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Dynamic Screening with Limited Commitment

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DYNAMIC SCREENING WITH LIMITED COMMITMENT

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ABSTRACT: We examine a model of dynamic screening and price discrimination in which the seller has limited commitment power. Two cohorts of anonymous, patient, and risk-neutral buyers arrive over two periods. Buyers in the first cohort arrive in period one, are privately informed about the distribution of their values, and then privately learn the value realizations in period two. Buyers in the second cohort are "last-minute shoppers" that already know their values upon their arrival in period two. The seller can fully commit to a long-term contract with buyers in the first cohort, but cannot commit to the future contractual terms that will be offered to second-cohort buyers. The expected second-cohort contract serves as an endogenous type-dependent outside option for first-cohort buyers, reducing the seller's ability to extract rents via sequential contracts. We derive the seller-optimal equilibrium and show that the seller mitigates this effect by inducing some first-cohort buyers to strategically delay their time of contracting—the seller manipulates the timing of contracting in order to endogenously generate a commitment to maintaining high future prices. The seller's optimal contract pools low types, separates high types, and induces intermediate types to delay contracting.

KEYWORDS: Asymmetric information, Dynamic mechanism design, Limited commitment, Sequential screening, Type-dependent participation.

JEL CLASSIFICATION: C73, D82, D86.

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1. Introduction

In many contracting settings, agents have private information that changes over time. Recent advances in dynamic mechanism design have highlighted the benefits of using dynamic contracts in such settings. Different short- and long-term prices, option contracts, and introductory offers are all methods by which a principal can provide incentives for agents to reveal new private information over time; by doing so, a principal is able to make contingent decisions that extract greater rents than those generated by unconditional, static contracts. One of the basic intuitions arising from this literature is that by contracting in the earliest stages of a relationship, when her informational disadvantage is at its smallest, a principal can relax the participation and incentive constraints she faces. Thus, early contracting leads to more effective price discrimination and smaller information rents. This intuition arises in large part from the assumption that the principal is able to determine the timing of contracting. In many settings, however, such an assumption need not be justified: in many markets, agents are "born" or enter the market at different times, and they are frequently able to time their transactions or delay entry into contractual relationships. Moreover, a principal may not be able to prevent such delays and treat different agent cohorts differently.

This strategic delay by agents in the timing of contracting is even more of a concern when the principal has limited commitment power. In particular, we have in mind settings in which the principal can commit to fully enforceable long-term contracts that bind (with some restrictions) her bilateral relationship with individual agents, but cannot to commit in advance to the contractual terms that may be offered in future periods. This form of limited commitment, in addition to being of natural theoretical interest, also arises in a variety of real-world settings. For instance, consider the market for airline tickets. Each ticket sold for future travel is a long-term contract, complete with a commitment to its provisions for future refundability and exchangeability. The features of tickets that may be sold in the future (including prices, fare classes, and other terms and conditions) are not advertised or made available, nor is there any presumption that that an airline is pre-committed to the details of those tickets. Potential ticket buyers, on the other hand, face uncertainty about their value for traveling at the date in question. They must therefore decide whether to purchase a ticket immediately and take advantage of its option-like features (canceling the ticket if their realized value is low), or instead postpone their purchase in hopes of more advantageous contracting opportunities in the future. Optimal ticketing schemes must take this strategic timing of contracting into account, accounting for buyers' option value of postponing purchases and the impact of such behavior on the seller's ability to extract rents from different cohorts of buyers.

With this in mind, the present work studies the role of limited commitment in dynamic screening with strategic agents. We construct a simple two-period model in order to isolate the role of limited commitment in a transparent fashion. This model features a monopolist that faces two cohorts of buyers that arrive over two periods; all consumption occurs at the end of the second period. Each buyer in the first cohort (which arrives in the first period) initially has private information regarding the distribution from which her private value is drawn, but does not learn the realized value until the second period. Buyers in the second cohort arrive in period two, and already know their private value (which is drawn from a commonly known distribution). We assume that buyers are anonymous, so that the seller is unable to distinguish in period two between first-cohort buyers who postponed contracting and second-cohort buyers who have just arrived to the market. Thus, the seller cannot prevent first-cohort buyers from contracting in period two.

A straightforward strategic tension arises in this setting. Since buyers in the first cohort learn their values over time, the seller has a strong incentive to sequentially screen these buyers using (dynamic) option contracts. The seller would also like to sell to buyers in the second cohort by offering a (static) "last-minute price" contract in period two. We assume that the seller *can* credibly commit to a sequential contract offered in the first period; she *cannot* commit at that time, however, to contracts that she might offer in the second period. This second-period contract affects the seller's ability to screen cohort-one buyers and extract rents: the period-two contract serves as an endogenous outside option for cohort-one buyers. If the seller can commit to a relatively high period-two price, this outside option becomes less attractive, and the seller's profits from cohort-one buyers increase. With limited commitment, however, competition between the period-one seller and her future self increases the rents left to buyers and reduces the seller's profits.

A simple thought experiment is helpful in illustrating the interplay between the seller's limited commitment and the possibility of strategic delay by buyers. Suppose that the mass of cohort two is small, and suppose further that the monopoly price that corresponds to this cohort (in isolation) is low. If cohort-one buyers anticipate this low price, then waiting until the second period to contract is a very attractive outside option, and the seller's ability to extract rents in period one is reduced. Note, however, that the small mass of the second cohort implies that their contribution to total profits is also small; therefore, a seller with full commitment power could (relatively costlessly) commit to forgoing profits from the second-period cohort by charging an excessively high second-period price, thereby reducing the option value of strategic delay for cohort-one buyers and increasing overall profits. A seller with limited commitment power, on the other hand, would be unable to carry out the threat to maintain a high price in period two. Because the mass of cohort two is small, however, small changes in the composition of the set of buyers contracting in the second period can have a large impact on the distribution of buyers' values. In particular, the seller has a strong incentive to postpone contracting and encourage delay by some cohort-one buyers in order to generate stronger period-two demand. This delayed contracting by a subset of buyers generates (via sequential rationality) a higher period-two price and, hence, a commensurately lower period-one outside option—appropriate "management" of demand across the two periods yields the seller some measure of endogenous commitment power. Our main result identifies the subset of buyers that the seller induces to delay in her optimal contract, thereby characterizing this endogenous commitment and allowing an exploration of the tradeoffs required to extract rents in a dynamic limited-commitment environment.

In solving the seller's optimal contracting problem, we cannot resort to the revelation principle due to the lack of commitment power. Instead, we follow the approach of Riley and Zeckhauser (1983) and Skreta (2006) and solve for the seller-optimal equilibrium of a dynamic contracting game by focusing on equilibrium outcomes (as opposed to the equilibrium strategies implementing those outcomes). In particular, we solve for the optimal direct revelation mechanism with the additional constraints imposed by sequential rationality and then show that there exists a perfect Bayesian equilibrium of the general game that implements its outcomes. In doing so, we explicitly account for the possibility of strategic delay by buyers in the first cohort. So as to focus on the seller-optimal equilibrium, however, we allow the *seller* to "suggest" which buyers participate in the initial mechanism (just as a Myersonian mediator may "recommend" actions to agents). We then verify that these suggestions are indeed optimal from the *buyers'* perspective and are therefore compatible with individual rationality.

As a benchmark, we consider the optimal contract when the seller is able to commit in advance to the contract offered in the second period and buyers are unable to delay contracting. Here, the seller simply treats the two cohorts separately, and standard tools immediately yield the optimal contracts. The seller uses a set of sequential screening contracts as in Courty and Li (2000) to maximize profits from the first cohort, while offering the optimal monopoly price to buyers in the second cohort.

In order to isolate the role played by strategic delay in this dynamic setting, we then consider the setting where first-cohort buyers *can* delay their purchases and the seller cannot exclude these buyers from purchasing in the second period. The seller is still able, however, to commit in advance to the period-two contract. The seller thus faces a tradeoff between maximizing profits from second-cohort buyers and reducing the outside option available to first-cohort buyers. Note, however, that this outside option is type-dependent, as cohort-one buyers with different initial types have different preferences over future contracts: buyers with high expected future values derive greater expected utility from delaying their contracting than buyers with low expected future values. We characterize the seller's optimal contracts when faced with this type-dependent "participation" constraint, and show that there is no delayed contracting in the seller-optimal equilibrium. Instead, the seller's optimal contract pools "low" types by offering them a contract that replicates the fixed price offered in the second period, while "high" types are screened using contracts that induce a strict preference for immediate contracting. Moreover, the second-period price is higher than the monopoly price that would be offered in the benchmark no-delay case: relaxing the participation constraint for first-cohort buyers increases the seller's ability to discriminate among them, and this is more profitable than the marginal increase in profits gained from second-cohort buyers.

Finally, we examine the two-period game in which the seller cannot commit to the contract offered in the second period and buyers are able to delay contracting. Our main result characterizes properties of the seller-optimal equilibrium in this setting. We show that it is profitable for the seller to induce delayed contracting by a subset of cohort-one buyers. By incentivizing some buyers to postpone contracting, the seller is able to alter the distribution of buyer values in the second period (and therefore the sequentially rational optimal price). We show that the set of types that delay contracting must be an interval, and that this interval is typically an interior subset of the set of all types; that is, both "low" and "high" types will choose to contract in the initial period (with pooling of the low types and separation of the high types), while "intermediate" types delay contracting until the second period. Thus, the seller endogenously commits to a less appealing period-two contract by inducing the strategic delay of a subset of cohort-one buyers.

This insight serves as a complement to the findings in the literature that long-term contracts can be used by sellers in dynamic environments to increase profits. In particular, our result shows that the *absence* of contracts with some buyers can be a useful tool for changing the composition of the buyer population in future periods and thereby constraining the seller's *own* future behavior. This sheds new light on the role of commitment power in dynamic settings, and on the underlying sources of that commitment.

In addition, note that the optimal contract with limited commitment features an interesting non-monotonicity: in sharp contrast to most optimal contracting results in both the static and dynamic mechanism design literatures, where exclusion typically follows a simple cutoff rule, the set of buyers in our setting that contract in the first period is disjoint. This non-monotonicity reduces the seller's ability to "separate" types and price discriminate in the first period. Thus, demand management, though valuable in raising future prices and creating endogenous commitment, entails an additional deadweight loss relative to the full commitment benchmark. Indeed, the potential for endogenous commitment arising from delayed contracting highlights the complications that arise due to limited commitment, but also suggests that studying such models can lead to rich predictions and insights that further our understanding of dynamic contracting in real-world settings.

The present work contributes to the literature on optimal dynamic mechanism design.[1] This literature focuses on characterizing revenue-maximizing dynamic contracts for a principal facing agents with evolving private information. Typically, the principal is endowed with full commitment power and observes agent arrivals, enabling her to commit to excluding agents that do not contract immediately. Thus, in contrast to our model, all agents receive their (exogenously determined) reservation utility if they attempt to delay contracting, thereby incentivizing contracting upon arrival. Baron and Besanko (1984) were the first to study such problems and point out the crucial role of the "informativeness" of initial-period private information about future types in determining the optimal distortions away from efficiency used to reduce information rents. More recently, Pavan, Segal, and Toikka (2012) derive a dynamic envelope formula that is necessarily satisfied in general dynamic environments, and also identify some sufficient conditions for incentive compatibility.[2]

Our model is most closely related to the now-canonical work of Courty and Li (2000), who demonstrate the utility of sequential screening when buyers' private information may evolve between contracting and consumption.[3] We extend their model in several important ways: we introduce a second cohort of informed buyers who arrive in period two; we allow cohort-one buyers to postpone contracting until the second period; and we relax the assumption that the principal

[1]There is also an extensive literature (see Athey and Segal (2007, 2012); Bergemann and Välimäki (2010); Gershkov and Moldovanu (2009); Said (2012); and Skrzypacz and Toikka (2013), among others) on efficient dynamic mechanism design. See Bergemann and Said (2011) and Vohra (2012) for surveys of both literatures.

[2]The present work employs a dynamic "first-order approach" in characterizing incentive compatibility. While we show that our assumptions on primitives justify this approach, other settings may require a more careful accounting of global incentive compatibility constraints; see Battaglini and Lamba (2012).

[3]Boleslavsky and Said (2013) show that sequential screening becomes "progressive" screening when buyers' values are subject to additional (correlated) shocks over time. In contrast, Krähmer and Strausz (2012) show that there is no benefit to sequential screening when buyers have ex post participation constraints or limited (ex post) liability.

has unlimited commitment power. The first two of these features are shared with the work of Ely, Garrett, and Hinnosaar (2012), who show that a capacity-constrained seller with full commitment power can benefit from "overbooking" (selling more units than capacity); units can then be repurchased from low-value buyers and reallocated. By committing to biasing the reallocation stage away from late-arriving buyers, the seller is able to incentivize early purchases. Full commitment also plays an important role in Deb (2011) and Garrett (2012), who also consider models where agents' incentives to delay contracting influence the optimal contracts.[4]

Since our seller cannot commit in advance to contracts that might be offered in the second period, this paper also ties into the broader literature on mechanism design without commitment. Although Bester and Strausz (2001) extend the revelation principle to such problems, their results are restricted to settings with finite type spaces and static private information. As described above, we avoid working with arbitrary message spaces by adapting the equilibrium outcome approach of Skreta (2006) to our framework with evolving private information. In related work, Battaglini (2005) shows that, in a setting where types evolve according to a Markov process, the optimal full-commitment contract is often renegotiation-proof. Moreover, Battaglini (2007) demonstrates that, when private information evolves stochastically, optimal contracts continue to (partially) separate types, even when a renegotiation-proofness constraint binds. As in the present work, the additional private information that agents learn over time alleviates the "ratchet effect," as the principal's ability to extract rents in any period is limited by the need to provide incentives for continued information revelation, even when the seller knows the history of private types. In our model, however, the ratchet effect is only partially alleviated—the long-term contracts offered in the seller-optimal equilibrium are not renegotiation-proof. Nonetheless, these contracts require partial commitment only on the seller's part, as buyers may end the contractual relationship at any time. Indeed, buyers are even free to terminate their initial contract and participate in the second-period contract offered to new entrants; in the seller-optimal equilibrium, however, buyers do not benefit from this ability to anonymously re-contract with the seller in the second period.

2. Model

2.1. *Environment*

We consider a monopoly seller of some good who faces a continuum of privately informed buyers. The seller has a constant marginal cost of production $c \geq 0$ and faces no capacity constraints on the quantity that she may sell. There are two time periods $t \in \{1, 2\}$. In each period t, a cohort of anonymous and risk-neutral buyers arrives. Anonymity implies that, outside a contractual relationship, the seller is unable to distinguish in period two between buyers from each cohort. Each buyer has single-unit demand, and all consumption occurs at the end of period two. For simplicity, we assume that neither the seller nor the buyers discount the future; our results remain essentially unchanged in the presence of discounting, however.

[4]There is also a growing body of work examining the incentives for delay when buyers' private information does not change over time—see, for example, Board (2008); Gershkov, Moldovanu, and Strack (2013); and Pai and Vohra (2013). Deb and Pai (2013) and Mierendorff (2011) show that bunching may be necessary when buyers' exit times are private.

The first cohort of buyers consists of a unit mass of agents that arrive in period one. Each such buyer has an initial private type $\lambda \in \Lambda := [\underline{\lambda}, \bar{\lambda}]$, where λ is distributed according to the distribution F with continuous and positive density f. In period two, each buyer then learns her value $v \in \mathbf{V} := [\underline{v}, \bar{v}]$, where v is drawn from the conditional distribution $G(\cdot|\lambda)$ with continuous and positive density $g(\cdot|\lambda)$. We assume that G is twice continuously differentiable, and further that the family of distributions $\{G(\cdot|\lambda)\}_{\lambda\in\Lambda}$ is ordered by first-order stochastic dominance: $\partial G(v|\lambda)/\partial\lambda \leq 0$ for all $v \in \mathbf{V}$ and $\lambda \in \Lambda$.

The second cohort of buyers consists of a mass $\gamma > 0$ of new entrants that arrive in period two. Each such buyer already knows, upon her arrival, her private value $v \in \mathbf{V}$. We assume that values for cohort-two buyers are drawn from the commonly known distribution H with continuous and positive density h. We will denote by p_H the monopoly price corresponding to this late-arriving cohort of buyers; that is,

$$p_H := \max\{\operatorname*{argmax}_p\{(p-c)(1-H(p))\}\},$$

where we choose the largest such price if there are multiple maximizers. Notice that, by definition, we have $p_H \in (c, \bar{v})$. In addition, we assume that there exists some $\hat{\mu} \in \Lambda$ such that

$$p_H = \max\{\operatorname*{argmax}_p\{(p-c)(1-G(p|\hat{\mu}))\}\};$$

that is, the second-cohort monopoly price also corresponds to the monopoly price for some cohort-one initial type $\hat{\mu}$. Note, however, that we do not require $H = G(\cdot|\hat{\mu})$, so the distribution of cohort-two buyers' values need not correspond to that of any type of cohort-one buyer.

2.2. *Contracts*

Due to our assumption of limited commitment, we cannot directly appeal to the revelation principle. Instead, we must consider mechanisms with more general message spaces. Note that we restrict attention throughout to deterministic contracts and mechanisms.

In the initial period, the seller offers, and fully commits to, a dynamic mechanism to cohort-one buyers. Such a mechanism is a game form $D = \{M_{11}, M_{12}, \tau_{11}, \tau_{12}, a_1\}$, where M_{11} is a set of period-one messages; $M_{12}(m_{11})$ is a set of period-two messages that can depend on $m_{11} \in M_{11}$; $\tau_{11}(m_{11})$ is a period-one transfer; $\tau_{12}(m_{11}, m_{12})$ is a period-two transfer; and $a_1(m_{11}, m_{12}) \in \{0, 1\}$ is the eventual allocation in period two. We impose the restriction that there exist $m_{11} \in M_{11}$ and $m_{12} \in M_{12}$ that correspond to non-participation in the dynamic mechanism—buyers are not compelled to participate in the seller's mechanism in the first period, and are also free to exit the mechanism in the second period.

In the second period, the seller offers, and fully commits to, a *static* mechanism $S = \{M_{22}, \tau_{22}, a_2\}$ offered to cohort-two buyers *and* cohort-one buyers that rejected the initial dynamic mechanism D. Here, M_{22} is the set of possible (period-two) messages; $\tau_{22}(m_{22})$ is a (period-two) transfer; and $a_2(m_{22}) \in \{0, 1\}$ is the resulting allocation. Note that the fact that the mechanism S is offered to cohort-one buyers not participating in period one corresponds to our anonymity assumption: the seller is unable to distinguish between these cohort-one buyers and newly arrived cohort-two buyers. Since the game ends after period two, it is without loss of generality to assume that static mechanism S offered in period two is, in fact, a direct revelation mechanism. Such a mechanism may, in principle, attempt to discriminate across cohorts or initial-period types λ. Notice, however, that this dimension of private information is payoff-irrelevant in the second period—in period two, buyers' payoffs depend only on their realized values. This implies that, conditional on contracting in the second period, the seller is unable to screen across cohorts or initial types, and so it is without loss to assume that the mechanism S only conditions on buyers' reported values, so that $M_{22} = \mathbf{V}$.

We do not permit the seller to offer contracts in the first period that are contingent on the seller's choice of contract in the second period. In particular, note that no elements of the second-period static mechanism S are arguments of any elements of the first-period dynamic mechanism D. For instance, the seller cannot offer a contract in the first period that promises extremely large payments to buyers if, in the second period, the seller proposes any contract other than the full-commitment-optimal contract. Similarly, "most-favored nation" and "best-price guarantee" clauses are prohibited. Since the seller can fully commit to the terms of the period-one contract, such clauses serve as mechanisms to allow the seller to also commit to the terms of the period-two contract, thereby implicitly endowing the seller with full commitment power and violating the spirit of our limited commitment exercise.[5]

Strategy profiles are defined in the standard way: they are a choice of an action at each information set. Similarly, beliefs at each information set are defined in the usual way. These jointly generate outcomes: allocations and payments conditional on buyers' types (λ) and values (v). For buyers that contract in the initial period, these are

$$\{p_{11}(\lambda), p_{12}(v,\lambda), q_1(v,\lambda)\}_{v\in\mathbf{V},\lambda\in\Lambda},$$

where p_{11} is the period-one payment; p_{12} is the period-two payment; and q_1 is the allocation. For buyers that contract in the second period, these are denoted by

$$\{p_{22}(v), q_2(v)\}_{v\in\mathbf{V}},$$

where p_{22} is the period-two payment and q_2 is the allocation.

Working directly with the underlying mechanism design game is quite intractable, as the set of possible contracts is large and unwieldy. Instead, we search for optimal outcomes, with the additional restriction that they are implementable in a perfect Bayesian equilibrium of the "full" underlying game. We follow the approach of Skreta (2006) in considering the additional restrictions that sequential rationality imposes on a standard dynamic mechanism design problem and verifying that our resulting contract is indeed implementable. Thus, our analysis proceeds *as if* the seller uses direct revelation mechanisms for cohort-one buyers.

We must also account, however, for the potential for strategic delay by cohort-one buyers. Therefore, the period-one mechanism includes a participation decision $x : \Lambda \to [0,1]$, where $x(\lambda)$ denotes the probability with which type λ buyers contract immediately, and $(1 - x(\lambda))$ is the probability that a type λ buyer delays contracting until the second period. Since there is a continuum of buyers, these probabilities do not generate any aggregate uncertainty about the set

[5]See Board (2008) and Butz (1990) for analyses of best-price guarantees in dynamic durable-goods monopoly models.

of buyers who ultimately delay contracting. Therefore, $x(\lambda)$ and $(1 - x(\lambda))$ also correspond to the fractions of type λ buyers that contract immediately or delay until the second period, respectively.

Our method of solving for the optimal contract involves the *seller* choosing $x(\lambda)$, although in the underlying game it is the *buyers* who choose their time of contracting; we will impose the restriction, however, that the seller's choice of $x(\lambda)$ is consistent with rational behavior (with respect to correct expectations) on the buyers' part.[6] This contrast is deliberate: by analyzing the delay decision as an explicit choice by the seller, we are able to determine the seller-optimal equilibrium. This is essentially the approach of Jullien (2000), who characterizes the optimal contract for a principal facing an agent with an exogenously given type-dependent participation constraint—the principal chooses which agent types to include or exclude from her mechanism. In the present work, however, the option value of strategic delay is endogenously determined, so excluding some buyers in the first period has an additional impact on the seller's problem.

2.3. *Payoffs and Constraints*

The seller's expected payoff is simply the sum of profits derived from three groups: cohort-one buyers who contract in the first period; cohort-one buyers who delay until period two; and cohort two buyers. Thus, the seller's profits may be expressed as

$$\begin{aligned}\Pi := &\iint\limits_{\Lambda\times\mathbf{V}} x(\lambda)(p_{11}(\lambda) + p_{12}(v,\lambda) - cq_1(v,\lambda))dG(v|\lambda)dF(\lambda) \\ &\quad + \iint\limits_{\Lambda\times\mathbf{V}} (1 - x(\lambda))(p_{22}(v) - cq_2(v))dG(v|\lambda)dF(\lambda) + \gamma \int\limits_{\mathbf{V}} (p_{22}(v) - cq_2(v))dH(v).\end{aligned} \tag{1}$$

Her objective is then to maximize her payoff above. This problem is, of course, subject to a variety of incentive compatibility, individual rationality, and sequential rationality constraints.

We begin by considering the second-period constraints. First, all cohort-one buyers that contract in the initial period must prefer truthful reporting of their value in period two to any misreport. That is, we must have

$$U_{12}(v,\lambda) := q_1(v,\lambda)v - p_{12}(v,\lambda) \geq q_1(v',\lambda)v - p_{12}(v',\lambda) \text{ for all } v,v' \in \mathbf{V} \text{ and } \lambda \notin x^{-1}(0). \tag{IC$_{12}$}$$

A similar requirement holds for buyers in cohort two, as well as cohort-one buyers that delay contracting. Thus, we must have

$$U_{22}(v) := q_2(v)v - p_{22}(v) \geq q_2(v')v - p_{22}(v') \text{ for all } v,v' \in \mathbf{V}. \tag{IC$_{22}$}$$

In addition, these buyers' participation must be voluntary, and so the optimal contract must satisfy

$$U_{22}(v) \geq 0 \text{ for all } v \in \mathbf{V}. \tag{IR$_{22}$}$$

Since commitment in our model is both limited and one-sided, cohort-one buyers that contract in the first period are free to end their relationship with the seller and exit the contract. Indeed, due to anonymity, these buyers are also free to avail themselves of the second-period contract if they so choose. When the second-period contract participation constraint (IR$_{22}$) holds, the option

[6]This is akin to ensuring obedience to the mediator's recommended actions in Myerson (1986).

of re-contracting is always at least as attractive as exiting the relationship entirely; thus, we may express the second-period participation constraint for initial-period contracts as

$$U_{12}(v,\lambda) \geq U_{22}(v) \text{ for all } v \in \mathbf{V} \text{ and } \lambda \notin x^{-1}(0). \tag{IR$_{12}$}$$

Finally, in the absence of full commitment power, the second-period contract must be sequentially rational given the seller's beliefs; in particular, the seller chooses a mechanism in the second period that maximizes profits, given the set of buyers that delay contracting. Therefore, we must have

$$\{p_{22}, q_2\} \in \operatorname*{argmax}_{\hat{p},\hat{q}} \left\{ \begin{array}{c} \displaystyle\iint\limits_{\Lambda\times\mathbf{V}} (1-x(\lambda))(\hat{p}(v)-c\hat{q}(v))dG(v|\lambda)dF(\lambda) \\ \displaystyle + \gamma \int\limits_{\mathbf{V}} (\hat{p}(v)-c\hat{q}(v))dH(v) \end{array} \right\} \tag{SR}$$
$$\text{subject to (IC}_{22}\text{) and (IR}_{22}\text{).}$$

Note that the seller's beliefs about buyers' values in the second period are described by a smooth and well-behaved distribution, regardless of the properties of the set of buyers that delay contracting. Even if this set is disjoint, the induced second-period distribution of values has full support on $\mathbf{V}$. This follows from the evolution of buyers' private information over time as buyers learn their values. Moreover, note that the problem in (SR) is unaffected by measure-zero changes to the set of buyers that postpone contracting, as such changes do not alter the seller's beliefs about the resulting distribution of values. Therefore, individual buyers' decisions about delay will not affect the contract offered in period two.

There is also a set of constraints that must be satisfied in period one. For all $\lambda, \lambda' \in \Lambda$, we define

$$U_{11}(\lambda,\lambda') := x(\lambda')\left(-p_{11}(\lambda') + \int_{\mathbf{V}} U_{12}(v,\lambda')dG(v|\lambda)\right) + (1-x(\lambda'))\left(\int_{\mathbf{V}} U_{22}(v)dG(v|\lambda)\right).$$

(Note that there is no need to consider compound deviations if the second-period constraints (IC$_{12}$) are satisfied, as the initial-period type is irrelevant to the buyer's incentives in period two.) Then incentive compatibility in the initial period requires that

$$U_{11}(\lambda) := U_{11}(\lambda,\lambda) \geq U_{11}(\lambda,\lambda') \text{ for all } \lambda, \lambda' \in \Lambda. \tag{IC$_{11}$}$$

In addition, buyers must be incentivized to participate in the seller's mechanism in period one. Individual rationality in the initial period requires that

$$U_{11}(\lambda) \geq 0 \text{ for all } \lambda \in \Lambda. \tag{IR$_{11}$}$$

Moreover, note that the possibility of strategic delay implies that buyers have an outside option given by expectations about the contract offered in period two: each cohort-one buyer's expected payoff must always be at least as large as that obtained by delaying contracting. Recall, however, that buyers' future values depend on their initial type; therefore, the value of strategic delay varies with λ. Thus, the second-period contract effectively serves as a type-dependent participation constraint for cohort-one buyers. Moreover, rationality requires that type λ buyers must be indifferent between immediate contracting and delay whenever $x(\lambda) \in (0,1)$. Thus, the seller's contract must satisfy

$$U_{11}(\lambda) \geq \int_{\mathbf{V}} U_{22}(v)dG(v|\lambda) \text{ for all } \lambda \in \Lambda, \text{ with equality if } x(\lambda) < 1. \tag{SD}$$

Finally, notice that the option value of strategic delay is sufficient to make the "standard" participation constraint (IR_{11}) redundant whenever the period-two contract induces voluntary participation (that is, when (IR_{22}) is satisfied).

2.4. *Simplifying the Constraints*

Before proceeding to the solution to the seller's optimal contracting problem, it will be helpful to first simplify some of the constraints. We begin by considering the period-two incentive compatibility constraints. Note that these constraints are "easy," in the sense that standard arguments may be used to simplify them. In particular, note that constraints (IC_{12}) and (IC_{22}) are essentially static incentive compatibility constraints, and are therefore equivalent to the usual envelope and monotonicity conditions. (The proof follows standard techniques, and is therefore omitted.)

LEMMA 1. *The period-two incentive compatibility constraints (IC_{12}) for buyers contracting in period one are satisfied if, and only if, for all $\lambda \notin x^{-1}(0)$,*

$$\frac{\partial U_{12}(v,\lambda)}{\partial v} = q_1(v,\lambda) \textit{ almost everywhere, and} \tag{IC'_{12}}$$

$$q_1(v,\lambda) \textit{ is nondecreasing in } v. \tag{MON_{12}}$$

The period-two incentive compatibility constraints (IC_{22}) for buyers contracting in period two are satisfied if, and only if,

$$\frac{\partial U_{22}(v)}{\partial v} = q_2(v) \textit{ almost everywhere, and} \tag{IC'_{22}}$$

$$q_2(v) \textit{ is nondecreasing in } v. \tag{MON_{22}}$$

Note that Lemma 1 implies that, since the underlying mechanisms are deterministic, the allocations must follow cutoff rules. More specifically, there exists some function $k : \Lambda \to \mathbf{V}$ and a constant $\alpha \in \mathbf{V}$ such that

$$q_1(v,\lambda) = \begin{cases} 0 & \text{if } v < k(\lambda), \\ 1 & \text{if } v \geq k(\lambda); \end{cases} \text{ and } q_2(v) = \begin{cases} 0 & \text{if } v < \alpha, \\ 1 & \text{if } v \geq \alpha. \end{cases} \tag{2}$$

Incentive compatibility in period one is slightly less straightforward. Note that a cohort-one buyer's initial-period private type λ does *not* affect her payoffs directly. Instead, the impact of this initial type is purely informational: changes in λ yield different future preferences over allocations, but do not affect a buyer's ex post flow payoffs. That said, however, an envelope argument implies that a cohort-one buyer's expected payoff may be expressed as a function of the "effective" allocation rule

$$\bar{q}(v,\lambda) := x(\lambda)q_1(v,\lambda) + (1 - x(\lambda))q_2(v) \tag{3}$$

alone. Moreover, we can leverage the stochastic dominance order on $\{G(\cdot|\lambda)\}_{\lambda\in\Lambda}$ to show that this effective allocation rule must be monotone in both its arguments. Indeed, the period-one envelope condition we derive (with details in the appendix) and monotonicity of $\bar{q}$ are also sufficient for incentive compatibility.[7]

Lemma 2. *The incentive compatibility constraints* (IC_{11})*,* (IC_{12})*, and* (IC_{22}) *are satisfied for all buyers if, and only if,*

$$U_{11}'(\lambda) = -\int_{\mathbf{V}} \bar{q}(v,\lambda) G_\lambda(v|\lambda) d\lambda \text{ almost everywhere, and} \tag{IC'_{11}}$$

$$\bar{q}(v,\lambda) \text{ is nondecreasing in both } v \text{ and } \lambda, \tag{MON_{11}}$$

where G_λ denotes the partial derivative of G with respect to λ.

With this result in hand, we return to the seller's objective function. In particular, note that the seller's payoff in Equation (1) may be rewritten as

$$\Pi = \iint_{\Lambda \times \mathbf{V}} \left[\bar{q}(v,\lambda)(v-c) - U_{11}(\lambda)\right] dG(v|\lambda)dF(\lambda) + \gamma \int_{\mathbf{V}} \left(q_2(v)(v-c) - U_{22}(v)\right) dH(v).$$

When the incentive compatibility constraints (IC_{11}), (IC_{12}), and (IC_{22}) are satisfied, standard techniques can be used to rewrite this objective function into a more "usable" form—in particular, a form in which only the allocation rules q_1 and q_2 appear. Thus, the seller's payoff from an incentive compatible contract is

$$\iint\limits_{\Lambda \times \mathbf{V}} \bar{q}(v,\lambda)\varphi_1(v,\lambda) dG(v|\lambda)dF(\lambda) - U_{11}(\underline{\lambda}) + \gamma \left(\int_{\mathbf{V}} q_2(v)\psi_2(v)dH(v) - U_{22}(\underline{v}) \right), \tag{4}$$

where we define

$$\varphi_1(v,\lambda) := v - c + \frac{G_\lambda(v|\lambda)}{g(v|\lambda)} \frac{1-F(\lambda)}{f(\lambda)} \text{ and } \psi_2(v) := v - c - \frac{1-H(v)}{h(v)}.$$

Note that $\psi_2(v)$ is simply the standard Myerson (1981) virtual value for second-cohort buyers. The initial $(v-c)$ term reflects a buyer's contribution to the social surplus, while the remaining term corresponds to the distortions arising from incentive compatibility; in particular, the hazard rate $(1-H(v))/h(v)$ appears because rents received to a buyer with some value v must also accrue to buyers with higher values.

Similarly, $\varphi_1(v,\lambda)$ is the *dynamic* analog of the virtual value. The initial $(v-c)$ term is simply a cohort-one buyer's ex post contribution to the social surplus, while the remaining term corresponds to the distortions arising from incentive compatibility. The hazard rate of first-period types $(1-F(\lambda))/f(\lambda)$ appears because contracting begins in the first period; for any realized value v, any rents received by a buyer with initial type λ must also accrue to buyers with larger initial types. Meanwhile, the $G_\lambda(v|\lambda)/g(v|\lambda)$ term is the Baron and Besanko (1984) "informativeness measure" that captures the responsiveness of a buyer's second-period value to changes in her initial-period type.[8] This term is zero if type and value are independent, and large if small changes in λ have a large impact on $G(v|\lambda)$. Moreover, recall that $G_\lambda \leq 0$ due to first-order stochastic dominance; thus, as is standard, any information rents are paid *to* the buyer instead of *by*

[7] Similar to a result found in Krähmer and Strausz (2011), sufficiency here relies on the deterministic nature of the underlying mechanisms.

[8] Pavan, Segal, and Toikka (2012) refer to this ratio as the "impulse response" function.

her, thereby reducing the virtual surplus. Throughout what follows, we assume the monotonicity of the cohort-one virtual value.

ASSUMPTION 1. *The cohort-one virtual value $\varphi_1(v,\lambda)$ is increasing in both v and λ.*

Note that Assumption 1 is strictly stronger than the standard assumption that the distribution of private information at the time of contracting is log-concave; indeed, it is a joint assumption on both the distribution F of initial types λ and the conditional distributions G of values. As in much of the previous literature in dynamic mechanism design, we employ this assumption in order to justify our "local" first-order approach to incentive compatibility.[9] The assumption is, however, an easily verified condition on primitives that is satisfied in a wide variety of economic environments of interest.

3. DYNAMIC SCREENING WITH FULL COMMITMENT

In this section, we present the benchmark contracts for a setting in which the seller has the ability to fully commit to her future contracts. We proceed in three steps: first, we present the optimal contract when cohort-one buyers are unable to delay contracting; second, we discuss the optimal contract when the second-period contract is exogenously fixed; and finally, we present the unconstrained full-commitment optimal contract.

3.1. *Dynamic Screening without Strategic Delay*

Suppose that cohort-one buyers are unable to delay contracting or re-contract in period two. When this is the case, the seller is able to treat the two cohorts of buyers separately, with no regard to the potential impact of the second-period contract on buyers in the first cohort.

To make this more clear, we consider the seller's formal optimization problem. In order to account for the impossibility of delay, we require that $x(\lambda) = 1$ for all $\lambda \in \Lambda$. (Thus, $\bar{q}(v,\lambda) = q_1(v,\lambda)$ for all $(v,\lambda) \in \mathbf{V} \times \Lambda$.) Making use of the expression for the seller's payoff in Equation (4), we can write the seller's problem as

$$\max_{q_1,q_2,p_{11},p_{12},p_{22}} \left\{ \begin{array}{c} \displaystyle\iint_{\Lambda\times\mathbf{V}} q_1(v,\lambda)\varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) - U_{11}(\underline{\lambda}) \\ \displaystyle + \gamma\left(\int_{\mathbf{V}} q_2(v)\psi_2(v)dH(v) - U_{22}(\underline{v})\right) \end{array} \right\} \qquad (\mathcal{P}^{ND})$$

subject to (MON_{11}), (MON_{12}), (MON_{22}), (IR_{11}), and (IR_{22}).

Note first that the non-negativity of allocations combines with the conditions (IC'_{11}) and (IC'_{22}) to simplify the participation constraints. Since $U'_{11}(\lambda) \geq 0$ for all $\lambda \in \Lambda$ (as $q_1 \geq 0$ and $G_\lambda \leq 0$), the participation constraint (IR_{11}) for cohort-one buyers in period one reduces to $U_{11}(\underline{\lambda}) \geq 0$; meanwhile, as is standard, the participation constraint (IR_{22}) for cohort-two buyers reduces to

[9] Analogous conditions were first imposed by Baron and Besanko (1984) and Besanko (1985). Meanwhile, Pavan, Segal, and Toikka (2012) develop a variety of sufficient conditions guaranteeing the validity of the first-order approach in general dynamic environments. In the absence of such conditions, Battaglini and Lamba (2012) show that local incentive compatibility need not yield global incentive compatibility.

$U_{22}(\underline{v}) \geq 0$. Clearly, both of these inequalities must bind in any solution to the problem ($\mathcal{P}^{ND}$), as the terms are merely additive constants.

Recall that, by Lemma 1, the allocation rules q_1 and q_2 must be cutoff rules. To pin down q_2, note that incentive compatibility of the cutoff rule, when combined with the binding cohort-two participation constraint, requires that the period-two mechanism correspond to a simple price; clearly, the cohort-two monopoly price p_H is then optimal. Moreover, note that the seller's objective function is linear in q_1. Thus, setting aside the remaining cohort-one constraints momentarily, the seller should simply sell the good only to those cohort-one buyers with nonnegative virtual values. We thus define the cutoffs $k^{ND}(\lambda)$ and α^{ND} by

$$\varphi_1(k^{ND}(\lambda), \lambda) = 0 \text{ and } \alpha^{ND} := p_H. \tag{5}$$

(To ensure that these cutoffs are well-defined, we set $k^{ND}(\lambda) := \underline{v}$ if $\varphi_1(\underline{v}, \lambda) > 0$.) The optimal allocation rules can then be written as

$$q_1^{ND}(v, \lambda) := \begin{cases} 0 & \text{if } v < k^{ND}(\lambda), \\ 1 & \text{if } v \geq k^{ND}(\lambda); \end{cases} \text{ and } q_2^{ND}(v) := \begin{cases} 0 & \text{if } v < \alpha^{ND}, \\ 1 & \text{if } v \geq \alpha^{ND}. \end{cases} \tag{6}$$

It only remains to determine the payment rule associated with these allocations. As is standard in dynamic mechanism design problems, there may be many payment schemes implementing the first-cohort allocation rule above. We choose to focus on a particularly natural set of option contracts, where buyers pay a premium in the initial period in order to decrease the strike price they face in the second period. Moreover, this strike price is chosen to yield interim individual rationality. Thus, we let

$$p_{12}^{ND}(v, \lambda) := q_1^{ND}(v, \lambda) k^{ND}(\lambda) \text{ and } p_{22}^{ND}(v) := q_2^{ND}(v, \lambda) \alpha^{ND}. \tag{7}$$

Finally, the initial period price p_{11}^{ND} is determined using the envelope condition (IC'_{11}); we define

$$p_{11}^{ND}(\lambda) := \int_{\mathbf{V}} \int_{\underline{v}}^{v} q_1^{ND}(v', \lambda) dv' dG(v|\lambda) + \int_{\underline{\lambda}}^{\lambda} \int_{\mathbf{V}} q_1^{ND}(v, \mu) G_\lambda(v|\mu) dv d\mu. \tag{8}$$

It is straightforward to show that this contract is incentive compatible in the initial period, and hence optimal.

THEOREM 1. *Suppose that Assumption 1 is satisfied. Then $\{q_1^{ND}, p_{11}^{ND}, p_{12}^{ND}\}$ and $\{q_2^{ND}, p_{22}^{ND}\}$, as defined in Equations (6), (7), and (8), are optimal contracts that solve problem ($\mathcal{P}^{ND}$).*

It is important to note that this cohort-one contract (depicted in Figure 1) is identical to that of Courty and Li (2000); since the seller has full commitment power and the buyers are unable to delay contracting, the presence of a second cohort of buyers does not affect the contract offered to the first cohort.[10]

3.2. *Dynamic Screening with Strategic Delay and a Fixed Period-Two Price*

We now present the optimal contract for the case in which strategic delay of contracting is permitted. We assume that the seller can commit in advance to her second-period contract, but

[10] Our proof differs slightly from that of Courty and Li (2000), and so is included in the appendix for completeness.

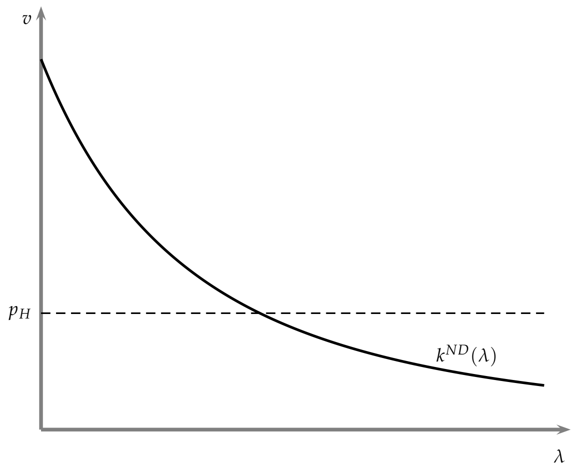

FIGURE 1. Optimal Contract without Strategic Delay

(as an intermediate step) we exogenously fix that contract and examine the incentives of the first-cohort buyers in isolation. (We will later allow the seller to optimally choose her second-period contract.) The seller's problem is then to

$$\max_{x,q_1,q_2,p_{11},p_{12},p_{22}} \left\{ \begin{array}{c} \iint_{\Lambda\times\mathbf{V}} [x(\lambda)q_1(v,\lambda) + (1-x(\lambda)q_2(v)]\, \varphi_1(v,\lambda) dG(v|\lambda) dF(\lambda) - U_{11}(\underline{\lambda}) \\ + \gamma \left(\int_{\mathbf{V}} q_2(v)\psi_2(v) dH(v) - U_{22}(\underline{v}) \right) \end{array} \right\} \quad (\mathcal{P}^{FC})$$

subject to (MON_{11}), (MON_{12}), (MON_{22}), (IR_{11}), (IR_{12}), (IR_{22}), and (SD).

So suppose that, for some exogenously given $\alpha \geq c$, the second-period contract is given by

$$q_2^{FC}(v) = \begin{cases} 0 & \text{if } v < \alpha, \\ 1 & \text{if } v \geq \alpha; \end{cases} \text{ and } p_{22}^{FC}(v) = q_2^{FC}(v)\alpha. \quad (9)$$

Thus, the seller is constrained to offer the good at the fixed price α in period two, without regard to that price's optimality or sequential rationality. Note, however, that such a contract *does* satisfy the incentive compatibility and individual rationality constraints (IC_{22}) and (IR_{22}). Thus, the seller now faces the "restricted" problem

$$\max_{x,q_1,p_{11},p_{12}} \left\{ \begin{array}{c} \iint_{\Lambda\times\mathbf{V}} \left[x(\lambda)q_1(v,\lambda) + (1-x(\lambda)q_2^{FC}(v)\right] \varphi_1(v,\lambda) dG(v|\lambda) dF(\lambda) - U_{11}(\underline{\lambda}) \\ + \gamma \left(\int_{\mathbf{V}} q_2^{FC}(v)\psi_2(v) dH(v) - U_{22}(\underline{v}) \right) \end{array} \right\} \quad (\mathcal{R}^{FC})$$

subject to (MON_{11}), (MON_{12}), (IR_{11}), (IR_{12}), and (SD).

Recall that the second-period contract serves as an outside option for each buyer in cohort one—each buyer can avail herself of the second-period price α instead of contracting in the initial period. Since different initial-period types value that contract differently, this implies that the second-period contract serves as a type-dependent participation constraint for buyers in cohort one; this is precisely the role of constraint (SD), which bounds cohort-one buyers' payoffs from below by the value of the second-period contract. Understanding the role played by this constraint is crucial to a determination of the optimal contract. The following lemma (with proof in the appendix) helps characterize when that constraint binds.

LEMMA 3. *If constraint (SD) binds at some $\hat{\lambda} \in \Lambda$ in the solution to ($\mathcal{R}^{FC}$), then it must bind at all $\lambda < \hat{\lambda}$.*

Thus, whenever the induced "participation constraint" arising from the possibility of strategic delay binds, it must bind on an interval that begins at the lowest possible type: we must have

$$U_{11}(\lambda) = U_{11}(\underline{\lambda}) - \int_{\underline{\lambda}}^{\lambda} \int_{\mathbf{V}} \bar{q}(v,\mu) G_{\lambda}(v|\mu) dv d\mu = \int_{\mathbf{V}} U_{22}(v) dG(v|\lambda)$$

for all λ in that interval (where we have used the envelope property from Lemma 2). Straightforward differentiation immediately implies that

$$\int_{\mathbf{V}} \bar{q}(v,\lambda) G_{\lambda}(v|\lambda) dv = -\int_{\mathbf{V}} U_{22}(v) \frac{\partial^2 G(v|\lambda)}{\partial v \partial \lambda} dv = \int_{\mathbf{V}} q_2^{FC}(v) G_{\lambda}(v|\lambda) dv$$

throughout the interval, where the second equality follows from integration by parts. Since both q_1 and q_2 are cutoff policies whenever (IC_{12}) and (IC_{22}) are satisfied, the expression above immediately implies that the cutoffs coincide. Thus, $q_1(v,\lambda) = q_2^{FC}(v)$ for all $\lambda \notin x^{-1}(0)$ at which constraint (SD) binds; that is, buyers in the interval where the constraint binds receive the same contract (guaranteeing allocation of the good in period two if their value is greater than α) regardless of whether or not they delay contracting.

Denote by $\hat{\lambda}$ the largest initial-period type for which the strategic delay constraint (SD) binds. Then, incorporating the cutoffs implied by Lemma 1, we may rewrite the problem ($\mathcal{R}^{FC}$) as

$$\max_{p_{11},p_{12},\hat{\lambda},k} \left\{ \begin{array}{c} \int_{\underline{\lambda}}^{\hat{\lambda}} \int_{\alpha}^{\bar{v}} \varphi_1(v,\lambda) dG(v|\lambda) dF(\lambda) + \int_{\hat{\lambda}}^{\bar{\lambda}} \int_{k(\lambda)}^{\bar{v}} \varphi_1(v,\lambda) dG(v|\lambda) dF(\lambda) \\ - \int_{\mathbf{V}} U_{22}(v) dG(v|\underline{\lambda}) + \gamma \left(\int_{\mathbf{V}} q_2^{FC}(v) \psi_2(v) dH(v) - U_{22}(\underline{v}) \right) \end{array} \right\}$$

subject to (MON_{11}) and (IR_{12}).

Note that constraint (IR_{11}) has been dispensed with and replaced by the binding constraint (SD) for the lowest period-one type $\underline{\lambda}$, as this buyer receives greater expected utility from the second-period contract than her reservation utility of zero. In addition, constraint (MON_{12}) has been replaced by the explicit inclusion of the cutoffs $k(\lambda)$. Finally, note that the monotonicity constraint (MON_{11}) requires $k(\lambda) \leq \alpha$ for all $\lambda \geq \hat{\lambda}$.

Define $\tilde{\lambda}(\alpha)$ by

$$\varphi_1(\alpha, \tilde{\lambda}(\alpha)) = 0, \tag{10}$$

and note that it is uniquely defined when φ_1 is increasing in both its arguments, as is imposed by Assumption 1. Notice that $\tilde{\lambda}(\alpha)$ is precisely the type for which pointwise maximization in ($\mathcal{P}^{ND}$), the problem without delay, would have led to $k^{ND}(\tilde{\lambda}(\alpha)) = \alpha$.

Now note that if $\hat{\lambda}$ is chosen such that $\hat{\lambda} > \tilde{\lambda}(\alpha)$, pointwise maximization of the objective function above yields cutoffs $k(\lambda)$ for all $\lambda > \hat{\lambda}$ satisfying $\varphi_1(k(\lambda),\lambda) = 0$. By Assumption 1, this implies that the monotonicity constraint (MON_{11}) is satisfied. However, profits can be increased by decreasing $\hat{\lambda}$, as there are buyers with positive virtual values (and hence positive profitability) that are being excluded. In particular, note that $\varphi_1(\alpha, \lambda) > 0$ for $\lambda \in (\tilde{\lambda}(\alpha), \hat{\lambda})$, so decreasing the cutoff for buyers with these values of λ is profitable.

Meanwhile, if $\hat{\lambda}$ is chosen such that $\hat{\lambda} < \tilde{\lambda}(\alpha)$, pointwise maximization yields cutoffs $k(\lambda) > \alpha$ for all $\lambda \in (\hat{\lambda}, \tilde{\lambda}(\alpha))$ (since $\varphi_1(\alpha, \lambda) < 0$ for these values of λ). Of course, this violates the monotonicity constraint (MON_{11}), which requires $k(\lambda) \leq \alpha$ for all $\lambda \geq \hat{\lambda}$. This implies that this constraint *must* bind for all $\lambda \leq \tilde{\lambda}(\alpha)$, at which pointwise maximization leaves the constraint slack. It is easy to see that this is the same as simply setting $\hat{\lambda} = \tilde{\lambda}(\alpha)$ and choosing $k(\lambda)$ optimally (without constraints) for $\lambda \geq \hat{\lambda}$, resulting in the no-delay-optimal cutoffs $k^{ND}(\lambda)$.

Thus, we have a candidate solution (which we will denote with the superscript FC) where

$$x^{FC}(\lambda) := 1;\ q_1^{FC}(v, \lambda) := \begin{cases} 0 & \text{if } v < k^{FC}(\lambda), \\ 1 & \text{if } v \geq k^{FC}(\lambda); \end{cases} \text{ and } k^{FC}(\lambda) := \begin{cases} \alpha & \text{if } \lambda \leq \tilde{\lambda}(\alpha), \\ k^{ND}(\lambda) & \text{if } \lambda > \tilde{\lambda}(\alpha). \end{cases} \tag{11}$$

Notice that no buyers are delayed, and the contracted allocation is everywhere at least as generous as that available by delay. In fact, we can also easily satisfy the "re-contracting" constraint (IR_{12}) by simply charging a price equal to the cutoff in the second period; that is, by setting

$$p_{12}^{FC}(v, \lambda) := q_1^{FC}(v, \lambda) k^{FC}(\lambda). \tag{12}$$

Finally, $p_{11}^{FC}(\lambda)$ is pinned down by integration of the envelope condition in Equation (IC'_{11}), where the constant of integration is simply the (strictly positive) utility that the lowest type $\underline{\lambda}$ receives from the contract above:

$$\begin{aligned} p_{11}^{FC}(\lambda) := \int_{\mathbf{V}} \int_{\underline{v}}^{v} q_1^{FC}(v', \lambda) dv' dG(v|\lambda) + \int_{\underline{\lambda}}^{\lambda} \int_{\mathbf{V}} q_1^{FC}(v, \mu) G_\lambda(v|\mu) dv d\mu \\ - \int_{\mathbf{V}} \int_{\underline{v}}^{v} q_2^{FC}(v') dv' dG(v|\underline{\lambda}). \end{aligned} \tag{13}$$

Thus, it only remains to verify that the payment rules above implement the optimal allocation. Given Assumption 1 on the monotonicity of φ_1 and the characterization of incentive compatibility in Lemma 2, however, we can show (with details in the appendix) that this is indeed the case.

THEOREM 2. *Suppose that the second-period contract $\{q_2^{FC}, p_{22}^{FC}\}$ corresponds to a fixed price $\alpha \geq c$, and suppose further that Assumption 1 is satisfied. Then the contract $\{x^{FC}, q_1^{FC}, p_{11}^{FC}, p_{12}^{FC}\}$ defined in Equations* (11), (12), *and* (13) *is an optimal contract that solves problem ($\mathcal{R}^{FC}$).*

Thus, the optimal contract (depicted in Figure 2) takes a particularly appealing form: no buyers delay contracting to the second period; all buyers with types below $\tilde{\lambda}(\alpha)$ are guaranteed the fixed price α at no additional upfront cost; and all buyers with types above $\tilde{\lambda}(\alpha)$ receive the same contract as they would if there were no delay permitted, but with a smaller upfront premium. In addition, notice that we may view the benchmark no-delay case discussed above as a special case of the above solution. In particular, by fixing $\alpha \geq \bar{v}$ (that is, fixing the second-period price to be no smaller than the upper bound of the support of values), cohort-one buyers are effectively dissuaded from delaying to the second period. This leads to an optimal contract for cohort-one buyers identical to that in the no-delay case.

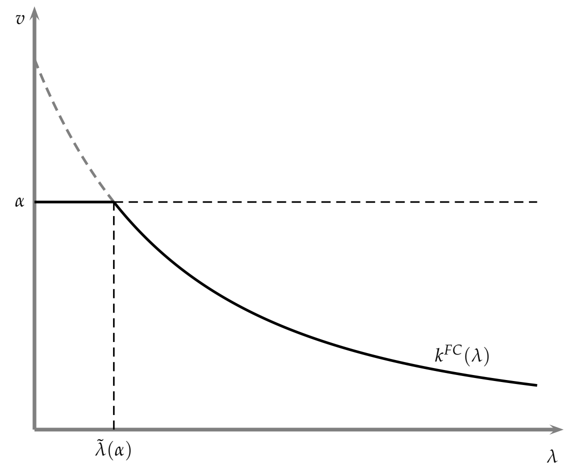

FIGURE 2. Optimal Contract with Strategic Delay and a Fixed Period-Two Price

3.3. *Dynamic Screening with Strategic Delay and an Optimal Period-Two Price*

We now consider the seller's problem *without* artifically restricting her second-period price. Notice first that the constraints (IC_{22}) and (IR_{22}) limit the seller to second-period contracts that correspond to a fixed price α. Moreover, Theorem 2 pins down the optimal contract offered to buyers in the first cohort for *any* second-period price α. In fact, we can write the seller's total profits (from both cohorts of buyers) for any second-period price α as

$$\begin{aligned}\Pi^{FC}(\alpha) := &\int_{\underline{\lambda}}^{\tilde{\lambda}(\alpha)} \int_{\alpha}^{\bar{v}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) dv d\lambda + \int_{\tilde{\lambda}(\alpha)}^{\bar{\lambda}} \int_{k^{ND}(\lambda)}^{\bar{v}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) dv d\lambda \\ &- \int_{\alpha}^{\bar{v}} (v-\alpha) g(v|\underline{\lambda}) dv + \gamma \left[(\alpha - c)(1 - H(\alpha)) \right].\end{aligned} \tag{14}$$

Using this expression, we can characterize (with details in the appendix) the optimal second-period price.

COROLLARY 1. *Suppose α^{FC} maximizes $\Pi^{FC}(\alpha)$ in Equation (14), and suppose that Assumption 1 is satisfied. Then the contracts $\{x^{FC}, q_1^{FC}, p_{11}^{FC}, p_{12}^{FC}\}$ and $\{q_2^{FC}, p_{22}^{FC}\}$ defined in Equations* (9), (11), (12), *and* (13) *(with $\alpha = \alpha^{FC}$) are optimal contracts for a seller with full commitment. Moreover, $\alpha^{FC} > p_H$.*

Thus, the seller optimally increases the second-period price above the price that would be charged if there were no strategic delay. Clearly, this leads to a decrease in profits from second-cohort buyers alone—in the absence of potential buyers from cohort one, the seller would simply charge p_H in the second period. This is, of course, offset by two effects. First, the outside option of *all* first-cohort buyers is reduced, as delayed contracting involves a higher price; this implies that the induced participation constraint (SD) is relaxed somewhat, reducing the rents left to cohort-one buyers. Second, the increased price improves the seller's ability to screen buyers in the first period by relaxing the monotonicity constraint (MON_{11}) implied by incentive compatibility. In particular, recall that the buyer's allocation rule, for any $\lambda \leq \tilde{\lambda}(\alpha)$, is identical to the outside option, while for any $\lambda \geq \tilde{\lambda}(\alpha)$, it has the same (optimal) distortions as when delay is not permitted. Since $\tilde{\lambda}$ is decreasing, this implies that raising the second-period contract price increases the seller's ability to impose additional distortions that reduce buyers' information rents.

Furthermore, notice that in the limit as γ shrinks to zero, the impact of cohort-two entrants on the seller's profits also shrinks to zero. This implies that the seller's tradeoff between reducing the outside option of cohort-one buyers and maximizing profits from cohort-two buyers disappears—there is no longer a second-period cost associated with reducing the value of strategic delay. Indeed, the proof of Corollary 1 demonstrates that, in the limit, we have $\partial\Pi^{FC}(\alpha)/\partial\alpha > 0$ for all $\alpha < \bar{v}$. Therefore, in the limit, the seller commits to $\alpha^{FC} \geq \bar{v}$ and no sales in the final period, thereby reducing the value of strategic delay to zero.

4. Dynamic Screening with Limited Commitment

We now consider the general solution to the problem with limited commitment. Returning to the objective function in Equation (4), we can write the seller's problem as

$$\max_{x,p_{11},p_{12},q_1,p_{22},q_2} \left\{ \begin{aligned} &\iint_{\Lambda\times\mathbf{V}} \left[x(\lambda)q_1(v,\lambda) + (1-x(\lambda))q_2(v)\right]\varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) - U_{11}(\underline{\lambda}) \\ &\quad + \gamma\left(\int_{\mathbf{V}} q_2(v)\psi_2(v)dH(v) - U_{22}(\underline{v})\right) \end{aligned} \right\} \tag{$\mathcal{P}^{LC}$}$$

subject to (MON_{11}), (MON_{12}), (MON_{22}), (IR_{11}), (SD), (IR_{12}), (IR_{22}), and (SR).

As in the case above where the seller can pre-commit to future contracts, the second-period incentive compatibility and individual rationality constraints (MON_{22}) and (IR_{22}) imply that the seller's period-two contract will always take the form of a price; that is, there exists some $\alpha \in \mathbf{V}$ such that

$$q_2^{LC}(v) := \begin{cases} 0 & \text{if } v < \alpha, \\ 1 & \text{if } v \geq \alpha; \end{cases} \text{ and } p_{22}^{LC}(v) := q_2(v)\alpha. \tag{15}$$

Notice, however, that the sequential rationality constraint (SR) implies that the seller's optimization problem involves an "internal" problem for the choice of the second-period price α. In particular, α must solve

$$\max_{\alpha'} \left\{ (\alpha' - c)\left[\int_{\Lambda} (1-x(\lambda))(1-G(\alpha'|\lambda))dF(\lambda) + \gamma(1-H(\alpha'))\right] \right\}. \tag{SR$'$}$$

Since $x(\lambda)$ is endogenously determined, this implies that Lemma 3—which describes the set of initial-period types for which the constraint (SD) binds *in an optimal contract*—no longer applies to the seller's problem with limited commitment. The following result, however, demonstrates (with details in the appendix) that Lemma 3 does in fact extend to the present setting with limited commitment.[11]

Lemma 4. *If constraint (SD) binds at some $\hat{\lambda} \in \Lambda$ in the solution to ($\mathcal{P}^{LC}$), then it must bind at all $\lambda < \hat{\lambda}$.*

[11]Recall that Lemmas 1 and 2 describe the properties and implications of the incentive compatibility constraints (IC_{11}), (IC_{12}), and (IC_{22}). As these results make no further assumptions about the optimization problem that the constraints apply to, they continue to hold in the present problem.

As was the case with a fixed second-period contract, we can use Lemma 4 to pin down the allocation rule for buyers that face a binding "induced" participation constraint and contract in period one. Since (SD) binds on an interval $[\underline{\lambda}, \hat{\lambda}]$, Lemma 2 then implies that all cohort-one buyers with types $\lambda \in [\underline{\lambda}, \hat{\lambda}] \setminus x^{-1}(0)$—that is, those buyers with initial types below $\hat{\lambda}$ who do not delay contracting—receive *exactly* their anticipated outside option via the optimal period-one contract. Thus, $\bar{q}(v, \lambda) = q_2(v)$ for all $\lambda \in [\underline{\lambda}, \hat{\lambda}]$. Moreover, recall that Lemma 1 implies that the first-period contract's allocation rule is a set of cutoff policies. We can therefore rewrite the seller's problem as

$$\max_{x, \hat{\lambda}, p_{11}, p_{12}, k} \left\{ \begin{array}{c} \int_{\underline{\lambda}}^{\hat{\lambda}} \int_{\alpha}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) + \int_{\hat{\lambda}}^{\bar{\lambda}} \int_{k(\lambda)}^{\bar{v}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda) \\ + \gamma(\alpha - c)(1 - H(\alpha)) - U_{11}(\underline{\lambda}) \end{array} \right\}$$

subject to (MON_{11}), (IR_{11}), (SD), (IR_{12}), and (SR′).

Note that the monotonicity constraint (MON_{11}) implies that we must have $k(\lambda) \leq \alpha$ for all $\lambda \geq \hat{\lambda}$. Fixing $\hat{\lambda}$ and $x(\cdot)$ (and hence, via the sequential rationality constraint (SR′), also fixing α), we can isolate the question of optimally choosing the cutoffs $k(\lambda)$. As was the case with the full-commitment problem ($\mathcal{R}^{FC}$), it is easy to see that the optimal allocation and cutoff are

$$q_1^{LC}(v, \lambda) := \begin{cases} 0 & \text{if } v < k^{LC}(\lambda), \\ 1 & \text{if } v \geq k^{LC}(\lambda); \end{cases} \text{ where } k^{LC}(\lambda) := \begin{cases} \alpha & \text{if } \lambda \leq \tilde{\lambda}(\alpha), \\ k^{ND}(\lambda) & \text{if } \lambda > \tilde{\lambda}(\alpha). \end{cases} \tag{16}$$

Therefore, if the choice of $\hat{\lambda}$ is such that $\hat{\lambda} \geq \tilde{\lambda}(\alpha)$, then k^{LC} immediately "jumps" down to the pointwise optimal (no-delay) cutoff k^{ND} to the right of $\hat{\lambda}$. On the other hand, if $\hat{\lambda} < \tilde{\lambda}(\alpha)$, we can "extend" $\hat{\lambda}$ to $\tilde{\lambda}(\alpha)$. Note, as before, that Assumption 1 implies that k^{LC} is decreasing, and so the induced allocation rule is increasing in λ.

We now turn to characterizing the set of buyers that delay contracting—which, via sequential rationality, in turn determines the period-two price. Any solution α to (SR′) must satisfy its associated first-order condition; with some rearrangement, this condition may be written as

$$\int_{\Lambda} (1 - x(\lambda)) \psi_1(\alpha, \lambda) g(\alpha|\lambda) dF(\lambda) + \gamma \psi_2(\alpha) h(\alpha) = 0, \tag{SR$''$}$$

where

$$\psi_1(v, \lambda) := v - c - \frac{1 - G(v|\lambda)}{g(v|\lambda)}$$

is the (static) virtual value for the conditional distribution of cohort-one values $G(\cdot|\lambda)$. In general, the first-order condition (SR″) is necessary but not sufficient. Moreover, even when (SR″) *is* sufficient, drawing meaningful conclusions about the prices generated by different delay decisions need not be possible: first-order stochastic dominance alone does not provide sufficient structure for meaningful analysis. We therefore make two additional assumptions about the environment:

ASSUMPTION 2. *The monopoly profit functions* $\{(p - c)(1 - G(p|\lambda))\}_{\lambda \in \Lambda}$ *and* $(p - c)(1 - H(p))$ *are strictly concave for all* $p \in [c, \bar{v}]$.

ASSUMPTION 3. *The monopoly price* $p_\lambda := \text{argmax}_p\{(p - c)(1 - G(p|\lambda))\}$ *is strictly increasing in* λ.

Assumption 2 guarantees the uniqueness of a solution to the second-period maximization problem (SR′) for any delay choices $x(\lambda)$; it also (via the Theorem of the Maximum) guarantees the continuity of that solution. Assumption 3, on the other hand, implies that the period two-price responds in the expected manner to changes in the set of delayed buyers: roughly speaking, delaying buyers with higher initial types increases the period-two price while delaying buyers with lower initial types decreases the period-two price.[12,13]

With this additional structure in hand, it is possible to show that it is without loss of generality to consider contracts that induce all buyers within an interval to delay contracting, while all other buyers contract immediately. Any contract in which delayed buyers are "dispersed"—that is, with either "gaps" in the set of delayed buyers or only "fractional" delay for some types—can be improved upon by concentrating the mass of delayed buyers and "closing" any gaps. Indeed, this can be done while leaving the induced second-period price unchanged. This implies that the seller's ability to price discriminate in the first period is improved *without* affecting the tradeoff between cohort-one outside options and period-two profits. It is easy to see that this implies an increase in the seller's profits. A complete proof of this result may be found in the appendix.

LEMMA 5. *Suppose Assumptions 2 and 3 are satisfied. Fix any period-one contract $\{x^*, q_1^*, p_{11}^*, p_{12}^*\}$ such that $\int_\Lambda (1 - x^*(\lambda))dF(\lambda) > 0$ with induced second-period price α^* and total profits Π^*. Then there exists some $\mu_1, \mu_2 \in \Lambda$ and another period-one contract $\{x^{**}, q_1^{**}, p_{11}^{**}, p_{12}^{**}\}$ with*

$$x^{**}(\lambda) = \begin{cases} 0 & \text{if } \lambda \in [\mu_1, \mu_2], \\ 1 & \text{otherwise;} \end{cases}$$

such that the induced second-period price is $\alpha^{**} = \alpha^*$ and total profits are $\Pi^{**} \geq \Pi^*$.

With this result in hand, we can therefore rewrite the seller's problem as

$$\max_{\mu_1,\mu_2,p_{11},p_{12}} \left\{ \begin{aligned} &\int_{\underline{\lambda}}^{\mu_2} \int_{\alpha}^{\bar{v}} \varphi_1(v,\lambda) dG(v|\lambda) dF(\lambda) + \int_{\mu_2}^{\bar{\lambda}} \int_{k^{LC}(\lambda)}^{\bar{v}} \varphi_1(v,\lambda) dG(v|\lambda) dF(\lambda) \\ &\qquad + \gamma(\alpha - c)(1 - G(\alpha|\hat{\mu})) - \int_{\alpha}^{\bar{v}} (v - \alpha) dG(v|\underline{\lambda}) \end{aligned} \right\} \tag{$\mathcal{R}^{LC}$}$$

subject to (IR_{12}) and (SR′′).

Notice that we have incorporated the optimal allocation and cutoffs defined in Equation (16); replaced $\hat{\lambda}$ with μ_2; and also replaced constraint (IR_{11}) with the binding constraint (SD) at $\underline{\lambda}$, as the non-negative option value of delayed contracting implies (IR_{11}) is satisfied. The only remaining constraints are the re-contracting constraint and sequential rationality constraints (IR_{12}) and (SR′′).

Before characterizing the optimal interval of delayed buyers, note that we may first determine the seller's optimal payment rules for buyers that contract in period one. In particular, let

$$p_{12}^{LC}(v,\lambda) := q_1^{LC}(v,\lambda) k^{LC}(\lambda). \tag{17}$$

[12] A sufficient condition that implies Assumption 3 is that the family of conditional distributions $\{G(\cdot|\lambda)\}_{\lambda\in\Lambda}$ is ordered by hazard rate dominance: if $\lambda > \lambda'$, then $g(v|\lambda)/(1 - G(v|\lambda)) \leq g(v|\lambda')/(1 - G(v|\lambda'))$ for all $v \in \mathbf{V}$.

[13] Note that Assumptions 2 and 3 are satisfied by a wide variety of distributional families. One natural example is the family of power distributions where $G(v|\lambda) = v^\lambda$ for $v \in \mathbf{V} = [0, 1]$ and $\lambda \in \Lambda \subseteq \mathbb{R}_+$.

This corresponds to simply charging each buyer a price equal to her (type-dependent) cutoff. Since $k^{LC}(\lambda) \leq \alpha$ for all λ, this also implies that the re-contracting constraint (IR_{12}) is satisfied. Moreover, $p_{11}^{LC}(\lambda)$ is determined by integration of the envelope condition in Equation (IC'_{11}), where the constant of integration is simply the (non-negative) utility received by the lowest type $\underline{\lambda}$:

$$\begin{aligned} p_{11}^{LC}(\lambda) := \int_{\mathbf{V}} \int_{\underline{v}}^{v} q_1^{LC}(v', \lambda) dv' dG(v|\lambda) + \int_{\underline{\lambda}}^{\lambda} \int_{\mathbf{V}} q_1^{LC}(v, \mu) G_\lambda(v|\mu) dv d\mu \\ - \int_{\mathbf{V}} \int_{\underline{v}}^{v} q_2^{LC}(v') dv' dG(v|\underline{\lambda}). \end{aligned} \tag{18}$$

Notice that this payment rule is the immediate analogue (in the limited-commitment setting) of $p_{11}^{FC}(\lambda)$ defined in Equation (13).

With these preliminaries in hand, we can proceed to our main result (a complete proof of which may be found in the appendix) characterizing the seller's optimal contract when she has limited commitment power.

THEOREM 3. *Suppose that Assumptions 1, 2, and 3 are satisfied. Let μ_1^{LC} and μ_2^{LC} solve problem ($\mathcal{R}^{LC}$), and define*

$$x^{LC}(\lambda) := \begin{cases} 0 & \text{if } \lambda \in [\mu_1^{LC}, \mu_2^{LC}], \\ 1 & \text{otherwise.} \end{cases}$$

Then

(1) *the contracts $\{x^{LC}, q_1^{LC}, p_{11}^{LC}, p_{12}^{LC}\}$ and $\{q_2^{LC}, p_{22}^{LC}\}$ defined in Equations (15), (16), (17), and (18) (with $\alpha = \alpha^{LC}$) are optimal contracts for a seller with limited commitment;*
(2) *the solution to (SR''), given μ_1^{LC} and μ_2^{LC}, is $\alpha^{LC} \in [p_H, \alpha^{FC}]$;*
(3) *if $\alpha^{LC} \in (p_H, \alpha^{FC})$, then the set of cohort-one buyers that delay contracting is an interval $[\mu_1^{LC}, \mu_2^{LC}]$ with $\hat{\mu} \leq \mu_1^{LC} < \mu_2^{LC}$; and*
(4) *the seller's profits are given by*

$$\Pi^{LC} := \Pi^{FC}(\alpha^{LC}) - \int_{\tilde{\lambda}(\alpha^{LC})}^{\max\{\tilde{\lambda}(\alpha^{LC}), \mu_2\}} \int_{k^{ND}(\lambda)}^{\alpha^{LC}} \varphi_1(v, \lambda) dG(v|\lambda) dF(\lambda). \tag{19}$$

There are several key features of the characterization in Theorem 3 (illustrated in Figure 3) to note. First, the seller delays an interval of cohort-one buyers; by doing so, she is able to increase the second-period price α^{LC} above the cohort-two monopoly price p_H. This has two effects: it relaxes the participation constraint (SD) induced by the possibility of strategic delay, thereby increasing the seller's profits from cohort-one buyers; but it also moves prices away from the cohort-two optimum, thereby decreasing the seller's profits from cohort-two buyers. Notice, however, that the induced second-period price α^{LC} is lower than the second-period price under full commitment α^{FC}. Thus, in the absence of commitment concerns, the seller would continue trading off cohort-two profits in order to decrease the implicit outside option available to cohort-one buyers. This tradeoff is not feasible when the seller has limited commitment, however; in order to continue increasing the period-two price, the seller needs to induce delayed contracting by additional buyers with higher types (as a cohort-one buyer's impact on the sequentially rational second-period price is increasing in her type). While doing so succeeds in reducing the endogenous outside option in

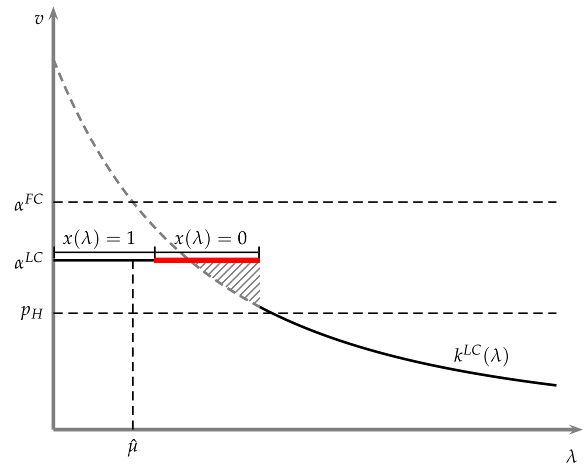

FIGURE 3. Optimal Contract with Limited Commitment

period one, it also reduces the seller's ability to screen buyers in the first period. In particular, each additional buyer that delays contracting until the second period is a buyer whom the seller cannot separate in the first period and sequentially screen. This reduced screening ability reflects the "deadweight loss" of limited commitment: in order to reduce the option value of strategic delay, the seller must inefficiently exclude buyers with initial types $\lambda \in (\tilde{\lambda}(\alpha^{LC}), \mu_2^{LC})$ and realized values $v \in (k^{ND}(\lambda), \alpha^{LC})$. If, on the other hand, the seller was able to fully commit in advance to a period-two price equal to α^{LC}, these buyers (indicated by the shaded region in Figure 3) would not be excluded. This deadweight loss (measured by the integral in Equation (19)) is, of course, in addition to the fact that α^{LC} is suboptimally low for a seller with full commitment power.

Note further that there is, in general, a gap between the type $\hat{\mu}$ corresponding to the cohort-two monopoly price and the lower bound of the interval of delayed types. To see why this is the case, notice that as we increase the mass of cohort-one buyers who delay contracting, we decrease the responsiveness of the sequentially rational second-period price to the composition of that set. Thus, increasing the second-period price is most efficiently achieved by inducing the delay of a relatively small set of buyers with relatively high monopoly prices (as opposed to a large set of buyers with low-to-intermediate monopoly prices). This implies that the optimal contract in the case of limited supply will generally display an interesting non-monotonicity: cohort-one buyers with relatively low and relatively high initial-period types will contract in the initial period, while intermediate initial-period types will delay contracting to the second period.

It is easy to see that the optimal contract described in Theorem 3 satisfies the "re-contracting" constraint (IR_{12}). Each buyer with initial type λ who contracts in period one faces a call option in period two, where the strike price for consumption is the cutoff $k^{LC}(\lambda)$. Since $k^{LC}(\lambda) \leq \alpha^{LC}$ for all λ, a buyer that exits her initial-period contract in period two and partakes in the second-period mechanism faces a higher price. Doing so is clearly suboptimal, and so buyers do not benefit from their ability to anonymously re-contract with the seller in period two.[14] This does

[14]Even if we implemented the optimal allocation using refund contracts, as in Courty and Li (2000), this would continue to be the case. In fact, we could even allow buyers to claim their refund and then purchase at the prevailing price in period two; in the optimal contract, there is no benefit to doing so.

not imply, however, that the optimal initial-period contracts are renegotiation-proof. Indeed, the seller has a strict incentive to renegotiate with type-λ buyers who face a strike price $k^{LC}(\lambda)$ greater than the λ-specific monopoly price (which solves $\psi_1(v,\lambda) = 0$). It is easy to see that the set of such buyers is generally nonempty: Theorem 3 shows that all initial-period types $\lambda < \hat{\mu}$ face a cutoff $k^{LC}(\lambda) = \alpha^{LC} > p_H$, while Assumption 3 implies that they have monopoly prices strictly lower than p_H. Thus, even though the seller is unable to commit to future contractual terms, her commitment to long-term contracts plays an important role in her rent-extraction ability.

Recall that we have approached this dynamic contracting problem in a "reduced form" approach that focuses on the allocations and payments that are generated by equilibrium behavior by the seller and buyers. We have not yet verified, however, that there indeed exists a perfect Bayesian equilibrium in the underlying game that implements the contracts described above. Suppose, however, that the seller offers the menu of call options

$$\mathcal{M} := \{(p_{11}^{LC}(\lambda), k^{LC}(\lambda))\}_{\lambda > \mu} \cup \{(0, \alpha^{LC})\}$$

in the first period, where each $(\pi_1, \pi_2) \in \mathcal{M}$ denotes an upfront premium π_1 that guarantees the period-two strike price π_2. If each cohort-one buyer expects that all buyers with initial-period types $\lambda \in [\mu_1^{LC}, \mu_2^{LC}]$ will delay contracting, then they will expect the seller to (rationally) set a price α^{LC} in the second period. Moreover, as each buyer is infinitesimal, a unilateral deviation (either to delay or to contract in the first period) will not affect the seller's second-period pricing problem (SR). Since constraints (IC_{11}) and (SD) are satisfied in the contract described in Theorem 3, this implies that it is, in fact, optimal for buyers with $\lambda < \mu_1^{LC}$ to choose the $(0, \alpha^{LC})$ option; for buyers with $\lambda \in [\mu_1^{LC}, \mu_2^{LC}]$ to delay contracting; and for buyers with $\lambda > \mu_2^{LC}$ to choose the $(p_{11}^{LC}(\lambda), k^{LC}(\lambda))$ option. Thus, the seller will indeed set her second-period equal to α^{LC}. Finally, the strike prices have been chosen to implement the optimal cohort-one allocation rule q_1^{LC}. Thus, the sequence of contracts described in Theorem 3 are indeed implementable in a perfect Bayesian equilibrium.

This equilibrium construction highlights one potential shortcoming of the optimal contract described in Theorem 3. In equilibrium, "low" buyers that contract in the first period and "intermediate" buyers that delay contracting until period two receive contracts that are essentially identical; that is, members of each group are indifferent about their timing of contracting. If the seller is willing to sacrifice an arbitrarily small amount of profits, however, it is possible to provide *strict* incentives to these two groups of buyers. In particular, the seller can decrease the likelihood of allocation promised to low buyers while also providing them with a small subsidy; doing so provides a strict incentive to contract immediately. Intermediate buyers, however, derive greater benefit from allocation due to their increased likelihood of high values, and so prefer to delay contracting.

To see this more formally, fix any arbitrarily small $\epsilon > 0$, and define the allocation rule

$$q_1^{\epsilon}(v,\lambda) := \begin{cases} 0 & \text{if } v < k^{\epsilon}(\lambda), \\ 1 & \text{if } v \geq k^{\epsilon}(\lambda); \end{cases} \quad \text{where } k^{\epsilon}(\lambda) := \begin{cases} \alpha^{LC} + \epsilon & \text{if } \lambda \leq \mu_1^{LC}, \\ k^{ND}(\lambda) & \text{if } \lambda \geq \mu_2^{LC}. \end{cases}$$

This allocation rule (depicted in Figure 4) simply increases the cutoff for buyers that, in the opti-

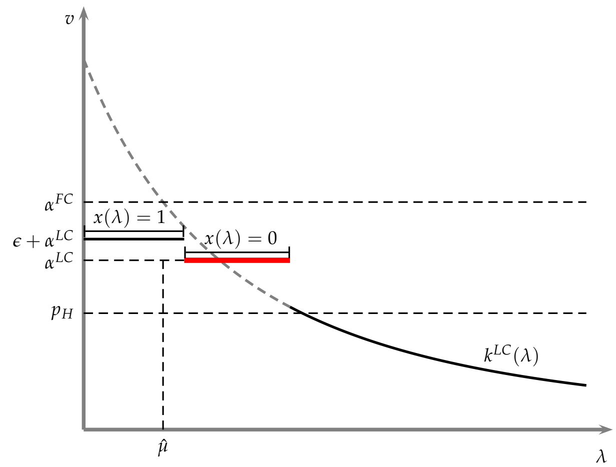

FIGURE 4. ϵ-Optimal Contract with Limited Commitment

mal contract, contract immediately while being indifferent about delay. Of course, k^ϵ is (weakly) decreasing, so initial-period incentive compatibility is maintained. Indeed, this is true even when we continue to charge a second-period price $p^\epsilon_{12}(v,\lambda) = q^\epsilon_1(v,\lambda)k^\epsilon(\lambda)$ equal to the cutoff—the first-period payment rule must be modified. In particular, we can subsidize all cohort-one buyers with $\lambda < \mu^{LC}_1$ by an amount

$$\delta := \int_{\alpha^{LC}}^{\alpha^{LC}+\epsilon} (1 - G(v|\mu^{LC}_1))dv,$$

which is chosen to make type μ^{LC}_1 exactly indifferent between contracting immediately and delaying until period two. Thus, if a buyer with initial-period type λ chooses the contract intended for types $\lambda' \in [\underline{\lambda}, \mu^{LC}_1)$, her expected payoff is

$$\begin{aligned}\delta + \int_{\mathbf{V}} (q^\epsilon_1(v,\lambda')v - p^\epsilon_{12}(v,\lambda'))dG(v|\lambda) &= \delta + \int_{\alpha^{LC}+\epsilon}^{\bar{v}} (v - \alpha^{LC} - \epsilon)dG(v|\lambda) \\ &= \int_{\alpha^{LC}}^{\alpha^{LC}+\epsilon} (1 - G(v|\mu^{LC}_1))dv + \int_{\alpha^{LC}+\epsilon}^{\bar{v}} (1 - G(v|\lambda))dv.\end{aligned}$$

Meanwhile, if this buyer postpones contracting to the second period, her expected payoff is

$$\int_{\mathbf{V}} (q^{LC}_2(v)v - p^{LC}_{22}(v))dG(v|\lambda) = \int_{\alpha^{LC}}^{\bar{v}} (v - \alpha^{LC})dG(v|\lambda) = \int_{\alpha^{LC}}^{\bar{v}} (1 - G(v|\lambda))dv.$$

The former option is strictly preferred to the latter if, and only if,

$$\begin{aligned}0 &< \int_{\alpha^{LC}}^{\alpha^{LC}+\epsilon} (1 - G(v|\mu^{LC}_1))dv + \int_{\alpha^{LC}+\epsilon}^{\bar{v}} (1 - G(v|\lambda))dv - \int_{\alpha^{LC}}^{\bar{v}} (1 - G(v|\lambda))dv \\ &= \int_{\alpha^{LC}}^{\alpha^{LC}+\epsilon} (1 - G(v|\mu^{LC}_1))dv - \int_{\alpha^{LC}}^{\alpha^{LC}+\epsilon} (1 - G(v|\lambda))dv = \int_{\alpha^{LC}}^{\alpha^{LC}+\epsilon} [G(v|\lambda) - G(v|\mu^{LC}_1)]dv.\end{aligned}$$

Of course, this inequality holds whenever $\lambda < \mu^{LC}_1$; the subsidy δ is more than sufficient to compensate these buyers for their less attractive allocation. On the other hand, buyers with initial-period types $\lambda \geq \mu^{LC}_1$ still prefer to sort as they did in the limited-commitment optimal contract, and their expected payoff remains unchanged. Thus, it is easy to see that the seller's loss from this contract can be made arbitrarily small by taking ϵ to zero; that is, it is possible to provide strict incentives for the "best" timing of purchases using an ϵ-optimal contract for arbitrarily small $\epsilon > 0$.

5. Concluding Remarks

A critical assumption in our analysis is our assumption, as is common in the dynamic mechanism design literature, that the distribution of values $G(\cdot|\lambda)$ for any type λ first-order stochastically dominates that of any lower type $\lambda' < \lambda$. A natural alternative ordering is one in which higher types face greater uncertainty about their realized values (in the sense of second-order stochastic dominance). Indeed, Courty and Li (2000) also examined a special case of second-order stochastic dominance in which the family of distributions $\{G(\cdot|\lambda)\}_{\lambda\in\Lambda}$ are rotation ordered.[15] They showed that, under certain additional conditions on the allocation rule, monotonicity of the optimal cutoffs $k^{ND}(\cdot)$ continues to be sufficient for incentive compatibility and hence optimality. The additional assumptions ensure that the cutoffs $k^{ND}(\cdot)$ are in regions where the cumulative distributions are ordered in a well-behaved manner similar to first-order stochastic dominance.

Unfortunately, our analysis in the present work does not in general extend to such type spaces. (In the special case where both $k^{ND}(\cdot)$ and p_H are larger than the distributional rotation point, our results extend immediately. In this case, all the relevant allocations occur in a region where types are ordered "as though" by first-order stochastic dominance.) One difficulty arises in identifying the set of types for which the strategic delay constraint (SD) binds in an optimal contract—even for a fixed period-two price. In particular, finding an analogue to Lemma 3 is difficult because there is no general relationship governing the relative rankings of the cutoffs $k^{ND}(\cdot)$ and the period-two price α; the utility from either may be increasing, decreasing, or simply non-monotone in λ. An additional complication arises in fully characterizing initial-period incentive compatibility: without the structure imposed by first-order stochastic dominance, general necessary and sufficient conditions for incentive compatibility are not known. This precludes the "first-order approach" to solving for optimal contracts, thereby necessitating a consideration of "global" incentive compatibility constraints as in Battaglini and Lamba (2012).

There are a number of other natural generalizations of our model. One such extension is to allow for uncertain market conditions in the second period; for example, suppose that the mass γ of cohort-two entrants is randomly drawn from some distribution. When the seller can observe the realization of this draw before choosing her period-two mechanism, the price charged in the second period becomes (from the perspective of a cohort-one buyer) a random variable. Clearly, there exists some deterministic "certainty equivalent" contract for each initial-period type λ such that the buyer receives exactly the same utility from contracting immediately instead of delaying contracting to period two. However, first order stochastic dominance is not sufficient to guarantee that this certainty equivalent price is well-behaved and monotonically decreasing in λ. Since incentive compatibility requires monotonicity in the effective allocation rule, this implies that there

[15] In the rotation order, all distributions $G(v|\lambda)$ pass through a single point $z \in \mathbf{V}$. Moreover, $G(v|\lambda)$ is increasing in λ for all $v < z$ and decreasing in λ for all $v > z$. For more on this order, see Johnson and Myatt (2006).

can no longer be an interval of types contracting in the initial period for whom the "induced participation constraint" binds.[16] This, in turn, complicates the seller's revenue management problem as she now must provide rents (beyond this endogenous outside option) to low initial types to prevent them from delaying contracting until the second period.

A second natural extension of our framework provides a tractable model to study a seller without commitment who has a *limited* supply of the good. Since we have a continuum of buyers in each period, there is no aggregate uncertainty regarding the distribution of values of cohort one buyers in the second period. Thus, for a given contract, the seller knows exactly the quantity promised to cohort-one buyers who contract in the first period. This precludes over-sale situations in which the seller promises a greater quantity than her available supply. Moreover, it is easy to see that the optimal second-period price is simply determined by market-clearing. Despite this apparent simplicity, however, the case of limited supply introduces some additional complexity: in addition to determining which cohort-one buyer purchases should be postponed to the second period, the seller must also determine the constraint on period-two supply remaining after period-one contracting. In particular, offering generous contracts in the first period ensures that there is greater scarcity and increased competition in the second period, thereby inducing higher prices.[17] Understanding the tradeoffs involved when the seller has an additional avenue for endogenously generating commitment power is certainly an interesting question, but is also one that is beyond the scope of the present work.

One limitation of our approach is that all consumption occurs at the end of period two, and we do not permit multiple periods of consumption. In practice, there are many dynamic contracting environments in which buyers' valuations evolve and payoffs accrue over multiple periods. In such settings, the private information at any point of time is correlated with future preferences (as in our model), while remaining payoff-relevant in the present. Pavan, Segal, and Toikka (2012) develop useful tools for such a setting under the assumptions that the seller has full commitment power and that agents cannot strategically delay contracting. Relaxing these assumptions would almost surely yield rich predictions. We leave such questions, however, for future work.

[16]Similar complications can arise in our setting when we consider stochastic (as opposed to deterministic) contracts.

[17]This is similar to the intuition (in a model with fully persistent private information) of Dilme and Li (2012), where the seller holds occasional fire-sales in order to endogenously commit to future supply restrictions.

APPENDIX

PROOF OF LEMMA 2. We begin by proving the necessity of condition (IC'_{11}). So suppose that the incentive compatibility constraints (IC_{11}), (IC_{12}), and (IC_{22}) are satisfied, and consider the expected payoff $U_{11}(\lambda, \lambda')$ of a cohort-one buyer with initial type λ who reports λ'. Recall that

$$U_{11}(\lambda, \lambda') := x(\lambda') \left[-p_{11}(\lambda') + \int_{\mathbf{V}} U_{12}(v, \lambda') dG(v|\lambda) \right] + (1 - x(\lambda')) \left[\int_{\mathbf{V}} U_{22}(v) dG(v|\lambda) \right].$$

It is straightforward to see that

$$\begin{aligned}
\frac{\partial U_{11}(\lambda, \lambda')}{\partial \lambda} &= x(\lambda') \int_{\mathbf{V}} U_{12}(v, \lambda') \frac{\partial^2 G(v|\lambda)}{\partial \lambda \partial v} dv + (1 - x(\lambda')) \int_{\mathbf{V}} U_{22}(v) \frac{\partial^2 G(v|\lambda)}{\partial \lambda \partial v} dv \\
&= x(\lambda') \left[U_{12}(v, \lambda') G_\lambda(v|\lambda) \big|_{v=\underline{v}}^{\bar{v}} - \int_{\mathbf{V}} \frac{\partial U_{12}(v, \lambda')}{\partial v} G_\lambda(v|\lambda) dv \right] \\
&\quad + (1 - x(\lambda')) \left[U_{22}(v) G_\lambda(v|\lambda) \big|_{v=\underline{v}}^{\bar{v}} - \int_{\mathbf{V}} \frac{\partial U_{22}(v)}{\partial v} G_\lambda(v|\lambda) dv \right] \\
&= - \int_{\mathbf{V}} \left(x(\lambda') \frac{\partial U_{12}(v, \lambda')}{\partial v} + (1 - x(\lambda')) \frac{\partial U_{22}(v)}{\partial v} \right) G_\lambda(v|\lambda) dv,
\end{aligned}$$

where we have made use of the fact that $G_\lambda(\underline{v}|\mu) = G_\lambda(\bar{v}|\mu) = 0$ for all $\mu \in \Lambda$. Moreover, we can apply the result of Lemma 1 to conclude that

$$\frac{\partial U_{11}(\lambda, \lambda')}{\partial \lambda} = - \int_{\mathbf{V}} \left[x(\lambda') q_1(v, \lambda') + (1 - x(\lambda')) q_2(v) \right] G_\lambda(v|\lambda) dv = - \int_{\mathbf{V}} \bar{q}(v, \lambda') \frac{G_\lambda(v|\lambda)}{g(v|\lambda)} dG(v|\lambda).$$

Note, however, that the constraint (IC_{11}) implies that $U_{11}(\lambda, \lambda) = \max_{\lambda'} \{U_{11}(\lambda, \lambda')\}$. Since $U_{11}(\lambda) = U_{11}(\lambda, \lambda)$, the envelope theorem (see Milgrom and Segal (2002), for instance) implies that $U'_{11}(\lambda) = \partial U_{11}(\lambda, \lambda') / \partial \lambda |_{\lambda' = \lambda}$, immediately yielding the expression in Equation (IC'_{11}).

To see that condition (MON_{11}) is a necessary implication of incentive compatibility, note that the cutoff property described in Equation (2) immediately implies that the effective allocation rule $\bar{q}(v, \lambda)$ is nondecreasing in v for all $\lambda \in \Lambda$. To see that $\bar{q}(v, \lambda)$ is also nondecreasing in λ for all $v \in \mathbf{V}$, note that the envelope properties described in Lemma 1 imply that the second-period payoffs have the standard revenue equivalence property. Thus, for all $\lambda \notin x^{-1}(0)$ and all $v \in \mathbf{V}$,

$$U_{12}(v, \lambda) = \max\{v - k(\lambda), 0\} - U_{12}(\underline{v}, \lambda) \text{ and } U_{22}(v) = \max\{v - \alpha, 0\} - U_{22}(\underline{v}).$$

Fix any $\lambda, \lambda' \notin x^{-1}(0)$. Then ($\text{IC}_{11}$) implies that

$$0 \leq U_{11}(\lambda) - U_{11}(\lambda, \lambda') = p_{11}(\lambda') - p_{11}(\lambda) + \int_{\mathbf{V}} \left(U_{12}(v, \lambda) - U_{12}(v, \lambda') \right) dG(v|\lambda) \text{ and}$$

$$0 \leq U_{11}(\lambda') - U_{11}(\lambda', \lambda) = p_{11}(\lambda) - p_{11}(\lambda') + \int_{\mathbf{V}} \left(U_{12}(v, \lambda') - U_{12}(v, \lambda) \right) dG(v|\lambda').$$

Adding these two inequalities and simplifying yields

$$0 \leq \int_{\mathbf{V}} \left(\max\{v - k(\lambda), 0\} - \max\{v - k(\lambda'), 0\} \right) d[G(v|\lambda) - G(v|\lambda')].$$

Notice that $\max\{v - k(\lambda), 0\} - \max\{v - k(\lambda'), 0\}$ is increasing in v whenever $k(\lambda) < k(\lambda')$. Since the family of distributions $\{G(\cdot|\mu)\}_{\mu \in \Lambda}$ is ordered by first-order stochastic dominance, we must have $(k(\lambda) - k(\lambda'))(\lambda - \lambda') \leq 0$; otherwise, a contradiction obtains.

Similarly, fix any $\lambda \in x^{-1}(0)$ and any $\lambda' \notin x^{-1}(0)$. Then (IC$_{11}$) implies that

$$0 \leq U_{11}(\lambda) - U_{11}(\lambda, \lambda') = p_{11}(\lambda') + \int_{\mathbf{V}} \left(U_{22}(v) - U_{12}(v, \lambda')\right) dG(v|\lambda) \text{ and}$$
$$0 \leq U_{11}(\lambda') - U_{11}(\lambda', \lambda) = -p_{11}(\lambda') + \int_{\mathbf{V}} \left(U_{12}(v, \lambda') - U_{22}(v)\right) dG(v|\lambda').$$

Adding these two inequalities and simplifying yields

$$0 \leq \int_{\mathbf{V}} \left(\max\{v - \alpha, 0\} - \max\{v - k(\lambda'), 0\}\right) d[G(v|\lambda) - G(v|\lambda')].$$

As above, $\max\{v-\alpha,0\} - \max\{v-k(\lambda'),0\}$ is an increasing function whenever $\alpha < k(\lambda')$ and is a decreasing function whenever $\alpha > k(\lambda')$. Due to the stochastic order on $\{G(\cdot|\lambda)\}$, we must therefore have $(\alpha - k(\lambda'))(\lambda - \lambda') \leq 0$.

Thus, we may conclude that the cutoffs associated with the allocation rules q_1 and q_2 are (weakly) decreasing in λ. This monotonicity implies that $\bar{q}(v, \lambda)$ is nondecreasing in λ for all $v \in \mathbf{V}$, thereby establishing the necessity of (MON$_{11}$).

To prove the converse, suppose that conditions (IC$'_{11}$) and (MON$_{11}$) are satisfied. Note that, with deterministic allocations, we may write the allocation rules q_1 and q_2 as cutoff rules, as in Equation (2), where (MON$_{11}$) implies that the "effective" cutoff $\bar{k}(\lambda) := x(\lambda)k(\lambda) + (1 - x(\lambda))\alpha$ is decreasing. In addition, note that (IC$'_{11}$) implies that we may write

$$U_{11}(\lambda) - U_{11}(\lambda') = \int_{\lambda'}^{\lambda} U'_{11}(\mu)d\mu = \int_{\lambda}^{\lambda'} \int_{\mathbf{V}} \bar{q}(v, \mu) G_\lambda(v|\mu) dv d\mu.$$

Now fix arbitrary $\kappa_{11} \in \mathbb{R}$, $\kappa_{12} \in \mathbb{R}^\Lambda$, and $\kappa_{22} \in \mathbb{R}$, and define

$$\begin{aligned} &p_{22}(v) := q_2(v)\alpha + \kappa_{22},\ p_{12}(v, \lambda) := q_1(v, \lambda)k(\lambda) + \kappa_{12}(\lambda), \text{ and} \\ &p_{11}(\lambda) := \int_{\mathbf{V}} U_{12}(v, \lambda) dG(v|\lambda) - \int_{\underline{\lambda}}^{\lambda} U'_{11}(\mu) d\mu - \kappa_{11}. \end{aligned} \tag{20}$$

Clearly, the second-period incentive compatibility conditions (IC$_{12}$) and (IC$_{22}$) are satisfied given (MON$_{11}$) and the payment rules above. Thus, it remains to be shown that (IC$_{11}$) is also satisfied. To see that this is the case, notice that the payment rules above allow us to write

$$\begin{aligned} U_{11}(\lambda, \lambda') &= x(\lambda') \left[U_{11}(\lambda') - \int_{\mathbf{V}} U_{12}(v, \lambda') dG(v|\lambda') + \int_{\mathbf{V}} U_{12}(v, \lambda') dG(v|\lambda)\right] \\ &\qquad + (1 - x(\lambda')) \left[\int_{\mathbf{V}} U_{22}(v) dG(v|\lambda)\right] \\ &= x(\lambda') \left[U_{11}(\lambda') - \int_{\mathbf{V}} q_{12}(v, \lambda')(v - k(\lambda')) d[G(v|\lambda') - G(v|\lambda)]\right] \\ &\qquad + (1 - x(\lambda')) \left[\int_{\mathbf{V}} q_2(v)(v - \alpha) dG(v|\lambda) - \kappa_{22}\right] \\ &= x(\lambda') \left[U_{11}(\lambda') + \int_{k(\lambda')}^{\bar{v}} [G(v|\lambda') - G(v|\lambda)] dv\right] + (1 - x(\lambda')) \left[\int_{\alpha}^{\bar{v}} [1 - G(v|\lambda)] dv - \kappa_{22}\right]. \end{aligned}$$

So fix any $\lambda, \lambda' \in \Lambda$, and suppose that $x(\lambda') > 0$. Then

$$\begin{aligned}
U_{11}(\lambda) - U_{11}(\lambda,\lambda') &= U_{11}(\lambda) - U_{11}(\lambda') - \int_{k(\lambda')}^{\bar{v}} [G(v|\lambda') - G(v|\lambda)]dv \\
&= \int_{\lambda}^{\lambda'} \int_{\mathbf{V}} \bar{q}(v,\mu) G_{\lambda}(v|\mu) dv d\mu - \int_{k(\lambda')}^{\bar{v}} [G(v|\lambda') - G(v|\lambda)]dv \\
&\geq \int_{\lambda}^{\lambda'} \int_{\mathbf{V}} \bar{q}(v,\lambda') G_{\lambda}(v|\mu) dv d\mu - \int_{k(\lambda')}^{\bar{v}} [G(v|\lambda') - G(v|\lambda)]dv \\
&= \int_{k(\lambda')}^{\bar{v}} \int_{\lambda}^{\lambda'} G_{\lambda}(v|\mu) d\mu dv - \int_{k(\lambda')}^{\bar{v}} [G(v|\lambda') - G(v|\lambda)]dv = 0,
\end{aligned}$$

where the second equality follows from applying the Fundamental Theorem of Calculus to (IC'_{11}), and the inequality follows from the fact that, by (MON_{11}), $\bar{q}(v,\mu)$ is increasing in μ. Similarly, suppose that $x(\lambda') = 0$. Then

$$\begin{aligned}
U_{11}(\lambda) - U_{11}(\lambda,\lambda') &= U_{11}(\lambda) - \int_{\alpha}^{\bar{v}} [1 - G(v|\lambda)]dv + \kappa_{22} \\
&= U_{11}(\lambda') - \int_{\lambda'}^{\lambda} \int_{\mathbf{V}} \bar{q}(v,\mu) G_{\lambda}(v|\mu) dv d\mu - \int_{\alpha}^{\bar{v}} [1 - G(v|\lambda)]dv + \kappa_{22} \\
&= \int_{\lambda}^{\lambda'} \int_{\mathbf{V}} \bar{q}(v,\mu) G_{\lambda}(v|\mu) dv d\mu - \int_{\alpha}^{\bar{v}} [G(v|\lambda') - G(v|\lambda)]dv \\
&\geq \int_{\lambda}^{\lambda'} \int_{\mathbf{V}} \bar{q}(v,\lambda') G_{\lambda}(v|\mu) dv d\mu - \int_{\alpha}^{\bar{v}} [G(v|\lambda') - G(v|\lambda)]dv \\
&= \int_{\alpha}^{\bar{v}} \int_{\lambda}^{\lambda'} G_{\lambda}(v|\mu) d\mu dv - \int_{\alpha}^{\bar{v}} [G(v|\lambda') - G(v|\lambda)]dv = 0,
\end{aligned}$$

where the second equality follows from (IC'_{11}) and the Fundamental Theorem of Calculus; the third equality from the fact that $U_{11}(\lambda') = \int_{\mathbf{V}} U_{22}(v) dG(v|\lambda')$ since $x(\lambda') = 0$; and the inequality from the fact that, since (MON_{11}) holds, $\bar{q}(v,\mu)$ is increasing in μ.

Thus, $U_{11}(\lambda) \geq U_{11}(\lambda,\lambda')$ for all $\lambda, \lambda' \in \Lambda$, implying that the initial-period incentive compatibility constraint (IC_{11}) is satisfied. □

PROOF OF THEOREM 1. Constraints (IC'_{11}), (IC'_{12}), and (IC'_{22}) are satisfied by construction, as these constraints have been incorporated into the objective function in problem ($\mathcal{P}^{ND}$). Meanwhile, Assumption 1 implies that the monotonicity conditions (MON_{11}) and (MON_{12}) are also satisfied. (Constraint (MON_{22}) is trivially satisfied.) Invoking Lemma 2, there exist payment rules that implement the contract above. Indeed, it is easy to verify that the payments in Equations (7) and (8) equal those defined in Equation (20), but with $\kappa_{11} = 0$, $\kappa_{12}(\lambda) = 0$ for all $\lambda \in \Lambda$, and $\kappa_{22} = 0$. Thus, the contracts $\{q_1^{ND}, p_{11}^{ND}, p_{12}^{ND}\}$ and $\{q_2^{ND}, p_{22}^{ND}\}$ solve the seller's problem. □

PROOF OF LEMMA 3. Denote the solution to ($\mathcal{R}^{FC}$) using a * superscript, and suppose that the hypothesis is false; that is, suppose there exists some $\tilde{\mu} < \hat{\lambda}$ such that $U_{11}^*(\tilde{\mu}) > \int_{\mathbf{V}} \max\{v - \alpha, 0\} dG(v|\tilde{\mu})$. Recalling that U_{11}^* is absolutely continuous (due to Lemma 2), this also implies that (SD) is slack for all μ in some neighborhood of $\tilde{\mu}$. Notice that, of course, this slack can only exist if $x^*(\mu) = 1$.

Moreover, notice that (SD) binding at $\hat{\lambda}$ implies that $\bar{q}^*(v,\hat{\lambda}) \geq q_2(v)$. To see why this is the case, note that if

$$(U^*)'(\hat{\lambda}) = -\int_{\mathbf{V}} \bar{q}^*(v,\hat{\lambda}) G_\lambda(v|\hat{\lambda}) dv < -\int_{\mathbf{V}} q_2(v) G_\lambda(v|\hat{\lambda}) dv = \left[\frac{\partial}{\partial \lambda} \int_{\mathbf{V}} U_{22}(v) dG(v|\lambda)\right]_{\lambda=\hat{\lambda}},$$

then the binding (SD) constraint $U^*_{11}(\hat{\lambda}) = \int_{\mathbf{V}} U_{22}(v) dG(v|\hat{\lambda})$ implies that the same constraint is violated in the interval $(\hat{\lambda}, \hat{\lambda}+\epsilon)$, where $\epsilon > 0$ is sufficiently small.

Now define an alternative mechanism (which we will denote using a ** superscript) with $x^{**}(\lambda) = x^*(\lambda)$ for all $\lambda \in \Lambda$, and for all $\lambda \in (x^*)^{-1}(1)$

$$q_1^{**}(v,\lambda) := \begin{cases} q_1^*(v,\lambda) & \text{if } \lambda > \hat{\lambda}, \\ q_2(v) & \text{if } \lambda \leq \hat{\lambda}, \end{cases}$$

$$p_{11}^{**}(\lambda) := \begin{cases} p_{11}^*(\lambda) & \text{if } \lambda > \hat{\lambda}, \\ 0 & \text{if } \lambda \leq \hat{\lambda}, \end{cases}$$

$$p_{12}^{**}(v,\lambda) := \begin{cases} p_{12}^*(v,\lambda) & \text{if } \lambda > \hat{\lambda}, \\ p_{22}(v) & \text{if } \lambda \leq \hat{\lambda}. \end{cases}$$

Note that this new contract inherits all the required properties of the optimal contract—the set of delayed buyers (and profitability thereof) is unchanged, the contract is incentive compatible in period two (since the mechanisms in period two correspond to prices), and the contract is incentive compatible in period one. To see that this last claim is true, notice that incentives for $\lambda > \hat{\lambda}$ are not affected by the change in contracts. In addition, $\bar{q}^{**}$ is monotone in λ since $\bar{q}^*$ was (and since $\bar{q}^*(v,\hat{\lambda}) \geq q_2(v)$, as shown above), and so it easy to verify that buyers with $\lambda < \hat{\lambda}$ are also incentivized to report their types truthfully. Finally, since (SD) is satisfied for all $\lambda \in \Lambda$, it must be the case that the contract is also individually rational in the first period.

Now notice that standard techniques allow us to write the seller's expected profits from each initial-period type λ as

$$\begin{aligned} \Pi_{11}(\lambda) &= x(\lambda)\left[p_{11}(\lambda) + \int_{\mathbf{V}} (p_{12}(v,\lambda) - cq_1(v,\lambda)) dG(v|\lambda)\right] \\ &\qquad + (1-x(\lambda)) \int_{\mathbf{V}} (p_2(v) - cq_2(v)) dG(v|\lambda) \\ &= \int_{\mathbf{V}} (v-c)\bar{q}(v,\lambda) dG(v|\lambda)) - U_{11}(\hat{\lambda}) - \int_{\hat{\lambda}}^{\lambda} U_1'(\mu) d\mu \\ &= \int_{\mathbf{V}} (v-c)\bar{q}(v,\lambda) dG(v|\lambda) - U_1(\hat{\lambda}) + \int_{\hat{\lambda}}^{\lambda} \int_{\mathbf{V}} \bar{q}(v,\mu) G_\lambda(v|\mu) dv d\mu. \end{aligned}$$

Since $q_1^*(v,\lambda) = q_1^{**}(v,\lambda)$ for all $\lambda > \hat{\lambda}$, $x^* = x^{**}$, and $U^*_{11}(\hat{\lambda}) = U^{**}_{11}(\hat{\lambda}) = \int_{\mathbf{V}} \max\{v-\alpha, 0\} dG(v|\hat{\lambda})$, we have $\Pi^{**}_{11}(\lambda) = \Pi^*_{11}(\lambda)$ for all $\lambda > \hat{\lambda}$.

Recalling the monotonicity result from Lemma 2, we have $k^*(\lambda) \geq \alpha$ for all $\lambda \leq \hat{\lambda}$. However, this inequality must be *strict* in some neighborhood of $\tilde{\mu}$ where $x^*(\lambda) = 1$; if not, we have

$(U_{11}^*)'(\lambda) = -\int_{\mathbf{V}} q_2(v) G_\lambda(v|\lambda) dv$ everywhere, implying that, in contradiction to our previous assumption, constraint (SD) binds everywhere. On the other hand, $k^{**}(\lambda) = \alpha$ for *all* $\lambda < \hat{\lambda}$ such that $x^{**}(\lambda) = 1$.

Therefore, we have $\Pi_1^{**}(\lambda) \geq \Pi_1^*(\lambda)$ for all $\lambda \leq \hat{\lambda}$, with *strict* inequality in a neighborhood of $\tilde{\mu}$. This follows from the facts that q_1^{**} is more generous than q_1^*, as it requires a lower cutoff for buyers' values; this additional generosity is efficient since $\alpha \geq c$; and the information rents are smaller since $G_\lambda \leq 0$ (and therefore we are *adding* more profits in the final term of Π_{11} above).

Of course, this contradicts the optimality of the original * contract. Thus, as desired, constraint (SD) must bind for all $\lambda < \hat{\lambda}$. □

Proof of Lemma 4. Notice that the proof of Lemma 3 extends to the present setting with limited commitment. That proof is by contradiction: if the constraint (SD) binds at some $\hat{\lambda} \in \Lambda$ but is slack for some $\tilde{\mu} < \hat{\lambda}$, then it is possible to construct an alternative first-period contract that increases the seller's profits while maintaining incentive compatibility. Since that construction does not alter the set of cohort-one buyers that delay contracting to period two, constraint (SR) is unaffected and the resulting period-two price is unchanged. Therefore, the strategic delay constraint (SD) is satisfied in the construction, and the result follows immediately. □

Proof of Theorem 2. Constraints (IC_{11}') and (IC_{12}') are satisfied by construction, as these constraints have been incorporated into the objective function in problem ($\mathcal{R}^{FC}$). Meanwhile, Assumption 1 implies that the monotonicity conditions (MON_{11}) and (MON_{12}) are also satisfied. Invoking Lemma 2, there exist payment rules that implement the contract above. Indeed, it is easy to verify that the payments in Equations (12) and (13) equal those defined in Equation (20), but with $\kappa_{11} = \int_{\mathbf{V}} q_2^{FC}(v) dG(v|\underline{\lambda})$, $\kappa_{12}(\lambda) = 0$ for all $\lambda \in \Lambda$, and $\kappa_{22} = 0$. Finally, it is trivial to verify that each buyer's expected utility from the cohort-one contract is no smaller than the expected value of delay, so constraint (SD) is also satisfied. Thus, the contract $\{x^{FC}, q_1^{FC}, p_{11}^{FC}, p_{12}^{FC}\}$ solves the seller's problem. □

Proof of Corollary 1. If α^{FC} maximizes $\Pi^{FC}(\alpha)$, then Theorem 2 immediately implies that the contracts $\{x^{FC}, q_1^{FC}, p_{11}^{FC}, p_{12}^{FC}\}$ and $\{q_2^{FC}, p_{22}^{FC}\}$ defined in Equations (9), (11), (12), and (13) (with $\alpha = \alpha^{FC}$) are optimal.

In addition, notice that we may rewrite the seller's profits $\Pi^{FC}(\alpha)$ in Equation (14) as $\Pi^{FC}(\alpha) = \Pi_1^{FC}(\alpha) + \Pi_2^{FC}(\alpha)$, where

$$\Pi_1^{FC}(\alpha) := \int_{\underline{\lambda}}^{\tilde{\lambda}(\alpha)} \int_{\alpha}^{\bar{v}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) dv d\lambda + \int_{\tilde{\lambda}(\alpha)}^{\bar{\lambda}} \int_{k^{ND}(\lambda)}^{\bar{v}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) dv d\lambda$$
$$- \int_{\alpha}^{\bar{v}} (v-\alpha) g(v|\underline{\lambda}) dv \text{ and}$$
$$\Pi_2^{FC}(\alpha) := \gamma \left[(\alpha - c)(1 - H(\alpha)) \right]$$

are the profits derived from cohort-one and cohort-two buyers, respectively. Moreover,

$$\begin{aligned}\frac{\partial \Pi_1^{FC}(\alpha)}{\partial \alpha} &= \frac{\partial \tilde{\lambda}(\alpha)}{\partial \alpha}\int_{\alpha}^{\bar{v}} \varphi_1(v,\tilde{\lambda}(\alpha))g(v|\tilde{\lambda}(\alpha))f(\tilde{\lambda}(\alpha))dv - \int_{\underline{\lambda}}^{\tilde{\lambda}(\alpha)} \varphi_1(\alpha,\lambda)g(\alpha|\lambda)f(\lambda)d\lambda \\ &\quad - \frac{\partial \tilde{\lambda}(\alpha)}{\partial \alpha}\int_{k^{ND}(\tilde{\lambda}(\alpha))}^{\bar{v}} \varphi_1(v,\tilde{\lambda}(\alpha))g(v|\tilde{\lambda}(\alpha))f(\tilde{\lambda}(\alpha))dv + (\alpha-\alpha)g(\alpha|\underline{\lambda}) + \int_{\alpha}^{\bar{v}} g(v|\underline{\lambda})dv \\ &= 1 - G(\alpha|\underline{\lambda}) - \int_{\underline{\lambda}}^{\tilde{\lambda}(\alpha)} \varphi_1(\alpha,\lambda)g(\alpha|\lambda)dF(\lambda),\end{aligned}$$

where the second equality follows from the fact that $k^{ND}(\tilde{\lambda}(\alpha)) = \alpha$. Assumption 1 implies that, for any α, $\varphi_1(\alpha,\lambda) \leq 0$ for all $\lambda \leq \tilde{\lambda}(\alpha)$. Therefore, $\partial\Pi_1^{FC}(\alpha)/\partial\alpha \geq 0$ for all α (with strict inequality for $\alpha < \bar{v}$). Since $\Pi_2^{FC}(\alpha)$ is, by definition, maximized at p_H, Π_1^{FC} strictly increasing immediately implies that $\alpha^{FC} > p_H$. □

PROOF OF LEMMA 5. Consider the * contract, and define $\bar{\mu} := \sup\{\lambda | x^*(\lambda) < 1\}$ to be the largest initial-period type that may delay contracting. In addition, note that (by Assumptions 2 and 3) the first-order condition in (SR$''$) implies the existence of some $\mu^* \in \Lambda$ such that

$$\alpha^* = \operatorname*{argmax}_{\alpha}\{(\alpha - c)(1 - G(\alpha|\mu^*))\}.$$

Moreover, Assumptions 2 and 3 also imply that $\psi_1(\alpha^*,\lambda)g(\alpha^*|\lambda) < 0$ for all $\lambda > \mu^*$, and that $\psi_1(\alpha^*,\lambda)g(\alpha^*|\lambda) > 0$ for all $\lambda < \mu^*$.

So define

$$X_1 := \int_{\underline{\lambda}}^{\mu^*} (1 - x^*(\lambda))\psi_1(\alpha^*,\lambda)g(\alpha^*|\lambda)dF(\lambda) \text{ and } X_2 := \int_{\mu^*}^{\bar{\mu}} (1 - x^*(\lambda))\psi_1(\alpha^*,\lambda)g(\alpha^*|\lambda)dF(\lambda).$$

Note that $X_1 \geq 0 \geq X_2$, and that $X_1 + X_2 + \gamma\psi_2(\alpha^*)h(\alpha^*) = 0$. In addition, define

$$Y_1(\mu) := \int_{\mu}^{\mu^*} \psi_1(\alpha^*,\lambda)g(\alpha^*|\lambda)dF(\lambda) \text{ and } Y_2(\mu) := \int_{\mu^*}^{\mu} \psi_1(\alpha^*,\lambda)g(\alpha^*|\lambda)dF(\lambda).$$

It is easy to see that $Y_1(\mu)$ is strictly increasing, and that $Y_1(\mu^*) = 0$ and $Y_1(\underline{\lambda}) \geq X_1$. This implies that there exists a unique $\mu_1 \in [\underline{\lambda}, \mu^*]$ such that $Y_1(\mu_1) = X_1$. Similarly, $Y_2(\mu)$ is strictly decreasing, $Y_2(\mu^*) = 0$, and $Y_2(\bar{\mu}) \leq X_2$. Hence, there exists a unique $\mu_2 \in [\mu^*, \bar{\mu}]$ such that $Y_2(\mu_2) = X_2$.

Thus, we may define x^{**} by

$$x^{**}(\lambda) := \begin{cases} 0 & \text{if } \lambda \in [\mu_1, \mu_2], \\ 1 & \text{otherwise.} \end{cases}$$

By construction, α^* solves the first-order condition (SR$''$) induced by x^{**}, implying that $\alpha^{**} = \alpha^*$. Thus, the option value of strategic delay is unchanged for all buyers, leaving constraint (SD) unaffected. Since Lemma 4 implies that this constraint must bind for all $\lambda \leq \mu_2$, we must have $\Pi^{**}(\lambda) = \Pi^*(\lambda)$ for all such λ. However, $\Pi^{**}(\lambda) \geq \Pi^*(\lambda)$ for all $\lambda > \mu_2$, since we now have greater freedom in choosing $\hat{\lambda}$ in the seller's optimization problem. In particular, note that $\hat{\lambda}$ cannot be lower than the upper bound of the set of delayed cohort-one buyers; thus, by decreasing this upper bound from $\bar{\mu}$ to μ_2, we have relaxed an implicit constraint on the seller's problem. □

PROOF OF THEOREM 3. Since μ_1^{LC} and μ_2^{LC} solve problem ($\mathcal{R}^{LC}$), Theorem 2 immediately implies that the proposed contract is optimal.

It is easy to see that we cannot have $\alpha^{LC} < p_H$. Delaying some cohort-one buyers in order to reduce the second-period price below p_H has two effects: it reduces the profits derived from cohort-two buyers (which are, by definition, maximized at p_H), and it also increases the utility that must be promised to *all* cohort-one buyers while decreasing the seller's ability to price discriminate. Each of these forces decreases overall profits, and so we must have $\alpha^{LC} \geq p_H$.

Abusing notation slightly, denote by $\alpha(\mu_1, \mu_2)$ the second-period price induced when the set of delayed cohort-one buyers (which Lemma 5 implies we can take, without loss, to be an interval) is $[\mu_1, \mu_2]$. We can then write the seller's profits from the objective function in ($\mathcal{R}^{LC}$) as

$$\begin{aligned}
\Pi^{LC}(\mu_1,\mu_2) :=& \int_{\underline{\lambda}}^{\mu_2}\int_{\alpha(\mu_1,\mu_2)}^{\bar{v}} \varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) + \int_{\mu_2}^{\bar{\lambda}}\int_{\min\{\alpha(\mu_1,\mu_2),k^{ND}(\lambda)\}}^{\bar{v}} \varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) \\
&+ \gamma(\alpha(\mu_1,\mu_2)-c)(1-H(\alpha(\mu_1,\mu_2))) - \int_{\alpha(\mu_1,\mu_2)}^{\bar{v}}(v-\alpha(\mu_1,\mu_2))dG(v|\underline{\lambda}) \\
=& \int_{\underline{\lambda}}^{\tilde{\lambda}(\alpha(\mu_1,\mu_2))}\int_{\alpha(\mu_1,\mu_2)}^{\bar{v}} \varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) \\
&+ \int_{\tilde{\lambda}(\alpha(\mu_1,\mu_2))}^{\max\{\tilde{\lambda}(\alpha(\mu_1,\mu_2)),\mu_2\}}\int_{\alpha(\mu_1,\mu_2)}^{\bar{v}} \varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) \\
&+ \int_{\max\{\tilde{\lambda}(\alpha(\mu_1,\mu_2)),\mu_2\}}^{\bar{\lambda}}\int_{k^{ND}(\lambda)}^{\bar{v}} \varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda) \\
&+ \gamma(\alpha(\mu_1,\mu_2)-c)(1-H(\alpha(\mu_1,\mu_2))) - \int_{\alpha(\mu_1,\mu_2)}^{\bar{v}}(v-\alpha(\mu_1,\mu_2))dG(v|\underline{\lambda}).
\end{aligned}$$

We can rearrange this expression to yield

$$\Pi^{LC}(\mu_1,\mu_2) = \Pi^{FC}(\alpha(\mu_1,\mu_2)) - \int_{\tilde{\lambda}(\alpha(\mu_1,\mu_2))}^{\max\{\tilde{\lambda}(\alpha(\mu_1,\mu_2)),\mu_2\}}\int_{k^{ND}(\lambda)}^{\alpha(\mu_1,\mu_2)} \varphi_1(v,\lambda)dG(v|\lambda)dF(\lambda),$$

where Π^{FC} is the full-commitment profit function as defined in Equation (14). Notice that whenever $\mu_2 > \tilde{\lambda}(\alpha(\mu_1,\mu_2))$, the integrand in the last line above is always positive, as k^{ND} is decreasing in λ, and $\varphi_1(v,\lambda) \geq 0$ whenever $v \geq k^{ND}(\lambda)$. We may use this expression to evaluate the seller's profits when she induces a second-period price $\alpha^{LC} > \alpha^{FC}$.

Define $\Gamma := \{\mu \in \Lambda : \text{there exists } \mu' \leq \mu \text{ with } \alpha(\mu',\mu) = \alpha^{FC}\}$. If Γ is empty, then it is not possible to induce a second-period price α^{FC}. Since $\alpha(\mu_1,\mu_2)$ is continuous and $\alpha(\underline{\lambda},\underline{\lambda}) = p_H < \alpha^{FC}$, this implies that $\alpha^{LC} \leq \max_{\mu_1,\mu_2}\{\alpha(\mu_1,\mu_2)\} < \alpha^{FC}$. On the other hand, if Γ is nonempty, we may define $\mu_2^{FC} := \min\{\mu \in \Gamma\}$; that is, μ_2^{FC} is the smallest upper bound of the set of delayed buyers that is compatible with inducing a second-period price α^{FC}. In addition, we implicitly define μ_1^{FC} by $\alpha(\mu_1^{FC},\mu_2^{FC}) = \alpha^{FC}$.

Note that if $\mu_2^{FC} \leq \tilde{\lambda}(\alpha^{FC})$, the seller is able to replicate the effective allocation and payment rules from the full commitment case. This implies that the payoff of the (less-constrained) full-commitment problem is achievable even with limited commitment, and so $\alpha^{LC} = \alpha^{FC}$. So suppose instead that $\mu_2^{FC} > \tilde{\lambda}(\alpha^{FC})$, and note that Assumptions 2 and 3 imply that inducing $\alpha^{LC} > \alpha^{FC}$

requires $\mu_2^{LC} > \mu_2^{FC}$. Then

$$\begin{aligned}
\Pi^{LC}(\mu_1^{FC}, \mu_2^{FC}) &= \Pi^{FC}(\alpha^{FC}) - \int_{\tilde{\lambda}(\alpha^{FC})}^{\mu_2^{FC}} \int_{k^{ND}(\lambda)}^{\alpha^{FC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) dv d\lambda \\
&= \Pi^{FC}(\alpha^{FC}) - \int_{k^{ND}(\mu_2^{FC})}^{\alpha^{FC}} \int_{\tilde{\lambda}(v)}^{\mu_2^{FC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) d\lambda dv, \text{ and} \\
\Pi^{LC}(\mu_1^{LC}, \mu_2^{LC}) &= \Pi^{FC}(\alpha^{LC}) - \int_{\tilde{\lambda}(\alpha^{LC})}^{\mu_2^{LC}} \int_{k^{ND}(\lambda)}^{\alpha^{LC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) dv d\lambda \\
&= \Pi^{FC}(\alpha^{LC}) - \int_{k^{ND}(\mu_2^{LC})}^{\alpha^{LC}} \int_{\tilde{\lambda}(v)}^{\mu_2^{LC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) d\lambda dv,
\end{aligned}$$

where the second equality in each expression follows from reversing the order of integration. Subtracting the two expressions above and then rearranging the integrals, we have

$$\begin{aligned}
\Pi^{LC}(\mu_1^{FC}, \mu_2^{FC}) - \Pi^{LC}(\mu_1^{LC}, \mu_2^{LC}) = \Pi^{FC}(\alpha^{FC}) - \Pi^{FC}(\alpha^{LC}) &+ \int_{\alpha^{FC}}^{\alpha^{LC}} \int_{\tilde{\lambda}(v)}^{\mu_2^{FC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) d\lambda dv \\
&+ \int_{k^{ND}(\mu_2^{LC})}^{\alpha^{LC}} \int_{\mu_2^{FC}}^{\mu_2^{LC}} \varphi_1(v,\lambda) g(v|\lambda) f(\lambda) d\lambda dv.
\end{aligned}$$

By definition, $\Pi^{FC}(\alpha^{FC}) \geq \Pi^{FC}(\alpha^{LC})$. Moreover, it is straightforward to verify that each of the integrands is positive, and therefore $\Pi^{LC}(\mu_1^{FC}, \mu_2^{FC}) > \Pi^{LC}(\mu_1^{LC}, \mu_2^{LC})$; thus, the seller with limited commitment induces a second-period price $\alpha^{LC} \leq \alpha^{FC}$.

Now suppose that $\alpha^{LC} \in (p_H, \alpha^{FC})$, and that the set of buyers that delay contracting is the (nonempty) interval $[\mu_1^{LC}, \mu_2^{LC}]$. Since $\alpha^{LC} > p_H$, Assumption 2 implies that $\psi_2(\alpha^{LC})h(\alpha^{LC}) > 0$. Similarly, since $\psi_1(p_H, \hat{\mu})g(p_H|\hat{\mu}) = 0$ (by the definition of $\hat{\mu}$), Assumption 2 also implies that $\psi_1(\alpha^{LC}, \hat{\mu})g(\alpha^{LC}|\hat{\mu}) > 0$. Finally, notice that the first-order condition (SR$''$) may be rewritten as

$$\int_{\mu_1^{LC}}^{\mu_2^{LC}} \psi_1(\alpha^{LC}, \lambda) g(\alpha^{LC}|\lambda) dF(\lambda) + \gamma \psi_2(\alpha^{LC}) h(\alpha^{LC}) = 0.$$

So suppose that $\mu_1^{LC} < \hat{\mu}$. Assumptions 2 and 3 then imply that $\psi_1(\alpha^{LC}, \mu_1^{LC})g(\alpha^{LC}|\mu_1^{LC}) > 0$. In addition, the first-order condition above implies that $\psi_1(\alpha^{LC}, \mu_2^{LC})g(\alpha^{LC}|\mu_2^{LC}) < 0$. Therefore, there exists some $\mu^* \in [\mu_1^{LC}, \mu_2^{LC}]$ such that $\psi_1(\alpha^{LC}, \mu^*)g(\alpha^{LC}|\mu^*) = 0$. So define

$$X := \int_{\mu_1^{LC}}^{\hat{\mu}} \psi_1(\alpha^{LC}, \lambda) g(\alpha^{LC}, \lambda) dF(\lambda) \text{ and } Y(z) := \int_{\hat{\mu}}^{z} \psi_1(\alpha^{LC}, \lambda) g(\alpha^{LC}, \lambda) dF(\lambda),$$

and note that $X > 0$, $Y(\hat{\mu}) = 0$, $Y(\mu_2^{LC}) = -(\gamma\psi_2(\alpha^{LC})h(\alpha^{LC}) + X)<0$, and $Y'(z) < 0$ for all $z > \hat{\mu}$. Therefore, there must exist some $\mu' \in (\hat{\mu}, \mu_2^{LC})$ such that $Y(\mu') = -\gamma\psi_2(\alpha^{LC})h(\alpha^{LC})$. Thus, $\alpha(\hat{\mu}, \mu') = \alpha^{LC} = \alpha(\mu_1^{LC}, \mu_2^{LC})$, implying that the seller can achieve the same second-period price by delaying a (strictly) smaller subset of buyers. If $\mu' \geq \tilde{\lambda}(\alpha^{LC})$, the seller is now able to increase profits by separating cohort-one buyers with types in (μ', μ_2^{LC}), contradicting the optimality of the proposed contract. On the other hand, if $\mu' < \tilde{\lambda}(\alpha^{LC})$, then the seller can increase the second-period price towards the full-commitment price α^{FC} by expanding the delay set "rightwards" *without* sacrificing the ability to screen. This again contradicts the optimality of the proposed contract. Thus, we must have $\mu_1^{LC} \geq \hat{\mu}$. □

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