{E}(Y_{1-d}|D = d, X_1, X_0)$ *for* $d = 1, 0$*.*

Assumption 5 has the simple interpretation of requiring nonnegative (resp. nonpositive) conditional average treatment effects for the treated (resp. untreated). Under Assumption 5, omitting conditioning variables for ease of notation,

$$\begin{aligned} f_d &= \mathbb{E}[Y_d] \\ &= \mathbb{E}[Y_d|D=d]\mathbb{P}(D=d) + \mathbb{E}[Y_d|D=1-d]\mathbb{P}(D=1-d) \\ &\leq \mathbb{P}[Y=1,D=d] + \mathbb{E}[Y_{1-d}|D=1-d]\mathbb{P}(D=1-d) \\ &= \mathbb{P}(Y=1,D=d) + \mathbb{P}(Y=1,D=1-d). \end{aligned}$$

Moreover, if $f_d > 0$ and $f_{1-d} = 0$, $\mathbb{P}(Y=1,D=1-d) = 0$. This implies that

$$\mathbb{P}(1,d|x_1,x_0) \leq \mathbb{E}[Y_d|x_1,x_0] \leq \mathbb{P}(1,d|x_1,x_0) + \mathbb{P}(1,1-d|x_1,x_0)$$

characterizes the empirical content of the potential outcomes model $Y = Y_1D + Y_0(1-D)$ in all generality (i.e., without weak separability and without assumptions on the dimension of unobservable heterogeneity). It also shows that the simple binary Roy model has no empirical content relative to (f_1, f_0) beyond Assumption 4. More precisely, the identified set for (f_1, f_0) under Assumption 5 is the same as under Assumption 1-4. Indeed, bounds (2.1) and (2.2) still hold under Assumption 5. They are also sharp, since Assumption 4 implies Assumption 5.

Corollary 1. *The identified set for (f_1, f_0) under Assumptions 1-3 and 5 is characterized by inequalities (2.1) and (2.2).*

In case of exclusion restrictions, an immediate corollary to Proposition 1 gives conditions for identification of the outcome equations. This identification result is related to Heckman (1990)'s identification at infinity in the following sense: in the special case of a single index model, where $f_0(x_0) = \phi(x_0\beta)$, where ϕ is a distribution function and β is a conformable vector of parameters, if $x_{0j} \to -\infty$ for some element x_{0j} of x_0 such that $\beta_j > 0$, then $f_0(x_0) \to 0$.

Corollary 2 (Identification)**.** *Under Assumptions 1-3 and 5, the following hold (writing $\omega = (z, x_1, x_0)$ as before).*

a. *If there is $x_0 \in Dom(X_0)$ such that $f_0(x_0) = 0$, then f_1 is identified over $Dom(X_1)$.*
b. *If there is $x_1 \in Dom(X_1)$ such that $f_1(x_1) = 0$, then f_0 is identified over $Dom(X_0)$.*
a'. *Take $x_1 \in Dom(X_1)$. If there is $x_0 \in Dom(X_0)$ such that $\mathbb{P}(1,0|x_1,x_0) = 0$, then $f_1(x_1)$ is identified.*
b'. *Take $x_0 \in Dom(X_0)$. If there is $x_1 \in Dom(X_1)$ such that $\mathbb{P}(1,1|x_1,x_0) = 0$, then $f_0(x_0)$ is identified.*

The existence of valid instruments or exclusion restrictions is often problematic in applications of discrete choice models. However, in the Roy model of sectorial choice with sector specific unobserved heterogeneity, it is natural to expect some sector specific observed heterogeneity as well. Such sector specific observed heterogeneity would provide exclusion restrictions in the form of variables affecting outcome equation for Y_d without affecting outcome equation for Y_{1-d}. Such exclusion restrictions would yield intersection bounds in Proposition 1. Of course, even if it exists, sector specific observed heterogeneity may not satisfy a. or b. of Corollary 2. However, the availability of an exclusion restriction as in a. or b. of Corollary 2 is consistent with the spirit of a model of sector specific heterogeneity.

Although a source of variation in selection indicator D, which does not affect outcome equations conditionally on treatment, is incompatible with the structural assumption 4, the effect of an instrument Z satisfying Assumption 3 may be entertained in an extention of Assumption 5:

Assumption 6 (Instrumental Variables)**.** *$E(Y_d - Y_{1-d}|D = d, X_1, X_0, Z) \geq 0$ for $d = 1, 0$.*

The latter assumption strengthens non negativity of treatment effect on the treated, now required to hold conditionally on instrument Z, which narrows the bounds on the average structural functions according to the following straightforward corollary of Proposition 1.

Corollary 3 (Instrumental variable bounds)**.** *Under Assumptions 1-3 and Assumption 6 in Model (1.1), the following inequalities characterize the identified set for (f_1, f_0).*

$$\sup_{x_0,z} \mathbb{P}(1, 1|x_1, x_0, z) \leq \quad f_1(x_1) \quad \leq \inf_{x_0,z} \Big[\mathbb{P}(1, 1|x_1, x_0, z) + \mathbb{P}(1, 0|x_1, x_0, z)\Big] \tag{2.3}$$

$$\sup_{x_1,z} \mathbb{P}(1, 0|x_1, x_0, z) \leq \quad f_0(x_0) \quad \leq \inf_{x_1,z} \Big[\mathbb{P}(1, 0|x_1, x_0, z) + \mathbb{P}(1, 1|x_1, x_0, z)\Big] \tag{2.4}$$

where the infima and suprema are taken over the domains of the excluded variables X_1 or X_0 and Z as indicated and when they exist.

In addition to narrowing the bounds, the availability of an instrument Z recovers testability of the model, i.e., possibility of upper and lower bounds crossing, even in the absence of variables X_1 or X_0 that selectively affect a single sector.

2.1.2. *Bounds on the joint distribution of sector specific heterogeneity.* The bounds proposed in Proposition 1 are joint sharp bounds on the structural functions, hence on treatment effects. To derive them, we treated the joint distribution of sector specific heterogeneity as a nuisance parameter. One may also be interested in the empirical content of the model relative to sector specific heterogeneity. Since the distributions of u_1 and u_0 are both normalized and assumed uniform on $[0, 1]$, the joint distribution satisfies Fréchet bounds:

$$\max(f_1(x_1) + f_0(x_0) - 1, 0) \leq \mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) \leq \min(f_1(x_1), f_0(x_0)).$$

The relevant question, therefore, is whether the Roy model assumption on selection, i.e., Assumption 4, holds empirical content relative to the distribution of unobserved sector specific heterogeneity beyond Fréchet bounds. In Figure 1, $\mathbb{P}(Y = 1)$ is equal to the L-shaped region on the left side of the graph. The area of the left vertical band is f_1 and the area of the lower horizontal band is f_0. These two bands overlap on the lower left rectangle, whose area is equal to $\mathbb{P}(u_1 \leq f_1, u_0 \leq f_0)$. Hence $f_1 + f_0 = \mathbb{P}(Y = 1) + \mathbb{P}(u_1 \leq f_1, u_0 \leq f_0)$. Adding conditioning variables, we have the following bounds on the joint distribution of sector specific heterogeneity:

$$\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) = f_1(x_1) + f_0(x_0) - \mathbb{P}(Y = 1|x_1, x_0). \tag{2.5}$$

This yields a sharper lower bound than the Fréchet bounds whenever $\mathbb{P}(Y = 1|x_1, x_0) < 1$. In particular, perfect negative correlation between u_1 and u_0 is ruled out. Note however, that the above constraint no longer holds when we replace the Roy selection hypothesis, i.e., Assumption 4, by Assumption 5. Hence the conclusion that the latter two assumptions hold the same empirical content is valid when considering empirical content relative to the structural functions and the treatment effects, but not when considering empirical content relative to the joint distribution of unobserved heterogeneity.

2.2. **Extended binary Roy model.**

2.2.1. *Extended selection assumption.* Assumption 4 is very restrictive and recent research by Haultfoeuille and Maurel (2013) and Bayer, Khan, and Timmins (2011) on the Roy model with continuous outcomes has focused on an extended version according to the terminology of Heckman and Vytlacil (1999), where selection depends on $Y_1^* - Y_0^*$ and a function of observable variables $g(Z, X_1, X_0)$ sometimes called "non pecuniary component". We now investigate the implications of this selection assumption in the binary case.

Assumption 7 (Observable heterogeneity in selection)**.** $Y_1^* - Y_0^* > g(Z, X_1, X_0) \Rightarrow D = 1$ *and* $Y_1^* - Y_0^* < g(Z, X_1, X_0) \Rightarrow D = 0$ *for some unknown function* g *of the vectors of observable variables* Z*,* X_1 *and* X_0*.*

Assumption 7 includes separability of the structural selection function in $Y_1^* - Y_0^*$ and $g(Z, X_1, X_0)$. The more general case without separability of the selection function is considered in Section 2.2.5. Under Assumptions 1-3 and 7, we may still characterize the empirical content of the model graphically, in Figures 3 and 4. We drop Z, X_1 and X_0 from the notation in the discussion below. For each value of (u_1, u_0), the outcome is uniquely determined by f_1, f_0 and g. Again, the missing piece to compute the likelihood of outcomes $\mathbb{P}(i, j)$, $i, j = 1, 0$, is the copula for (u_1, u_0). From the knowledge of the probabilities of horizontal and vertical bands in the (u_1, u_0) space, we can derive the sharp bounds on structural parameters f_1, f_0 and g. Four cases are considered below to explain the bounds, which are derived formally in Proposition 2.

a. Case where $g \geq f_1$ in Figure 4. The probability of outcome $(Y = 1, D = 0)$ is seen to be exactly equal to the area of the lower horizontal band. Hence $f_0 = \mathbb{P}(1, 0)$ is identified. Moreover, the area of the horizontal band $(f_0, f_0 - f_1 + g)$ is smaller than the probability

FIGURE 3. Characterization of the empirical content of the extended binary Roy model in the unit square of the (u_1, u_0) space in case $0 \leq g < f_1$.

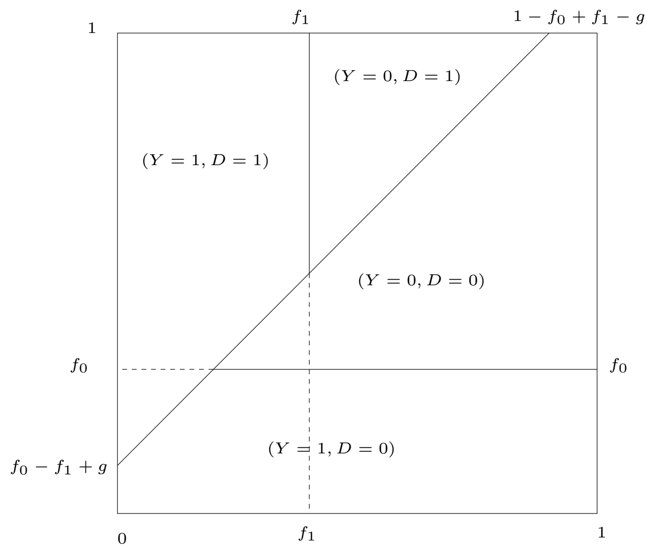
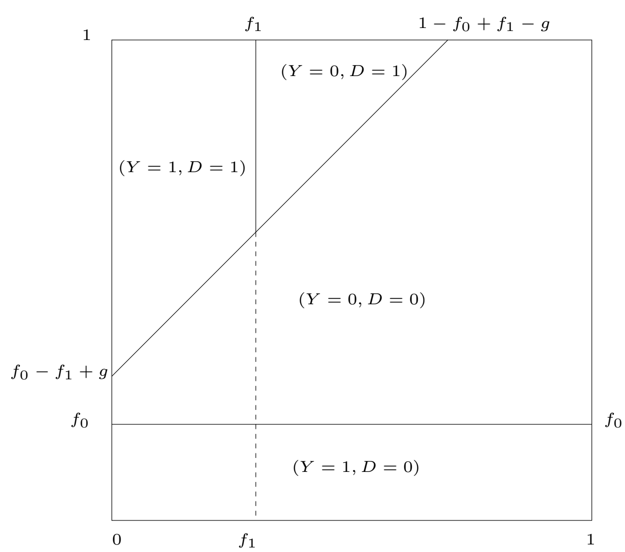


FIGURE 4. Characterization of the empirical content of the extended binary Roy model in the unit square of the (u_1, u_0) space in case $g \geq f_1$.



of outcome $(Y = 0, D = 0)$. Hence $g \leq f_1 + \mathbb{P}(0,0)$. Similar reasoning yields $\mathbb{P}(1,1) \leq f_1 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0,0)$.

b. Case where $0 \leq g < f_1$ in Figure 3. The area of the lower horizontal band $(0, f_0 - f_1 + g)$ is smaller than the probability of outcome $(Y = 1, D = 0)$. Hence $g \leq f_1 - f_0 + \mathbb{P}(1, 0)$. Moreover, the area of the horizontal band $(0, f_0)$ is larger than the probability of outcome $(Y = 1, D = 0)$ and smaller than the probability of outcome $(Y = 1)$. Hence $\mathbb{P}(1, 0) \leq f_0 \leq \mathbb{P}(Y = 1)$. Finally, $\mathbb{P}(1, 1) \leq f_1 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0, 0)$ still holds.

c. Case where $-f_0 < g \leq 0$. Similarly to Case b., we obtain bounds $g \geq f_1 - f_0 + \mathbb{P}(1, 1)$, $\mathbb{P}(1, 0) \leq f_0 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0, 1)$ and $\mathbb{P}(1, 1) \leq f_1 \leq \mathbb{P}(Y = 1)$.

d. Case where $g \leq -f_0$. Similarly to Case a., $f_1 = \mathbb{P}(1, 1)$ is identified and $\mathbb{P}(1, 0) \leq f_0 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0, 1)$ and $g \geq -f_0 - \mathbb{P}(0, 1)$.

In addition, in both cases a. and b., where $g > f_1 - f_0$, corresponding to Figures 4 and 3, the marginal constraint on u_1 fixes the probability mass in the thin right vertical band to $f_0 - f_1 + g$. Hence the maximum probability mass that can be shifted to the left of f_1 is $p_{11} + p_{10} + p_{00} - (f_0 - f_1 + g)$, so that we have the additional constraint $f_0 \leq p_{11} + p_{10} + p_{00} - g$. Symmetrically, in case $g < f_1 - f_0$, we have the constraint $f_1 \leq g + p_{11} + p_{10} + p_{00}$. Since $g > f_1 - f_0$ also implies $f_1 \leq g + p_{11} + p_{10} + p_{00}$ and $g < f_1 - f_0$ also implies $f_0 \leq p_{11} + p_{10} + p_{00} - g$, the two constraints $f_0 \leq p_{11} + p_{00} + p_{10} - g$ and $f_1 \leq g + p_{11} + p_{10} + p_{01}$ always hold. Proposition 2 shows validity of the bounds discussed above for the triplet $(f_1(x_1), f_0(x_0), g(\omega))$.

Proposition 2 (Bounds for the extended binary Roy model)**.** *Under Assumptions 1-3 and 7, the following bounds for (f_1, f_0, g) hold (writing $\omega = (z, x_1, x_0)$ as before).*

$$\begin{aligned} f_1(x_1) \quad &\in \quad \Big[\sup_{z,x_0} \mathbb{P}(1,1|\omega), \inf_{z,x_0} [\mathbb{P}(1,1|\omega) + \mathbb{P}(0,0|\omega)1\{g(\omega) > 0\} \\ &\qquad + \min[\min(\mathbb{P}(1,0|\omega), f_0(x_0) + g(\omega))1\{g(\omega) > -f_0(x_0)\}), \\ &\qquad\qquad g(\omega) + \mathbb{P}(1,0|\omega) + \mathbb{P}(0,1|\omega)]]\Big], \\ f_0(x_0) \quad &\in \quad \Big[\sup_{z,x_1} \mathbb{P}(1,0|\omega), \inf_{z,x_1} [\mathbb{P}(1,0|\omega) + \mathbb{P}(0,1|\omega)1\{g(\omega) < 0\} \\ &\qquad + \min[\min(\mathbb{P}(1,1|\omega), f_1(x_1) - g(\omega))1\{g(\omega) < f_1(x_1)\}), \\ &\qquad\qquad \mathbb{P}(1,1|\omega) + \mathbb{P}(0,0|\omega) - g(\omega)]]\Big] \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} g(\omega) \in \Big(&\Big[-f_0(x_0) - \mathbb{P}(0,1|\omega), -f_0(x_0)\Big] \cup \Big[f_1(x_1) - f_0(x_0) - \mathbb{P}(1,1|\omega), \\ &\quad f_1(x_1) - f_0(x_0) + \mathbb{P}(1,0|\omega)\Big] \cup \Big[f_1(x_1), f_1(x_1) + \mathbb{P}(0,0|\omega)\Big]\Big) \\ \cap \Big[&f_1(x_1) - \mathbb{P}(1,1|\omega) - \mathbb{P}(1,0|\omega) - \mathbb{P}(0,1|\omega), \mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega) + \mathbb{P}(0,0|\omega) - f_0(x_0)\Big] \end{aligned} \tag{2.7}$$

where the infima and suprema are taken over the domain of Z, X_1 or X_0 as indicated and when they arise.

2.2.2. *Identification implications of exclusion restrictions.* Simple identification conditions can be derived for f_1 and f_0 from the bounds of Proposition 2 under exclusion restrictions. However, it can be seen immediately that exclusion restrictions cannot identify $g(\,)$, since it would require $\mathbb{P}(Y=1,D=1|\omega)$, $\mathbb{P}(Y=0,D=1|\omega)$, $\mathbb{P}(Y=1,D=0|\omega)$ and $\mathbb{P}(Y=0,D=0|\omega)$ to simultaneously equal zero.

Corollary 4 (Identification)**.** *Under Assumptions 1-3 and 7, the following hold (writing $\omega=(z,x_1,x_0)$ as before).*

a. *If there is $z\in Dom(Z)$ and $x_0\in Dom(X_0)$ such that $g(\omega)\le -f_0(x_0)$, then $f_1(x_1)=\mathbb{P}(1,1|\omega)$ is identified.*
b. *If there is $z\in Dom(Z)$ and $x_1\in Dom(X_1)$ such that $g(\omega)\ge f_1(x_1)$, then $f_0(x_0)=\mathbb{P}(1,0|\omega)$ is identified.*
a'. *Take $x_1\in Dom(X_1)$. If there is $x_0\in Dom(X_0)$ or $z\in Dom(Z)$ such that $\mathbb{P}(1,0|\omega)=\mathbb{P}(0,0|\omega)=0$, then $f_1(x_1)$ is identified.*
b'. *Take $x_0\in Dom(X_0)$. If there is $x_1\in Dom(X_1)$ or $z\in Dom(Z)$ such that $\mathbb{P}(1,1|\omega)=\mathbb{P}(0,1|\omega)=0$, then $f_0(x_0)$ is identified.*

2.2.3. *Sharp bounds in the extended Roy model.* When the object of interest is treatment parameters only, the three dimensional identification region defined by the sharp bounds on (f_1,f_0,g) is projected on the two-dimensional space (f_1,f_0) as follows.

Proposition 3 (Sharp bounds for the extended Roy model)**.** *Under Assumptions 1-3 and 7, the identified set for (f_1,f_0) is characterized by the following bounds, where $\lambda\in\{0,1\}$ and $\varepsilon>0$ is arbitrarily small:*

$$\begin{array}{rcccl}\sup\limits_{z,x_0}\mathbb{P}(1,1|\omega) & \le & f_1(x_1) & \le & \inf\limits_{z,x_0}\left[\mathbb{P}(1,1|\omega)+\mathbb{P}(1,0|\omega)+\lambda\max(0,\mathbb{P}(0,0|\omega)-\varepsilon)\right] \\ \sup\limits_{z,x_1}\mathbb{P}(1,0|\omega) & \le & f_0(x_0) & \le & \inf\limits_{z,x_1}\left[\mathbb{P}(1,0|\omega)+\mathbb{P}(1,1|\omega)+(1-\lambda)\max(0,\mathbb{P}(0,1|\omega)-\varepsilon)\right]\end{array} \tag{2.8}$$

The binary λ ensures joint sharpness of the bounds on f_1 and f_0. It reflects the fact that in Figure 3 the diagonal sperating the $D=1$ from the $D=0$ regions is either on the right of the point (f_1,f_0), in which case the sharper bound on f_1 holds, or on the left of point (f_1,f_0), in which case the sharper bound on f_0 holds, but not both at the same time. The presence of $\varepsilon>0$ in the bounds reflects the fact that if, as in Figure 3, the diagonal is on the right of the point (f_1,f_0), i.e., $\lambda=1$, then f_1 could only attain the upper bound $p_{11}+p_{10}+p_{00}$ if all the mass corresponding to Region $(Y=0,D=0)$ was shifted into the triangle below the diagonal, above f_0 and left of f_1. However, this is impossible since the vertical band on the right has non zero mass by the uniform marginal constraint on u_1. Hence, the presence of ε in the bounds is due to the linearity of the boundary between Regions $D=0$ and $D=1$. This explains why it disappears in the nonseparable case of the next section.

If the object of interest is the non pecuniary component g, the three dimensional identification region is projected on the one-dimensional space for g into the single interval $[-\mathbb{P}(1,1|\omega)-$

$\mathbb{P}(1,0|\omega), \mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega)]$, since the bounds in (2.6) cross at those values. In the presence of instruments (or exclusion restrictions), the projections on (f_1, f_0) and on g can be much tighter and the projection on (f_1, f_0) may even be reduced to a point, as in Corollary 4.

2.2.4. *Testing the Roy selection assumption.* As we have just seen, in the absence of exclusion restrictions, the identified region always contains the hyperplane $g = 0$, so that it is impossible to test the classical Roy selection hypothesis. However, in the presence of exclusion restrictions, the hypothesis $g(\omega) = 0$ may become testable. There is a non zero non pecuniary component in the selection equation if the hyperplane $g(\omega) = 0$ does not intersect the three dimensional identification region for $(f_1(x_1), f_0(x_0), g(\omega))$ defined by the bounds in Proposition 2. This implies the crossing of the intersection bounds in Proposition 1, in the sense that

$$\sup_{x_0} \mathbb{P}(1,1|x_1,x_0) \quad > \quad \inf_{x_0} \Big[\mathbb{P}(1,1|x_1,x_0) + \mathbb{P}(1,0|x_1,x_0)\Big]$$

or

$$\sup_{x_1} \mathbb{P}(1,0|x_1,x_0) \quad > \quad \inf_{x_1} \Big[\mathbb{P}(1,0|x_1,x_0) + \mathbb{P}(1,1|x_1,x_0)\Big]$$

so that by Proposition 1, the simple Roy model is rejected. In practice, the test for the existence of a non pecuniary component would be carried out by constructing a confidence region according to the methods proposed in Chernozhukov, Lee, and Rosen (2013) or Andrews and Shi (2013) and checking, whether the hyperplane $g(\omega) = 0$ intersects the confidence region. If it does, we fail to reject the hypothesis of existence of a non pecuniary component and if it doesn't, we reject the hypothesis at significance level equal to 1 minus the confidence level chosen for the confidence region. The hypotheses $g \geq 0$ or $g \leq 0$ may be tested in the same way.

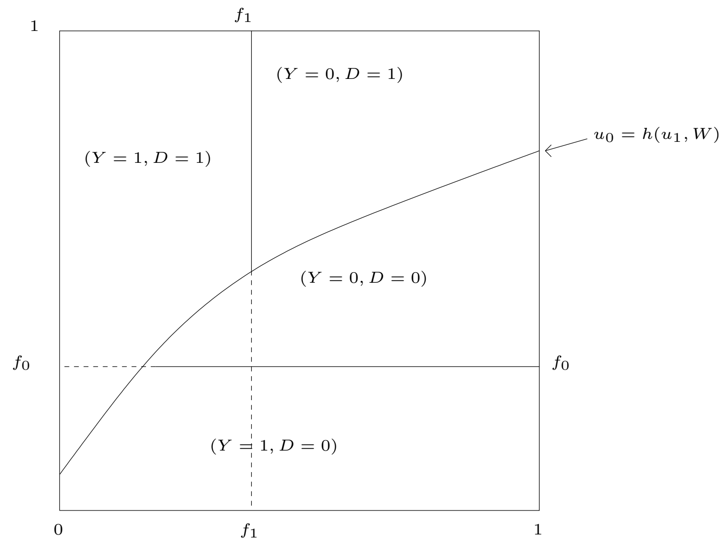
2.2.5. *Sharp bounds without separability of the selection function.* The same arguments can be applied to derive the empirical content of the model where the selection equation generalizes Assumption 7 with the following.

Assumption 8 (Nonseparable selection function)**.** *Suppose the selection rule is $u_0 > h(u_1, W) \Rightarrow D = 1$ and $u_0 < h(u_1, W) \Rightarrow D = 0$, with h strictly increasing in u_1, for all W.*

Assumption 7 is a special case of Assumption 8, where $h(u_1, W) = u_1 + f_0(X_0) - f_1(X_1) + g(W)$. The identified region for the pair $(f_1(x_1), f_0(x_0))$ is obtained in the same way as the separable case except that f_1 attains $\mathbb{P}(1,1)+\mathbb{P}(1,0)+\mathbb{P}(0,0)$ and f_0 attains $\mathbb{P}(1,1)+\mathbb{P}(1,0)+\mathbb{P}(0,0)$. This occurs because the nonlinearity of the curve separating region $D = 1$ from region $D = 0$ allows all the mass corresponding to $\mathbb{P}(0,0)$ to be shifted on the left of f_1, as in Figure 5.

Proposition 4 (Sharp bounds for the extended Roy model without separability)**.** *Under Assumptions 1-3 and 8, the identified set for (f_1, f_0) is characterized by the following inequalities, where λ*

FIGURE 5. Characterization of the empirical content of the binary Roy model in the unit square of the (u_1, u_0) space without separability of the selection function.



takes the values 1 *or* 0*:*

$$\begin{array}{rclcl} \sup_{z,x_0} \mathbb{P}(1,1|\omega) & \leq & f_1(x_1) & \leq & \inf_{z,x_0} [\mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega) + \lambda \mathbb{P}(0,0|\omega)] \\ \sup_{z,x_1} \mathbb{P}(1,0|\omega) & \leq & f_0(x_0) & \leq & \inf_{z,x_1} [\mathbb{P}(1,0|\omega) + \mathbb{P}(1,1|\omega) + (1-\lambda) \mathbb{P}(0,1|\omega)] \end{array} \tag{2.9}$$

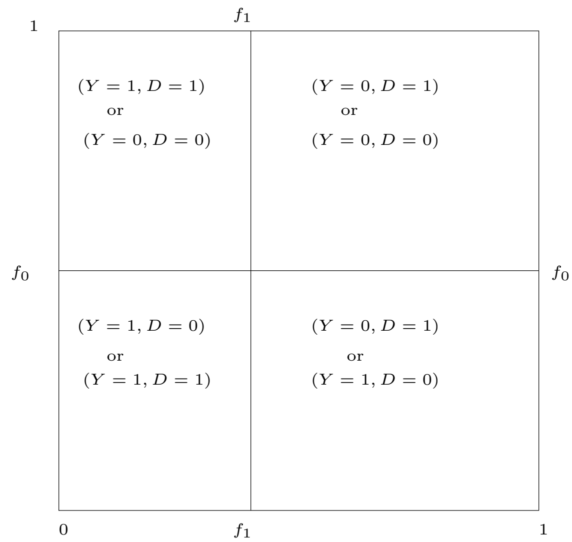
In this context, however, the Roy selection assumption, i.e., Assumption 4, may no longer be tested with the strategy developed above.

3. Sharp bounds for the generalized binary Roy model

So far, we have assumed that selection occurs on the basis of success probability and other observable variables. We now turn to the general case, where unobservable heterogeneity, beyond $u_0 - u_1$, may play a role in sectorial selection. Knowledge of (u_1, u_0) now no longer uniquely determines the outcome $(Y = i, D = j)$ as seen in Figure 6. Multiplicity of equilibria and lack of coherence of the model can be dealt with, however, with the optimal transportation approach of Galichon and Henry (2011) (or equivalently with the random set approach of Beresteanu, Molchanov, and Molinari (2011) as in Chesher, Rosen, and Smolinski (2013)), as shown in the proof of Theorem 1 below.

Theorem 1 (Sharp bounds for the generalized Roy model)**.** *Under Assumptions 1-3, the empirical content of the model is characterized by inequalities (3.1)-(3.3) below (writing* $\omega = (z, x_1, x_0)$ *as*

Figure 6. Characterization of the empirical content of the generalized binary Roy model in the unit square of the (u_1, u_0) space.



before).

$$\sup_{z,x_0} \mathbb{P}(1,1|\omega) \quad \leq f_1(x_1) \leq \quad 1 - \sup_{z,x_0} \mathbb{P}(0,1|\omega), \tag{3.1}$$

$$\sup_{z,x_1} \mathbb{P}(1,0|\omega) \quad \leq f_0(x_0) \leq \quad 1 - \sup_{z,x_1} \mathbb{P}(0,0|\omega) \tag{3.2}$$

and

$$\begin{aligned} & \sup_z \max\Big(0, f_0(x_0) - \mathbb{P}(1,0|\omega) - \mathbb{P}(0,1|\omega), f_1(x_1) - \mathbb{P}(1,1|\omega) - \mathbb{P}(0,0|\omega)\Big) \\ \leq \quad & \mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) \\ \leq \quad & \inf_z \min\Big(\mathbb{P}(Y=1|\omega), f_1(x_1) + f_0(x_0) - \mathbb{P}(Y=1|\omega)\Big). \end{aligned} \tag{3.3}$$

Theorem 1 is not an operational characterization of the empirical content of the model since the sharp bounds involve the unknown quantity $\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0)$, which, by the normalization of Assumption 2, is exactly the copula of (u_1, u_0). In the case of total ignorance about the copula of (u_1, u_0), after plugging Fréchet bounds $\max(f_1(x_1) + f_0(x_0) - 1, 0) \leq \mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) \leq \min(f_1(x_1), f_0(x_0))$, inequalities (3.3) are shown to be redundant. Hence we have the following.

Corollary 5. *The identified set for (f_1, f_0) under Assumption 1-3 is characterized by inequalities (3.1) and (3.2).*

In order to sharpen those bounds, we may consider restrictions on the copula for (u_1, u_0) or restrictions on the selection equation. We consider both strategies in turn.

3.1. **Restrictions on selection.** Consider the following selection model, where selection depends on $Y_1^* - Y_0^*$ and $g(Z, X_1, X_0)$ and selection specific unobserved heterogeneity v, which is separable and which is independent of (resp. dependent on) sector specific unobserved heterogeneity (u_1, u_0) under Assumption 9 (resp. Assumption 10). As before, write $W = (Z, X_1, X_0)$.

Assumption 9. $Y_1^* - Y_0^* > g(W) + v \Rightarrow D = 1$ *and* $Y_1^* - Y_0^* < g(W) + v \Rightarrow D = 0$, *with* $v \perp\!\!\!\perp (u_1, u_0, W)$ *and* $\mathbb{E}v = 0$ *(without loss of generality).*

With $v \perp\!\!\!\perp (u_1, u_0)$, we have $\mathbb{P}(u_d \leq g(z, x_1, x_0) + v + f_1(x_1) - f_0(x_0)|z, x_1, x_0) = \mathbb{E}_v\mathbb{E}[1\{u_d \leq g(z, x_1, x_0) + v - f_1(x_1) + f_0(x_0)\}|z, x_1, x_0, v] = \max(0, g(z, x_1, x_0) - f_1(x_1) + f_0(x_0))$ and it is shown in Corollary 6 that the bounds on $g(\,)$ derived in Section 2 remain valid.

Corollary 6. *Under Assumptions 1-3 and 9, (2.7) holds.*

As for the bounds on (f_1, f_0), (2.6) remain valid under specific domain restrictions for v.

Assumption 10. $Y_1^* - Y_0^* > g(W) + v \Rightarrow D = 1$ *and* $Y_1^* - Y_0^* < g(W) + v \Rightarrow D = 0$, *with* $v \perp\!\!\!\perp W$, $\mathbb{E}v = 0$ *(without loss of generality).*

Note that Assumption 10 is equivalent to assuming the selection equation $D = 1\{h(W) > \eta\}$ with η arbitrarily dependant on (u_1, u_0). Indeed, one can take $h(W) = f_1(X_1) - f_0(X_0) - g(W)$ and $\eta = v + u_1 - u_0$.

Corollary 7. *Under Assumption 1-3 and 10, (3.1) and (3.2) are sharp bounds for the pair* (f_1, f_2).

From Corollary 7, we conclude that the separability of the selection specific unobserved heterogeneity term has no empirical content, in the sense that the identified set for (f_1, f_0) is identical to the case, where there is no information on selection. This is related to the lack of empirical content of LATE in Kitagawa (2009) and it is in sharp contrast with the case of no sector specific heterogeneity in Shaikh and Vytlacil (2011), Jun, Pinkse, and Xu (2010) and Mourifié (2011), where the ordering between f_1 and f_0 can be used as identifying information. Indeed, if $f_1 \leq f_0$, we have $f_1 = \mathbb{P}(Y = 1, D = 1) + \mathbb{P}(u_1 \leq f_1, D = 0) \leq \mathbb{P}(Y = 1, D = 1) + \mathbb{P}(u_1 \leq f_0, D = 0)$. The last term is equal to $\mathbb{P}(Y = 1, D = 0)$ if $u_1 = u_0$ but is not identified in the case with sector specific unobserved heterogeneity.

3.2. **Restrictions on the joint distribution of sector specific heterogeneity.**

3.2.1. *Parametric restrictions on the copula.* In case the copula for (u_1, u_0) is parameterized with parameter vector θ, sharp bounds are obtained straightforwardly by replacing $\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0)$ with the parametric version $F(f_1(x_1), f_0(x_0); \theta)$ in (3.3).

3.2.2. *Perfect correlation.* In the case of perfect correlation between the two sector specific unobserved heterogeneity variables, $\mathbb{P}(u_1 \leq f_1(x_0), u_0 \leq f_0(x_0)) = \min(f_1(x_1), f_0(x_0))$ so that the sharp bounds of Theorem 1 specialize to (3.1), (3.2), $\min(f_1(x_1), f_0(x_0)) \leq \inf_z \mathbb{P}(Y = 1|z, x_1, x_0)$ and $\sup_z \mathbb{P}(Y = 1|z, x_1, x_0) \leq \max(f_1(x_1), f_0(x_0))$, which are the bounds derived in Chiburis (2010).

3.2.3. *Independence.* In the special case, where the two sector specific errors are independent of each other $u_1 \perp\!\!\!\perp u_0$, sharp bounds can be derived from Theorem 1 and $\mathbb{P}(u_1 \leq f_1(x_0), u_0 \leq f_0(x_0)) = \mathbb{P}(u_1 \leq f_1(x_1))\mathbb{P}(u_0 \leq f_0(x_0)) = f_1(x_1)f_0(x_0)$. The sharp bounds obtained allow formal tests of the hypothesis of independence of the two unobserved heterogeneity components. This would not be achievable based only on Fréchet bounds (as noted by Tsiatis (1975) in the case of competing risks), as we always have $f_0 + f_1 - 1 \leq f_0 f_1 \leq \min(f_1, f_0)$ when $0 \leq f_1, f_0 \leq 1$.

3.2.4. *Factor structure.* Theorem 1 also allows us to characterize the empirical content of the factor model for sector specific unobserved heterogeneity proposed in Aakvik, Heckman, and Vytlacil (2005).

Assumption 11 (Factor model)**.** *Sector specific unobserved heterogeneity has factor structure $u_d = \alpha_d u + \eta_d$, $d = 1, 0$, with $\mathbb{E}u = 0$, $\mathbb{E}u^2 = 1$ (without loss of generality) and $\eta_1 \perp\!\!\!\perp \eta_0 | u$. η_d is uniformly distributed on $[0, 1]$ for $d = 1, 0$, conditionally on u.*

This factor specification for sector specific unobserved heterogeneity is particularly appealing in applications to the effects of employment programs. Success in securing a job depends on common unobservable heterogeneity in talent and motivation and sector specific noise. Under Assumptions 1, 3 and 11, we still have $\mathbb{E}[Y_d|z, x_1, x_0] = f_d(x_d)$ and

$$\begin{aligned}
\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) &= \mathbb{E}_u \mathbb{P}(\eta_1 \leq f_1(x_1) - \alpha_1 u, \eta_0 \leq f_0(x_0) - \alpha_0 u | x_1, x_0, u) \\
&= \mathbb{E}_u \mathbb{P}(\eta_1 \leq f_1(x_1) - \alpha_1 u | x_1, u) \mathbb{P}(\eta_0 \leq f_0(x_0) - \alpha_0 u | x_1, x_0, u) \\
&= f_1(x_1) f_0(x_0) + \alpha_1 \alpha_0.
\end{aligned}$$

Hence we can obtain sharp bounds on parameters f_1, f_0, α_1 and α_0 as follows.

Corollary 8 (Sharp bounds for the factor model)**.** *Under Assumptions 1, 3 and 11, the empirical content of the model is characterized by (3.1), (3.2) and (writing $\omega = (z, x_1, x_0)$ as before)*

$$\begin{aligned}
& \sup_z \max\Big(0, f_0(x_0) - \mathbb{P}(1,0|\omega) - \mathbb{P}(0,1|\omega), f_1(x_1) - \mathbb{P}(1,1|\omega) - \mathbb{P}(0,0|\omega)\Big) \\
\leq\ & f_1(x_1) f_0(x_0) + \alpha_1 \alpha_0 \\
\leq\ & \inf_z \min\Big(\mathbb{P}(Y = 1|\omega), f_1(x_1) + f_0(x_0) - \mathbb{P}(Y = 1|\omega)\Big)
\end{aligned}$$

We recover the case of independent sector specific heterogeneity variables, when $\alpha_1 = \alpha_0 = 0$.

4. Bounds on the effects of Swan-Ganz catheterization

4.1. **Swan-Ganz catherization.** Catherization is a common surgical procedure consisting in inserting a tube through an artery to monitor the heart and enhance diagnostic. It can be traced back to the mid nineteenth century with Claude Bernard and was first performed on a human subject (himself) by Werner Forßmann in 1929. Swan-Ganz catheterization refers to the more recent pulmonary artery catheterization procedure introduced by Jeremy Swan and William Ganz. Its benefits are controversial and have been the object of intense empirical scrutiny in the last 20 years. We analyze the data set collected by Connors, Speroff, Dawson, and Thomas (1996) on catherer use and survival outcomes.

4.2. **Data.** The data was collected from intensive care units in 5 major US hospitals, namely Beth Israel in Boston, the Duke University Medical Center, MetroHealth Medical Center in Clevelad OH, St Joseph's Hospital in Marshfield WI and the University of California Medical Center. The data set includes 5735 patients admitted between 1989 and 1994 with one of the following life threatening diseases: acute respiratory failure (ARF), congestive heart failure (CHF), chronic obstructive pulmonary disease (COPD), cirrhoris (Cirrhosis), nontraumatic coma (Coma), multi-organ system failure (MOSF) with malignancy or with sespis. As detailed in Connors, Speroff, Dawson, and Thomas (1996), "exclusion criteria included age less than 18 years, death or discharge within 48 hours, inability to speak English, acute psychiatric disorders, pregnancy, acquired immunodeficiency syndrome (AIDS), acute burns, and head trauma or other trauma." Patients were classified as treated if they were catheterized within 24 hours and the outcome variable is an indicator of survival for at least 6 months after catheterization.

Data on observed heterogeneity includes age, gender, race, education, income, health insurance status, income and weight. Summary statistics for treated and untreated populations are given in Table 1. The last column gives the p-value in a test of the hypothesis that the mean characteristic for the treated and untreated are equal. The hypothesis is rejected at the 1% level for all observable characteristics except for age, race and health insurance status. Almost half the patients are admitted with acute respiratory failure, while the rest are evenly distributed among remaining ailments. Coma, conjestive heart failure, cirrhosis and pulmonary patients have relatively low rates of catheterization, whereas the rate in case of sepsis is relatively high. Younger, male, educated and richer patients have higher rates of catheterization than older, female, less educated and poorer patients.

4.3. **Former results.** Connors, Speroff, Dawson, and Thomas (1996), Hirano and Imbens (2001) and Li, Racine, and Wooldridge (2008) find clear evidence of adverse effects of catheterization with methodologies that rule out treatment selection on unobservable variables. Altonji, Elder, and Taber (2008) mitigates those results with restrictive forms of selection on unobservables, while Bhattacharya, Shaikh, and Vytalcil (2008) derive bounds on treatment effect under monotone treatment assumptions and propose week-end ICU admission as an instrument for selection to sharpen the bounds. Their results are more nuanced but broadly in line with previous findings. Bhattacharya, Shaikh, and Vytlacil (2011) propose sharper bounds under an assumption that rules out the Roy model with treatment specific unobservable heterogeneity model we consider here. They show that, with the help of the instrument mentioned above, they can identify the sign of treatment effect and again find negative effects of catheterization.

4.4. **Methodology.** The premise of the Roy model in the context of Catheterization treatment is that medical staff in charge of treatment decision observe all relevant characteristics of the patient, only parts of which are observed by the econometrician, and make the best decision on such observations. Unobserved (to the econometrician) characteristics of the patient that are relevant to the latter's survival under catheterization are possibly correlated, but not identical to the characteristics relevant to survival without treatment. The Roy model implies, therefore, that the patients are well dispatched. Hence, negative results on catheterization treatment effects conditionaly on observed variables are due to unobserved heterogeneity. We allow for the instrument proposed in Bhattacharya, Shaikh, and Vytlacil (2011), so that are starting point is Assumption 6 and we compute the sharp bounds (2.3)-(2.4) of Corollary 3. The variable Z in the corollary, i.e., the instrument, is an indicator of week-end ICU admission, which is assumed to affect treatment selection but not outcomes conditional on treatment, reflecting the fact that the composition of dispatching staff is different on week-ends, whereas surgeons are the same. For more details on the validity of the instrument, see Bhattacharya, Shaikh, and Vytlacil (2011). In the case of COPD, we find that the bounds (2.4) cross, hence the Roy model is rejected and we recompute bounds (2.8) with $\lambda = 0$ instead of (2.3)-(2.4).

We conduct inference on the sharp bounds above with the Chernozhukov, Kim, Lee, and Rosen (2013) STATA implementation of the intersection bounds procedure of Chernozhukov, Lee, and Rosen (2013). Conditional probabilities are computed parametrically (in order to accommodate multiple continuous covariates), with a probit procedure and bounds are computed with the CKLRSTATA 2013 clr2bound command. To the best of our knowledge, this is the first implementation of this procedure.

4.5. **Empirical findings.** Results are collected in Tables 2 to 8. Each line corresponds to a category with respect to cited binary explanatory variables and the bounds are reported for individuals of average age, weight and education level. In each line of each table, the upper line gives bounds for the probability of survival conditionally on being catheterized and the lower line gives bounds for the probability of survival conditionally on not being catheterized.

Bounds for the probability of survival for treated and non treated conditionally on observed covariates are quite wide, except in the case of COPD patients, where the bounds for the probability of survival without treatment cross, so that the benchmark model is rejected. Consider first the patients with other ailments. A first striking observation is that lower bounds for the probability of survival with catherization are systematically dominated by the lower bounds for the probability of survival without catheterization. The upper bounds are the same with and without catheterization (they are theoretically equal) except in the case of numbers in bold, which we shall return to below.

In many cases, the bounds for the probability of survival after catheterization include zero, so that the hypothesis that the survival probability is zero cannot be rejected. In all those cases, we also tested directly the hypothesis that the probability $\mathbb{P}(1,1|\omega)$ of being treated and surviving is zero and failed to reject. If that probability is zero, the probability of surviving conditionally on not being catheterized is identified and we recomputed (intervals in bold) the confidence interval for the latter. The [−] sign in those cases indicated that the sign of the conditional average treatement effect is identified and negative. This occurs in all of the following cases.

- For patients with nontraumatic coma:
  - all low income individuals with private insurance,
  - black low income men without private insurance.
- For patients with MOSF with malignancy, all low income individuals EXCEPT:
  - black low income women without private insurance,
  - black low income men with private insurance.
- For patients with Cirrhosis, all low income individuals EXCEPT:
  - white low income females with private insurance.

In only one case, namely high income white men without private insurance admitted with MOSF with malignancy, the situation is reversed. The probability of not being catheterized and surviving is zero and hence the probability of susrvival conditionally on catheterization is identified and the sign of the conditional average treatment effect is identified and positive. Finally, in one case, low income white men with private insurance admitted with MOSF with malignancy, the probability of survival is zero, hence the average treatment effect is trivially identified and equal to zero.

In the case of chronic obstructive pulmonary disease patients, we derive bounds for the extended Roy model with negative $g(Z)$. Lower bounds for the probability of survival without treatment are much larger than lower bounds for the probability of survival with treatment. In the case of low income black men with private insurance, the probability of being treated and surviving is zero, hence the probability of survival conditionally on not being catheterized is identified (boldface numbers in the table).

Conclusion

We have derived sharp bounds in the simple, extended and generalized binary Roy models, including a factor specification proposed by Aakvik, Heckman, and Vytlacil (2005). The bounds are simple enough to lend themselves to existing inference methods for intersection bounds as in Chernozhukov, Lee, and Rosen (2013) and Andrews and Shi (2013). The methods introduced here can be applied to the derivation of nonparametric sharp bounds for the Tobit version of the Roy model as well as in other binary models with several unobserved heterogeneity dimensions, such as entry and participation games. An application to the catherization data of Connors, Speroff, Dawson, and Thomas (1996) shows that negative effects of catheterization consistently reported in past studies may be due to the failure to account for treatment specific unobserved factors.

Appendix A. Proofs

In all the proofs, we use the notation $\omega = (z, x_1, x_0)$. When there is no ambiguity, we shall write $f_1 = f_1(x_1)$, $f_0 = f_0(x_0)$ and $g = g(\omega)$.

A.1. **Proof of Proposition 1.**

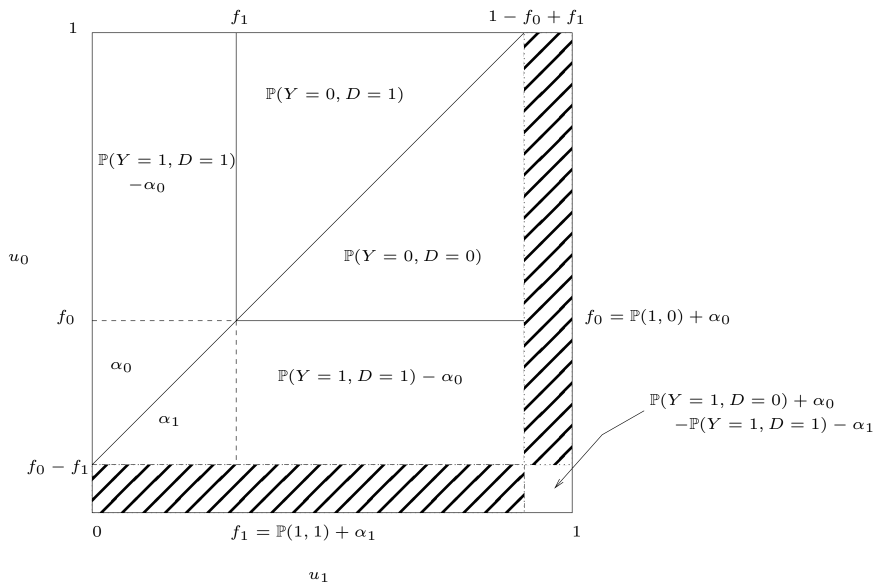
A.1.1. *Validity of the bounds.* See main text.

A.1.2. *Sharpness of the bounds.* To show the sharpness of the joint bounds for $f_1(x_1)$ and $f_0(x_0)$, it is sufficient to construct joint distributions for the unobserved heterogeneity vector (u_0^*, u_1^*) such that each point in the identified region is attained and which is compatible with the observed data in the following sense:

(1) $\mathbb{P}(u_0^* \leq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = \mathbb{P}(Y = 1, D = 0|\omega)$,
(2) $\mathbb{P}(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = \mathbb{P}(Y = 1, D = 1|\omega)$,
(3) $\mathbb{P}(u_0^* \geq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = \mathbb{P}(Y = 0, D = 0|\omega)$,
(4) $\mathbb{P}(u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = \mathbb{P}(Y = 0, D = 1|\omega)$.

The identified region is a rectangle and its extreme points are $(f_1(x_1) = P(Y = 1, D = 1|\omega), f_0(x_0) = P(Y = 1, D = 0|\omega))$, $(f_1(x_1) = P(Y = 1|\omega), f_0(x_0) = P(Y = 1|\omega))$, $(f_1(x_1) = P(Y = 1|\omega), f_0(x_0) = P(Y = 1, D = 0|\omega))$ and $(f_1(x_1) = P(Y = 1, D = 1|\omega), f_0(x_0) = P(Y = 1|\omega))$. We construct a joint distribution for (u_1, u_0) such that $f_1(x_1) = \mathbb{P}(1, 1|\omega) + \alpha_1$ and $f_0(x_0) = \mathbb{P}(1, 0|\omega) + \alpha_0$, for any (α_1, α_0) satisfying $0 < \alpha_1 \leq \mathbb{P}(1, 0|\omega)$ and $0 < \alpha_0 \leq \mathbb{P}(1, 1\omega)$. We consider the case where $f_0 - f_1 > 0$. The case, where $f_0 - f_1 \leq 0$ is treated symmetrically. The extreme case where $(\alpha_0 = 0,\ \alpha_1 > 0)$, $(\alpha_1 = 0,\ \alpha_0 > 0)$ and $(\alpha_0 = 0, \alpha_1 = 0)$ will be derived below with a small modification of the following distribution. We propose the following candidate as a potential joint distribution:

Figure 7.



Denote $\delta = \mathbb{P}(1,0|\omega) + \alpha_0 - \mathbb{P}(1,1|\omega) - \alpha_1 > 0$ and denote $\mathbb{P}(i,j|\omega) = p_{ij}$.

$$h^1(t,s) = \mathbb{P}(u_0^* \le s, u_1^* \le t) = 0, \quad \text{if } \; 0 \le t \le 1-\delta, s \le \delta, \tag{A.1}$$

$$h^2(t,s) = \mathbb{P}(\delta \le u_0^* \le s, 1-\delta \le u_1^* \le t) = 0, \quad \text{if } \; t \ge 1-\delta, s \ge \delta, \tag{A.2}$$

$$h^3(t,s) = \mathbb{P}(p_{10} + \alpha_0 \le u_0^* \le s, u_1^* \le t) = \frac{a(t,s)}{a(p_{11}+\alpha_1, 1)}(p_{11} - \alpha_0), \tag{A.3}$$

$$\text{if } \; 0 \le t \le p_{11} + \alpha_1, s \ge p_{10} + \alpha_0,$$

$$h^4(t,s) = \mathbb{P}(u_0^* \le s, 1-\delta \le u_1^* \le t) = \frac{(t-(1-\delta))s}{\delta}, \quad \text{if } \; t \ge 1-\delta, s \le \delta, \tag{A.4}$$

$$h^5(t,s) = \mathbb{P}(\delta \le u_0^* \le s, p_{11} + \alpha_1 \le u_1^* \le t) = \frac{b(t,s)}{b(1-\delta, p_{10}+\alpha_0))}(p_{11} - \alpha_0), \tag{A.5}$$

$$\text{if } \; p_{11} + \alpha_1 \le t \le 1-\delta, \delta \le s \le p_{10} + \alpha_0,$$

$$h^6(t,s) = \mathbb{P}(\delta \le u_0^* \le s, u_1^* \le t, u_0^* \le u_1^* + \delta) = \frac{t(p_{11}+\alpha_1 + s - (t+\delta))}{(p_{11}+\alpha_1)^2} 1\{s > \delta\}\alpha_1 \tag{A.6}$$

$$\text{if } \; t \le p_{11} + \alpha_1, \delta \le s \le t + \delta,$$

$$h^7(t,s) = \mathbb{P}(\delta \le u_0^* \le s, u_1^* \le t, u_0^* \ge u_1^* + \delta) = \frac{(s-\delta)(p_{11}+\alpha_1 + t - (s-\delta))}{(p_{11}+\alpha_1)^2} 1\{t > 0\}\alpha_0 \tag{A.7}$$

$$\text{if } \; t \le s - \delta, \delta \le s \le p_{10} + \alpha_0,$$

$$\begin{aligned} h^8(t,s) &= \mathbb{P}(p_{10}+\alpha_0 \le u_0^* \le s, p_{11}+\alpha_1 \le u_1^* \le t, u_0^* \le u_1^* + \delta) \\ &= \frac{(t-(p_{11}+\alpha_1))((1-p_{10}-\alpha_0) + s - (t+\delta))}{((1-p_{10}-\alpha_0))^2} 1\{s > p_{10}+\alpha_0\} p_{00} \end{aligned} \tag{A.8}$$

$$\text{if } \; p_{11} + \alpha_1 \le t \le 1-\delta, p_{10} + \alpha_0 \le s \le t + \delta,$$

$$\begin{aligned} h^9(t,s) &= \mathbb{P}(p_{10}+\alpha_0 \le u_0^* \le s, p_{11}+\alpha_1 \le u_1^* \le t, u_0^* \ge u_1^* + \delta) \\ &= \frac{(s-(p_{10}+\alpha_0))((1-p_{10}-\alpha_0) + t - (s-\delta))}{((1-p_{10}-\alpha_0))^2} 1\{t > p_{11}+\alpha_1\} p_{01} \end{aligned} \tag{A.9}$$

$$\text{if } \; p_{11} + \alpha_1 \le t \le s - \delta, p_{10} + \alpha_0 \le s \le 1,$$

where

$$a(t,s) = t(s-(\mathbb{P}(1,0|\omega)+\alpha_0)),$$
$$b(t,s) = (t-(\mathbb{P}(1,1|\omega)+\alpha_1))(s-\delta),$$

It is easy to verify that this function is a joint distribution such as the marginals are uniform distribution over [0,1] and which is compatible with the observed data (i.e., respects Conditions 1 to 4) when $f_1(x_1) = \mathbb{P}(1,1|\omega)+\alpha_1$ and $f_0(x_0) = \mathbb{P}(1,0|\omega)+\alpha_0$, for any (α_1,α_0) satisfying $0 \leq \alpha_1 \leq \mathbb{P}(1,0|\omega)$ and $0 \leq \alpha_0 \leq \mathbb{P}(1,1\omega)$.

Let us verify now that the marginals are uniform.

- $\mathbb{P}(u_1^* \leq t)$ for $t \leq p_{11}+\alpha_1$, use the equalities (A.1), (A.6), (A.7) and (A.3).
  $\mathbb{P}(u_1^* \leq t) = h^1(t,\delta)+h^6(t,t+\delta)+h^7(t,p_{10}+\alpha_0)+h^3(t,1) = t.$
- $\mathbb{P}(p_{11}+\alpha_1 \leq u_1^* \leq t)$ for $p_{11}+\alpha_1 \leq t \leq 1-\delta$, use (A.1), (A.5), (A.8) and (A.9).
  $\mathbb{P}(p_{11}+\alpha_1 \leq u_1^* \leq t) = h^1(t,\delta)+h^5(t,p_{10}+\alpha_0)+h^8(t,t+\delta)+h^9(t,1) = t-(p_{11}+\alpha_1).$
- $\mathbb{P}(1-\delta \leq u_1^* \leq t)$. for $1-\delta \leq t \leq 1$, use (A.4) and (A.2).
  $\mathbb{P}(1-\delta \leq u_1^* \leq t) = h^4(t,\delta)+h^2(t,1) = t-(1-\delta).$
- $\mathbb{P}(u_0^* \leq s)$ for $s \leq \delta$, use (A.1), (A.4).
  $\mathbb{P}(u_0^* \leq s) = h^1(1-\delta,s)+h^4(1,s) = s.$
- $\mathbb{P}(\delta \leq u_0^* \leq s)$ for $\delta \leq s \leq p_{10}+\alpha_0$, use (A.7), (A.6), (A.5) and (A.2).
  $\mathbb{P}(\delta \leq u_0^* \leq s) = h^7(s-\delta,s)+h^6(p_{11}+\alpha_1,s)+h^5(1-\delta,s)+h^2(1,s) = s-\delta.$
- $\mathbb{P}(p_{10}+\alpha_0 \leq u_0^* \leq s)$ for $p_{10}+\alpha_0 \leq s \leq 1$, use (A.3), (A.9), (A.8) and (A.2).
  $\mathbb{P}(p_{10}+\alpha_0 \leq u_0^* \leq s) = h^3(p_{11}+\alpha_1,s)+h^9(s-\delta,s)+h^8(1-\delta,s)+h^2(1,s) = s-(p_{10}+\alpha_0).$

This ensures that the marginals are indeed uniform. Now, let verify that equations (1) to (4) are verified.

(1) $\mathbb{P}(u_0^* \leq f_0, u_1^* \geq u_0^*+f_1-f_0) = \mathbb{P}(u_0^* \leq p_{10}+\alpha_0, u_1^* \geq u_0^*+\delta) = h^1(1-\delta,\delta)+h^4(1,\delta)+h^6(p_{11}+\alpha_1, s+p_{11}+\alpha_1)+h^5(1-\delta,p_{10}+\alpha_0)+h^2(1,p_{10}+\alpha_0) = p_{10}$
(2) $\mathbb{P}(u_1^* \leq f_1, u_1^* \leq u_0^*+f_1-f_0) = h^7(p_{10}+\alpha_0-\delta,p_{10}+\alpha_0)+h^3(p_{11}+\alpha_1,1) = p_{11},$
(3) $\mathbb{P}(u_0^* \geq f_0, u_1^* \geq u_0^*+f_1-f_0) = h^8(1-\delta,1) = p_{00},$
(4) $\mathbb{P}(u_1^* \geq f_1, u_1^* \leq u_0^*+f_1-f_0) = h^8(1,1-\delta) = p_{01}$

The case where $\delta = 0$ i.e $f_0-f_1 = 0$ can be treated with the same construction except that we have to remove the equation (A.1), (A.2) and (A.3). Consider now the case where ($\alpha_0 = 0$, $\alpha_1 > 0$). We propose the same distribution as before except for (A.3), (A.5) and (A.7). The intuition is to remove the mass from the area of α_0 (A.7) and add it on the area delimited by (A.3), (A.5).

$$h^3(t,s) = \mathbb{P}(p_{10}+\alpha_0 \le u_0^* \le s, u_1^* \le t) = \frac{p_{11}}{(p_{11}+\alpha_1)(1-p_{10})}t(s-p_{10}),$$
$$\text{if } \ 0 \le t \le p_{11}+\alpha_1, s \ge p_{10}+\alpha_0,$$
$$h^5(t,s) = \mathbb{P}(\delta \le u_0^* \le s, p_{11}+\alpha_1 \le u_1^* \le t) = \frac{p_{11}}{(p_{11}+\alpha_1)(1-p_{10})}(t-(p_{11}+\alpha_1))(s-\delta),$$
$$\text{if } \ p_{11}+\alpha_1 \le t \le 1-\delta, \delta \le s \le p_{10}+\alpha_0,$$
$$h^7(t,s) = 0 \ \text{ if } \ t \le s-\delta, \delta \le s \le p_{10}+\alpha_0.$$

The same method can be applied to the two remains cases.

A.2. **Proof of Proposition 2.** To show validity of the bounds, we drop all the conditioning variables $\omega = (z, x_1, x_0)$ from the notation. We have $D = 1 \Rightarrow Y_0^* + g \le Y_1^* \Rightarrow 1\{Y_0^* + g \ge 0\} \le 1\{Y_1^* \ge 0\} \Rightarrow 1\{Y_0^* + g \ge 0\}1\{D = 1\} \le 1\{Y_1^* \ge 0\}1\{D = 1\} \Rightarrow \mathbb{E}[1\{Y_0^* + g \ge 0\}|D = 1] \le \mathbb{E}[1\{Y_1^* \ge 0\}|D = 1] \Rightarrow \mathbb{E}[1\{Y_0^* + g \ge 0\}|D = 1] \le \mathbb{E}[Y_1|D = 1]$. We can easily derive equivalent inequalities when $D = 0$. Hence, if $D = 1\{Y_1^* > Y_0^* + g\}$ then $\mathbb{E}[1\{Y_0^* + g \ge 0\}|D = 1] \le \mathbb{E}[Y_1|D = 1]$ and $\mathbb{E}[Y_1|D = 0] \le \mathbb{E}[1\{Y_0^* + g \ge 0\}|D = 0]$. Hence, when $g \ge 0$, $\mathbb{E}[Y_0|D = 1] \le \mathbb{E}[Y_1|D = 1]$ and when $g \le 0$, $\mathbb{E}[Y_1|D = 0] \le \mathbb{E}[Y_0|D = 0]$. Finally, if $g = 0$ we have $\mathbb{E}[Y_d|D = d] \ge \mathbb{E}[Y_d|D = 1-d]$ where $d \in \{0, 1\}$. Those inequalities allow us to construct the sharp bounds for f_1 and f_0 in the case where $D = 1\{Y_1^* > Y_0^* + g\}$. Indeed, $f_1 = \mathbb{E}[Y_1] = \mathbb{E}[Y_1, D = 1] + \mathbb{E}[Y_1|D = 0]P(D = 0)$ and $f_0 = \mathbb{E}[Y_0] = \mathbb{E}[Y_0, D = 0] + \mathbb{E}[Y_0|D = 1]P(D = 1)$. Now, if $g \ge 0$, then $P(Y = 1, D = 1) \le f_1 \le P(Y = 1, D = 1) + P(D = 0)$ and $P(Y = 1, D = 0) \le f_0 \le P(Y = 1)$. On the other hand, if $g \le 0$, $P(Y = 1, D = 1) \le f_1 \le P(Y = 1)$ and $P(Y = 1, D = 0) \le f_0 \le P(Y = 1, D = 0) + P(D = 1)$. Finally, $f_0 = \mathbb{E}[1\{u_0 \le f_0\}1\{u_1 \ge u_0 + f_1 - f_0 - g\}] + \mathbb{E}[1\{u_0 \le f_0\}1\{u_1 \le u_0 + f_1 - f_0 - g\}]$. Hence, if $g \ge f_1$, then $\{u_1 \le u_0 + f_1 - f_0 - g\} \Rightarrow \{u_0 \ge f_0\}$ and $f_0(X_0) \le \mathbb{E}[1\{u_0 \le f_0\}1\{u_1 \ge u_0 + f_1 - f_0 - g\}] + \mathbb{E}[1\{u_0 \le f_0\}1\{u_0 \ge f_0\}] \le \mathbb{E}[1\{u_0 \le f_0\}1\{u_1 \ge u_0 + f_1 - f_0 - g\}] = P(Y = 1, D = 0)$.

Now the bounds for g can be obtained as follows.

- If $g + f_0 - f_1 \ge 0$ and $g \le f_1$, then $\{u_0 \le g + f_0 - f_1\} \Rightarrow \{u_0 \le u_1 + g + f_0 - f_1\}$ and $\{u_0 \le g + f_0 - f_1\} \Rightarrow \{u_0 \le f_0\}$. So $\{u_0 \le g + f_0 - f_1\} \Rightarrow \{u_0 \le u_1 + g + f_0 - f_1\} \cap \{u_0 \le f_0\}$. Hence $g + f_0 - f_1 = P(u_0 \le g + f_0 - f_1) \le P(\{u_0 \le u_1 + g + f_0 - f_1\} \cap \{u_0 \le f_0\}) = P(Y = 1, D = 0)$.
- If $g + f_0 - f_1 \ge 0$ and $g \ge f_1$, then $\{u_0 \le g + f_0 - f_1\} \Rightarrow \{u_0 \le u_1 + g + f_0 - f_1\}$, hence $g + f_0 - f_1 = P(u_0 \le g + f_0 - f_1) \le P(\{u_0 \le u_1 + g + f_0 - f_1\}) = P(D = 0)$. As $f_0 = P(Y = 1, D = 0)$ we have $g - f_1 \le P(Y = 0, D = 0)$.
- If $g + f_0 - f_1 \le 0$ and $g \ge -f_0$, then by similar arguments, we have $g + f_0 - f_1 \ge -P(Y = 1, D = 1)$.
- If $g + f_0 - f_1 \le 0$ and $g \le -f_0$, then $g + f_0 \ge -P(Y = 0, D = 1)$.

Finally, the validity of bounds $f_1(x_1) - \mathbb{P}(1, 1|\omega) - \mathbb{P}(1, 0|\omega) - \mathbb{P}(0, 1|\omega) \le g(\omega) \le \mathbb{P}(1, 1|\omega) + \mathbb{P}(1, 0|\omega) + \mathbb{P}(0, 0|\omega) - f_0(x_0)$ is shown formally in the main text. This completes the proof.

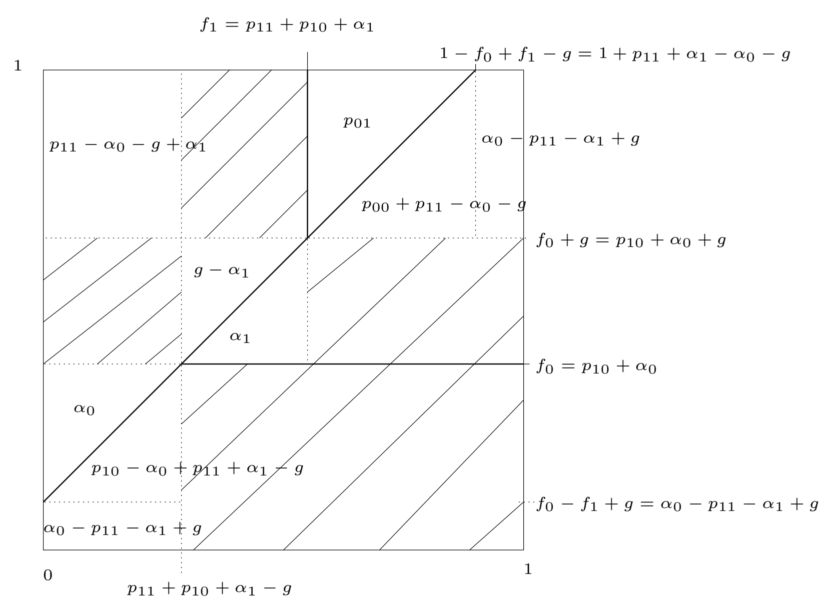
A.3. **Proof of Proposition 3.** For convenience of notation, we drop conditioning variables and write p_{ij} for $\mathbb{P}(Y = i, D = j|\omega)$. We consider the case $p_{ij} > 0$ and take $0 \le \alpha_1 < p_{00}$ and $0 \le \alpha_0 \le p_{11}$ arbitrarily and construct a joint distribution for (u_1, u_0) with uniform marginals and such that $f_1 = p_{11} + p_{10} + \alpha_1$ and $f_0 = p_{10} + \alpha_0$. Figure 8 shows a distribution of mass, which yields this for any g satisfying $\alpha_1 < g \le f_1$ in the case $f_0 - f_1 + g > 0$. The other case is treated symmetrically. The shaded areas are zero probability regions for (u_1, u_0). There remains to distribute mass in each of the designated areas of the partition of the unit square in Figure 8 so as to ensure that the marginals u_1 and u_0 are uniformly distributed. The case, where $f_1 = p_{11} + \alpha_1$ and $f_0 = p_{10} + \alpha_0$, with $0 \leq \alpha_1 < p_{10}$ and $0 \leq \alpha_0 \leq p_{11}$ arbitrary, is obtained from Proposition 1 by simply setting $g = 0$.

As was detailed in the proof of Proposition 1, the suitable marginals can be obtained by distributing mass uniformly on the three rectangles in Figure 8 and distributing mass on the three pairs of triangles in the following way: we explain the mass distribution on the middle pair of triangles in Figure 8. The other two are treated similarly.

$$\begin{aligned}\mathbb{P}(p_{11} + p_{10} + \alpha_1 - g \leq u_1^* < p_{11} + p_{10} + \alpha_1 - g + t, p_{10} + \alpha_0 \leq u_0^* < p_{10} + \alpha_0 + s, u_0^* \leq u_1^*)\\ = \alpha_1 t(1 + s - t),\\ \mathbb{P}(p_{11} + p_{10} + \alpha_1 - g \leq u_1^* < p_{11} + p_{10} + \alpha_1 - g + t, p_{10} + \alpha_0 \leq u_0^* < p_{10} + \alpha_0 + s, u_0^* > u_1^*)\\ = (g - \alpha_1)s(1 + t - s),\end{aligned}$$

for $0 \leq t \leq g$ and $0 \leq s \leq g$. The proof is completed by checking that the marginal distributions are indeed uniform.

Figure 8.



A.4. **Proof of Proposition 4.** The proof is exactly identical to the that of Proposition 3 except that $h(u_1, \omega)$ can be chosen as in Figure 5 so that all the mass $\mathbb{P}(0, 0|\omega)$ can be shifted on the left of $f_1(x_1)$ and therefore we can no longer restrict α_1 to be strictly positive. The case $\alpha_1 = 0$ is also attained. The result follows immediately.

A.5. **Proof of Theorem 1.** Under Assumptions 1-3, the model can be equivalently written $(Y, D) \in G((u_1, u_0)|W)$ almost surely conditionally on $W = (Z, X_1, X_0)$, where G is a multi-valued mapping, which to (u_1, u_0) associates $(y, d) = G((u_1, u_0)|W) = \{(1, 1), (1, 0)\}$ if $u_1 \le f_1(x_1)$ and $u_0 \le f_0(x_0)$, $\{(0, 1), (1, 0)\}$ if $u_1 > f_1(x_1)$ and $u_0 \le f_0(x_0)$, $\{(1, 1), (0, 0)\}$ if $u_1 \le f_1(x_1)$ and $u_0 > f_0(x_0)$ and $\{(0, 1), (0, 0)\}$ if $u_1 > f_1(x_1)$ and $u_0 > f_0(x_0)$. Hence Theorem 1 of Galichon and Henry (2011) applies and the empirical content of the model is characterized by the collection of inequalities $P(A|W) \le P((u_1, u_0) : G((u_1, u_0)|W) \text{ hits } A|W)$ for each subset A of $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ (i.e., 16 inequalities). The only non redundant inequalities are $P(1, 1|W) \le f_1(X_1)$, $P(1, 0|W) \le f_0(X_0)$, $P(0, 1|W) \le 1 - f_1(X_1)$, $P(0, 0|W) \le 1 - f_0(X_0)$, $P(Y = 0|W) \le 1 - P(u_1 \le f_1(X_1), u_0 \le f_0(X_0)|X_1, X_0)$, $P(Y = 1|W) \le 1 - P(u_1 > f_1(X_1), u_0 > f_0(X_0)|X_1, X_0)$, $P(0, 0|W) + P(1, 1|W) \le P(u_1 \le f_1(X_1), u_0 \le f_0(X_0)|X_1, X_0) + P(u_0 > f_0(X_0)|X_0)$ and $P(0, 1|W) + P(1, 0|W) \le P(u_1 \le f_1(X_1), u_0 \le f_0(X_0)|X_1, X_0) + P(u_1 > f_1(X_1)|X_1)$. After some manipulation, the result follows.

A.6. **Proof of Corollary 6.** We show that the bounds (2.7) for g remain valid. We drop conditioning variables from the notation throughout this section.

- If $g+v+f_0-f_1 \ge 0$ and $g+v \le f_1$, then $\{u_0 \le g+v+f_0-f_1\} \Rightarrow \{u_0 \le u_1+g+v+f_0-f_1\}$ and $\{u_0 \le g+v+f_0-f_1\} \Rightarrow \{u_0 \le f_0\}$. So $\{u_0 \le g+v+f_0-f_1\} \Rightarrow \{u_0 \le u_1+g+v+f_0-f_1\} \cap \{u_0 \le f_0\}$. Therefore $P(u_0-v \le g+f_0-f_1) \le P(\{u_0 \le u_1+g+v+f_0-f_1\} \cap \{u_0 \le f_0\}) = P(Y = 1, D = 0)$.
- If $g+v+f_0-f_1 \ge 0$ and $g+v \ge f_1$, then $\{u_0 \le g+v+f_0-f_1\} \Rightarrow \{u_0 \le u_1+g+v+f_0-f_1\}$. Therefore $P(u_0-v \le g+f_0-f_1) \le P(\{u_0 \le u_1+g+v+f_0-f_1\}) = P(D = 0)$.
- If $g+v+f_0-f_1 \le 0$ and $g+v \ge -f_0$, then $\{u_1 \le f_1-f_0-g-v\} \Rightarrow \{u_1 \le u_0+f_1-f_0-g-v\}$ and $\{u_1 \le f_1-f_0-g-v\} \Rightarrow \{u_1 \le f_1\}$. So $\{u_1 \le f_1-f_0-g-v\} \Rightarrow \{u_1 \le u_0+f_1-f_0-g-v\} \cap \{u_1 \le f_1\}$. Therefore $P(u_1+v \le f_1-f_0-g) \le P(\{u_1 \le u_0+f_1-f_0-g-v\} \cap \{u_1 \le f_1\}) = P(Y = 1, D = 1)$.
- If $g+v+f_0-f_1 \le 0$ and $g+v \le -f_0$, then $\{u_1 \le f_1-f_0-g-v\} \Rightarrow \{u_1 \le u_0+f_1-f_0-g-v\}$. Hence $P(u_1+v \le f_1-f_0-g) \le P(u_1 \le u_0+f_1-f_0-g-v) = P(D = 1)$.

Now, since $v \perp\!\!\!\perp (u_0, u_1)$, we have: $P(u_0 \le g+v+f_0-f_1) = \mathbb{E}_v[\mathbb{E}[1\{u_0 \le g+v+f_0-f_1\}|v]] = \mathbb{E}_v[g+v+f_0-f_1] = g+f_0-f_1$. Then, we get the following:

- If $g+v+f_0-f_1 \ge 0$ and $g+v \le f_1$, then $g+f_0-f_1 \le P(Y = 1, D = 0)$.
- If $g+v+f_0-f_1 \ge 0$ and $g+v \ge f_1$, then $g-f_1 \le P(Y = 0, D = 0)$.
- If $g+v+f_0-f_1 \le 0$ and $g+v \ge -f_0$, then $g+f_0-f_1 \ge -P(Y = 1, D = 1)$.
- If $g+v+f_0-f_1 \le 0$ and $g+v \le -f_0$, then $g+f_0 \ge -P(Y = 0, D = 1)$.

which completes the proof.

A.7. **Proof of Corollary 7.** Our goal here is to show that the following bounds are sharp for f_0 and f_1.

$$\begin{aligned} P(Y = 1, D = 1|\omega) &\le f_1(x_1) \le P(Y = 1, D = 1|\omega) + P(D = 0|\omega) \\ P(Y = 1, D = 0 \mid \omega) &\le f_0(x_0) \le P(Y = 1, D = 0|\omega) + P(D = 1|\omega). \end{aligned}$$

The previous results show that the lower bounds are sharp. Now, to show that these bounds are sharp for $f_1(x_1)$ it is sufficient to construct a joint distribution (u_0^*, u_1^*) such that $f_1(x_1)$ equals $P(Y = 1, D =$

$1|\omega) + P(D = 0|\omega)$ and $f(0, x) = P(Y = 1, D = 0|\omega) + P(D = 1|\omega)$ and which is compatible with the observed data in the following sense:

(1) $P(u_0^* \leq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) = P(Y = 1, D = 0|\omega)$
(2) $P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) = P(Y = 1, D = 1|\omega)$
(3) $P(u_0^* \geq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) = P(Y = 0, D = 0|\omega)$
(4) $P(u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) = P(Y = 0, D = 1|\omega)$

Define the following joint distribution (u_0^*, u_1^*, v^*) such that $u_0^* + u_1^* \leq f_1(x_1) + f_0(x_0)$ and $2v^* = 3u_0^* - 3u_1^* - 3f_0(x_0) + 3f_1(x_1) - 2g(\omega)$. Under the condition that $u_0^* + u_1^* \leq f_1(x_1) + f_0(x_0)$, we have $\{u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v^*\} \Rightarrow \{u_1^* \leq f_1(x_1)\}$ and $\{u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v^*\} \Rightarrow \{u_0^* \leq f_0(x_0)\}$. Hence,

$$\begin{aligned} f_1(x_1) &= P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) \\ &\qquad + P(u_1^* \leq f_1(x_1), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) \\ &= P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) \\ &\qquad + P(u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega) - v|\omega) \\ &= P(Y = 1, D = 1|\omega) + P(D = 0|\omega). \end{aligned}$$

With the same strategy, we can also show $f_0(x_0) = P(Y = 1, D = 0|\omega) + P(D = 1|\omega)$.

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Table 1. Descriptive statistics

	Not Catheterized		Catheterized		p-value
	Mean	sd	Mean	sd	
Age	61.76	17.28	60.74	15.63	0.025
Male	54%	0.5	59%	0.49	0.000
Black	16.5%	0.37	15.3%	0.36	0.255
Years Education	11.57	3.13	11.86	3.16	0.000
Private Insurance	48.1%	0.22	55%	0.24	0.114
Income > 11K	41%	0.49	48%	0.5	0.000
Weight (kg)	72.47	19.67	77.73	19.49	0.000
Coma	10%	0.29	4%	0.2	0.000
CHF	7%	0.25	1%	0.29	0.000
MOSF with malignancy	7%	0.25	7%	0.26	0.518
MOSF with sepsis	15%	0.36	32%	0.47	0.000
Acute Respiratory Failure	45%	0.5	42%	0.49	0.031
Cirrhosis	5%	0.22	2%	0.15	0.000
COPD	11%	0.32	3%	0.16	0.000
N	3551		2184		-

Table 2. Nontraumatic coma

Male	**Black**	**Private Insurance**	**Income > 11k**	**coma**	**Lower bound**	**Upper bound**	**ATE sign**
0	0	0	0	1	0.0113 0.1491	0.3091 0.3091	
0	0	0	1	1	0.0570 0.1558	0.3683 0.3683	
0	0	1	0	1	0.0000 [**0,1650**	0.2588 **0,2437**]	[-]
0	0	1	1	1	0.0409 0.1885	0.3170 0.3170	
0	1	0	0	1	0.0039 0.1340	0.2933 0.2933	
0	1	1	0	1	0.0000 [**0,1475**	0.2462 **0,2358**]	[-]
0	1	1	1	1	0.0274 0.1695	0.3062 0.3062	
1	0	0	0	1	0.0003 0.1087	0.2569 0.2569	
1	0	0	1	1	0.0463 0.1158	0.3157 0.3157	
1	0	1	0	1	0.0000 [**0,1243**	0.2074 **0,2036**]	[-]
1	0	1	1	1	0.0290 0.1484	0.2648 0.2648	
1	1	0	0	1	0.0000 [**0,0757**	0.2419 **0,1991**]	[-]
1	1	0	1	1	0.0377 0.0987	0.3022 0.3022	
1	1	1	0	1	0.0000 [**0,1059**	0.1957 **0,1966**]	[-]
1	1	1	1	1	0.0155 0.1285	0.2550 0.2550	

[] : Means that f_d is point identified.
[−][+] [0] : Means the sign of the ATE is identified.

Table 3. Congestive heart failure

Male	**Black**	**Private Insurance**	**Income > 11k**	**chf**	**Lower bound**	**Upper bound**
0	0	0	0	1	0.1365 0.2060	0.5167 0.5167
0	0	0	1	1	0.1819 0.2134	0.5743 0.5743
0	0	1	0	1	0.1129 0.2380	0.4658 0.4658
0	0	1	1	1	0.1588 0.2467	0.5221 0.5221
0	1	0	0	1	0.1255 0.1887	0.4959 0.4959
0	1	1	0	1	0.0995 0.2187	0.4474 0.4474
0	1	1	1	1	0.1444 0.2258	0.5057 0.5058
1	0	0	0	1	0.1268 0.1686	0.4641 0.4641
1	0	0	1	1	0.1724 0.1764	0.5214 0.5214
1	0	1	0	1	0.1026 0.2009	0.4132 0.4132
1	0	1	1	1	0.1489 0.2103	0.4689 0.4689
1	1	1	0	1	0.0891 0.1805	0.3958 0.3958
1	1	1	1	1	0.1343 0.1881	0.4536 0.4536

[] : Means that f_d is point identified.
[−][+] [0] : Means the sign of the ATE is identified.

Table 4. Multi-organ system failure with malignancy

Male	**Black**	**Private Insurance**	**Income $>$ 11k**	**mosf_malign**	**Lower bound**	**Upper bound**	**ATE sign**
0	0	0	0	1	0.0000 [**0.000**	0.1432 **0.1001**]	[-]
0	0	1	0	1	0.0000 [**0.0378**	0.1040 **0.0977**]	[-]
0	0	1	1	1	0.0274 0.0629	0.1657 0.1657	
0	1	1	0	1	0.0000 [**0.0171**	0.0907 **0.0919**]	[-]
0	1	1	1	1	0.0113 0.0409	0.1546 0.1546	
1	0	0	0	1	0.000 [**0.0517**	0.0907 **0.0579**]	[-]
1	0	0	1	1	[**0.0278** 0.0000	**0.1207**] 0.1533	[+]
1	0	1	0	1	0.0000 [**0.0000**	0.0515 **0.0556**]	[-]
1	0	1	1	1	0.0178 0.0228	0.1125 0.1125	
1	1	0	0	1	0.000 [**0.0690**	0.0738 **0.0487**]	[-]
1	1	1	0	1	[**0.000** [**0.000**	**0.0396**] **0.0511**]	[0]
1	1	1	1	1	0.0015 0.0000	0.1029 0.1029	

[] : Means that f_d is point identified.
[$-$][+] [0] : Means the sign of the ATE is identified.

Table 5. Multi-organ system failure with sepsis

Male	**Black**	**Private Insurance**	**Income > 11k**	**mosf_sept**	**Lower bound**	**Upper bound**
0	0	0	0	1	0.1591 0.1344	0.4377 0.4377
0	0	0	1	1	0.2036 0.1426	0.4968 0.4966
0	0	1	0	1	0.1402 0.1737	0.3831 0.3831
0	0	1	1	1	0.1852 0.1827	0.4413 0.4413
0	1	0	0	1	0.1467 0.1189	0.4182 0.4182
0	1	0	1	1	0.1903 0.1256	0.4791 0.4790
0	1	1	0	1	0.1244 0.1542	0.3687 0.3687
0	1	1	1	1	0.1680 0.1613	0.4293 0.4293
1	0	0	0	1	0.1538 0.0937	0.3887 0.3887
1	0	0	1	1	0.1981 0.1024	0.4472 0.4472
1	0	1	0	1	0.1338 0.1331	0.3341 0.3341
1	0	1	1	1	0.1792 0.1432	0.3914 0.3914
1	1	0	0	1	0.1409 0.0776	0.3699 0.3699
1	1	0	1	1	0.1847 0.0848	0.4303 0.4303
1	1	1	0	1	0.1178 0.1123	0.3212 0.3212
1	1	1	1	1	0.1617 0.1203	0.3810 0.3810

[] : Means that f_d is point identified.
[−][+] [0] : Means the sign of the ATE is identified.

TABLE 6. Acute respiratory failure

Male	**Black**	**Private Insurance**	**Income > 11k**	**arf**	**Lower bound**	**Upper bound**
0	0	0	0	1	0.1175 0.2334	0.4883 0.4883
0	0	0	1	1	0.1632 0.2403	0.5481 0.5481
0	0	1	0	1	0.1059 0.2787	0.4437 0.4437
0	0	1	1	1	0.1515 0.2867	0.5026 0.5026
0	1	0	0	1	0.1081 0.2177	0.4730 0.4730
0	1	0	1	1	0.1527 0.2234	0.5340 0.5340
0	1	1	0	1	0.0916 0.2584	0.4346 0.4346
0	1	1	1	1	0.1358 0.2644	0.4957 0.4957
1	0	0	0	1	0.1100 0.1944	0.4377 0.4377
1	0	0	1	1	0.1558 0.2018	0.4969 0.4969
1	0	1	0	1	0.0986 0.2398	0.3927 0.3927
1	0	1	1	1	0.1445 0.2488	0.4504 0.4504
1	1	0	0	1	0.1002 0.1779	0.4234 0.4234
1	1	0	1	1	0.1452 0.1840	0.4838 0.4838
1	1	1	0	1	0.0835 0.2180	0.3855 0.3855
1	1	1	1	1	0.1280 0.2249	0.4459 0.4459

[] : Means that f_d is point identified.
[−][+] [0] : Means the sign of the ATE is identified.

Table 7. Cirrhosis

Male	**Black**	**Private Insurance**	**Income > 11k**	**cirrho**	**Lower bound**	**Upper bound**	**ATE sign**
0	0	0	0	1	0.0000 0.1778	0.3581 0.3581	
0	0	0	1	1	0.0387 0.1848	0.4158 0.4158	
0	0	1	0	1	0.0000 [**0.1879**	0.3134 **0.3070**]	[-]
0	0	1	1	1	0.0177 0.2132	0.3702 0.3702	
0	1	0	0	1	0.0000 [**0.1423**	0.3359 **0.2952**]	[-]
0	1	1	0	1	0.0000 [**0.1690**	0.2935 **0,2975**]	[-]
0	1	1	1	1	0.0001 0.1930	0.3518 0.3518	
1	0	0	0	1	0.0000 [**0.1211**	0.3101 **0.2666**]	[-]
1	0	0	1	1	0.0328 0.1466	0.3673 0.3673	
1	0	1	0	1	0.0000 [**0.1493**	0.2657 **0,2674**]	[-]
1	0	1	1	1	0.0117 0.1750	0.3219 0.3219	
1	1	0	0	1	0.0000 [**0.1034**	0.2885 **0,2560**]	[-]
1	1	1	0	1	0.0000 [**0.1297**	0.2466 **0.2586**]	[-]
1	1	1	1	1	0.0000 0.1540	0.3044 0.3044	

[] : Means that f_d is point identified.
[−][+] [0] : Means the sign of the ATE is identified.

Table 8. Chronic obstructive pulmonary disease

Male	Black	Private Insurance	Income > 11k	copd	Lower bound	Upper bound	ATE sign
0	0	0	0	1	0.0320 0.3389	0.5337 0.6029	
0	0	0	1	1	0.0754 0.3498	0.5958 0.6532	
0	0	1	0	1	0.0175 0.3681	0.4843 0.5611	
0	0	1	1	1	0.0608 0.3801	0.5454 0.6194	
0	1	0	0	1	0.0191 0.3250	0.5173 0.5801	
0	1	1	0	1	0.0000 0.3520	0.4708 0.5407	
0	1	1	1	1	0.0419 0.3626	0.5337 0.5915	
1	0	0	0	1	0.0242 0.3005	0.4848 0.5767	
1	0	0	1	1	0.0679 0.3119	0.5464 0.6265	
1	0	1	0	1	0.0090 0.3293	0.4356 0.5355	
1	0	1	1	1	0.0527 0.3420	0.4962 0.5842	
1	1	1	0	1	0.0000 [**0.3125**	0.4231 **0.516**]	[-]
1	1	1	1	1	0.0335 0.3237	0.4855 0.5609	

[] : Means that f_d is point identified.
[−][+] [0] : Means the sign of the ATE is identified.