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NONPARAMETRIC SHARP BOUNDS FOR PAYOFFS IN 2 ÃŠ 2 GAMES

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Abstract. We derive the empirical content of Nash equilibrium in 2×2 games of perfect information, including duopoly entry and coordination games. The derived bounds are nonparametric intersection bounds and are simple enough to lend themselves to existing inference methods. Implications of pure strategy Nash equilibrium and of exclusion restrictions are also derived. Without further assumptions, the hypothesis of Nash equilibrium play is not falsifiable. However, nontrivial bounds hold for the extent of potential monopoly advantage or free riding incentives.

Keywords: participation games, partial identification, intersection bounds.

JEL subject classification: C25, C72, D43

Introduction

The empirical analysis of full information game theoretic models has emerged as a leading way to learn about strategic interactions between economic agents and to estimate, for example, the

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extent of monopoly advantage in imperfect competitive environments or free riding incentives in cooperation settings. Beside the numerous applications in industrial organization, as evidenced by the recent survey in Bajari, Hong, and Nekipelov (2012), areas of impact include labor economics, as in Bjorn and Vuong (1984) and Kooreman (1994), social interactions, as in Soetevent and Kooreman (2007), family economics, as in Engers and Stern (2002), or development economics, as in Méango (2012). The empirical approach to models of multiperson simultaneous decisions goes back at least to Bjorn and Vuong (1984) and was popularized in the field of industrial organization by Bresnahan and Reiss (1990, 1991) and Berry (1992) among others. In those cases, attention was restricted to specific parametric utilities or profits and unobserved heterogeneity types. Coherency of the model, in the sense of Heckman (1978) and Gouriéroux, Laffont, and Monfort (1980), was obtained by removing multiplicity of predicted outcomes in the game (Bjorn and Vuong (1984) assume an ad-hoc uniform equilibrium selection device, whereas Bresnahan and Reiss (1991) coarsen the outcome space). The multiplicity issue was addressed head-on by Jovanovic (1989) and Tamer (2003) and both Galichon and Henry (2006, 2011) and Beresteanu, Molchanov, and Molinari (2008, 2011) propose characterizations of the empirical content of Nash equilibrium play in models with simultaneous decisions by multiple agents, while retaining the parametric framework for payoffs and unobserved types. Much of the empirical content in the latter characterizations, however, rests on the specific parametric assumptions maintained, some of which may be structurally motivated, but others, especially parametric assumptions on unobserved type distributions, are entirely ad-hoc. Kline and Tamer (2012) seem to be the first to remove parametric assumptions and consider sharp bounds in full information games, but their focus, however, is best response functions, which may be of interest in their own right, but which are not the focus of the literature, generally interested in recovering payoff functions (utilities and profits). Aradillas-Lopez (2011) considers nonparametric bounds on predicted probabilities of strategy profiles under asymmetric information. Neither considers nonparametric sharp bounds on payoff functions in full information games as we do here.

Within the class of two person games with binary strategies in full information, we consider the identification problem, where the distribution of realized decisions is known by the analyst, who assumes that such realizations emerge from Nash equilibrium play (in pure or mixed strategies) in the game. Hence we adopt a pure revealed preference approach to the model of interaction and analyze the empirical content of maximizing behavior as in Henry and Mourifié (2012), with the additional complication that the dummy endogenous variable is the result of a simultaneous decision by a second agent. Based on the characterization of the empirical content of games with Shapley regular core in Galichon and Henry (2011), we derive sharp nonparametric bounds on payoffs and unobserved heterogeneity distributions. Additional constraints on the order of payoffs, to consider games of complements or games of susbstitutes, and on type distributions, to evaluate shape and other distributional restrictions, can be easily added to see how they shrink the identified region. One of the main arguments for allowing agents to randomize in the empirical analysis of games, as in Bajari, Hong, and Ryan (2010), Beresteanu, Molchanov, and Molinari (2008, 2011), Bajari, Hahn, Hong, and Ridder (2011) and Galichon and Henry (2011), is almost sure existence of equilibrium in mixed strategies, whereas existence of equilibrium in pure strategies only is not garanteed. This argument is only relevant in case of parametric assumptions on the unobserved heterogeneity (or type) distribution, but fails to sway in the framework entertained here, as regions of the type space may well have zero probability. We therefore analyze the implications of restricting play to pure strategies and derive sharp bounds on payoffs and type distributions in that case too. Considering type distributions as nuisance infinite dimensional parameters and projecting the identified region allows us then to derive sharp nonparametric bounds on the payoff functions themselves. We find that the hypothesis of Nash equilibrium play is not falsifiable in this framework, as the identified region is never empty. Rejection of the model becomes possible under the assumption of an exclusion restriction, namely variation in the payoff of a player that leaves the opponent's profit unchanged. In the latter case, the bounds become intersection bounds, as in Chernozhukov, Lee, and Rosen (2009) and they can cross. We also find that, without additional prior information, we cannot identify, whether the game is of complements or substitutes. However, we obtain non trivial sharp bounds on monopoly advantage and free-riding incentives, when they arise.

The importance of deriving the empirical content of discrete game specification without ad hoc distributional assumptions is also crucial to answer fundamental questions such as the empirical content of equilibrium when mixed strategies are allowed, the testability of independant randomizability, of complete information and simultaneity. They cannot be addressed in the usual framework with parametrically specified unobservables. We do find for instance that we cannot test whether players randomize independently or use correlated strategies.

The remainder of the paper is organized as follows. Section 1 derives the analytical framework, the games analyzed, their equilibrium correspondences and the objects of interest. Section 2 derives joint sharp bounds for payoff functions and type distributions, treating the equilibrium selection mechanism as a nuisance parameter. Section 3 considers implications of pure strategy play and derives the projection of the identified set to obtain sharp bounds for the payoff functions. Sharp bounds are also given for monopoly advantage and free riding incentives. The last section concludes.

1. Analytical framework

We shall be concerned with the following econometric model.

$$Y_i = 1\{\Pi_i(Y_{3-i}, X_i) > \varepsilon_i\} \text{ and } \varepsilon_i \sim U[0,1],\ i = 1,2, \tag{1.1}$$

where $1\{A\} = 1$ if A is true and zero otherwise, $Y = (Y_1, Y_2)$ is a pair of observed binary outcome variables, $\Pi = (\Pi_1, \Pi_2)$ are unknown functions of Y_{3-i}, observable random vectors $X = (X_1, X_2)$ and unobservable random variables $\varepsilon = (\varepsilon_1, \varepsilon_2)$. We assume that the only source of endogeneity in the econometric model is the interaction between players and the simultaneous choice. Hence, we assume that observable heterogeneity variables are exogenous.

Assumption 1 (Exogeneity)**.** *The following exogeneity assumption holds:* $(X_1, X_2) \perp\!\!\!\perp (\varepsilon_1, \varepsilon_2)$ *and for ease of notation, we shall drop all components that are common to* X_1 *and* X_2 *and relabel* X_i *as the vector of observable heterogeneity variables (if they arise) that affect* Π_i *but are excluded from* Π_{3-i}.

We give two structural interpretation of this model within the range of noncooperative games of perfect information with 2 players and 2 strategies each.

1.1. **General** 2×2 **games.** In a first structural interpretation of Model (1.1) we consider general 2×2 games of perfect information with payoff structure given in Table 1, which is common knowledge to the two players. Working under assumptions that rule out ties, the best response of Player 1 to $Y_2 = 1$ is $Y_1 = 1$ if $\tilde\Pi_1(1,1,X_1) - \tilde\Pi_1(0,1,X_1) > [\tilde\varepsilon_1(0,1) - \tilde\varepsilon_1(1,1)]$ and zero otherwise, whereas the best response to $Y_2 = 0$ is $Y_1 = 1$ if $\tilde\Pi_1(1,0,X_1) - \tilde\Pi_1(0,0,X_1) > [\tilde\varepsilon_1(0,0) - \tilde\varepsilon_1(1,0)]$ and zero otherwise. Best responses for Player 2 are obtained symmetrically. Assuming that the unobserved heterogeneity differences $\tilde\varepsilon_1(1,Y_2) - \tilde\varepsilon_1(0,Y_2)$ and $\tilde\varepsilon_2(Y_1,1) - \tilde\varepsilon_2(Y_1,0)$ are independent of the opponent's action and are absolutely continuous with respect to Lebesgue measure and setting $\Pi_1(Y_2,X_1) = \tilde\Pi_1(1,Y_2,X_1) - \tilde\Pi_1(0,Y_2,X_1)$, $\Pi_2(Y_1,X_2) = \tilde\Pi_2(Y_1,1,X_2) - \tilde\Pi_2(Y_1,0,X_2)$, $\varepsilon_1 = -\tilde\varepsilon_1(1,Y_2) + \varepsilon_1(0,Y_2)$ and $\varepsilon_2 = -\tilde\varepsilon_2(Y_1,1) + \varepsilon_2(Y_1,0)$, we obtain Model (1.1), where $\varepsilon_i \sim U[0,1]$ is without loss of generality.

1.2. **Participation games.** In a second structural interpretation of Model (1.1), we consider the special case of 2×2 participation games, where a player's payoff when she chooses not to participate

TABLE 1. Payoff structure of 2×2 games.

	1	0
1	$\tilde{\Pi}_1(1,1,X_1) + \tilde{\varepsilon}_1(1,1), \tilde{\Pi}_2(1,1,X_2) + \tilde{\varepsilon}_2(1,1)$	$\tilde{\Pi}_1(1,0,X_1) + \tilde{\varepsilon}_1(1,0), \tilde{\Pi}_2(1,0,X_2) + \tilde{\varepsilon}_2(1,0)$
0	$\tilde{\Pi}_1(0,1,X_1) + \tilde{\varepsilon}_1(0,1), \tilde{\Pi}_2(0,1,X_2) + \tilde{\varepsilon}_2(0,1)$	$\tilde{\Pi}_1(0,0,X_1) + \tilde{\varepsilon}_1(0,0), \tilde{\Pi}_2(0,0,X_2) + \tilde{\varepsilon}_2(0,0)$

is independent of the opponent's behavior and can therefore be normalized to zero. Each player has 2 strategies and 3 different payoffs. For each player, the 3 different payoffs can be ranked in 3! distinct ways. Hence there are 36 classes of ordinally equivalent such 2×2 participation games (but only 7 strategically distinct classes of games as we shall see). The payoff structure as in Table 2, which is common knowledge to the two players.

Assuming that the profit functions are weakly separable in ε_i, $i = 1, 2$, and the latter are absolutely continuous with respect to Lebesgue measure, the game can be summarized by Model (1.1) without loss of generality (see Vytlacil (2002)).

1.3. **Implications of each structural interpretation.** Depending on the chosen structural interpretation, the analyst will be able to answer different empirical questions. Two questions of particular relevance in 2×2 game theoretic modeling of economic interactions are the price of competition and the extent of free riding incentives. 2×2 games are applied to the empirical analysis of imperfect competition since at least Bresnahan and Reiss (1990) and Berry (1992). Two questions of particular interest arise: whether the two players (firms) are complements or substitutes and the extent of the monopoly advantage if they are substitutes. Both questions can be answered (partially)

TABLE 2. Payoff structure of 2×2 participation games.

	1	0
1	$\Pi_1(1, X_1, \varepsilon_1), \Pi_2(1, X_2, \varepsilon_2)$	$\Pi_1(0, X_1, \varepsilon_1), 0$
0	$0, \Pi_2(0, X_2, \varepsilon_2)$	$0, 0$

if the quantities

$$\tilde{\Pi}_1(1,0,X_1) + \tilde{\varepsilon}_1(1,0) - [\tilde{\Pi}_1(1,1,X_1) + \tilde{\varepsilon}_1(1,1)]$$
$$\tilde{\Pi}_2(0,1,X_2) + \tilde{\varepsilon}_2(0,1) - [\tilde{\Pi}_2(1,1,X_2) + \tilde{\varepsilon}_2(1,1)]$$

are (partially) identified. Now, with the structural interpretation of participation games in Section 1.2, we have

$$\begin{aligned}
\tilde{\Pi}_1(1,0,X_1) + \tilde{\varepsilon}_1(1,0) - [\tilde{\Pi}_1(1,1,X_1) + \tilde{\varepsilon}_1(1,1)] &= \Pi_1(0,X_1) - \Pi_1(1,X_1)\\
\tilde{\Pi}_2(0,1,X_2) + \tilde{\varepsilon}_2(0,1) - [\tilde{\Pi}_2(1,1,X_2) + \tilde{\varepsilon}_2(1,1)] &= \Pi_2(0,X_2) - \Pi_2(1,X_2)
\end{aligned}$$

and we shall derive sharp bounds on $\Pi = (\Pi_1(1,X_1), \Pi_1(0,X_1), \Pi_2(1,X_2), \Pi_2(0,X_2)$. 2×2 games are also used to model the provision of public goods. In that context, the extent of free riding incentives is of particular empirical relevance and it is measured by the following quantities.

$$\tilde{\Pi}_1(0,1,X_1) + \tilde{\varepsilon}_1(0,1) - [\tilde{\Pi}_1(1,1,X_1) + \tilde{\varepsilon}_1(1,1)]$$
$$\tilde{\Pi}_2(1,0,X_2) + \tilde{\varepsilon}_2(1,0) - [\tilde{\Pi}_2(1,1,X_2) + \tilde{\varepsilon}_2(1,1)].$$

Under both the structural interpretations of Sections 1.1 and 1.2, we have the following.

$$\begin{aligned}
\tilde{\Pi}_1(0,1,X_1) + \tilde{\varepsilon}_1(0,1) - [\tilde{\Pi}_1(1,1,X_1) + \tilde{\varepsilon}_1(1,1)] &= \varepsilon_1 - \Pi_1(1)\\
\tilde{\Pi}_2(1,0,X_2) + \tilde{\varepsilon}_2(1,0) - [\tilde{\Pi}_2(1,1,X_2) + \tilde{\varepsilon}_2(1,1)] &= \varepsilon_2 - \Pi_2(1)
\end{aligned}$$

and we shall derive sharp bounds on $\Pi_1(1)$ and $\Pi_2(1)$.

1.4. **Equilibrium.** We assume, as is customary, that players choose the strategy that maximizes their payoff in pure or mixed strategy Nash equilibrium (see Aradillas-Lopez and Tamer (2008) for some discussion of the empirical content of other notions of rationality in games). We distinguish four cases, according to the ordering between $\Pi_i(1,X_i)$ and $\Pi_i(0,X_i)$.

(1) Duopoly entry game: $\Pi_i(1,X_i) \leq \Pi_i(0,X_i)$, $i = 1,2$.
(2) Coordination game: $\Pi_i(0,X_i) \leq \Pi_i(1,X_i)$, $i = 1,2$.
(3) Asymmetric game 1: $\Pi_1(0,X_1) \leq \Pi_1(1,X_1)$ and $\Pi_2(1,X_2) \leq \Pi_2(0,X_2)$.
(4) Asymmetric game 2: $\Pi_1(1,X_1) \leq \Pi_1(0,X_1)$ and $\Pi_2(0,X_2) \leq \Pi_2(1,X_2)$.

In each case, the equilibrium correspondence is represented on the unit square as a function of the pair $(\varepsilon_1, \varepsilon_2)$ in Figures 1-3 and Case (4) can be obtained from Case (3) by permuting the two players.

Definition 1 (Equilibrium correspondence)**.** *The equilibrium correspondence, denoted $G(\varepsilon, X, \Pi)$, is the set of equilibria of the game for a given values of (ε, X, Π). It is a subset of the simplex on $\{(1,1),(1,0),(0,1),(0,0)\}$ and its elements are non degenerate probabilities in case the equilibrium is in mixed strategies and degenerate probabilities in case the equilibrium is in pure strategies.*

The equilibrium has similar features in the duopoly, coordination and asymmetric games. When

$$\varepsilon \notin [\min(\Pi_1(1,X_1),\Pi_1(0,X_1)),\max(\Pi_1(1,X_1),\Pi_1(0,X_1))]$$
$$\times[\min(\Pi_2(1,X_2),\Pi_2(0,X_2)),\max(\Pi_2(1,X_2),\Pi_2(0,X_2))],$$

there is a unique equilibrium in pure strategies. For instance, when $\varepsilon_i > \max(\Pi_i(1,X_i),\Pi_i(0,X_i))$, $i = 1,2$, the game is a Prisoner's Dilemma. When, on the other hand,

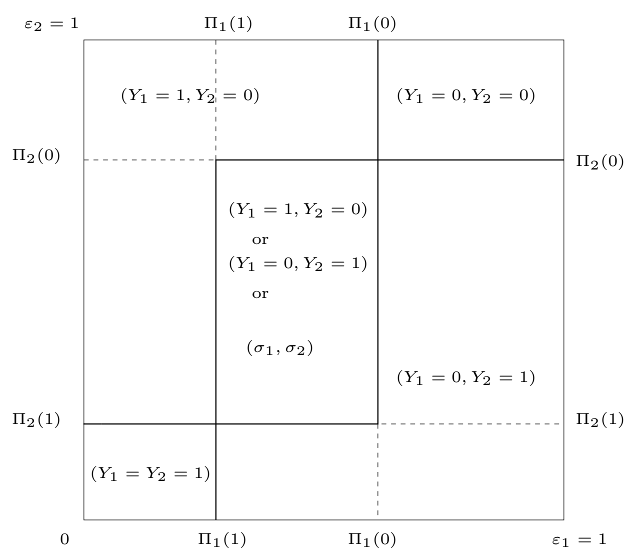
$$\varepsilon \in [\min(\Pi_1(1,X_1),\Pi_1(0,X_1)),\max(\Pi_1(1,X_1),\Pi_1(0,X_1))]$$
$$\times[\min(\Pi_2(1,X_2),\Pi_2(0,X_2)),\max(\Pi_2(1,X_2),\Pi_2(0,X_2))],$$

there is always one equilibrium in mixed strategies. There is also two equilibria in pure strategies in the case of duopoly entry and coordination. For instance, when $\Pi_i(1) < \varepsilon_i < \Pi_i(0)$, $i = 1,2$, we have a game of Chicken (or public good provision) and when $\Pi_i(0) < \varepsilon_i < \Pi_i(1)$, $i = 1,2$, we have a Battle of the Sexes.

1.5. **Object of inference.** The analyst observes the realized strategy profile and realized values of the heterogeneity variables X_1 and X_2. However, realized values of heterogeneity variables ε_1 and ε_2 are not observed and the payoff functions Π_1 and Π_2 are unknown and are the object of inference. The model is incomplete in two respects:

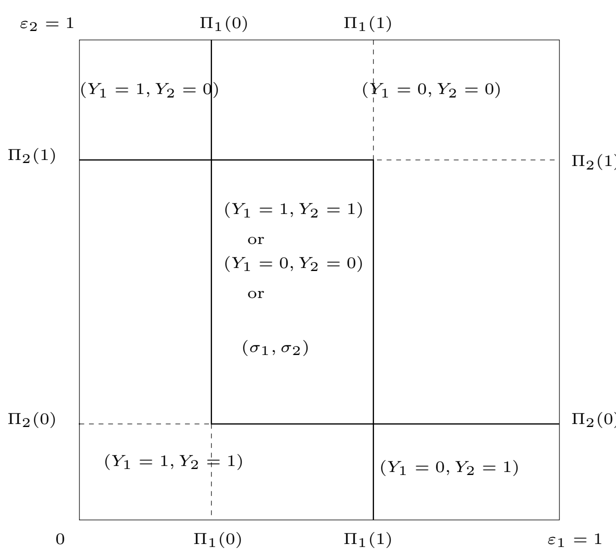
(1) The marginal distributions of the unobserved heterogeneity variables ε_1 and ε_2 are normalized. However, the joint distribution of $(\varepsilon_1, \varepsilon_2)$, which we shall denote $C(\varepsilon_1, \varepsilon_2)$ (since

FIGURE 1. Equilibrium correspondence in the duopoly entry case. For each value of the pair $(\varepsilon_1, \varepsilon_2)$, the predicted equilibria are given. In the central rectangle, corresponding to values of unobserved heterogeneity such that $\Pi_i(1) \leq \varepsilon_i \leq \Pi_i(0)$, for $i = 1, 2$, three equilibria are predicted, including two in pure strategies, $(Y_1 = 1, Y_2 = 0)$ and $(Y_1 = 0, Y_2 = 1)$ and one in mixed strategies, with Player i participating with probability $\sigma_i(\varepsilon_{3-i}) = (\Pi_{3-i}(0) - \Pi_{3-i}(1))^{-1}(\Pi_{3-i}(0) - \varepsilon_{3-i})$. In the rest of the $(\varepsilon_1, \varepsilon_2)$ space, single pure strategy Nash equilibria are predicted for each value of the unobserved heterogeneity pair $(\varepsilon_1, \varepsilon_2)$.



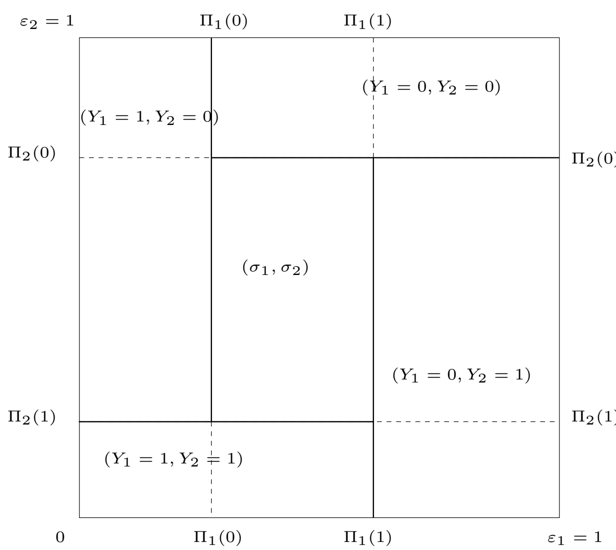
it is equal to the copula, given the uniform normalization) is unknown. This implies that although the probability of any horizontal or any vertical band in Figures 1-3 is predicted by the model, the probability of other rectangles are not. This means in particular that the likelihood of observing, say, $(Y_1 = 1, Y_2 = 1)$ in the duopoly entry case of Figure 1 is not pinned down by the model.

FIGURE 2. Equilibrium correspondence in the coordination case. For each value of the pair $(\varepsilon_1, \varepsilon_2)$, the predicted equilibria are given. In the central rectangle, corresponding to values of unobserved heterogeneity such that $\Pi_i(0) \leq \varepsilon_i \leq \Pi_i(1)$, for $i = 1, 2$, three equilibria are predicted, including two in pure strategies, $(Y_1 = 1, Y_2 = 1)$ and $(Y_1 = 0, Y_2 = 0)$ and one in mixed strategies, with Player i participating with probability $\sigma_i(\varepsilon_{3-i}) = (\Pi_{3-i}(1) - \Pi_{3-i}(0))^{-1}(\varepsilon_{3-i} - \Pi_{3-i}(0))$. In the rest of the $(\varepsilon_1, \varepsilon_2)$ space, single pure strategy Nash equilibria are predicted for each value of the unobserved heterogeneity pair $(\varepsilon_1, \varepsilon_2)$.



(2) In each of the three Figures 1-3, multiple equilibria arise in the central region of the $(\varepsilon_1, \varepsilon_2)$ space. This implies that, short of additional information about the equilibrium selection mechanism, the model delivers multiple predictions for the strategy profile, only one of which is actually realized.

FIGURE 3. Equilibrium correspondence in the asymmetric case. For each value of the pair $(\varepsilon_1, \varepsilon_2)$, the predicted equilibria are given. In the central rectangle, corresponding to values of unobserved heterogeneity such that $\Pi_1(0) \leq \varepsilon_1 \leq \Pi_1(1)$ and $\Pi_2(1) \leq \varepsilon_2 \leq \Pi_2(0)$, a single equilibrium in mixed strategies is predicted, with Player i participating with probability $\sigma_i(\varepsilon_{3-i}) = (\Pi_{3-i}(0) - \Pi_{3-i}(1))^{-1}(\Pi_{3-i}(0) - \varepsilon_{3-i})$. In the rest of the $(\varepsilon_1, \varepsilon_2)$ space, single pure strategy Nash equilibria are predicted for each value of the unobserved heterogeneity pair $(\varepsilon_1, \varepsilon_2)$.



Model incompleteness results here, as we shall see, in partial identification of the payoff functions, the joint distribution of unobserved heterogeneity and the equilibrium selection mechanism. Throughout the paper, we shall treat the equilibrium selection mechanism as a nuisance parameter and concentrate on the derivation of the empirical content of the model, when no additional assumption is maintained about equilibrium selection. We shall proceed in two steps.

(1) First we define and characterize the identified set for the distribution of unobserved heterogeneity and for the payoff functions (jointly). This will be achieved in Section 2 with an application of the characterization of the identified set for Shapley regular games in Galichon and Henry (2011).

(2) Second we treat both the equilibrium selection mechanism and the distribution of unobserved heterogeneity as nuisance parameters and we derive in Section 3 the identified set for the payoff functions as the projection of the joint identified set obtained in Point (1).

2. Identified set for payoffs and heterogeneity distribution

In order to define and characterize the empirical content of Nash equilibrium play in 2×2 games of perfect information, we first clarify the observability structure and the structural elements to be identified.

Definition 2 (True frequencies)**.** *The probabilities of each of the four strategy profiles* $(Y_1 = j_1, Y_2 = j_2)$*, for* $j_1, j_2 = 1, 0$*, (as would be obtained from an infinite sample of i.i.d. replications of the game) are called* true frequencies *and denoted* $P(Y_1 = j_1, Y_2 = j_2 | X_1, X_2)$ *or* $P((j_1, j_2)|X_1, X_2)$ *for* $j_1, j_2 = 1, 0$*. We shall assume throughout this (partial) identification analysis that the true frequencies are known.*

Knowing the true frequencies of strategy profiles, we seek to characterize all the informational content of Nash equilibrium play with a finite collection of inequalities involving payoff functions $\Pi_i(j, X_i)$, $i = 1, 2$ and $j = 1, 0$ and the joint distribution of unobserved heterogeneity denoted:

$$C(u_1, u_2) = P(\varepsilon_1 \leq u_1, \varepsilon_2 \leq u_2), \; \forall (u_1, u_2) \in [0, 1]^2. \tag{2.1}$$

The notation $C(u_1, u_2)$ is chosen in reference to the fact that, given the uniform normalization of the marginals, C is also the copula of the pair $(\varepsilon_1, \varepsilon_2)$.

The inequalities characterizing the empirical content of the model will be sharp in the sense that all (C, Π_1, Π_2) that satisfy them are compatible with Nash equilibrium play in the 2×2 perfect information game specification. We define the identified set as in Beresteanu, Molchanov, and Molinari (2011).

Definition 3 (Identified set)**.** *The identified set for payoff functions and unobserved heterogeneity distribution is the collection of values of* (C, Π_1, Π_2) *such that there exists a probability* $\sigma \mapsto \mu(\sigma|\varepsilon, X, \Pi)$ *(an equilibrium selection mechanism) on the equilibrium correspondence* $G(\varepsilon, X, \Pi)$ *satisfying for each strategy profile* $(Y_1 = j_1, Y_2 = j_2)$, $j_1, j_2 = 1, 0$,

$$P((j_1, j_2)|X) = \int_{[0,1]^2} \left\{ \int_{G(\varepsilon,X,\Pi)} \sigma((j_1, j_2), \varepsilon, \Pi)\mu(\sigma|\varepsilon, X, \Pi) \right\} dC(\varepsilon_1, \varepsilon_2),$$

where $P((j_1, j_2)|X)$ *is the true frequency of* $(Y_1 = j_1, Y_2 = j_2)$.

Definition 3 is a rephrasing of the fact that there is a way to complete the model so that predicted probabilities are equal to true frequencies. Applying Theorem 5 of Galichon and Henry (2011) for Shapley regular games and removing redundant inequalities yields the characterization of the identified set given in Theorem 1 (see Appendix A for the proof). First, we need some additional notation relative to the probabilities of each strategy profile under mixed strategies.

Lemma 1 (Profile probabilities under mixed strategies)**.** *The probability that Player* i *participates in case the mixed strategy equilibrium is selected is* $\sigma_i(\varepsilon_{3-i}, \Pi_{3-i}) = (\Pi_{3-i}(0) - \Pi_{3-i}(1))^{-1}(\Pi_{3-i}(0) - \varepsilon_{3-i})$ *and the predicted probability of strategy profile* (j_1, j_2) *is* $\Sigma_{j_1,j_2}(C, \Pi)$ *with:*

$$\begin{aligned}
\Sigma_{11}(C,\Pi) &= \left| \int_{\Pi_1(1)}^{\Pi_1(0)} \int_{\Pi_2(1)}^{\Pi_2(0)} \sigma_1(\varepsilon_2, \Pi_2)\sigma_2(\varepsilon_1, \Pi_1) dC(\varepsilon_1, \varepsilon_2) \right|, \\
\Sigma_{00}(C,\Pi) &= \left| \int_{\Pi_1(1)}^{\Pi_1(0)} \int_{\Pi_2(1)}^{\Pi_2(0)} (1 - \sigma_1(\varepsilon_2, \Pi_2))(1 - \sigma_2(\varepsilon_1, \Pi_1)) dC(\varepsilon_1, \varepsilon_2) \right|, \\
\Sigma_{10}(C,\Pi) &= \left| \int_{\Pi_1(1)}^{\Pi_1(0)} \int_{\Pi_2(1)}^{\Pi_2(0)} \sigma_1(\varepsilon_2, \Pi_2)(1 - \sigma_2(\varepsilon_1, \Pi_1)) dC(\varepsilon_1, \varepsilon_2) \right|, \\
\Sigma_{01}(C,\Pi) &= \left| \int_{\Pi_1(1)}^{\Pi_1(0)} \int_{\Pi_2(1)}^{\Pi_2(0)} (1 - \sigma_1(\varepsilon_2, \Pi_2))\sigma_2(\varepsilon_1, \Pi_1) dC(\varepsilon_1, \varepsilon_2) \right|.
\end{aligned} \tag{2.2}$$

With this notation, we can state the characterization of the identified set.

Theorem 1 (Identified set)**.** (C, Π) *belongs to the identified set if and only if one of the following holds for almost all values of* X*. For ease of exposition, we denote* $P(i,j) = P(Y_1 = i, Y_2 = j | X_1, X_2)$ *and* $\Pi_i(j, X_i) = \Pi_i(j)$*,* $i = 1, 2$*, and* $j = 1, 0$*.*

(1) *(Duopoly entry)* $\Pi_i(1) \leq \Pi_i(0)$*,* $i = 1, 2$*, and*

$$\begin{aligned}
&C(\Pi_1(1), \Pi_2(1)) \leq P(1,1) \leq C(\Pi_1(1), \Pi_2(1)) + \Sigma_{11}(C, \Pi)\\
&1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0), \Pi_2(0)) \leq P(0,0) \leq 1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0), \Pi_2(0)) + \Sigma_{00}(C, \Pi),\\
&\Pi_2(0) + [C(\Pi_1(0), \Pi_2(1)) - C(\Pi_1(1), \Pi_2(1))] - C(\Pi_1(0), \Pi_2(0)) \leq P(0,1) \leq \Pi_2(0) - C(\Pi_1(1), \Pi_2(0)),\\
&\Pi_1(0) + [C(\Pi_1(1), \Pi_2(0)) - C(\Pi_1(1), \Pi_2(1))] - C(\Pi_1(0), \Pi_2(0)) \leq P(1,0) \leq \Pi_1(0) - C(\Pi_1(0), \Pi_2(1)).
\end{aligned} \tag{2.3}$$

(2) *(Coordination game)* $\Pi_i(1) \geq \Pi_i(0)$*,* $i = 1, 2$*, and*

$$\begin{aligned}
&\Pi_2(0) - C(\Pi_1(1), \Pi_2(0)) \leq P(0,1) \leq \Pi_2(0) - C(\Pi_1(1), \Pi_2(0)) + \Sigma_{01}(C, \Pi),\\
&\Pi_1(0) - C(\Pi_1(0), \Pi_2(1)) \leq P(1,0) \leq \Pi_1(0) - C(\Pi_1(0), \Pi_2(1)) + \Sigma_{10}(C, \Pi),\\
&C(\Pi_1(0), \Pi_2(1)) + [C(\Pi_1(1), \Pi_2(0)) - C(\Pi_1(0), \Pi_2(0))] \leq P(1,1) \leq C(\Pi_1(1), \Pi_2(1)),\\
&1 - \Pi_1(0) - \Pi_2(0) + [C(\Pi_1(0), \Pi_2(1)) - C(\Pi_1(1), \Pi_2(1))] + C(\Pi_1(1), \Pi_2(0))\\
&\qquad\qquad \leq P(0,0) \leq 1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0), \Pi_2(0)).
\end{aligned} \tag{2.4}$$

(3) *(Asymmetric case 1)* $\Pi_1(1) \geq \Pi_1(0)$*,* $\Pi_2(1) \leq \Pi_2(0)$ *and*

$$\begin{aligned}
&P(1,1) = C(\Pi_1(1), \Pi_2(1)) + \Sigma_{11}(C, \Pi)\\
&P(0,0) = 1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0), \Pi_2(0)) + \Sigma_{00}(C, \Pi)\\
&P(0,1) = \Pi_2(0) - C(\Pi_1(1), \Pi_2(0)) + \Sigma_{01}(C, \Pi),\\
&P(1,0) = \Pi_1(0) - C(\Pi_1(0), \Pi_2(1)) + \Sigma_{10}(C, \Pi).
\end{aligned} \tag{2.5}$$

(4) *(Asymmetric case 2) The constraints of Case (3) hold after permutation of the two players.*

Consider the duopoly entry case. All other cases are derived in the same way. The equilibrium correspondence of the game is represented in Figure 1. The observation of strategy profile $(Y_1 = 1, Y_2 = 1)$ is rationalizable as the result of a pure strategy equilibrium in region $\varepsilon \in [0, \Pi_1(1)] \times [0, \Pi_2(1)]$ with probability $C(\Pi_1(1), \Pi_2(1))$ or as the result of a mixed strategy equilibrium in region

$\varepsilon \in [\Pi_1(1), \Pi_1(0)] \times [\Pi_2(1), \Pi_2(0)]$ with probability Σ_{11} if the equilibrium in mixed strategies is selected. Hence the true frequency $P(1,1)$ is at least equal to $C(\Pi_1(1), \Pi_2(1))$ if the equilibrium in mixed strategies is never selected and at most equal to $C(\Pi_1(1), \Pi_2(1)) + \Sigma_{11}$ if the equilibrium in mixed strategies is always selected. Hence we recover the bounds on the first line of (2.3). The same reasoning applies to strategy profile $(Y_1 = 0, Y_2 = 0)$ to yield the second line of (2.3).

The observation of strategy profile $(Y_1 = 0, Y_2 = 1)$ can be rationalized as the result of a pure strategy equilibrium in the lower right L-shaped region or as the result of a pure strategy equilibrium or a mixed strategy equilibrium in region $\varepsilon \in [\Pi_1(1), \Pi_1(0)] \times [\Pi_2(1), \Pi_2(0)]$. The maximum rationalizable true frequency $P(0,1)$ is therefore obtained when the pure strategy equilibrium $(Y_1 = 0, Y_2 = 1)$ is always selected in region $\varepsilon \in [\Pi_1(1), \Pi_1(0)] \times [\Pi_2(1), \Pi_2(0)]$. The resulting upper bound is equal to $P(\varepsilon_1 \geq \Pi_1(1), \varepsilon_2 \leq \Pi_2(0))$, which is equal to the right-hand side on the third line of (2.3). The minimum rationalizable true frequency $P(0,1)$ is obtained when the pure strategy equilibrium $(Y_1 = 1, Y_2 = 0)$ is always selected so that $(Y_1 = 0, Y_2 = 1)$ never occurs in region $\varepsilon \in [\Pi_1(1), \Pi_1(0)] \times [\Pi_2(1), \Pi_2(0)]$. The resulting lower bound is the probability of the lower left L-shaped region, whose probability is equal to the left-hand side of Line 3 of (2.3). The same reasoning applies to true frequency $P(1,0)$ and Line 4 of (2.3).

Note that additional constraints can be derived from the analysis of the game. In particular, the maximum rationalizable frequency $P(0,1)$ is obtained when the pure strategy equilibrium $(Y_1 = 0, Y_2 = 1)$ is always selected in the region with multiple equilibria. This implies of course that the other equilibria are never selected, which constrains the rationalizable frequency $P(Y_1 = 1, Y_2 = 0)$. Hence $P(0,1) + P(1,0)$ is bounded above by $1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(1), \Pi_2(1)) - C(\Pi_1(0), \Pi_2(0))$. However, the latter constraint on (C, Π) is redundant, as it is implied by the combination of $P(1,1) \geq C(\Pi_1(1), \Pi_2(1))$ and $P(0,0) \geq 1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0), \Pi_2(0))$.

This shows that true frequencies that are rationalizable as Nash equilibrium strategy profiles of the 2×2 game necessarily satisfy inequalities in (2.3-2.5). The proof of Theorem 1 in Appendix A shows the converse, namely that true frequencies that satisfy inequalities (2.3-2.5) are rationalizable as Nash equilibrium strategy profiles of the 2×2 game. Hence, the bounds of Theorem 1 are sharp.

3. Empirical content of equilibrium in pure strategies

When equilibria in mixed strategies are ruled out, $\Sigma_{j_1 j_2} = 0$ for $j_1, j_2 = 1, 0$ and the lower bounds in each of the inequalities in (2.3)-(2.5) are redundant. Hence we have the following result.

Theorem 2 (Identified set for (C, Π) with only pure strategies)**.** (C, Π) *belongs to the identified set if and only if one of the following holds for almost all values of* X*. For ease of exposition, we denote* $P(i,j) = P(Y_1 = i, Y_2 = j | X_1, X_2)$ *and* $\Pi_i(j, X_i) = \Pi_i(j)$*,* $i = 1, 2,$ *and* $j = 1, 0$*.*

(1) *(Duopoly entry)* $\Pi_i(1) \leq \Pi_i(0)$*,* $i = 1, 2,$ *and*

$$\begin{aligned} P(1,1) &= C(\Pi_1(1), \Pi_2(1)) \\ P(0,0) &= 1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0), \Pi_2(0)), \\ P(0,1) &\leq \Pi_2(0) - C(\Pi_1(1), \Pi_2(0)), \\ P(1,0) &\leq \Pi_1(0) - C(\Pi_1(0), \Pi_2(1)). \end{aligned} \tag{3.1}$$

(2) *(Coordination game)* $\Pi_i(1) \geq \Pi_i(0)$*,* $i = 1, 2,$ *and*

$$\begin{aligned} P(0,1) &= \Pi_2(0) - C(\Pi_1(1), \Pi_2(0)), \\ P(1,0) &= \Pi_1(0) - C(\Pi_1(0), \Pi_2(1)), \\ P(1,1) &\leq C(\Pi_1(1), \Pi_2(1)), \\ P(0,0) &\leq 1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0), \Pi_2(0)). \end{aligned} \tag{3.2}$$

(3) *(Asymmetric case 1)* $\Pi_1(1) \geq \Pi_1(0)$, $\Pi_2(1) \leq \Pi_2(0)$ *and*

$$\begin{aligned} P(1,1) &= C(\Pi_1(1), \Pi_2(1)) \\ P(0,0) &= 1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0), \Pi_2(0)) \\ P(0,1) &= \Pi_2(0) - C(\Pi_1(1), \Pi_2(0)), \\ P(1,0) &= \Pi_1(0) - C(\Pi_1(0), \Pi_2(1)). \end{aligned} \tag{3.3}$$

(4) *(Asymmetric case 2) The constraints of Case (3) hold after permutation of the two players.*

The results of Theorem 2 can be applied in several ways. We describe two polar cases. On the one hand, we may add assumptions on the joint distribution of firm specific unobserved heterogeneity $(\varepsilon_1, \varepsilon_2)$, positing (1) a parametric copula, (2) perfect correlation of $(\varepsilon_1, \varepsilon_2)$, as in the case of an industry-wide shock or (3) independence of ε_1 and ε_2 as in the case of purely idiosyncratic shocks. A combination of the latter two cases can also be entertained in the form of (4) a factor model. These implications are detailed in Section 3.1. On the other hand, we may acknowledge total ignorance of the joint distribution of firm specific unobserved heterogeneity and project the identified region of Theorem 3 to obtain nonparametric sharp bounds on the payoff functions only. We describe this in Section 3.2.

3.1. **Restrictions on the joint distribution of firm specific heterogeneity.** We consider first refinements of the bounds of Theorem 2 based on a variety of assumptions on the joint distribution of firm specific unobserved heterogeneity.

3.1.1. *Parametric restrictions on the copula.* In the case where the copula for $(\varepsilon_1, \varepsilon_2)$ is parameterized with parameter vector θ, sharp bounds are obtained straightforwardly by replacing $C(\varepsilon_1, \varepsilon_2)$ with the parametric version $C(\varepsilon_1, \varepsilon_2, \theta)$ in Lemma 1 and Theorems 1 and 2. Parameterizing the copula $C(\varepsilon_1, \varepsilon_2)$ while leaving the marginal distributions of ε_1 and ε_2 unrestricted yields nonparametric bounds, akin to those derived by Aradillas-Lopez (2010) in the case of incomplete information games.

3.1.2. *Perfect correlation.* The case of perfect correlation between the two firm specific unobserved heterogeneity components is also of interest, as it corresponds to an industry-wide productivity shock in industrial organization applications. In that case, the copula attains its Fréchet upper bounds $C(\varepsilon_1, \varepsilon_2) = \min(\varepsilon_1, \varepsilon_2)$ so that the sharp bounds of Theorem 2 in case of duopoly entry yield $P(1,1) = \min(\Pi_1(1), \Pi_2(1))$, $P(0,0) = \min(1 - \Pi_1(0), 1 - \Pi_2(0))$, $P(0,1) \leq \max(\Pi_2(0) - \Pi_1(1), 0)$ and $P(1,0) \leq \max(\Pi_1(0) - \Pi_2(1), 0)$. Similar sharp bounds for the three others cases may be easily derived.

3.1.3. *Independence.* In the other polar case, where the two firms specific unobserved heterogeneity components are purely idiosyncratic shocks, $\varepsilon_1 \perp\!\!\!\perp \varepsilon_0$ and sharp bounds are derived from Theorem 2 by simply setting $C(\varepsilon_1, \varepsilon_2) = \varepsilon_1 \varepsilon_2$.

3.1.4. *Factor structure.* Intermediate cases between the two polar cases of industry-wide shock and idiosyncratic shocks can also be entertained with a simple factor model for the pair of unobserved heterogeneities $(\varepsilon_1, \varepsilon_2)$. Suppose unobserved heterogeneity has factor structure $\varepsilon_d = \alpha_d \varepsilon + \eta_d$, $d = 1, 2$, with $\mathbb{E}\varepsilon = 0$, $\mathbb{E}\varepsilon^2 = 1$ (without loss of generality) and $\eta_1 \perp\!\!\!\perp \eta_2 | \varepsilon$. η_d is uniformly distributed on $[0,1]$ for $d = 1, 2$, conditionally on ε. This factor specification achieves a decomposition of unobserved heterogeneity components into an industry common shock ε and a purely idiosyncratic shock η_d, $d = 1, 2$. We recover the case of purely idiosyncratic firm specific unobserved heterogeneity, when $\alpha_1 = \alpha_0 = 0$. By iterated expectations, we find for each $i, j = 1, 0$:

$$
\begin{aligned}
C(\Pi_1(i, x_1), \Pi_2(j, x_2) | x_1, x_2) &= \mathbb{P}(\varepsilon_1 \leq \Pi_1(i, x_1), \varepsilon_2 \leq \Pi_2(j, x_2) | x_1, x_2) \\
&= \mathbb{E}_\varepsilon \mathbb{P}(\eta_1 \leq \Pi_1(i, x_1) - \alpha_1 \varepsilon, \eta_2 \leq \Pi_2(j, x_2) - \alpha_2 \varepsilon | x_1, x_2, \varepsilon) \\
&= \mathbb{E}_\varepsilon \mathbb{P}(\eta_1 \leq \Pi_1(i, x_1) - \alpha_1 \varepsilon | x_1, \varepsilon) \mathbb{P}(\eta_2 \leq \Pi_2(j, x_2) - \alpha_2 \varepsilon | x_1, x_2, \varepsilon) \\
&= \Pi_1(i, x_1) \Pi_2(j, x_2) + \alpha_1 \alpha_2,
\end{aligned}
$$

from which sharp bounds can be derived for the payoff functions and the pair (α_1, α_2).

3.2. **Sharp bounds on the payoff functions.** From the identified set for (Π, C) we can derive sharp bounds for the payoff functions alone using Fréchet bounds on C in each of the four cases. Consider the duopoly entry case for instance. Line 1 of (3.1) yields $P(1,1) = C(\Pi_1(1), \Pi_2(1)) \leq \min(\Pi_1(1), \Pi_2(2))$ (Fréchet bound). Similarly, Line 2 of (3.1) yields $1-P(0,0) \geq \max(\Pi_1(0), \Pi_2(0))$. Since $\Pi_2(0) \geq \Pi_2(1)$, we have $C(\Pi_1(1), \Pi_2(0)) \geq C(\Pi_1(1), \Pi_2(1))$ and Lines 1 and 3 of (3.1) combined yield $P(1,1) + P(0,1) \leq \Pi_2(0) - [C(\Pi_1(1), \Pi_2(0)) - C(\Pi_1(1), \Pi_2(1))] \leq \Pi_2(0)$. Similarly, Lines 1 and 4 yield $P(1,1) + P(1,0) \leq \Pi_1(0)$. Finally, $P(0,1) + P(1,1) = 1 - P(1,0) - P(0,0) \geq \Pi_2(0) - [C(\Pi_1(0), \Pi_2(0)) - C(\Pi_1(0), \Pi_2(1))] \geq \Pi_2(0) - [\Pi_2(0) - \Pi_2(1)] = \Pi_2(1)$ and similarly $P(1,0) + P(1,1) \geq \Pi_1(1)$. We therefore have the validity of the following bounds for the duopoly entry case:

$$\begin{array}{ccccccccc} P(1,1) & \leq & \Pi_1(1) & \leq & P(1,1)+P(1,0) & \leq & \Pi_1(0) & \leq & 1-P(0,0), \\ P(1,1) & \leq & \Pi_2(1) & \leq & P(1,1)+P(0,1) & \leq & \Pi_2(0) & \leq & 1-P(0,0). \end{array}$$

For the coordination case, the same method (see the proof of Theorem 3) yields:

$$\begin{array}{ccccccccc} P(1,0) & \leq & \Pi_1(0) & \leq & P(1,1)+P(1,0) & \leq & \Pi_1(1) & \leq & 1-P(0,1), \\ P(0,1) & \leq & \Pi_2(0) & \leq & P(1,1)+P(0,1) & \leq & \Pi_2(1) & \leq & 1-P(1,0). \end{array}$$

and finally for the asymmetric cases:

$$\begin{array}{ccccccccc} P(1,0) & \leq & \Pi_1(0) & \leq & P(1,1)+P(1,0) & \leq & \Pi_1(1) & \leq & 1-P(0,1), \\ P(1,1) & \leq & \Pi_2(1) & \leq & P(1,1)+P(0,1) & \leq & \Pi_2(0) & \leq & 1-P(0,0), \end{array}$$

and similarly after permutation of the two players. We can now formally characterize the joint sharp bounds on payoff functions when only pure strategies are entertained.

Theorem 3 (Sharp bounds for payoff functions). Π *belongs to the identified set if and only if (3.4) and (3.5) below hold.*

$$\begin{aligned}
\min\left(\Pi_1(1,x_1),\Pi_1(0,x_1)\right) &\leq \inf_{x_2}\Big(P(1,1|x_1,x_2)+P(1,0|x_1,x_2)\Big)\\
\max\left(\Pi_1(1,x_1),\Pi_1(0,x_1)\right) &\geq \sup_{x_2}\Big(P(1,1|x_1,x_2)+P(1,0|x_1,x_2)\Big)\\
\min\left(\Pi_2(1,x_2),\Pi_2(0,x_2)\right) &\leq \inf_{x_1}\Big(P(1,1|x_1,x_2)+P(0,1|x_1,x_2)\Big)\\
\max\left(\Pi_2(1,x_2),\Pi_2(0,x_2)\right) &\geq \sup_{x_1}\Big(P(1,1|x_1,x_2)+P(0,1|x_1,x_2)\Big)
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\sup_{x_2} P(1,1|x_1,x_2) &\leq \Pi_1(1,x_1) \leq \inf_{x_2}\Big(1-P(0,1|x_1,x_2)\Big)\\
\sup_{x_2} P(1,0|x_1,x_2) &\leq \Pi_1(0,x_1) \leq \inf_{x_2}\Big(1-P(0,0|x_1,x_2)\Big)\\
\sup_{x_1} P(1,1|x_1,x_2) &\leq \Pi_2(1,x_2) \leq \inf_{x_1}\Big(1-P(1,0|x_1,x_2)\Big)\\
\sup_{x_1} P(0,1|x_1,x_2) &\leq \Pi_2(0,x_2) \leq \inf_{x_1}\Big(1-P(0,0|x_1,x_2)\Big).
\end{aligned} \tag{3.5}$$

In the case without excluded variables, it is immediately apparent from the bounds of Theorem 3 that the sign of $\Pi_i(1)-\Pi_i(0)$ is not identified, hence we cannot determine from the data only, whether the game is a duopoly entry game, a game of cooperation or an asymmetric game. With exclusion restrictions, however, it becomes possible to identify the class of games if bounds cross in all cases except one. An example is the case when $\sup_{x_2}\left(P(1,0|x_1,x_2)+P(1,1|x_1,x_2)\right) > \inf_{x_2}\left(1-P(0,1|x_1,x_2)\right)$ and $\sup_{x_1}\left(P(0,1|x_1,x_2)+P(1,1|x_1,x_2)\right) > \inf_{x_1}\left(1-P(1,0|x_1,x_2)\right)$, so that cooperation and asymmetric games are rejected, whereas $\sup_{x_2}\left(P(1,0|x_1,x_2)+P(1,1|x_1,x_2)\right) \leq \inf_{x_2}\left(1-P(0,0|x_1,x_2)\right)$ and $\sup_{x_1}\left(P(0,1|x_1,x_2)+P(1,1|x_1,x_2)\right) \leq \inf_{x_1}\left(1-P(0,0|x_1,x_2)\right)$, so that the duopoly entry game is not rejected.

In the case without excluded variable, the bounds on the payoff functions $\Pi_i(1)$ and $\Pi_i(0)$ can be reduced to a point, but may never cross, so that the hypothesis of Nash equilibrium play is not falsifiable. If, on the other hand, there is an exclusion restriction, hence variation in the payoff of one player that leaves the other player's payoff unchanged, the bounds may cross and the joint assumption of Nash equilibrium play and the exclusion restriction may be rejected. For instance, if $\inf_{x_2}(P(1,0|x_1,x_2)+P(1,1|x_1,x_2)) < \min(\sup_{x_2}(P(1,1|x_1,x_2)), \sup_{x_2}(P(1,0|x_1,x_2)))$ then the bounds cross in all cases and the model is rejected.

Sharp bounds on monopoly advantage can also be easily derived from the bounds of Theorem 3. Indeed, considering Player 1 only for simplicity, monopoly advantage is $|\Pi_1(0,x_1)-\Pi_1(1,x_1)| \leq 1-\min\left(\sup_{x_2} P(0,0|x_1,x_2), \sup_{x_2} P(0,1|x_1,x_2)\right) - \sup_{x_2} P(1,1|x_1,x_2)$. If we assume a priori that the game is duopoly entry, then the bounds on monopoly advantage simplify to $\Pi_1(0,x_1)-\Pi_1(1,x_1) \leq 1-\sup_{x_2} P(0,0|x_1,x_2) - \sup_{x_2} P(1,1|x_1,x_2)$. Bounding free-riding incentives $\varepsilon_1 - \Pi_1(1)$ (free riding incentives of Player 1) is a little more involved, since they involve the unobserved heterogeneity component ε. We may however apply the bounds of Theorem 2 to derive joint sharp bounds on the distribution of the pair $(\varepsilon_1-\Pi_1(1), \varepsilon_2-\Pi_2(1))$.

Conclusion

This paper contributed to the literature on the empirical analysis of game theoretic models of economic interactions by providing sharp bounds on nonparametrically specified payoff functions and type distributions. This complements results of Kline and Tamer (2012) who derive sharp bounds on best response functions. The bounds obtained lend themselves to standard partial identification inference methods, and therefore allow nonparametric inference on utility functions, profit functions, unobserved heterogeneity distributions and more specific quantities such as the extent of monopoly advantage in duopoly entry games and free riding incentives in cooperation games. The method employed to derive sharp bounds on payoff functions only as a projection of the joint identified region for payoff functions and type distributions could be applied to higher dimensions to extend the present results to multiperson games with more complex strategy spaces. Other equilibrium concepts (Stackelberg, correlated strategies etc...) could also be entertained in future work.

Appendix A. Proofs

In all that follows, for ease of notation, we drop the conditioning variables and write $\Pi_i(Y_{3-i} = j, X_i) = \Pi_i(j)$ for $i = 1, 2$ and $j = 1, 0$ and $p_{j_1 j_2} = P(Y_1 = j_1, Y_2 = j_2 | X_1, X_2)$. We shall also use the following symmetries in the game. All results concerning the second asymmetric game can be obtained from results concerning the first asymmetric game after permutation of the two players. The coordination game is obtained from the duopoly entry game by relabeling. Hence all results for the coordination game can be obtained from the results for the duopoly entry game with the following conversion table: $\Pi_1(0)$ in the duopoly entry case is replaced by $1 - \Pi_1(1)$ and vice-versa. $\Pi_2(1)$ is replaced by $\Pi_2(0)$ and vice-versa. Σ_{j_1,j_2} is replaced by Σ_{1-j_1,j_2}. $P(j_1, j_2)$ is replaced by $P(1 - j_1, j_2)$. Finally, σ_1 is replaced by $1 - \sigma_1$.

A.1. **Proof of Theorem 1.** Dropping all explanatory variables from the notation, the equilibrium correspondence $\varepsilon = (\varepsilon_1, \varepsilon_2) \mapsto G(\varepsilon)$, namely the set of all Nash equilibria in mixed strategies, for a given value of $\varepsilon = (\varepsilon_1, \varepsilon_2)$, is represented in Figure 1 and formally defined by $G(\varepsilon) = \{(0, 0)\}$ if $\varepsilon_i > \Pi_i(0)$, $i = 1, 2$, $G(\varepsilon) = \{(1, 1)\}$ if $\varepsilon_i < \Pi_i(1)$, $i = 1, 2$, $G(\varepsilon) = \{(\sigma_1, \sigma_2)\}$ if $\Pi_i(1) < \varepsilon_i < \Pi_i(0)$, $i = 1, 2$, $G(\varepsilon) = \{(1, 0)\}$ if $\varepsilon_1 < \Pi_1(1)$ $\varepsilon_2 > \Pi_2(1)$ or $\varepsilon_1 < \Pi_1(0)$ $\varepsilon_2 > \Pi_2(0)$, with the convention that a degenerate mixed strategy is denoted as its realization.

For almost all values of ε, there is at most one equilibrium in non degenerate mixed strategies. Hence, by Lemma 2 of Galichon and Henry (2011), the game has a Shapley regular core (see for instance Definition 9 of Galichon and Henry (2011)) and Theorem 5 of Galichon and Henry (2011) applies. The identified set for payoff functions and type distributions is therefore characterized by $P(B) \leq \int \left(\max_{\sigma \in G(\epsilon)} \sigma(B)\right) dC(\epsilon)$, for all subsets B of the set of realized decision profiles $\{(0, 1), (1, 0), (0, 0), (1, 1)\}$. This induces the following inequalities.

$$
\begin{aligned}
P(1,1) &\leq C(\Pi_1(1),\Pi_2(1)) + \int_{\Delta} \sigma_1(u_2)\sigma_2(u_1)dC(u_1,u_2), && \text{(A.1)}\\
P(0,0) &\leq 1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0),\Pi_2(0))\\
&\quad + \int_{\Delta} (1-\sigma_1(u_2))(1-\sigma_2(u_1))dC(u_1,u_2), && \text{(A.2)}\\
P(0,1) &\leq \Pi_2(0) - C(\Pi_1(1),\Pi_2(0)),\\
P(1,0) &\leq \Pi_1(0) - C(\Pi_1(0),\Pi_2(1)),
\end{aligned}
$$

$$
\begin{aligned}
P(1,1) &\geq C(\Pi_1(1),\Pi_2(1)) && \text{(A.3)}\\
P(0,0) &\geq 1 - \Pi_1(0) - \Pi_2(0) + C(\Pi_1(0),\Pi_2(0)) && \text{(A.4)}\\
P(0,1) &\geq \Pi_2(0) - C(\Pi_1(1),\Pi_2(1)) - [C(\Pi_1(0),\Pi_2(0)) - C(\Pi_1(0),\Pi_2(1))], && \text{(A.5)}\\
P(1,0) &\geq \Pi_1(0) - C(\Pi_1(1),\Pi_2(1)) - [C(\Pi_1(0),\Pi_2(0)) - C(\Pi_1(1),\Pi_2(0))], && \text{(A.6)}
\end{aligned}
$$

and

$$P(0,0)+P(0,1) \leq 1-\Pi_1(0)+[C(\Pi_1(0),\Pi_2(0))-C(\Pi_1(1),\Pi_2(0))], \tag{A.7}$$

$$P(0,0)+P(1,0) \leq 1-\Pi_2(0)+[C(\Pi_1(0),\Pi_2(0))-C(\Pi_1(0),\Pi_2(1))], \tag{A.8}$$

$$P(1,1)+P(0,1) \leq \Pi_2(0)+[C(\Pi_1(1),\Pi_2(1))-C(\Pi_1(1),\Pi_2(0))], \tag{A.9}$$

$$P(1,1)+P(1,0) \leq \Pi_1(0)+[C(\Pi_1(1),\Pi_2(1))-C(\Pi_1(0),\Pi_2(1))], \tag{A.10}$$

$$P(1,0)+P(0,1) \leq \Pi_1(0)+\Pi_2(0)-C(\Pi_1(1),\Pi_2(1))-C(\Pi_1(0),\Pi_2(0)), \tag{A.11}$$

$$\begin{aligned} P(0,0)+P(1,1) \leq\ & C(\Pi_1(1),\Pi_2(1))+\int_\Delta \sigma_1(u_2)\sigma_2(u_1)dC(u_1,u_2) \\ & +1-\Pi_1(0)-\Pi_2(0)+C(\Pi_1(0),\Pi_2(0)) \\ & +\int_\Delta (1-\sigma_1(u_2))(1-\sigma_2(u_1))dC(u_1,u_2). \end{aligned} \tag{A.12}$$

Now, we will show that (A.7)-(A.12) are redundant. (A.3) and (A.6) jointly imply that $P(1,1)+P(1,0) \geq \Pi_1(0)-[C(\Pi_1(0),\Pi_2(0))-C(\Pi_1(1),\Pi_2(0))]$ so that $1-\Pi_1(0)+[C(\Pi_1(0),\Pi_2(0))-C(\Pi_1(1),\Pi_2(0))] \geq 1-P(1,1)-P(1,0)$, hence (A.7) holds. Similarly, (A.3) and (A.5) imply (A.8), (A.4) and (A.6) imply (A.9), (A.4) and (A.5) imply (A.10), (A.3) and (A.4) imply (A.11) and finally (A.1) and (A.2) imply (A.12). The result follows.

A.2. **Proof of Theorem 3.**

A.2.1. *Duopoly entry case.* Consider first the duopoly entry case, with $\Pi_i(0) \geq \Pi_i(1)$, $i=1,2$. The bounds are shown to hold in the main text as a corollary of Theorem 2. We show now that the bounds are jointly sharp. To do so, take any given true frequency profile $(p_{11},p_{10},p_{01},p_{00})$ and exhibit a joint distribution $C(\varepsilon_1,\varepsilon_2)$ and an equilibrium selection mechanism $\delta \in [0,1]$ (denoting the probability that $(Y_1=1,Y_2=0)$ is selected in the region of multiplicity) such that all Π can be rationalized.

Construction of the joint distribution. We construct the joint distribution in the following way. Assume $P(\varepsilon_1 \leq \Pi_1(1), \varepsilon_2 \leq \Pi_2(1)) = p_{11}$ and $P(\varepsilon_1 \geq \Pi_1(0), \varepsilon_2 \geq \Pi_2(0)) = p_{00}$. From the marginal constraints,

$P(\varepsilon_1 \leq \Pi_1(1)) = \Pi_1(1)$ and $P(\varepsilon_1 \geq \Pi_1(0) = 1 - \Pi_1(0)$. Hence we can choose s and t in $[0,1]$ such that the following hold.

$$P(\varepsilon_1 \leq \Pi_1(1), \varepsilon_2 \geq \Pi_2(0)) = (1-s)(\Pi_1(1) - p_{11}),$$

$$P(\varepsilon_1 \leq \Pi_1(1), \Pi_2(1) \leq \varepsilon_2 \leq \Pi_2(0)) = s(\Pi_1(1) - p_{11}),$$

$$P(\varepsilon_1 \geq \Pi_1(0), \varepsilon_2 \leq \Pi_2(1)) = (1-t)(1 - p_{00} - \Pi_1(0)),$$

$$P(\varepsilon_1 \geq \Pi_1(0), \Pi_2(1) \leq \varepsilon_2 \leq \Pi_2(0)) = t(1 - p_{00} - \Pi_1(0)).$$

The mass in the remaining regions is constrained accordingly. In particular, we have:

$$P(\Pi_1(1) \leq \varepsilon_1 \leq \Pi_1(0), \Pi_2(1) \leq \varepsilon_2 \leq \Pi_2(0)) = \Pi_2(0) - \Pi_2(1) - s(\Pi_1(1) - p_{11}) - t(1 - p_{00} - \Pi_1(0)).$$

This mass can be divided between $(Y_1 = 1, Y_2 = 0)$ and $(Y_1 = 0, Y_2 = 1)$ with an appropriate choice of equilibrium selection mechanism, in order to satisfy the following constraint.

$$\begin{aligned} p_{10} &= 1 - p_{00} - \Pi_2(0) + s(\Pi_1(1) - p_{11}) \\ &\qquad + \delta\Big(\Pi_2(0) - \Pi_2(1) - s(\Pi_1(1) - p_{11}) - t(1 - p_{00} - \Pi_1(0))\Big), \end{aligned} \tag{A.13}$$

with equilibrium selection parameter $\delta \in [0,1]$. There remains to show that equation (A.13) has a solution for $(s,t,\delta) \in [0,1]^3$.

Case $1 - p_{00} = \Pi_1(0)$**:** When $1 - p_{00} = \Pi_1(0)$, equation (A.13) becomes

$$p_{01} + p_{11} - \Pi_2(0) + \delta(\Pi_2(0) - \Pi_2(1)) + s(1-\delta)(\Pi_1(1) - p_{11}) = 0.$$

If $\Pi_1(1) = p_{11}$, then δ can be chosen equal to $(\Pi_2(0) - p_{01} - p_{11})/(\Pi_2(0) - \Pi_2(1))$ (or δ unrestricted in case $\Pi_2(0) = \Pi_2(1)$). If $(1-\delta)(\Pi_1(1) - p_{11}) > 0$, then

$$s = \frac{\Pi_2(0) - p_{01} - p_{11} - \delta(\Pi_2(0) - \Pi_2(1))}{(1-\delta)(\Pi_1(1) - p_{11})}$$

must be between 0 and 1. So we must have $\delta \leq (\Pi_2(0) - p_{01} - p_{11})/(\Pi_2(0) - \Pi_2(1))$ (no restriction if $\Pi_2(0) = \Pi_2(1)$) and

$$(\Pi_2(0) - p_{01} - p_{11}) - (\Pi_1(1) - p_{11}) \leq \delta\Big((\Pi_2(0) - \Pi_2(1)) - (\Pi_1(1) - p_{11})\Big). \tag{A.14}$$

We denote the latter $A \leq \delta B$. Since $\Pi_2(1) \leq p_{01} + p_{11}$, only three cases need to be considered:

(1) $0 < A \leq B$: the δ needs to be chosen larger than or equal to A/B. Combined with the above, it yields $0 < A/B \leq \delta \leq (A + \Pi_1(1) - p_{11})/(B + \Pi_1(1) - p_{11}) \leq 1$, which has solutions since $A \leq B$ and $\Pi_1(1) \geq p_{11}$.

(2) $A < 0 \leq B$: then (A.14) is always satisfied for $\delta \geq 0$ since the left-hand-side is negative and the right-hand-side is non negative.

(3) $A < B < 0$: then any $\delta \in [0, 1]$ satisfies (A.14) since $-A > -B$.

Case $1 - p_{00} - \Pi_1(0) > 0$: When $\delta(1 - p_{00} - \Pi_1(0)) > 0$, equation (A.13) can be rewritten:

$$t = \frac{p_{01} + p_{11} - \Pi_2(0) + \delta(\Pi_2(0) - \Pi_2(1)) + s(1 - \delta)(\Pi_1(1) - p_{11})}{\delta(1 - p_{00} - \Pi_1(0))}$$

so we need to show there exists $(s, \delta) \in [0, 1]^2$ such that

$$0 \leq p_{01} + p_{11} - \Pi_2(0) + \delta(\Pi_2(0) - \Pi_2(1)) + s(1 - \delta)(\Pi_1(1) - p_{11}) \leq \delta(1 - p_{00} - \Pi_1(0)). \tag{A.15}$$

Subcase $\Pi_1(1) = p_{11}$: We need to show the existence of $\delta \in [0, 1]$ such that

$$0 \leq \delta(\Pi_2(0) - \Pi_2(1)) - (\Pi_2(0) - p_{01} - p_{11}) \leq \delta(1 - p_{00} - \Pi_1(0)).$$

The left inequality is satisfied for

$$\frac{\Pi_2(0) - p_{01} - p_{11}}{\Pi_2(0) - \Pi_2(1)} \leq \delta \leq 1, \tag{A.16}$$

since $\Pi_2(1) \leq p_{01} + p_{11} \leq \Pi_2(0)$. The right inequality is equivalent to

$$-\Big(\Pi_2(0) - p_{01} - p_{11}\Big) \leq \delta\Big(1 - p_{00} - \Pi_1(0) - (\Pi_2(0) - \Pi_2(1))\Big),$$

which is true for any $\delta \geq 0$ if $1 - p_{00} - \Pi_1(0) \geq \Pi_2(0) - \Pi_2(1)$ and for any

$$0 \leq \delta \leq \frac{\Pi_2(0) - p_{01} - p_{11}}{\Pi_2(0) - \Pi_2(1) - (1 - p_{00} - \Pi_1(0))} \tag{A.17}$$

otherwise. (A.16) and (A.17) are compatible since $1 - p_{00} \geq \Pi_1(0)$.

Subcase $\Pi_1(1) > p_{11}$: (A.15) is equivalent to

$$\frac{-\Big(p_{01} + p_{11} - \Pi_2(0) + \delta(\Pi_2(0) - \Pi_2(1))\Big)}{(1-\delta)(\Pi_1(1) - p_{11})} \leq s$$
$$\leq \frac{\delta(1 - p_{00} - \Pi_1(0)) - \Big(p_{01} + p_{11} - \Pi_2(0) + \delta(\Pi_2(0) - \Pi_2(1))\Big)}{(1-\delta)(\Pi_1(1) - p_{11})}.$$

The latter admits a solution $s \in [0, 1]$ if and only if

$$\delta(1 - p_{00} - \Pi_1(0)) - \Big(p_{01} + p_{11} - \Pi_2(0) + \delta(\Pi_2(0) - \Pi_2(1))\Big) \quad \geq \quad 0 \tag{A.18}$$

$$\text{and} \quad -\Big(p_{01} + p_{11} - \Pi_2(0) + \delta(\Pi_2(0) - \Pi_2(1))\Big) \quad \leq \quad (1-\delta)(\Pi_1(1) - p_{11}). \tag{A.19}$$

(A.19) is equivalent to

$$\delta\Big((\Pi_1(1) - p_{11}) - (\Pi_2(0) - \Pi_2(1))\Big) \leq (\Pi_1(1) - p_{11}) - (\Pi_2(0) - p_{01} - p_{11}), \tag{A.20}$$

which we denote $\delta B \leq A$. Since $\Pi_2(1) \leq p_{01} + p_{11}$, we have $A \geq B$ and we need only consider the following three cases:

(1) If $0 < B \leq A$, (A.20) is satisfied for all $\delta \in [0, 1]$.

(2) If $B \leq 0 \leq A$, (A.20) is satisfied for all $\delta \geq 0$, since the left hand side is negative and the right-hand-side positive.

(3) If $B \leq A < 0$: (A.20) is satisfied for a choice of $\delta \geq A/B$, namely

$$\frac{(\Pi_2(0) - p_{01} - p_{11}) - (\Pi_1(1) - p_{11})}{(\Pi_2(0) - \Pi_2(1)) - (\Pi_1(1) - p_{11})} \leq \delta \leq 1. \tag{A.21}$$

(A.18) is equivalent to

$$\delta\Big(1 - p_{00} - \Pi_1(0) - (\Pi_2(0) - \Pi_2(1))\Big) \geq p_{01} + p_{11} - \Pi_2(0).$$

The right-hand-side is negative, so the statement is

(1) true for all $\delta \in [0,1]$ when $1 - p_{00} - \Pi_1(0) - (\Pi_2(0) - \Pi_2(1)) \geq 0$,

(2) true for all

$$0 \leq \delta \leq \frac{\Pi_2(0) - p_{01} - p_{11}}{(\Pi_2(0) - \Pi_2(1)) - (1 - p_{00} - \Pi_1(0))} \tag{A.22}$$

when $1 - p_{00} - \Pi_1(0) - (\Pi_2(0) - \Pi_2(1)) < 0$.

Note that there is a solution to both (A.21) and (A.22). Indeed, calling the left-hand-side of (A.21) $-A/(-B)$, with both numerator and denominator positive, we can write the right-hand-side of (A.22) as $[-A+(\Pi_1(1)-p_{11})]/(-B+(\Pi_1(1)-p_{11})-(1-p_{00}-\Pi_1(0)))$, which is larger than or equal to $-A/(-B)$. This completes the proof for the duopoly entry case.

A.2.2. *Coordination case:* As shown above, results for the coordination case can be obtained from results pertaining to the duopoly entry case by relabeling of payoff functions.

A.2.3. *Asymmetric cases:* From the identified set for (Π, C) we can derive sharp bounds for the payoff functions alone using Fréchet bounds on C. Line 1 of (3.3) yields $P(1,1) = C(\Pi_1(1), \Pi_2(1)) \leq \min(\Pi_1(1), \Pi_2(2)) \leq \Pi_2(1)$ (Fréchet bound). Similarly, Line 4 of (3.3) yields $P(1,0) = \Pi_1(0) - C(\Pi_1(0), \Pi_2(1)) \leq \Pi_1(0)$ and $1 - P(1,0) = 1 - \Pi_1(0) + C(\Pi_1(0), \Pi_2(1)) \geq \Pi_2(1)$ (Fréchet lower bound). Line 3 yields $1 - P(0,1) = 1 - \Pi_2(0) + C(\Pi_1(1), \Pi_2(0)) \geq \Pi_1(1)$. Since $\Pi_1(1) \geq \Pi_1(0)$, we have $C(\Pi_1(1), \Pi_2(1)) \geq C(\Pi_1(0), \Pi_2(1))$ and Lines 1 and 4 of (3.3) combined yield $P(1,1)+P(1,0) = \Pi_1(0)+[C(\Pi_1(1), \Pi_2(1))-C(\Pi_1(0), \Pi_2(1))] \geq \Pi_1(0)$. Similarly, since $\Pi_2(1) \leq \Pi_2(0)$, we have $C(\Pi_1(1), \Pi_2(1)) \leq C(\Pi_1(1), \Pi_2(0))$ and Lines 1 and 3 yield $P(1,1)+P(0,1) = \Pi_2(0)+[C(\Pi_1(1), \Pi_2(1))-C(\Pi_1(1), \Pi_2(0))] \leq \Pi_2(0)$. Finally, $P(0,1)+P(1,1) = \Pi_2(0)-[C(\Pi_1(1), \Pi_2(0))-C(\Pi_1(1), \Pi_2(1))] \geq \Pi_2(0)-[\Pi_2(0)-\Pi_2(1)] = \Pi_2(1)$ and similarly $P(1,0)+P(1,1) \geq \Pi_1(0)$.

We show now that the bounds are jointly sharp. To do so, take any given true frequency profile $(p_{11}, p_{10}, p_{01}, p_{00})$ and exhibit a joint distribution $C(\varepsilon_1, \varepsilon_2)$ such that all Π can be rationalized.

Construction of the joint distribution. We construct the joint distribution in the following way. Let $(s,t,u,v) \in [0,1]^4$ be such that the following hold.

$$P(\varepsilon_1 \leq \Pi_1(0), \varepsilon_2 \leq \Pi_2(1)) = (1-u)p_{11},$$

$$P(\Pi_1(0) \leq \varepsilon_1 \leq \Pi_1(1), \varepsilon_2 \leq \Pi_2(1)) = up_{11},$$

$$P(\varepsilon_1 \geq \Pi_1(1), \varepsilon_2 \leq \Pi_2(1)) = (1-t)p_{01},$$

$$P(\varepsilon_1 \geq \Pi_1(1), \Pi_2(1) \leq \varepsilon_2 \leq \Pi_2(0)) = tp_{01},$$

$$P(\varepsilon_1 \geq \Pi_1(1), \varepsilon_2 \geq \Pi_2(0)) = (1-s)p_{00},$$

$$P(\Pi_1(0) \leq \varepsilon_1 \leq \Pi_1(1), \varepsilon_2 \geq \Pi_2(0)) = sp_{00},$$

$$P(\varepsilon_1 \leq \Pi_1(0), \varepsilon_2 \geq \Pi_2(0)) = (1-v)p_{10},$$

$$P(\varepsilon_1 \leq \Pi_1(0), \Pi_2(1) \leq \varepsilon_2 \leq \Pi_2(0)) = vp_{10}.$$

Marginal constraints are given by $1 - \Pi_1(1) = p_{01} + (1-s)p_{00}$, $1 - \Pi_2(0) = p_{00} + (1-v)p_{10}$, $\Pi_1(0) = p_{10} + (1-u)p_{11}$ and $\Pi_2(1) = p_{11} + (1-t)p_{01}$.

and the solution for $(s,t,u,v) \in [0,1]^4$ is the following.

$$(s,t,u,v) = \left(\frac{\Pi_1(1) - p_{11} - p_{10}}{p_{00}}, \frac{p_{11} + p_{01} - \Pi_2(1)}{p_{01}}, \frac{p_{11} + p_{10} - \Pi_1(0)}{p_{11}}, \frac{\Pi_2(0) - p_{11} - p_{01}}{p_{10}} \right).$$

Note that $\Pi_1(0) = 0$ and $\Pi_1(1) = 1$ can only be reached if $p_{10} = p_{01} = 0$, which in turns forces $\Pi_2(1) = \Pi_2(0) = p_{11} = 1 - p_{00}$. Similarly, $\Pi_2(0) = 1$ and $\Pi_2(1) = 0$ can only be reached if $p_{11} = p_{00} = 0$, which in turns forces $\Pi_1(1) = \Pi_1(0) = p_{10} = 1 - p_{01}$.

The bounds for the second asymmetric game are obtained by permuting the two players and the result follows.

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