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Components of bull and bear markets: bull corrections and bear rallies

By John M Maheu, Thomas H McCurdy and Yong Song

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John M. Maheu[†] Thomas H. McCurdy[‡] Yong Song[§]

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Abstract

Existing methods of partitioning the market index into bull and bear regimes do not identify market corrections or bear market rallies. In contrast, our probabilistic model of the return distribution allows for rich and heterogeneous intra-regime dynamics. We focus on the characteristics and dynamics of bear market rallies and bull market corrections, including, for example, the probability of transition from a bear market rally into a bull market versus back to the primary bear state. A Bayesian estimation approach accounts for parameter and regime uncertainty and provides probability statements regarding future regimes and returns. A Value-at-Risk example illustrates the economic value of our approach.

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[†]Department of Economics, University of Toronto and RCEA, jmaheu@chass.utoronto.ca

[‡]Rotman School of Managment, University of Toronto & CIRANO, tmccurdy@rotman.utoronto.ca

[§]Department of Economics, University of Toronto, ysong1@gmail.com

1 Introduction

There is a widespread belief both by investors, policy makers and academics that low frequency trends do exist in the stock market. Traditionally these positive and negative low frequency trends have been labelled as bull and bear markets respectively. If these trends do exist, then it is important to extract them from the data to analyse their properties and consider their use as inputs into investment decisions and risk assessment. We propose a model that provides answers to typical questions such as, 'Are we in a bull market or a bear market rally?' or 'Will this bull market correction become a bear market?'.

Traditional methods of identifying bull and bear markets are based on an ex post assessment of the peaks and troughs of the price index. Formal dating algorithms based on a set of rules for classification are found, for example, in Gonzalez, Powell, Shi, and Wilson (2005), Lunde and Timmermann (2004) and Pagan and Sossounov (2003). Some of this work is related to the dating methods used to identify turning points in the business cycle (Bry and Boschan (1971)). A drawback is that a turning point can only be identified several observations after it occurs.

Ex post dating algorithms sort returns into a particular regime with probability zero or one. The data provides more information; investors may be interested in estimated probabilities associated with particular states. Such information can be used to answer questions such as 'How likely is it that the market could turn into a bear next month?'. Further, ex post dating methods cannot be used for statistical inference on returns or for investment decisions which require more information from the return distribution, such as changing risk assessments. For adequate risk management and investment decisions, we need a probability model for returns and one for which the distribution of returns changes over time.

For time series that tend to be cyclical, for example, due to business cycles, a popular model has been a two-state regime-switching model in which the states are latent and the mixing parameters are estimated from the available data. One popular parameterization is a Markov-switching (MS) model for which transitions between states are governed by a Markov chain. Hamilton (1989) applied a two-state MS model to quarterly U.S. GNP growth rates in order to identify business cycles and estimate 1st-order Markov transition probabilities associated with the expansion and recession phases of those cycles.

Stock markets are also perceived to have a cyclical pattern which can be captured with regime-switching models. For example, Hamilton and Lin (1996) relate business cycles and stock market regimes, Chauvet and Potter (2000) and Maheu and McCurdy (2000a) use a Markov-switching parameterization to analyze properties of bull and bear market regimes extracted from aggregate stock market returns.[1] The latter paper al-

[1] There are many other applications of regime-switching models to forcing processes for asset pricing

lows duration-dependent transition probabilities, as well duration-dependent intra-state dynamics for returns and volatilities. Lunde and Timmermann (2004) study duration dependence after sorting stock returns into either a bull or bear market using their dating algorithm. Ntantamis (2009) explores potential explanatory variables for stock market regimes' duration.

In a related literature that investigates cyclical patterns in a broader class of assets, Guidolin and Timmermann (2005) use a 3-state regime-switching model to identify bull and bear markets in monthly UK stock and bond returns and analyze implications for predictability and optimal asset allocation. Guidolin and Timmermann (2006) add an additional state in order to model the nonlinear joint dynamics of monthly returns associated with small and large cap stocks and long-term bonds.

In contrast to the existing literature, our objective is to use higher-frequency weekly data and to provide a real-time approach to identifying phases of the market that relate to investors' perceptions of primary and secondary trends in aggregate stock returns. Existing approaches do not explicitly model bull market corrections and bear market rallies. Separating short-term reversals from the primary trend in high-frequency market returns is an important empirical regularity that a model must capture for it to be able to account for market dynamics.

We propose a latent 4-state Markov-switching model for weekly stock returns. Our focus is on modeling the component states of bull and bear market regimes in order to identify and forecast bull, bull correction, bear and bear rally states. The bear and bear rally states govern the bear regime; the bull correction and bull states govern the bull regime. The model can accommodate short-term reversals (secondary trends) within each regime of the market. For example, in the bull regime it is possible to have a series of persistent negative returns (a bull correction), despite the fact that the expected long-run return (primary trend) is positive in that regime. Analogously, bear markets often exhibit persistent rallies which are subsequently reversed as investors take the opportunity to sell with the result that the average return in that regime is still negative.

It is important to note that our additional states allow for both intra and inter-regime transitions. A bear rally is allowed to move back to the bear state or to exit the bear regime by moving to a bull state. Likewise, a bull correction can move back to the bull state or exit the bull regime by transitioning to a bear state. This richer structure allows

models and to asset returns. Cecchetti, Lam, and Mark (1990), Kandel and Stambaugh (1990), Gordon and St-Amour (2000), Calvet and Fisher (2007), Lettau, Ludvigson, and Wachter (2008), Guidolin and Timmermann (2008) and David and Veronesi (2009), among others, derive implications of regime-switching for equilibrium asset prices. Examples for interest rates include Garcia and Perron (1996) and Ang and Bekaert (2002b). Applications that explore the implications of nonlinearities due to regimes switches for asset allocation and/or predictability of returns include Turner, Startz, and Nelson (1989), van Norden and Schaller (1997), Maheu and McCurdy (2000b), Perez-Quiros and Timmermann (2001), Ang and Bekaert (2002a), Guidolin and Timmermann (2007).

regimes to feature several episodes of their component states. For example, a bull regime can be characterized by a combination of bull states and bull corrections. Similarly, a bear regime can consist of several episodes of the bear state and the bear rally state, exactly as many investors feel we observe in the data. Because, the realization of states in a regime will differ over time, bull and bear regimes can be heterogenous over time. These important intra and inter-regime dynamics are absent in the existing literature.

Our Bayesian estimation approach accounts for parameter and regime uncertainty and provides probability statements regarding future regimes and returns. As noted above, each bear and bull regime has two states. We identify the model by imposing the long-run mean of returns to be negative in the bear regime and positive in the bull regime; while allowing for very different dynamics within each regime. We consider several versions of the model in which the variance dynamics are decoupled from the mean dynamics. We find that a model in which the states associated with the first and second moment are coupled provides the best fit to the data.

Applied to 125 years of data our model provides superior identification of trends in stock prices. One important difference with our specification is that the richer dynamics in each regime, facilitated by our 4-state model, allow us to extract bull and bear markets in higher frequency data. As we show, a problem with a two-state Markov-switching model applied to higher frequency data is that it results in too many switches between the high and low return states. In other words, it is incapable of extracting the low frequency trends in the market. In high frequency data it is important to allow for short-term reversals in the regime of the market. Relative to a two-state model we find that market regimes are more persistent and there is less erratic switching. According to Bayes factors, our 4-state model of bull and bear markets is strongly favored over several alternatives including a two-state model, and different variance dynamics.

Our results include probabilistic identification of bear, bear rally, bull correction and bull states – as well as the characteristics of the associated bear and bull market regimes. For instance, bull regimes have an average duration of just under 5 years, while the duration of a bull correction is 4 months on average and a bear rally is just over half a year. The cumulative return mean of the bull market state is 7.88% but bull corrections offset this by 2.13% on average. Average cumulative return in the bear market state is -12.4% but bear market rallies counteract that steep decline by yielding a cumulative return of 7.1% on average. Note that these states are combined into bull and bear market regimes in heterogenous patterns over time yielding an average cumulative return in the bull market regime of 33% while that for the bear market regime is about -10%. Also, although the average cumulative return in the bear rally state is not much less than that in the bull market state, the ex post Sharpe ratio for the latter is about 2.5 times larger. This result highlights the importance of also considering assessments of volatility associated with the alternative states, for example, when identifying bear market rallies versus bull markets.

Of primary importance is the fact that our model can tell us the probabilities of market states in real time, unlike dating algorithms. It can also produce out-of-sample forecasts. For example, the model identifies in real time a transition from a bull market correction to a bear market in early October 2008. The bear rally and bull correction states are critical to modeling turning points between regimes; our results show that most transitions between bull and bear regimes occur through these states. This is consistent with investors' perceptions.[2] Further, we find asymmetries in intra-regime dynamics, for example, a bull market correction returns to the bull market state more often than a bear market rally reverts to the bear state. These are important features that the existing literature on bull and bear markets ignores.

Our Markov-switching structure provides a full description of the return distribution. In an out-of-sample application, the probability statements concerning the predictive density of returns are used to generate Value-at-Risk forecasts. This provides a simple example of the economic value of our proposed model.

This paper is organized as follows. The next section describes the data, Section 3 discusses two alternative ex post market regime dating algorithms. We use one of these algorithms to sort actual data and data simulated from our candidate models in order to determine whether the latter can match commonly perceived features of bull and bear markets. Section 4 summarizes the benchmark 2-state model and develops our proposed 4-state specification. Estimation and model comparison are discussed in Section 5. Section 6 presents results including: parameter estimates; probabilistic identification of the market states and regimes; and Value-at-Risk forecasts. Section 7 concludes.

2 Data

We begin with 125 years of daily capital gain returns on a broad market equity index. Our source for the period 1926-2008 inclusive is the value-weighted return excluding dividends associated with the CRSP S&P 500 index.[3] The 1885:02-1925 daily capital gain returns are courtesy of Bill Schwert (see Schwert (1990)). For 2009-2010, we use the daily rates of change of the S&P 500 index level (SPX) obtained from Reuters.

Returns are converted to daily continuously compounded returns from which we construct weekly continuously compounded returns by cumulating daily returns from Wednesday close to Wednesday close of the following week. If a Wednesday is missing, we use Tuesday close. If the Tuesday is also missing, we use Thursday. Weekly realized variance (RV) is computed as the sum of daily (intra-week) squared returns.

[2]A Google search turns up such headlines as: 'Bull Market or Bear-Market Rally?', 'Genuine bull market, not a bear market rally', 'A bear rally in bull's clothing?', 'Bear market rally/Bull market beginning?', and many more.

[3]Note that this is the S&P 90 prior to March 4, 1957.

Weekly returns are scaled by 100 so they are percentage returns. Unless otherwise indicated, henceforth returns refer to weekly continuously compounded returns expressed as a percentage. We have 6498 weekly observations covering the period February 25, 1885 to January 20, 2010. Summary statistics are shown in table 1.

3 Bull and Bear Dating Algorithms

Ex post sorting methods for classification of stock returns into bull and bear phases are called dating algorithms. Such algorithms attempt to use a sequence of rules to isolate patterns in the data. A popular algorithm is that used by Bry and Boschan (1971) to identify turning points of business cycles. Pagan and Sossounov (2003) adapted this algorithm to study the characteristics of bull/bear regimes in monthly stock prices. First a criterion for identifying potential peaks and troughs is applied; then censoring rules are used to impose minimum duration constraints on both phases and complete cycles. Finally, an exception to the rule for the minimum length of a phase is allowed to accommodate 'sharp movements' in stock prices.

The Pagan and Sossounov (2003) BB algorithm is summarized in the appendix. There are alternative dating algorithms or filters for identifying turning points. For example, the Lunde and Timmermann (2004) (LT) algorithm identifies bull and bear markets using a cumulative return threshold of 20% to locate peaks and troughs moving forward.[4] They define a binary market indicator variable I_t which takes the value 1 if the stock market is identified by their algorithm to be in a bull state at time t and 0 if it is in a bear state. Our application of this LT dating algorithm is also summarized in the appendix.

The classification of our data into bull and bear regimes using these two filters is found in Table 2. There are several features to note. First, the sorting of the data is broadly similar but with important differences. For example, during the 1930s the BB approach finds many more switches between market phases than does the LT algorithm. More recently, both identify 1987-12 as a trough but the subsequent bull phase ends in 1990-06 for LT but 2000-03 for BB. The average bear duration is similar (66 weeks) while the average bull duration is quite different, 117.0 weeks (BB) versus 166.7 (LT). In other words, the different parameters and assumptions in the filtering methods can result in a different classification of market phases.

Although the ex post dating algorithms can filter the data to locate different regimes, they cannot be used for forecasting or inference. In addition, since the sorting rule focuses on the first moment, it does not characterize the full distribution of returns. The latter is required if we wish to derive features of the regimes that are useful for measuring

[4] Lunde and Timmermann (2004) explore alternative thresholds and also asymmetric thresholds for switching from bull versus from bear markets. For this description we use a threshold of 20%.

and forecasting risk. Also, as noted above, ex post dating algorithms sort returns into a particular regime with probability zero or one. However, the data provides more information allowing one to estimate probabilities associated with particular states.

Nevertheless, the dating algorithms are still very useful. For example, we use the LT algorithm to sort data simulated from our candidate parametric models in order to determine whether the latter can match commonly perceived features of bull and bear markets.

4 Models

In this section, we briefly review a benchmark two-state model, our proposed 4-state model, and some alternative specifications of the latter used to evaluate robustness of our best model.

4.1 Two-State Markov-Switching Model

The concept of bull and bear markets suggests cycles or trends that get reversed. Since those regimes are not observable, as discussed in Section 1, two-state latent-variable MS models have been applied to stock market data. A two-state 1st-order Markov model can be written

$$r_t|s_t \sim N(\mu_{s_t}, \sigma^2_{s_t}) \tag{4.1}$$

$$p_{ij} = p(s_t = j|s_{t-1} = i) \tag{4.2}$$

$i = 1, 2$, $j = 1, 2$. We impose $\mu_1 < 0$ and $\mu_2 > 0$ so that $s_t = 1$ is the bear market and $s_t = 2$ is the bull market.

Modeling of the latent regimes, regime probabilities, and state transition probabilities, allows explicit model estimation and inference. In addition, in contrast to dating algorithms or filters, forecasts are possible. Investors can base their investment decisions on the posterior states or the whole forecast density.

4.2 MS-4 to allow Bull Corrections and Bear Rallies

Consider the following general K-state first-order Markov-switching model for returns

$$r_t|s_t \sim N(\mu_{s_t}, \sigma^2_{s_t}) \tag{4.3}$$

$$p_{ij} = p(s_t = j|s_{t-1} = i) \tag{4.4}$$

$i = 1, ..., K$, $j = 1, ..., K$. We explore a 4-state model, $K = 4$, in order to focus on modeling potential phases of the aggregate stock market. Without any additional restrictions we cannot identify the model and relate it to market phases. Therefore, we consider the following restrictions. First, the states $s_t = 1, 2$ are assumed to govern the bear market; we label these states as the *bear regime*. The states $s_t = 3, 4$ are assumed to govern the bull market; these states are labeled the *bull regime*. Each regime has 2 states which allows for positive and negative periods of price growth within each regime. In particular we impose

$$\begin{aligned} \mu_1 &< 0 \text{ (bear market state)}, \\ \mu_2 &> 0 \text{ (bear market rally)}, \\ \mu_3 &< 0 \text{ (bull market correction)}, \\ \mu_4 &> 0 \text{ (bull market state)}. \end{aligned} \tag{4.5}$$

This structure can capture short-term reversals in market trends. Each state can have a different variance and can accommodate autoregressive heteroskedasticity in returns. In addition, conditional heteroskedasticity within each regime can be captured.

Consistent with the 2 states in each regime the full transition matrix is

$$P = \begin{pmatrix} p_{11} & p_{12} & 0 & p_{14} \\ p_{21} & p_{22} & 0 & p_{24} \\ p_{31} & 0 & p_{33} & p_{34} \\ p_{41} & 0 & p_{43} & p_{44} \end{pmatrix}. \tag{4.6}$$

This structure allows for several important features that are excluded in the smaller Markov-switching models in the literature. First, a bear regime can feature several episodes of the bear state and bear rally state, exactly as many investors feel we observe in the data. Similarly, the bull regime can be characterized by a combination of bull states and bull corrections. Because, the realization of states in a regime will differ over time, both bull and bear regime will tend to look heterogenous to some extent. For instance, based on returns, a bear regime lasting 5 periods made of the states

$$s_t = 1, s_{t+1} = 1, s_{t+2} = 1, s_{t+3} = 2, s_{t+4} = 2, s_{t+5} = 2, s_{t+6} = 4$$

will tend to look very different than

$$s_t = 1, s_{t+1} = 1, s_{t+2} = 1, s_{t+3} = 2, s_{t+4} = 1, s_{t+5} = 1, s_{t+6} = 4.$$

A second important contribution is that a bear rally is allowed to move either into the bull state or back to the bear state; analogously, a bull correction can move to a bear state or back to the bull state. These important inter and intra-regime dynamics are absent in the existing literature.

The unconditional probabilities associated with P can be solved (Hamilton (1994))

$$\pi = (A'A)^{-1}A'e \tag{4.7}$$

where $A' = [P' - I,\ \iota]$ and $e' = [0, 0, 0, 0, 1]$ and $\iota = [1, 1, 1, 1]'$.

Using the matrix of unconditional state probabilities given by (4.7), we impose the following conditions on long-run returns in the bear and bull regimes respectively[5],

$$E[r_t|\text{bear regime}, s_t = 1, 2] = \frac{\pi_1}{\pi_1 + \pi_2}\mu_1 + \frac{\pi_2}{\pi_1 + \pi_2}\mu_2 < 0 \tag{4.8}$$

$$E[r_t|\text{bull regime}, s_t = 3, 4] = \frac{\pi_3}{\pi_3 + \pi_4}\mu_3 + \frac{\pi_4}{\pi_3 + \pi_4}\mu_4 > 0. \tag{4.9}$$

We do not impose any constraint on the variances.

The equations (4.5) and (4.6), along with equations (4.8) and (4.9), serve to identify[6] bull and bear regimes. The bull (bear) regime has a long-run positive (negative) return. Each market regime can display short-term reversals that differ from their long-run mean. For example, a bear regime can display a bear market rally (temporary period of positive returns), even though its long-run return is negative. Similarly for the bull market.

4.3 Other Models for Robustness Checks

Besides the 4-state model we consider several other specifications and provide model comparisons among them. The dependencies in the variance of returns are the most dominate feature of the data. This structure may adversely dominate dynamics of the conditional mean. The following specifications are included to investigate this issue.

4.3.1 Restricted 4-State Model

This is identical to the 4-state model in Section 4.2 except that inside a regime the return innovations are homoskedastic. That is, $\sigma_1^2 = \sigma_2^2$ and $\sigma_3^2 = \sigma_4^2$. In this case, the variance within each regime is restricted to be constant although the overall variance of returns can change over time due to switches between regimes.

[5]Note that at this point we are abstracting from an equilibrium model of investor behavior. Investors cannot identify states with probability 1 so modeling investors' expected returns at each point is beyond the scope of this paper. Regimes or states may have negative expected returns for some limited period for a variety of reasons such as changes in risk premiums due to learning following breaks (Pastor and Stambaugh (2001)), different investment horizons (Guidolin and Timmermann (2005)), etc.

[6]Discrete mixture of distributions are subject to identification issues. Label switching occurs when the states and parameters are permuted but the likelihood stays the same. Our prior restrictions avoid this issue and identify the model. For more discussion on this see Frühwirth-Schnatter (2006).

4.3.2 Markov-Switching Mean and i.i.d. Variance Model

In this model, the mean and variance dynamics are decoupled. This is a robustness check to determine to what extent the variance dynamics might be driving the regime transitions. This specification is identical to the Markov-switching model in Section 4.2 except that only the conditional mean follows the Markov chain while the variance follows an independent i.i.d mixture. That is,

$$r_t|s_t = \mu_{s_t} + z_t \tag{4.10}$$

$$z_t \sim \sum_{i=1}^{L} \eta_i \mathbf{N}(0, \sigma_i^2), \quad \eta_i \geq 0, \; \sum_{i=1}^{L} \eta_i = 1 \tag{4.11}$$

$$p_{ij} = p(s_t = j|s_{t-1} = i), \quad i, j = 1, ..., K \tag{4.12}$$

For identification, $\sigma_1^2 < \sigma_2^2 < \cdots < \sigma_L^2$ is imposed along with the constraints used for the conditional mean in the previous section. We focus on the case $K = 4$ and $L = 4$, again to allow us to capture at least four phases of cycles for aggregate stock returns.

5 Estimation and Model Comparison

5.1 Estimation

In this section we discuss Bayesian estimation for the most general model introduced in Section 4.2 assuming there are K states, $k = 1, ..., K$. The other models are estimated in a similar way with minor modifications.

There are 3 groups of parameters $M = \{\mu_1, ..., \mu_K\}$, $\Sigma = \{\sigma_1^2, ..., \sigma_K^2\}$, and the elements of the transition matrix P. Let $\theta = \{M, \Sigma, P\}$ and given data $I_T = \{r_1, ..., r_T\}$ we augment the parameter space to include the states $S = \{s_1, ..., s_T\}$ so that we sample from the full posterior $p(\theta, S|I_T)$. Assuming conditionally conjugate priors $\mu_i \sim N(m_i, n_i^2)$, $\sigma_i^{-2} \sim G(v_i/2, w_i/2)$ and each row of P following a Dirichlet distribution, allows for a Gibbs sampling approach following Chib (1996). Gibbs sampling iterates on sampling from the following conditional densities given startup parameter values for M, Σ and P:

$S|M, \Sigma, P$

$M|\Sigma, P, S$

$\Sigma|M, P, S$

$P|M, \Sigma, S$

Sequentially sampling from each of these conditional densities results in one iteration of the Gibbs sampler. Dropping an initial set of draws to remove any dependence from startup values, the remaining draws $\{S^{(j)}, M^{(j)}, \Sigma^{(j)}, P^{(j)}\}_{j=1}^{N}$ are collected to estimate features of the posterior density. Simulation consistent estimates can be obtained as sample averages of the draws. For example, the posterior mean of the state dependent mean and standard deviation of returns are estimated as

$$\frac{1}{N}\sum_{j=1}^{N}\mu_k^{(j)}, \quad \frac{1}{N}\sum_{j=1}^{N}\sigma_k^{(j)}, \tag{5.1}$$

for $k = 1, ..., K$ and are simulation consistent estimates of $E[\mu_k|I_T]$ and $E[\sigma_k|I_T]$ respectively.

The first sampling step of $S|M, \Sigma, P$ involves a joint draw of all the states. Chib (1996) shows that this can be done by a so-called forward and backward smoother through the identity

$$p(S|\theta, I_T) = p(s_T|\theta, I_T)\prod_{t=1}^{T-1} p(s_t|s_{t+1}, \theta, I_t). \tag{5.2}$$

The forward pass is to compute the Hamilton (1989) filter for $t = 1, ..., T$

$$p(s_t = k|\theta, I_{t-1}) = \sum_{l=1}^{K} p(s_{t-1} = l|\theta, I_{t-1})p_{lk}, \quad k = 1, ..., K, \tag{5.3}$$

$$p(s_t = k|\theta, I_t) = \frac{p(s_t = k|\theta, I_{t-1})f(r_t|I_{t-1}, s_t = k)}{\sum_{l=1}^{K} p(s_t = l|\theta, I_{t-1})f(r_t|I_{t-1}, s_t = l)}, \quad k = 1, ..., K. \tag{5.4}$$

Note that $f(r_t|I_{t-1}, s_t = k)$ is the normal pdf $N(\mu_k, \sigma_k^2)$. Finally, Chib (1996) has shown that a joint draw of the states can be taken sequentially from

$$p(s_t|s_{t+1}, \theta, I_t) \propto p(s_t|\theta, I_t)p(s_{t+1}|s_t, P), \tag{5.5}$$

where the first term on the right-hand side is from (5.4) and the second term is from the transition matrix. This is the backward step and runs from $t = T - 1, T - 2, ..., 1$. The draw of s_T is taken according to $p(s_T = k|\theta, I_T), k = 1, ..., K$.

The second and third sampling steps are straightforward and use results from the linear regression model. Conditional on S we select the data in regime k and let the number of observations of $s_t = k$ be denoted as T_k. Then $\mu_k|\Sigma, P, S \sim N(a_k, A_k)$,

$$a_k = A_k\left(\sigma_k^{-2}\sum_{t\in\{t|s_t=k\}} r_t + n_k^{-2}m_k\right), \quad A_k = (\sigma_k^{-2}T_k + n_k^{-2})^{-1}. \tag{5.6}$$

A draw of the variance is taken from

$$\sigma_k^{-2}|M, P, S \sim G\left((T_k + v_k)/2, \left(\sum_{t\in\{t|s_t=k\}}(r_t - \mu_k)^2 + w_k\right)/2\right) \tag{5.7}$$

Given the conjugate Dirichlet prior on each row of P, the final step is to sample $P|M, \Sigma, S$ from the Dirichlet distribution (Geweke (2005)).

An important byproduct of Gibbs sampling is an estimate of the smoothed state probabilties $p(s_t|I_T)$ which can be estimated as

$$p(\widehat{s_t = i}|I_T) = \frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{s_t=i}(S^{(j)}) \tag{5.8}$$

for $i = 1, ..., K$.

At each step, if a parameter draw violates any of the prior restrictions in (4.5), (4.6), (4.8) and (4.9), then it is discarded. For the 4-state model we set the independent priors as

$$\mu_1 \sim \mathbf{N}(-0.7, 1), \mu_2 \sim \mathbf{N}(0.2, 1), \mu_3 \sim \mathbf{N}(-0.2, 1), \mu_4 \sim \mathbf{N}(0.3, 1) \tag{5.9}$$

$$\sigma_i^{-2} \sim G(0.5, 0.05) \;\; \text{for } i = 1, 2, 3, 4 \tag{5.10}$$

$$\{p_{11}, p_{12}, p_{14}\} \sim Dir(8, 1.5, 0.5), \{p_{21}, p_{22}, p_{24}\} \sim Dir(1.5, 8, 0.5) \tag{5.11}$$

$$\{p_{31}, p_{33}, p_{34}\} \sim Dir(0.5, 8, 1.5), \{p_{41}, p_{43}, p_{44}\} \sim Dir(0.5, 1.5, 8). \tag{5.12}$$

These priors are informative but cover a wide range of empirically relevant parameter values.

5.2 Model Comparison

If the marginal likelihood can be computed for a model it is possible to compare models based on Bayes factors. Non-nested models can be compared as well as specifications with a different number of states. Note that the Bayes factor penalizes over-parameterized models that do not deliver improved predictions.[7] For the general Markov-switching model with K states, the marginal likelihood for model $\mathcal{M}_i$ is defined as

$$p(r|\mathcal{M}_i) = \int p(r|\mathcal{M}_i, \theta)p(\theta|\mathcal{M}_i)d\theta \tag{5.13}$$

which integrates out parameter uncertainty. $p(\theta|\mathcal{M}_i)$ is the prior and

$$p(r|\mathcal{M}_i, \theta) = \prod_{t=1}^{T} f(r_t|I_{t-1}, \theta) \tag{5.14}$$

[7]This is referred to as an Ockham's razor effect. See Kass and Raftery (1995) for a discussion on the benefits of Bayes factors.

is the likelihood which has S integrated out according to

$$f(r_t|I_{t-1},\theta) = \sum_{k=1}^{K} f(r_t|I_{t-1},\theta,s_t=k)p(s_t=k|\theta,I_{t-1}). \tag{5.15}$$

The term $p(s_t = k|\theta, I_{t-1})$ is available from the Hamilton filter. Chib (1995) shows how to estimate the marginal likelihood for MS models. His estimate is based on re-arranging Bayes' theorem as

$$\widehat{p(r|\mathcal{M}_i)} = \frac{p(r|\mathcal{M}_i,\theta^*)p(\theta^*|\mathcal{M}_i)}{p(\theta^*|r,\mathcal{M}_i)} \tag{5.16}$$

where θ^* is a point of high mass in the posterior pdf. The terms in the numerator are directly available above while the denominator can be estimated using additional Gibbs sampling runs.[8]

A log-Bayes factor between model $\mathcal{M}_i$ and $\mathcal{M}_j$ is defined as

$$\log(BF_{ij}) = \log(p(r|\mathcal{M}_i)) - \log(p(r|\mathcal{M}_j)). \tag{5.17}$$

Kass and Raftery (1995) suggest interpreting the evidence for $\mathcal{M}_i$ versus $\mathcal{M}_j$ as: not worth more than a bare mention for $0 \le \log(BF_{ij}) < 1$; positive for $1 \le \log(BF_{ij}) < 3$; strong for $3 \le \log(BF_{ij}) < 5$; and very strong for $\log(BF_{ij}) \ge 5$.

5.3 Predictive Density

An important feature of our probabilitic approach is that a predictive density of future returns can be computed that integrates out all uncertainty regarding states and parameters.

The predictive density for future returns based on current information at time t is computed as

$$p(r_t|I_{t-1}) = \int f(r_t|I_{t-1},\theta)p(\theta|I_{t-1})d\theta \tag{5.18}$$

which involved integrating out both state and parameter uncertainty using the posterior distribution $p(\theta|I_{t-1})$. From the Gibbs sampling draws $\{S^{(j)}, M^{(j)}, \Sigma^{(j)}, P^{(j)}\}_{j=1}^{N}$ based on data I_{t-1} we approximate the predictive density as

$$\widehat{p(r_t|I_{t-1})} = \frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{K} f(r_t|I_{t-1},\theta^{(i)},s_t=k)p(s_t=k|s_{t-1}^{(i)},\theta^{(i)}) \tag{5.19}$$

where $f(r_t|I_{t-1},\theta^{(i)},s_t=k)$ follows $N(\mu_k^{(i)},\sigma_k^{2(i)})$ and $p(s_t=k|s_{t-1}^{(i)},\theta^{(i)})$ is the transition

[8] The integrating constant in the prior pdf is estimated by simulation.

probability.

The predictive mean of a future state s_t can also be easily estimated by simulating from the distribution $p(s_t = k|s_{t-1}^{(i)}, \theta^{(i)})$ a state $s_t^{(j)}$ for each state and parameter draw $s_{t-1}^{(i)}, \theta^{(i)}$. The average of these draws, $\{s_t^{(j)}\}_{j=1}^N$ is an estimate of $E[s_t|I_{t-1}]$.

6 Results

6.1 Parameter Estimates and Implied Distributions

Model estimates for the 2-state Markov-switching (MS-2) model are found in Table 3. State 1 has a negative conditional mean along with a high conditional variance whereas state 2 displays a high conditional mean with a low conditional variance. Both regimes are very persistent. These results are consistent with the sorting of bull and bear regimes in Maheu and McCurdy (2000a) and Guidolin and Timmermann (2005).

Estimates for our proposed 4-state model (MS-4) are found in Table 4. All parameters are precisely estimated indicating that the data are quite informative. Recall that states $s_t = 1, 2$ capture the bear regime while states $s_t = 3, 4$ capture the bull regime. Each regime contains a state with a positive and a negative conditional mean. We label states 1 and 2 the bear and bear rally states respectively; states 3 and 4 are the bull correction and bull states.

Consistent with the MS-2 model, volatility is highest in the bear regime. In particular, the highest volatility occurs in the bear regime in state 1. This state also delivers the lowest average return. The highest average return and lowest volatility is in state 4 which is part of the bull regime. The bear rally state ($s_t = 2$) delivers a conditional mean of 0.23 and conditional standard deviation of 2.63. However, this mean is lower and the volatility higher than the bull positive growth state ($s_t = 4$). Analogously, the bull correction state ($s_t = 3$) has a larger conditional mean ($-0.13 > -0.94$) and smaller volatility ($2.18 < 6.01$) than the bear state 1.

All states display high persistence (p_{ii} is high for all i). However, the transition probabilities display some asymmetries. For example, the probability of a bear rally moving back to the bear state 1 ($p_{21} = 0.015$) is a little lower than changing regime to a bull market ($p_{24} = 0.019$). On the other hand, the probability of a bull correction returning to a bull market ($p_{34} = 0.051$) is considerably higher than changing regime to the bear state ($p_{31} = 0.010$).

Figure 1 displays the density of each of the 4 states. The differences in the illustrated densities are in accord with the parameter estimates in Table 4. Differences in the spreads of the densities are most apparent but the locations are also different. There is no suggestion from these plots that states 1 and 2 are the same or that states 3 and 4 are the same, as a two-state Markov-switching model would assume.

Integrating state 1 and 2 gives the bear regime and doing the same for states 3 and 4 produces the bull regime. These densities are shown in Figure 2. The bear regime has a mean slightly below 0 but with a much larger variance than the bull regime. The implied unconditional density of returns is a mixture of these two regimes and displayed in the middle of the figure. Table 5 reports the unconditional probabilties for the states. On average the market spends 0.157 of time in a bear rally while 0.304 in a bull correction. The most time is spent in the bull growth state 4. The unconditional probability of the bull regime is 0.773.

A comparison of the regime statistics implied by the parameter estimates for the MS-2 and MS-4 models is found in Table 6. The expected duration of regimes is much longer in the 4-state model. That is, by allowing heterogeneity within a regime in our 4-state model, we switch between bull and bear markets less frequently. For instance, in a MS-2 parameterization the bull market has a duration of only 82.6 weeks, about 18 months, while the richer MS-4 model has a bull duration of just under 5 years. As we will see below, there is much more switching between regimes in the MS-2 model.

In the 2-state model, the expected return and variance are fixed within a regime. In this case, the only source of intra-regime variance is return innovations. For example for the bear regime in the MS-2 model, the expected variance is $E[\text{Var}(r_t|s_t = 1)] = 19.6$. In contrast, the average variance for each regime in the 4-state model can be attributed to changes in the conditional mean as well as to the average conditional variance of the return innovations. For instance, the average variance of returns in the bear regime can be decomposed as $\text{Var}(r_t|s_t = 1, 2) = \text{Var}(E[r_t|s_t]|s_t = 1, 2) + E[\text{Var}(r_t|s_t)|s_t = 1, 2] = 0.31 + 16.1$, with a similar result for the bull regime. For the bull and bear phase, the mean dynamics account for a small share, 2% of the total variance.[9]

The MS-2 model assumes normality in both market regimes while the MS-4 shows that the data is at odds with this assumption. Skewness in present in bear markets while excess kurtosis is found in both bull and bear regimes. Overall the bear market deviates more from a normal distribution; it has thicker tails and captures more extreme events.

Table 7 summarizes features of the MS-4 parameterization for both the regimes and their component states derived from the posterior parameter estimates. The bear regime duration is 77.8 weeks, much shorter than the bull regime duration of 256.0 weeks. The average cumulative return in the bear (bull) regime[10] is -9.94 (33.0). The volatility in the bear market is more than twice that in the bull market. The third panel provides a breakdown of cumulative return means in each of the component states of the market regimes. The bear rally yields a cumulative return of 7.10 on average which partially

[9]This is computed as 0.31/(0.31+16.1) and 0.04/(0.04+2.89).

[10]This is equal to the expected return for the bear regime, given by Equation (4.8), times the expected duration for that regime which is $\begin{pmatrix} \frac{\pi_1}{\pi_1+\pi_2} \\ \frac{\pi_2}{\pi_1+\pi_2} \end{pmatrix}' \left[I_2 - \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \right]^{-2} \begin{pmatrix} p_{14} \\ p_{24} \end{pmatrix}$.

offsets the average decline of -12.4 in state 1. On the other hand the bull correction has a cumulative return mean of -2.13 which diminishes the average cumulative return of 7.88 in state 4. Note that these states are combined into bull and bear market regimes in heterogenous patterns over time yielding the statistics for regimes summarized in the first two panels of Table 7.

Although the stock market spends most of the time in the bull regime (states 3 and 4), in terms of individual states it is state 2 that has the longest duration while the shortest is state 1. The final panel of Table 7 records the conditional mean divided by the associated conditional standard deviation for each state, that is, estimates of μ_i/σ_i from Table 4. This is analogous to an ex post Sharpe ratio. State 4 provides the most favorable risk-return tradeoff followed by state 2, 3 and 1. Note that the Sharpe ratio in the bull state 4 is approximately 2.5 times larger than in the bear rally (state 2). In other words, even though the bear rally delivers a postive expected return, that return is much more variable than in the bull state.

6.2 Model Comparisons

One can conduct formal model comparisons based on the marginal likelihoods reported in Table 8. The constant mean and variance model performs the worst (has the lowest marginal likelihood). The next model has a constant mean but allows the variances to follow a 4-state i.i.d. mixture. Following this are models with a 2-state versus a 4-state Markov-switching conditional mean – both combined with a 4-state i.i.d. variance as in Section 4.3.2. In both cases, the additional dynamics that are introduced to the conditional mean of returns provides a significant improvement over the constant mean case with the same 4-state i.i.d. variance. However, all of these specifications are strongly dominated by their counterparts which allow a common 2 (or 4) state Markov chain to direct both conditional moments. These specifications capture persistence in the conditional variance.

Note that the log-Bayes factor between the 2-state MS and the 4-state MS in the conditional mean restricted to have only a 2-state conditional variance (Section 4.3.1) is large at $53.4 = -13849.9 - (-13903.3)$. This improved fit comes when additional conditional mean dynamics (going from 2 to 4 states) are added to the basic 2-state MS model. The best model is the 4-state Markov-switching model. The log-Bayes factor in support of the 4-state versus the 2-state model is $162.9 = -13740.4 - (-13903.3)$. The zero restrictions in the transition matrix (4.6) are also strongly supported by the data. For instance, the log-Bayes factor is $6.9 = -13740.4 - (-13747.3)$ in support of the MS-4 model with P matrix (4.6) as compared to a 4 state model with an unrestricted transition matrix (all 16 elements of P are estimated).

Overall, there is very strong evidence that the 4-state specification of Section 4.2

provides the best fit to weekly returns. The comparisons also show that this improved fit comes from improved fit to both the conditional mean and variance. Not only does our MS-4 model provide a better economic characterization of differences in stock market cycles but the model statistically dominates other alternatives.

The Markov-switching models specify a latent variable that directs low frequency trends in the data. As such, the regime characteristics from the population model are not directly comparable to the dating algorithms of Section 3. Instead, we consider the dating algorithm as a lens to view both the S&P500 data and data simulated from our preferred MS-4 model. Using parameter draws from the Gibbs sampler, we simulate return data from the model and then apply the LT dating algorithm to those simulated returns. This is done many times[11] and the average and 0.70 density intervals of these statistics are reported in Table 9 along with the statistics from the S&P500 data. Although our model provides a richer 4 state description of bull and bear markets it does account for all of the data statistics associated with a simpler 2 state view of the market using the LT dating algorithm.

6.3 Identification of Historical Turning Points in the Market

The dating of the market regimes using the LT dating algorithm are found in the top panel of Figure 3. The shaded portions under the cumulative return denote bull markets while the white portions of the figure are the bear markets. Below this panel is the smoothed probability of a bull market, $p(s_t = 3|I_T) + p(s_t = 4|I_T)$ for the 4-state model. The final plot in Figure 3 is the smoothed probability of a bull market, $p(s_t = 2|I_T)$ from the 2-state model. The 4-state model produces less erratic shifts between market regimes, closely matches the trends in prices, and generally corresponds to the dating algorithm. The 2-state model is less able to extract the low frequency trends in the market. In high frequency data it is important to allow intra-regime dynamics, such as short-term reversals.

Note that the success of our model should not be based on how well it matches the results from dating algorithms. Rather this comparison is done to show that the latent-state MS models can identify bull and bear markets with similar features to those identified by conventional dating algorithms. Beyond that, the Markov-switching models presented in this paper provide a superior approach to modeling stock market trends as they deliver a full specification of the distribution of returns along with latent market dynamics. Such an approach permits out-of-sample forecasting which we turn to in Section 6.4.

The following subsections discuss how our model identifies sub-regime dynamics using examples from various subperiods. There are several important points revealed by

[11] 10,000 simulations each of 6498 observations.

this dicussion. First, bear (bull) markets are persistent but are made of many regular transitions between states 1 and 2 (3 and 4). Second, in each of the examples the move between regimes occurs through either the bear rally or the bull correction state. In other words, these additional dynamics are critical to fully capturing turning points in stock market cycles. This is also borne out by our model estimates. The most likely route for a bear market to go to a bull market is through the bear rally state. Given that a bull market has just started, the probability is 0.9342 that the previous state was a bear rally[12], and only 0.0658 that it was a bear state. Similarly, given that a bear market has just started, the probability is 0.8663 that the previous state was a bull correction, and only 0.1337 that it was a bull state. The following subperiod descriptions provide examples of this richer specification of turning points plus frequent reversals within a regime.

6.3.1 1927-1939

Figure 4 displays the log-price and the realized volatility (square root of realized variance) in the top panel, the smoothed states of the MS-4 model in the second panel, and the posterior probability of the bull market, $p(s_t = 3|I_T) + p(s_t = 4|I_T)$, in the last panel.

Just before the crash of 1929 the model identifies a bull correction state. The transition from a bull to bear market occurs as a move from a bull market state to a bull correction state and then into the bear regime. For the week ending October 16 1929, there was a return of -3.348 and the market transitioned from the bull correction state into the bear market state with $p(s_t = 1|I_T) = 0.63$. This is further reinforced so that the next 5 weeks have essentially probability 1 for state 1.

As this figure shows, the remainder of this subperiod is decisively a bear market, but displays considerable heterogeneity in that there are several short-lived bear rallies. The high levels of realized volatility coincide with the high volatility in the bear market states. Periods of somewhat lower volatility are associated with the bear rally states.

In Figure 4, a strong bear rally begins in late November 1933 and lasts until August 25, 1937, at which time there is a move back into the bear market state. Realized volatility increases with this move into state 1.

6.3.2 1980-1985

In Figure 5, the market displays several moves between the bull market state and the bull correction state before a short-term move into a bear market in August of 1982. Once again the transition from a bull to bear market is through a bull correction state. However, the bear market that emerges has state 1 that lasts only about 4 weeks. This is followed by a bear rally that results in increased prices accompanied with substantial

[12] $p(s_t = 2|s_{t+1} = 4, s_t = 1 \text{ or } 2) \propto \frac{p_{24}\pi_2}{\pi_1+\pi_2}$, $p(s_t = 1|s_{t+1} = 4, s_t = 1 \text{ or } 2) \propto \frac{p_{14}\pi_1}{\pi_1+\pi_2}$

volatility. The bear rally turns into a bull market in late April of 1983, thereafter are periods of the bull market state and bull corrections.

6.3.3 1987 crash

Prior to the 1987 crash there is a dramatic run-up in stock prices with generally low volatility, as illustrated in the top panel of Figure 6. It is interesting to note that the model shows a great deal of uncertainty about the state of the market well before the crash. In the first week of October, just before the crash, the most likely state is the bull correction with $p(s_t = 3|I_T) = 0.37$. The bear state which starts the following week lasts for about 5 weeks after which a strong bear rally quickly emerges as of the week ending November 18, 1987. It is the bear rally state that exits into a bull market during the week of August 17, 1988. Prices resume their strong increase until they plateau with a bull correction beginning the week of October 4, 1989.

6.3.4 2006-2010

We conclude with an analysis of recent market activity in Figure 7. The bull market state turned into a bull correction in mid-July 2007, which persisted until an abrupt move into the bear market state in early September 2008. This transition was accompanied by a dramatic increase in realized volatility. According to our model, the bear market became a bear market rally in the third week of March 2009 where it stayed until mid-November 2009 when it moved into the bull market state. As noted earlier, the positive trend in returns during a bear market rally do not get interpreted as a bull market until the market volatility declines to levels more typical of bull markets.

6.4 Example Application

An industry standard measure of potential portfolio loss is the Value-at-Risk (VaR). $\text{VaR}_{(\alpha),t}$ is defined as the 100α percent quantile of the portfolio value or return distribution given information at time $t-1$. We compute $\text{VaR}_{(\alpha),t}$ from the predictive density of the MS-4 model as

$$p(r_t < \text{VaR}_{(\alpha),t}|I_{t-1}) = \alpha. \tag{6.1}$$

Given a correctly specified model, the probability of a return of $\text{VaR}_{(\alpha),t}$ or less is α.

To compute the Value-at-Risk from the MS-4 model we do the following. First, N draws from the predictive density are taken as follows: draw θ and s_{t-1} from the Gibbs sampler, a future state $\tilde{s}_t$ is simulated based on P and $\tilde{r}_t|\tilde{s}_t \sim N(\mu_{\tilde{s}_t}, \sigma^2_{\tilde{s}_t})$. The details are discussed in Section 5.3. From the resulting draws, the $\tilde{r}_t$ with rank $[N\alpha]$ is an estimate of $\text{VaR}_{(\alpha),t}$.

Figure 8 displays the conditional VaR from January 3, 2007 to January 20, 2010 predicted by the MS-4 model, as well as that implied by the normal benchmark for $\alpha = 0.05$. At each point the model is estimated based on information up to $t-1$. Similarly, the benchmark, $N(0, \sigma^2)$, sets σ^2 to the sample variance using I_{t-1}.

The normal benchmark overestimates the VaR for the early part of this subsample but starts to understate it at times, beginning in mid-2007, and then severely under estimates in the last few months of 2008. The MS-4 model provides a very different $\text{VaR}_{(.05),t}$ over time because it takes into account the predicted regime, as indicated by the middle and bottom panels of Figure 8 which show *forecasts* of the states and regimes respectively. Note that the potential losses, shown in the top panel, increase considerably in September and October 2008 as the model identifies a move from a bull to a bear market.

6.4.1 Real-time Identification of the Bear Market

This out-of-sample application also gives us an opportunity to assess in real time when our model identified a move into the bear regime. In Section 6.3.4 this was discussed in the context of the full sample smoothed estimates. We now consider the identification process that would have been historically available to investors using the model forecasts. This will differ from the previous results as we are using a smaller sample and updating estimates as new data arrives.

The second and third panel of Figure 8 report the predictive mean of the states and regimes. Prior to 2008, forecasts of the bull states occur the most, including some short episodes of bull corrections. In the first week of October 2008, the probability of a bull regime drops from 0.85 to essentially zero and remains there for some time. In other words, the model in real time detects a turning point in the first week of October 2008 from the bull to the bear regime. The first half of the bear regime that follows is characterized by the bear state while the second half is largely classified as a bear rally.

Toward the end of our sample there is a move from the bear market rally state to a bull market. In real time, in early December 2009 the model forecasts a move from the bear rally to the bull market state. For the week ending December 9, we have $p(s_t = 1|I_{t-1}) = 0.02$, $p(s_t = 2|I_{t-1}) = 0.17$, $p(s_t = 3|I_{t-1}) = 0.14$ and $p(s_t = 4|I_{t-1}) = 0.67$. The evidence for a bull market regime gradually strenthens; the last observation in our sample, January 20, 2010, has probabilities 0.01, 0.11, 0.07 and 0.81 for states 1,2,3 and 4, with the bull market state being the most likely.

7 Conclusion

This paper proposes a new 4-state Markov-switching model to identify the components of bull and bear market regimes in weekly stock market data. Bull correction and bull states govern the bull regime; bear rally and bear states govern the bear regime. Our probability model fully describes the return distribution while treating bull and bear regimes and their component states as unobservable.

A bear rally is allowed to move back to the bear state or to exit the bear regime by moving to a bull state. Likewise, a bull correction can move back to the bull state or exit the bull regime by transitioning to a bear state. This implies that regimes can feature several episodes of their component states. For example, a bull regime can be characterized by a combination of bull states and bull corrections. Similarly, a bear regime can consist of several episodes of the bear state and the bear rally state. Because the realization of states in a regime will differ over time, bull and bear regimes can be heterogenous over time. This richer structure, including both intra-regime and inter-regime dynamics, results in a richer characterization of market cycles.

Probability statements on regimes and future returns are available. Our model strongly dominates other alternatives. Model comparisons show that the 4-state specification of bull and bear markets is strongly favored over several alternatives including a two-state model, as well as various alternative specifications for variance dynamics. For example, relative to a two-state model, there is less erratic switching so that market regimes are more persistent.

We find that bull corrections and bear rallies are empirically important for out-of-sample forecasts of turning points and VaR predictions. For these out-of-sample applications, the model provides probability statements concerning the predictive density of returns. The probabilities are used in an example application that compares VaR forecasts to a normal benchmark model. The latter overestimates the VaR for much of the sample and then tends to understate it from mid-2007 to late 2009. The MS-4 specification has a very different $\text{VaR}_{(.05),t}$ over time becasue it takes into account forecasts of regime changes. The potential losses increased considerably in September and October of 2008 as the model identifies a move from a bull to a bear market.

8 Appendix: Ex Post Dating Algorithms

The Pagan and Sossounov (2003) adaptation of the Bry-Boschan (BB) algorithm can be summarized as follows:

1. Identify the peaks and troughs by using a window of 8 months.

2. Enforce alternation of phases by deleting the lower of adjacent peaks and the higher of adjacent troughs.

3. Eliminate phases less than 4 months unless changes exceed 20%.

4. Eliminate cycles less than 16 months.

Window width and phase duration constraints will depend on the particular series and will obviously be different for smoothed business cycle data than for stock prices. Pagan and Sossounov (2003) provide a detailed discussion of their choices for these constraints.

The Lunde and Timmermann (2004) dating algorithm defines a binary market indicator variable I_t which takes the value 1 if the stock market is in a bull state at time t and 0 if it is in a bear state. The stock price at the end of period t is labelled P_t. Our application of their dating algorithm can be summarized as: use a 6-month window to locate the initial local maximum or minimum.

Suppose we have a local maximum at time t_0, in which case we set $P_{t_0}^{\max} = P_{t_0}$.

1. Define stopping-time variables associated with a bull market as

$$\tau_{\max}(P_{t_0}^{\max}, t_0 \mid I_{t_0} = 1) = \inf\{t_0 + \tau : P_{t_0+\tau} \geq P_{t_0}^{\max}\}$$

$$\tau_{\min}(P_{t_0}^{\max}, t_0 \mid I_{t_0} = 1) = \inf\{t_0 + \tau : P_{t_0+\tau} \leq 0.8P_{t_0}^{\max}\}$$

2. One of the following happen.

   - If $\tau_{\max} < \tau_{\min}$, bull market continues, update the new peak value $P_{t_0+\tau_{\max}}^{\max} = P_{t_0+\tau_{\max}}$ discard previous peak at time t_0 and set $I_{t_0+1} = \cdots I_{t_0+\tau_{\max}} = 1$. Go to 2 above.
   - If $\tau_{\max} > \tau_{\min}$, we find a trough at time $t_0 + \tau_{\min}$ and we have been in a bear market from $t_0 + 1$ to $t_0 + \tau_{\min}$, $I_{t_0+1} = \cdots = I_{t_0+\tau_{\min}} = 0$. Record the value $P_{t_0+\tau_{\min}}^{\min} = P_{t_0+\tau_{\min}}$ and mark time t_0 as one peak. Go to 1 below for bear market.

On the other hand suppose t_0 is a local minimum.

1. Bear market stopping times are

$$\tau_{\min}(P_{t_0}^{\min}, t_0 \mid I_{t_0} = 0) = \inf\{t_0 + \tau : P_{t_0+\tau} \leq P_{t_0}^{\min}\}$$

$$\tau_{\max}(P_{t_0}^{\min}, t_0 \mid I_{t_0} = 0) = \inf\{t_0 + \tau : P_{t_0+\tau} \geq 1.2P_{t_0}^{\min}\}$$

2. One of the following happens.

   - If $\tau_{\min} < \tau_{\max}$, bear market continues, update the trough point forward, $P_{t_0+\tau_{\min}}^{\min} = P_{t_0+\tau_{\min}}$ discard previous trough value at time t_0 and set $I_{t_0+1} = \cdots = I_{t_0+\tau_{\min}} = 0$. Go to 1.
   - If $\tau_{\min} > \tau_{\max}$ we have a peak at $t_0 + \tau_{\max}$ and have been in a bull market from $t_0 + 1$ to $t_0 + \tau_{\max}$, $I_{t_0+1} = \cdots = I_{t_0+\tau_{\min}} = 1$. Record the value $P_{t_0+\tau_{\max}}^{\max} = P_{t_0+\tau_{\max}}$ and mark time t_0 as a trough and go to 1 above for the bull market.

This process is repeated until the last data point.

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Table 1: Weekly Return Statistics (1885-2010)[a]

N	Mean	standard deviation	Skewness	Kurtosis	J-B[b]
6498	0.085	2.40	-0.49	11.2	18475.5*

[a] Continuously compounded returns

[b] Jarque-Bera normality test: p-value = 0.00000

Table 2: BB and LT Dating Algorithm Turning Points

Troughs		Peaks		Troughs		Peaks	
BB[a]	LT[b]	BB	LT	BB	LT	BB	LT
1985-02					1940-06		1940-11
	1885-04			1942-05	1942-05	1943-07	
		1886-12		1943-12		1946-05	1946-05
1888-06		1890-06	1890-06	1948-02	1948-02	1948-06	
1890-12	1890-12	1892-03	1892-03	1949-06		1952-12	
1893-08	1893-08	1895-09	1895-09	1953-09		1956-07	1956-07
1896-08	1896-08	1897-09		1957-12	1957-12	1959-07	
1898-03		1899-04		1960-10			1961-12
1900-07		1902-09	1902-09	1962-07	1962-07	1966-02	1966-02
1903-10	1903-10		1906-01	1966-10	1966-10	1968-12	1968-12
		1906-10		1970-06	1970-06	1971-04	
1907-11	1907-11	1909-08	1909-08	1971-12		1973-01	1973-01
1910-08		1912-10		1974-10	1974-10	1976-09	
1914-12	1914-12	1916-11	1916-11	1978-03		1978-09	
1917-12	1917-12	1919-07	1919-07	1980-04		1980-11	1980-11
1921-06	1921-06	1929-09	1929-09	1982-08	1982-08	1983-06	
	1929-11		1930-04	1984-08		1987-08	1987-08
1932-06	1932-06		1932-09	1987-12	1987-12	1990-06	
	1933-03	1933-07	1933-07	1990-10		2000-03	2000-03
	1933-10		1934-02	2002-10	2002-10	2007-10	2007-10
1935-03	1935-03	1937-03	1937-03	2009-03	2009-03	2010-01	2010-01
1938-04	1938-04		1938-11				
	1939-04	1939-10	1939-10				

[a] BB: Bry and Boschan algorithm using Pagan and Sossounov parameters

[b] LT: Lunde and Timmermann algorithm

Table 3: MS-2-State Model Estimates

	mean	median	std	0.95 DI
μ_1	-0.46	-0.46	0.14	(-0.73, -0.20)
μ_2	0.20	0.20	0.02	(0.16, 0.25)
σ_1	4.42	4.42	0.13	(4.18, 4.69)
σ_2	1.64	1.64	0.02	(1.59, 1.69)
p_{11}	0.94	0.94	0.01	(0.92, 0.96)
p_{22}	0.99	0.99	0.002	(0.98, 0.99)

This table reports the posterior mean, median, standard deviation and 0.95 density intervals for model parameters.

Table 4: MS-4-State Model Estimates

	mean	median	std	95% DI
μ_1	-0.94	-0.92	0.27	(-1.50, -0.45)
μ_2	0.23	0.23	0.10	(0.04, 0.43)
μ_3	-0.13	0.12	0.08	(-0.31, -0.01)
μ_4	0.30	0.29	0.04	(0.22, 0.38)
σ_1	6.01	5.98	0.35	(5.41, 6.77)
σ_2	2.63	2.61	0.18	(2.36, 3.08)
σ_3	2.18	2.19	0.12	(1.94, 2.39)
σ_4	1.30	1.30	0.04	(1.20, 1.37)
p_{11}	0.921	0.923	0.020	(0.877, 0.955)
p_{12}	0.076	0.074	0.020	(0.042, 0.120)
p_{14}	0.003	0.001	0.004	(3e-6, 0.013)
p_{21}	0.015	0.014	0.007	(0.005, 0.031)
p_{22}	0.966	0.967	0.009	(0.945, 0.980)
p_{24}	0.019	0.018	0.006	(0.009, 0.034)
p_{31}	0.010	0.009	0.003	(0.004, 0.017)
p_{33}	0.939	0.943	0.018	(0.899, 0.965)
p_{34}	0.051	0.048	0.017	(0.027, 0.088)
p_{41}	0.001	0.0003	0.0007	(6e-7, 0.002)
p_{43}	0.039	0.037	0.012	(0.024, 0.067)
p_{44}	0.960	0.963	0.012	(0.933, 0.976)

The posterior mean, median, standard deviation and 0.95 density intervals for model parameters.

Table 5: Unconditional State Probabilites

	mean	0.95 DI
π_1	0.070	(0.035, 0.117)
π_2	0.157	(0.073, 0.270)
π_3	0.304	(0.216, 0.397)
π_4	0.469	(0.346, 0.579)

The posterior mean and 0.95 density intervals associated with the posterior distribution for π from Equation (4.7).

Table 6: Posterior Regime Statistics for MS-2 and MS-4 Models

	MS-2	MS-4
bear mean	-0.46	-0.13
	(-0.73, -0.20)	(-0.367, -0.005)
bear duration	18.2	77.8
	(13.2, 25.0)	(44.4, 134.6)
bear standard deviation	4.42	4.04
	(4.18, 4.69)	(3.51, 4.73)
bear variance from $\mathrm{Var}(E[r_t\|s_t]\|s_t = 1, 2)$	0.00	0.31
		(0.07, 0.68)
bear variance from $E[\mathrm{Var}(r_t\|s_t)\|s_t = 1, 2]$	19.6	16.1
	(17.5, 22.0)	(12.1, 22.0)
bear skewness	0	-0.42
		(-0.68, -0.20)
bear kurtosus	3	5.12
		(4.37, 5.93)
bull mean	0.20	0.13
	(0.16, 0.25)	(0.07, 0.18)
bull duration	82.6	256.0
	(59.1, 115.9)	(123.5, 509.6)
bull standard deviation	1.64	1.71
	(1.59, 1.69)	(1.59, 1.83)
bull variance from $\mathrm{Var}(E[r_t\|s_t]\|s_t = 3, 4)$	0.00	0.04
		(0.02, 0.09)
bull variance from $E[\mathrm{Var}(r_t\|s_t)\|s_t = 3, 4]$	2.69	2.89
	(2.54, 2.85)	(2.47, 3.30)
bull skewness	0	0.04
		(-0.11, 0.16)
bull kurtosus	3	3.77
		(3.51, 4.03)

The posterior mean and 0.95 density interval for regime statistics.

Table 7: Posterior State Statistics for the MS-4 Model

	mean	median	std	95% DI
Bear mean	-0.13	-0.11	0.10	(-0.367, -0.005)
Bear duration	77.8	74.0	23.1	(44.4, 134.6)
Bear cumulative return	-9.94	-8.28	7.89	(-29.6, -0.41)
Bear std	4.04	4.01	0.31	(3.51, 4.73)
Bull mean	0.13	0.13	0.03	(0.07, 0.18)
Bull duration	256.0	235.6	100.9	(123.5, 509.6)
Bull cumulative return	33.0	30.0	14.9	(12.9, 70.3)
Bull std	1.71	1.71	0.06	(1.59, 1.83)
s=1: cumulative return	-12.4	-11.8	4.49	(-23.0, -5.45)
s=2: cumulative return	7.10	6.97	3.10	(1.47, 13.8)
s=3: cumulative return	-2.13	-2.07	1.09	(-4.46, -0.27)
s=4: cumulative return	7.88	7.75	1.67	(5.02, 11.6)
s=1: duration	13.5	13.0	3.63	(8.13, 22.2)
s=2: duration	31.2	30.1	8.39	(18.3, 51.0)
s=3: duration	17.9	17.4	4.80	(9.91, 28.8)
s=4: duration	27.2	26.9	6.75	(14.9, 41.4)
s=1: μ_1/σ_1	-0.16	-0.15	0.04	(-0.25, -0.07)
s=2: μ_2/σ_2	0.09	0.09	0.04	(0.02, 0.17)
s=3: μ_3/σ_3	-0.06	-0.05	0.04	(-0.14, -0.01)
s=4: μ_4/σ_4	0.23	0.22	0.04	(0.17, 0.31)

This table report posterior statistics for various population moments.

Table 8: Log Marginal Likelihoods: Alternative Models

Model	$\log f(Y \mid \text{Model})$
Constant mean with constant variance	-14924.1
Constant mean with 4-state i.i.d variance	-14256.7
MS-4-state mean with 4-state i.i.d. variance (4.10 with $K = 4$)	-14036.4
MS-2-state (4.1)	-13903.3
MS-4-state mean with constant intra-regime variance ($\sigma_1^2 = \sigma_2^2$, $\sigma_3^2 = \sigma_4^2$)	-13849.9
MS-4-state (4.3) with unrestricted transition matrix P	-13747.3
MS-4-state (4.3)-(4.6)	-13740.4

Table 9: Dating-algorithm filtering of data and simulated data

	S&P	MS-4
Avg. number of bears	29	31.7
		(22, 42)[a]
Avg. bear duration	63.1	55.9
		(40.5, 74.7)
Avg. bear amplitude[b]	-45.0	-43.4
		(-52.7, -35.8)
Avg. bear return	-0.71	-0.80
		(-1.08, -0.57)
Avg. bear std	3.16	3.15
		(2.60, 3.73)
Avg. number of bulls	28	31.4
		(22, 42)
Avg. bull duration	166.7	158.5
		(103.0, 235.3)
Avg. bull amplitude	66.4	60.2
		(46.3, 80.0)
Avg. bull return	0.40	0.39
		(0.31, 0.48)
Avg. bull std	2.53	2.42
		(1.97, 2.91)

[a] 70% density interval

[b] Aggregate return over one regime

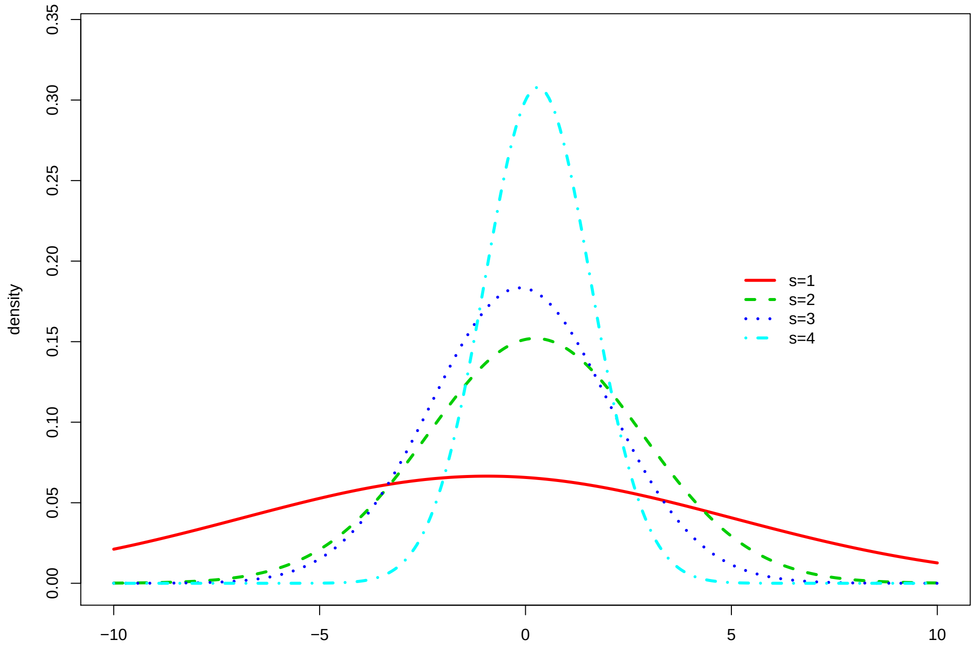

Figure 1: MS-4-States, State Densities

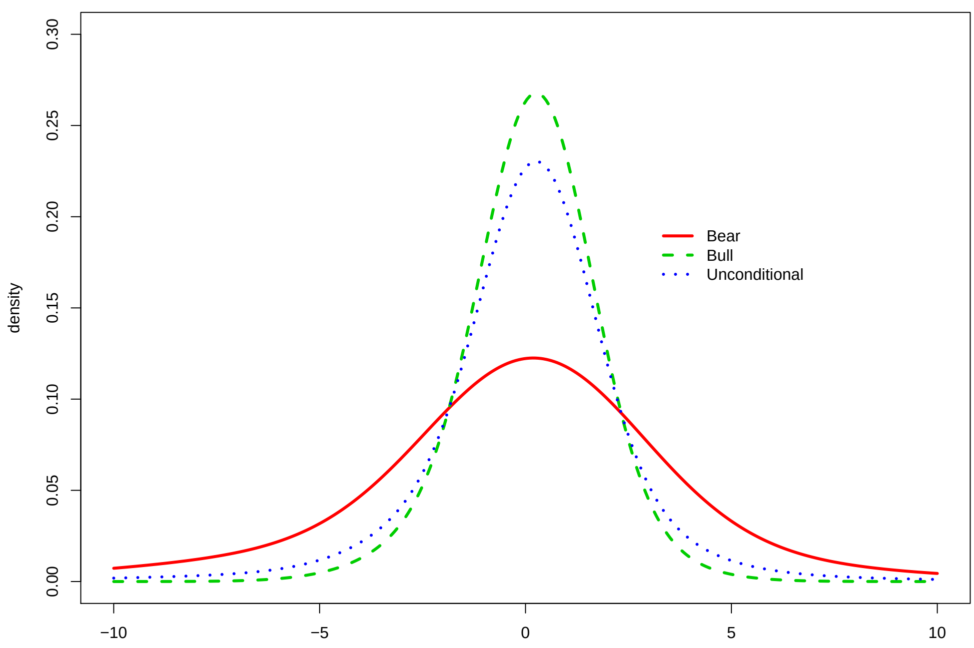

Figure 2: MS-4-States, Regime Densities

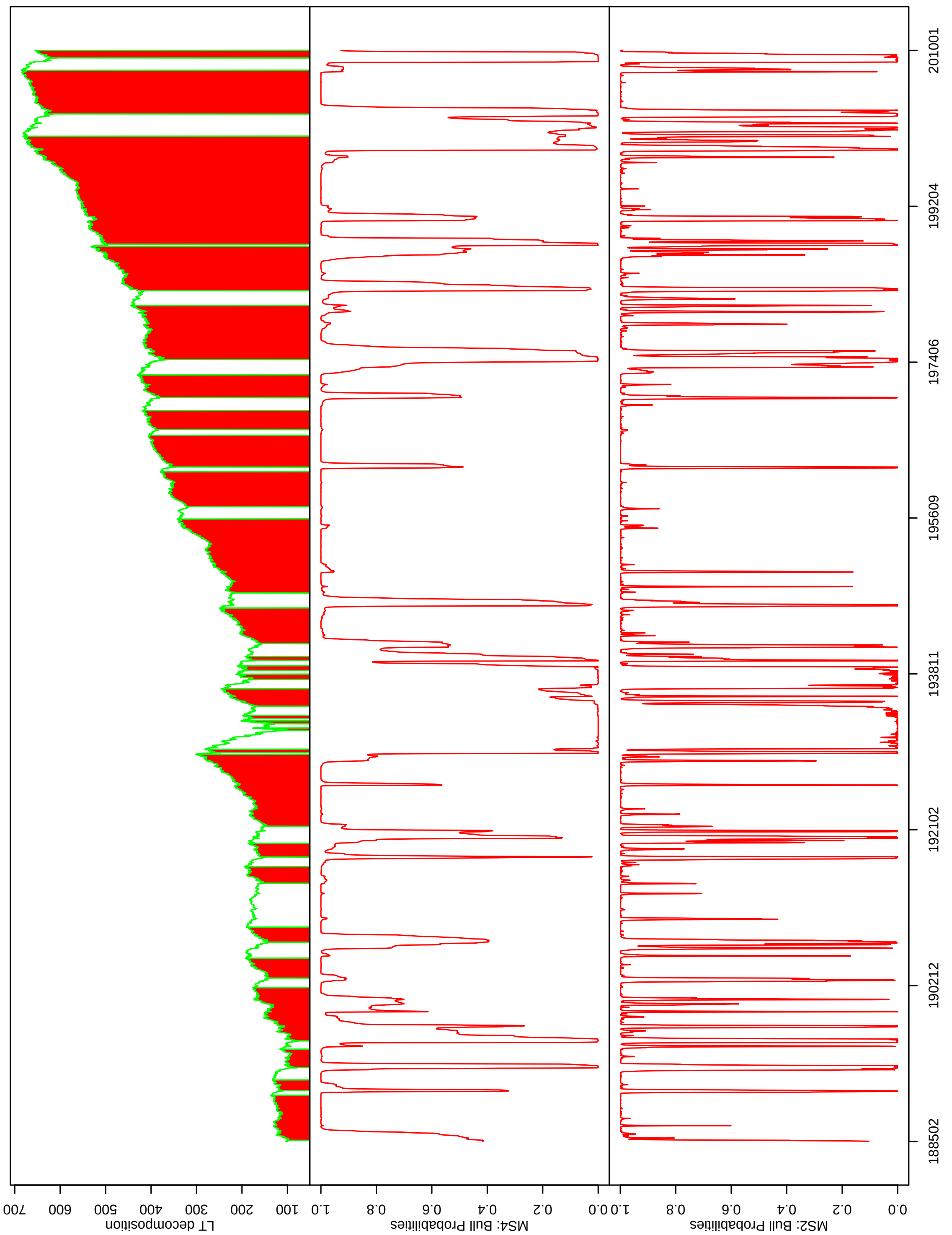

Figure 3: LT algorithm, MS-4 and MS-2

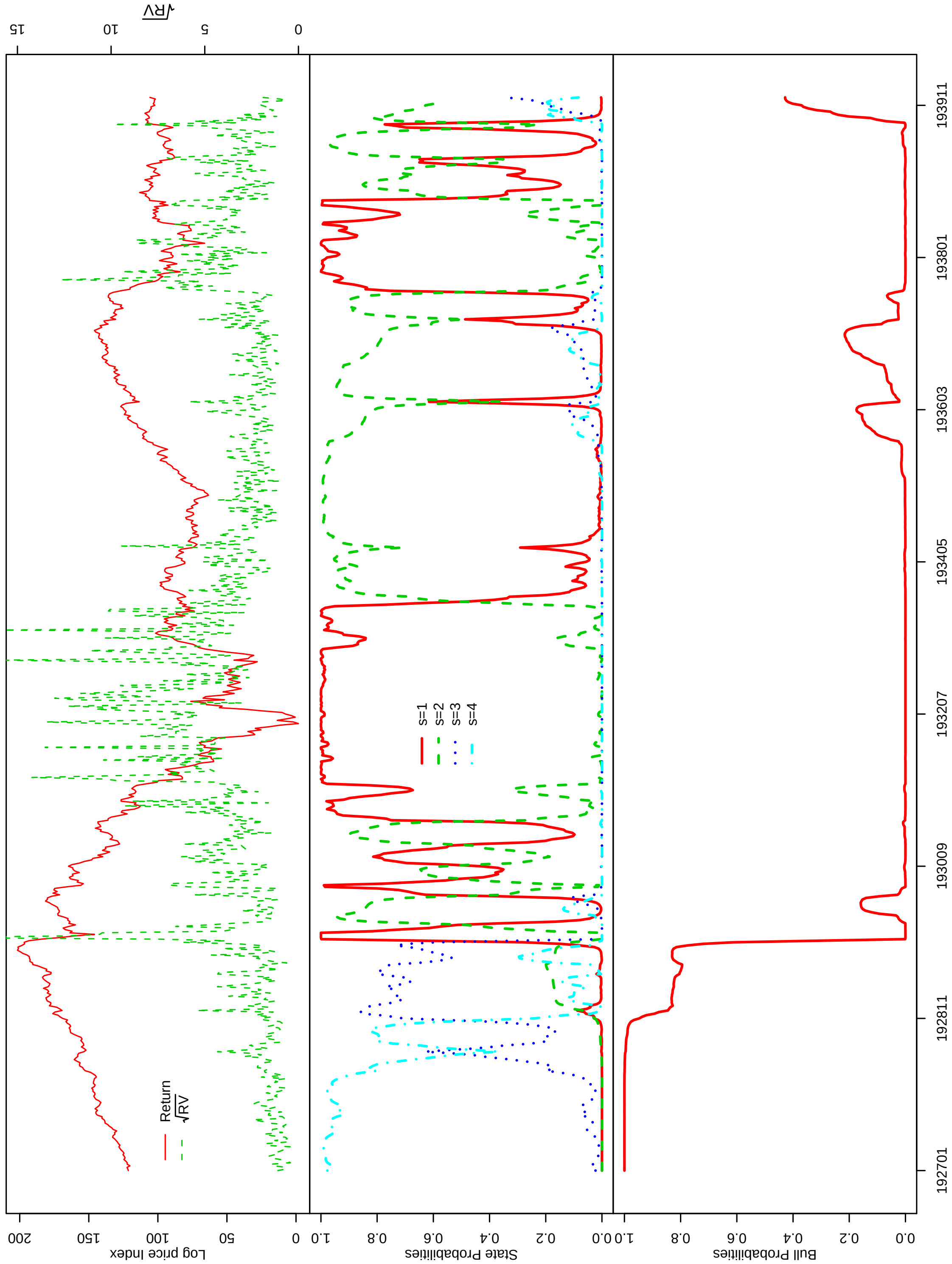

Figure 4: MS-4, 1927-1939

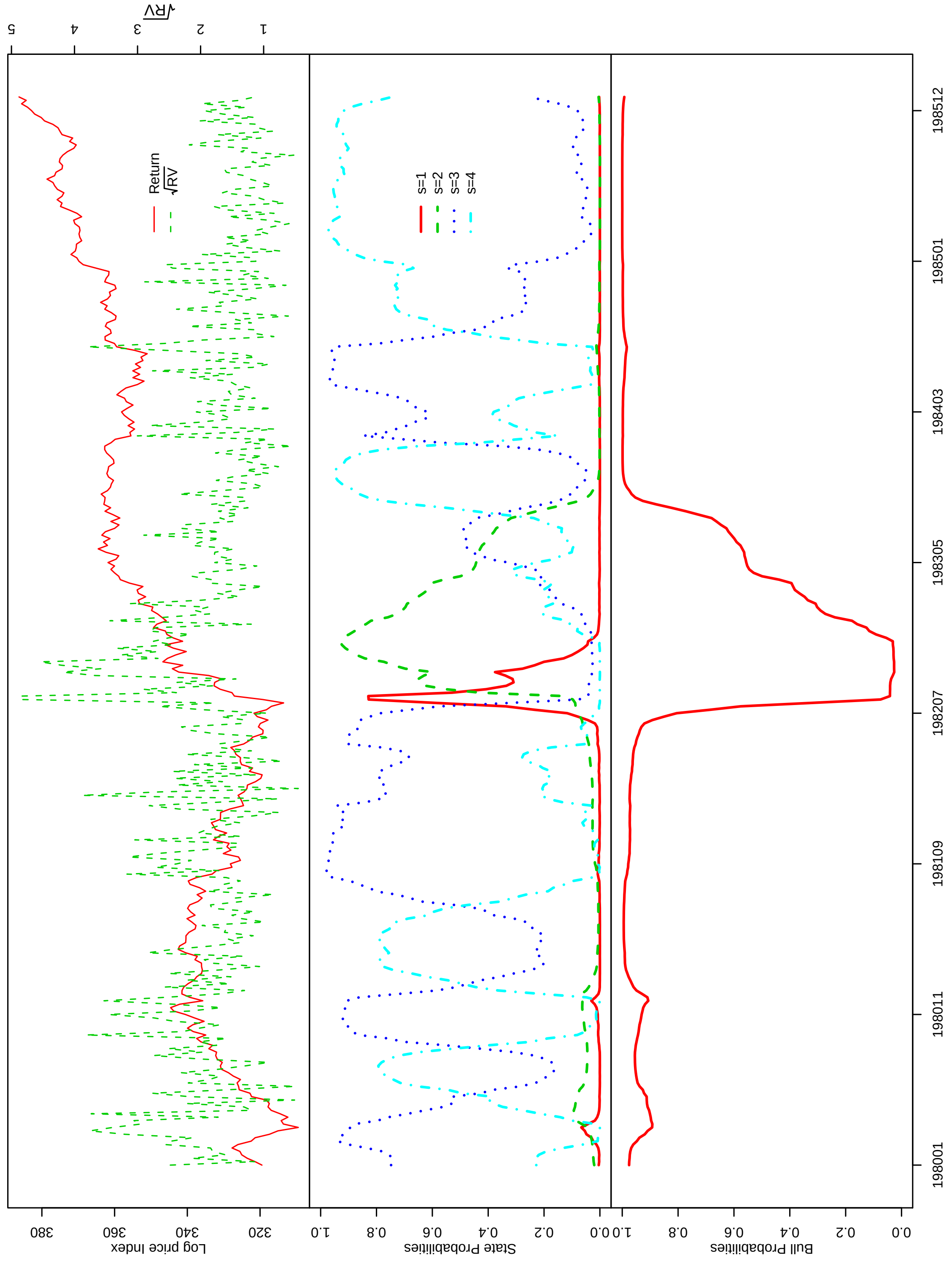

Figure 5: MS-4, 1980-1985

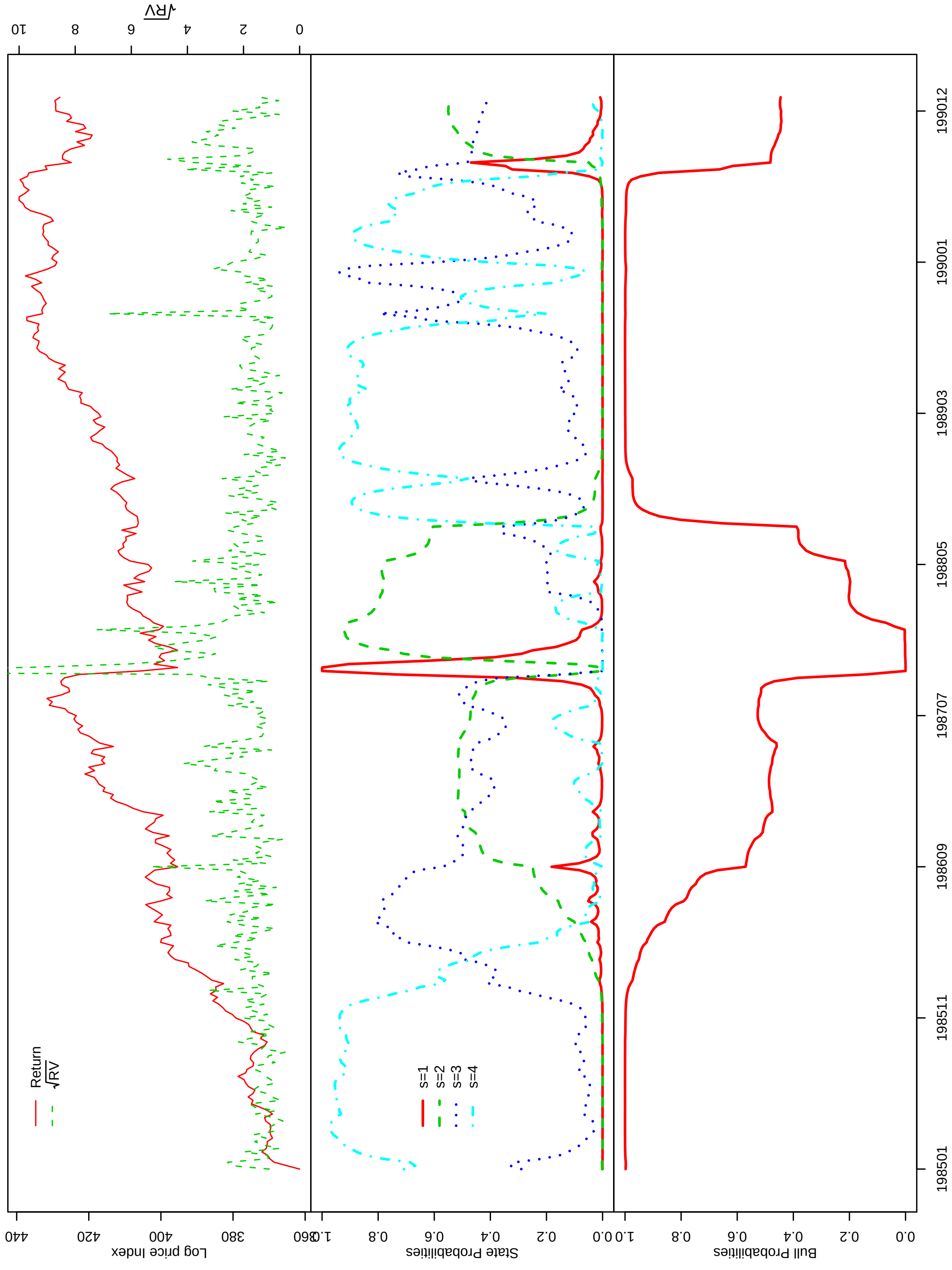

Figure 6: MS-4, 1985-1990

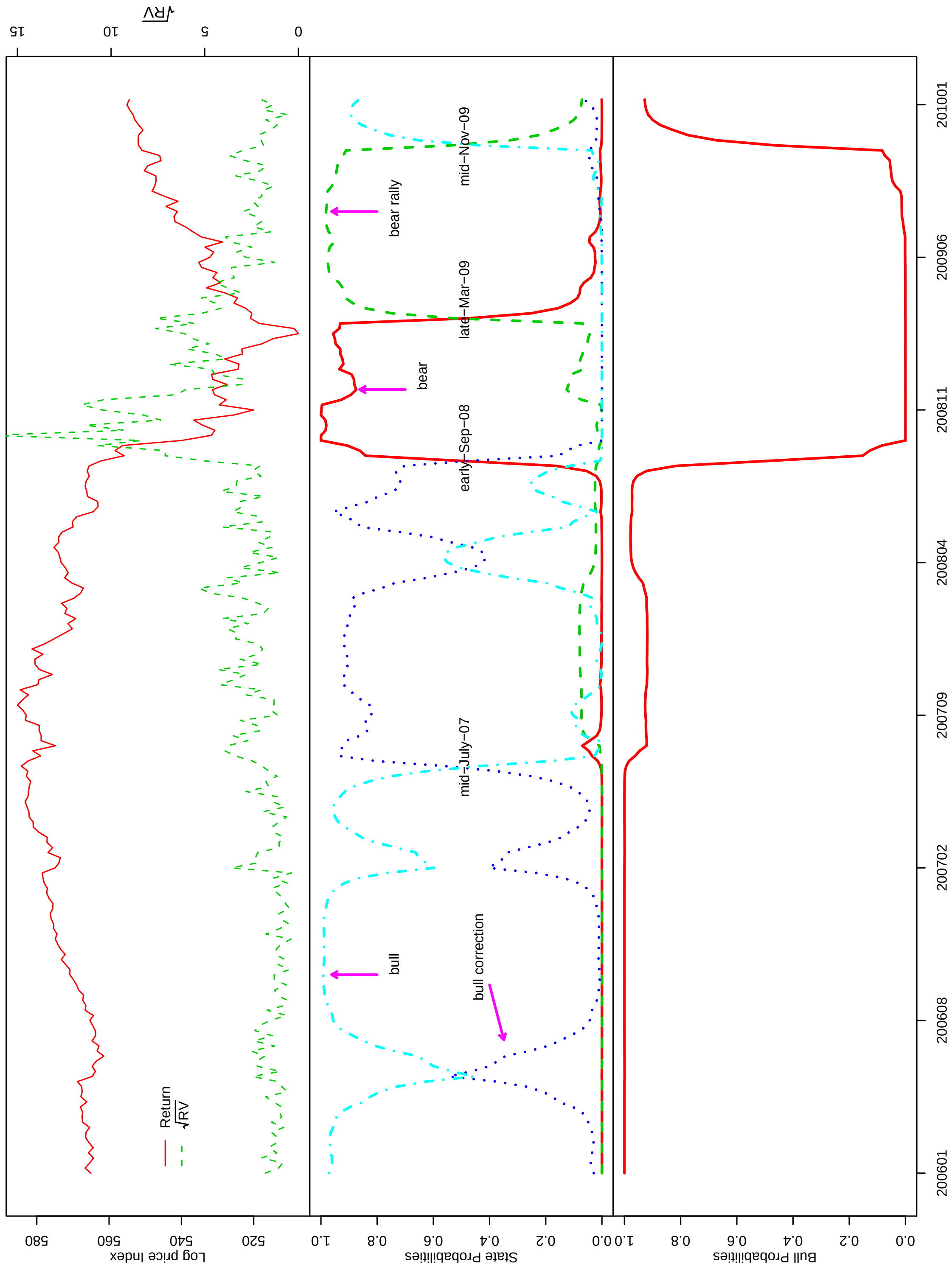

Figure 7: MS-4, 2006-2010

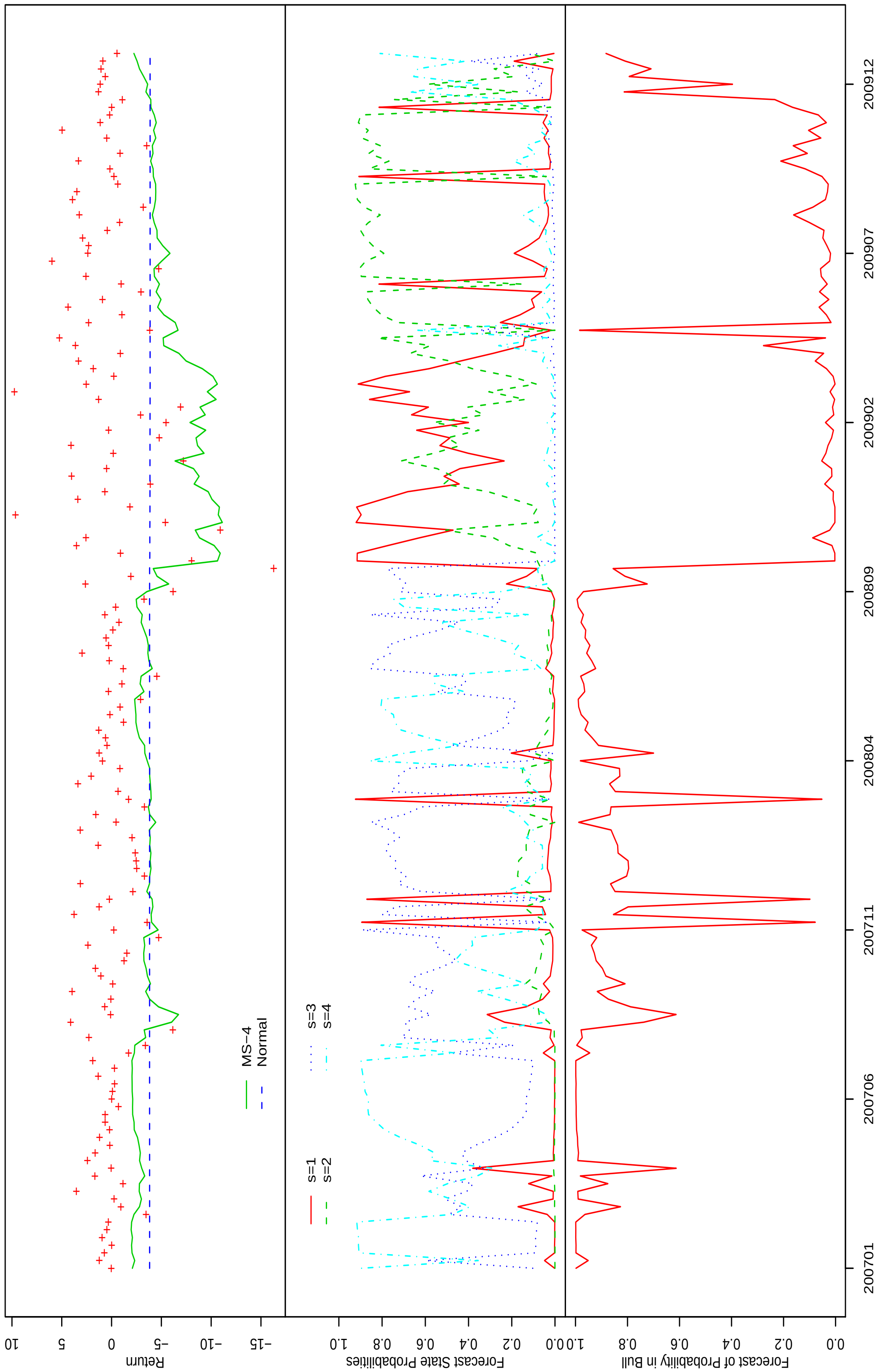

Figure 8: Value-at-Risk from MS-4 and Benchmark Normal distribution