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An Equilibrium Theory of Learning, Search and Wages

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Supplementary Material

Supplement to "An Equilibrium Model of Learning, Search and Wages"

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This supplementary appendix provides complete proofs of the lemmas and theorems presented in the paper.

A. Proofs of Theorem 3.1 and Two Lemmas

Proof of Theorem 3.1:

First, we prove existence of the equilibrium. The analysis leading to Theorem 3.1 has proven that if V obeys (3.7), then V and the optimal choices $G(\mu)$ exist. In Section 6 we characterize the steady state distribution of workers. Thus, for existence of an equilibrium it suffices to show that Assumption 2 is sufficient for all matches to be accepted, in which case V indeed obeys (3.7).

Consider a worker with beliefs $\mu \in M$ who obtains a match in submarket $x \in X$. Accepting the match yields the present value, $J_e(\phi(\mu), W(x))$, and rejecting the match yields $V(\phi(\mu))$. The worker strictly prefers to accept the match if and only if $J_e(\phi(\mu), W(x)) > V(\phi(\mu))$. Using (3.5), we can rewrite the latter condition as $W(x) > (r+\sigma)V(\phi(\mu))$. Since $W'(x) < 0$ and $\phi(\mu) \leq a_H$, a sufficient condition for this requirement for this condition to hold for all x and μ is

$$W(a_H^{-1}) > (r+\sigma)V(a_H).$$

Substituting $V(a_H)$ from (A.1) in Lemma A.1 below, we rewrite the condition as

$$(y-b)/c > [A + a_H x_H]\, \lambda'(x^*) - a_H \lambda(x_H),$$

where x^* is defined by $\lambda'(x^*) = a_H \lambda(a_H^{-1})$, and $x_H = g(a_H)$. Since the right-hand side of the inequality is maximized at $x_H = x^*$, Assumption 2 is sufficient for the inequality to hold and, hence, for all matches to be accepted.

Second, we prove that $g(\mu) > 0$ for all $g(\mu) \in G(\mu)$ and all $\mu \in M$. Let $\mu \in M$ be arbitrary beliefs and $g(\mu)$ an arbitrary selection from the set of optimal choices, $G(\mu)$. Suppose that $g(\mu) = 0$, contrary to theorem. In this case, (3.6) and (3.7) yield: $R(0, \mu) = V(\mu) = b/(r+\sigma)$. Consider a choice $x > 0$. Because it is always feasible for the worker to choose $x' = 0$ in the future, the future value function satisfies: $V(\mu') \geq b/(r+\sigma)$ for all posterior beliefs μ'. Thus, for all μ, the choice x yields at least the following payoff:

$$\tilde{R}(x, \mu) = \frac{x\mu}{A}\left[\frac{W(x)}{1-\sigma} + \frac{\delta b}{r+\sigma}\right] + (1 - x\mu)\frac{b}{r+\sigma}.$$

Note that $\tilde{R}(x, \mu)$ is differentiable and strictly concave in x. Substituting $W(.)$ from (3.10), we can verify that $\tilde{R}_1(0, \mu) > 0$ if and only if $(y-b)/c > A\lambda'(0)$. Since the latter condition is satisfied (see Remark 1), then $\max_x \tilde{R}(x, \mu) > \tilde{R}(0, \mu) = b/(r+\sigma)$. This is a contradiction. Therefore, $g(\mu) > 0$ for all μ.

Third, we prove that V is strictly increasing. Let $TV(\mu)$ denote the right-hand side of (3.7). Since T is a contraction mapping on the space of continuous functions on M (with the sup norm), it suffices to prove that T maps continuous and increasing functions on M into continuous and strictly increasing functions on M (see Stokey et al., 1989). Namely, we prove that $TV(\mu_a) > TV(\mu_b)$ for any continuous and increasing function V on M and for arbitrary $\mu_a, \mu_b \in M$, with $\mu_a > \mu_b$. Denote $g_i = g(\mu_i) \in G(\mu_i)$, where $i \in \{a, b\}$. We have:

$$\begin{aligned}
&R(g_a, \mu_a) - R(g_b, \mu_b) \geq R(g_b, \mu_a) - R(g_b, \mu_b) \\
&\geq g_b(\mu_a - \mu_b) \left\{ \tfrac{W(g_b)}{A(1-\sigma)} + \tfrac{\delta}{A} V(\phi(\mu_b)) - V(H(g_b, \mu_b)) \right\} \\
&> g_b(\mu_a - \mu_b) \left[V(\phi(\mu_b)) - V(H(g_b, \mu_b)) \right] \geq 0.
\end{aligned}$$

The first inequality comes from the fact that $g_i \in \arg\max_x R(x, \mu_i)$ and the second one from $V(H(g_b, \mu_a)) \geq V(H(g_b, \mu_b))$. The strict inequality uses the fact that $g_b > 0$ and that Assumption 2 implies $W(x) > (r + \sigma)V(\phi(\mu))$ for all x and μ (see above proof). The last inequality comes from $\phi(\mu_b) \geq H(g_b, \mu_b)$. Hence, $TV(\mu_a) > TV(\mu_b)$.

Finally, (weak) convexity of V follows from standard arguments (e.g., Nyarko, 1994, Proposition 3.2). Because a convex function is almost everywhere differentiable (see Royden, 1988, pp. 113-114), V is almost everywhere differentiable. *Q.E.D.*

The following lemmas are used in the proofs of other results:

Lemma A.1. *Denote $x_i = g(a_i)$, where $i \in \{H, L\}$. The following results hold: (i) The optimal choice x_i is unique and satisfies $R_1(x_i, a_i) \geq 0$, with strictly inequality only if $x_i = 1/a_H$. The value function satisfies:*

$$V(a_i) = \frac{Ab + a_i x_i W(x_i)}{(r + \sigma)\left[A + a_i x_i\right]}. \tag{A.1}$$

(ii) Condition (4.2) is necessary and sufficient for $x_H < 1/a_H$. Also, $x_L \geq x_H$, with strict inequality if $x_H < 1/a_H$. (iii) $\delta/A < V'(a_L^+)/V'(a_H^-)$ for all $\delta \leq \bar{\delta}$, where $\bar{\delta}$ is the smallest positive solution to $\Omega(\delta) = 0$ and Ω is defined as

$$\Omega(\delta) = \frac{r+\sigma}{1-\sigma}\left(\frac{r+\sigma}{1-\sigma} + \delta\right)^2 - \delta\left[\left(1 + \frac{a_L}{a_H}\right)\left(\frac{r+\sigma}{1-\sigma} + \delta\right) + \frac{a_L}{a_H}\right]. \tag{A.2}$$

Proof of Lemma A.1:

Part (i): Since the proofs are similar for the cases $i = H$ and $i = L$, we only give the proof for $i = H$. Noting that $\phi(a_H) = H(x, a_H) = a_H$ for all x, we get:

$$R(x, a_H) = \frac{x a_H}{A}\left[\frac{W(x)}{1-\sigma} + \delta V(a_H)\right] + (1 - x a_H)V(a_H),$$

and $(1 + r)V(a_H) = b + (1 - \sigma)\max_x R(x, a_H)$. Condition (iii) in (3.11) implies that $R(x, a_H)$ is strictly concave in x, and so the optimal choice, x_H, is unique. Since $R(x, a_H)$ is differentiable with respect to x, and since $x_H > 0$ by Theorem 3.1, then x_H satisfies the condition, $R_1(x_H, a_H) \geq 0$, with strictly inequality only if $x_H = 1/a_H$. The Bellman equation, $(1 + r)V(a_H) = b + (1 - \sigma)R(x_H, a_H)$, yields (A.1) for $i = H$.

Part (ii): From part (i), it is clear that $x_H < 1/a_H$ if and only if $R_1(a_H^{-1}, a_H) < 0$, which can be rewritten as

$$W(a_H^{-1}) + a_H^{-1}W'(a_H^{-1}) < (r + \sigma)V(a_H). \tag{A.3}$$

Substituting $V(a_H)$ from (A.1) and $W(x)$ from (3.10), we find that (A.3) is equivalent to

$$(y - b)/c < [A + a_H x_H]\,\lambda'(a_H^{-1}) - a_H\lambda(x_H).$$

The right-hand side is an increasing function of x_H, and its value at $x_H = 1/a_H$ is equal to the right-hand side of (4.2). Since $x_H \leq 1/a_H$, (4.2) is necessary for the above condition and, hence, necessary for $x_H < 1/a_H$. On the other hand, if $x_H = 1/a_H$, then $R_1(a_H^{-1}, a_H) \geq 0$, and $V(a_H)$ is given by (A.1) with $i = H$ and $x_H = 1/a_H$. Substituting this value of $V(a_H)$, we find that the condition $R_1(a_H^{-1}, a_H) \geq 0$ violates (4.2). Thus, (4.2) is also sufficient for $x_H < 1/a_H$.

The condition, $R_1(x_i, a_i) \geq 0$, holds for both $i = H$ and L. Inspecting this condition and using strict monotonicity of the value function, we can deduce that $x_L \geq x_H$, where the inequality is strict if $x_H < 1/a_H$.

Part (iii): First, we derive an upper bound on $V'(a_H^-)$ and a lower bound on $V'(a_L^+)$. These one-sided derivatives exist because V is continuous and convex (see Royden, 1988, pp. 113-114). Let $\varepsilon > 0$ be a sufficiently small number. For $V'(a_H^-)$, we can compute

$$\begin{aligned}\tfrac{1+r}{1-\sigma}\left[V(a_H) - V(a_H - \varepsilon)\right] &= R(x_H, a_H) - R(g(a_H - \varepsilon), a_H - \varepsilon)\\ &\leq R(x_H, a_H) - R(x_H, a_H - \varepsilon).\end{aligned}$$

Dividing by ε and taking the limit $\varepsilon \downarrow 0$, we obtain $(1 + r)V'(a_H^-) \leq (1 - \sigma)R_2(x_H, a_H^-)$. Using the facts $\phi(a_H) = a_H = H(x_H, a_H)$, $\phi'(a_H) = a_L/a_H$, and $H_2(x_H, a_H) = (1 - x_H a_L)/(1 - x_H a_H)$, we can compute:

$$R_2(x_H, a_H^-) = \frac{x_H}{(1-\sigma)A}\left[W(x_H) - (r + \sigma)V(a_H^-)\right] + \left[1 - \frac{(r+\sigma)x_H a_L}{(1-\sigma)A}\right]V'(a_H^-).$$

Substituting $V(a_H)$ from (A.1) and substituting the result into the inequality, $(1+r)V'(a_H^-) \leq (1-\sigma)R_2(x_H, a_H^-)$, we get:

$$V'(a_H^-) \leq \frac{Ax_H\,[W(x_H)-b]}{(r+\sigma)[A+x_Ha_L]\,[A+a_Hx_H]}.$$

Similarly, we can derive the following lower bound:

$$V'(a_L^+) \geq \frac{Ax_L\,[W(x_L)-b]}{(r+\sigma)[A+x_La_H]\,[A+a_Lx_L]}.$$

Next, we prove that $\delta/A < V'(a_L^+)/V'(a_H^-)$ for all $\delta \leq \bar{\delta}$. Substituting the above bounds on $V'(a_H^-)$ and $V'(a_L^+)$, we find that

$$\frac{V'(a_L^+)}{V'(a_H^-)} \geq \frac{x_L\,[W(x_L)-b]\,[A+x_Ha_L]\,[A+a_Hx_H]}{x_H\,[W(x_H)-b]\,[A+x_La_H]\,[A+a_Lx_L]}.$$

Recall that x_i satisfies $R_1(x_i, a_i) \geq 0$ for $i \in \{H, L\}$ and the value function satisfies $V(a_i) > b/(r+\sigma)$. Using these results, we can verify that $x[W(x)-b]$ is strictly increasing in x for $x \in [x_H, x_L]$. Because $x_L \geq x_H$ (see the proof above), $x_L\,[W(x_L)-b] \geq x_H[W(x_H)-b]$. Substituting this result and the facts that $x_H > 0$ and $x_L \leq 1/a_H$, we conclude that

$$\frac{V'(a_L^+)}{V'(a_H^-)} > \frac{A^2}{[A+1]\left[A+\frac{a_L}{a_H}\right]}.$$

Substituting this bound, we find that a sufficient condition for $\delta/A < V'(a_L^+)/V'(a_H^-)$ is $\Omega(\delta) \geq 0$, where $\Omega(\delta)$ is defined in (A.2). The function $\Omega(\delta)$ is quadratic and involves only the parameters of the model. Because $\Omega(0) > 0$, there exists $\bar{\delta} > 0$ such that $\Omega(\delta) \geq 0$ for all $\delta \in [0, \bar{\delta}]$. Thus, $\delta/A < V'(a_L^+)/V'(a_H^-)$ for all $\delta \in [0, \bar{\delta}]$. *Q.E.D.*

Lemma A.2. *For any given z, the functions $\mu V(\phi(\mu))$ and $(1+z\mu)V(H(-z,\mu))$ are convex in μ if $V(.)$ is convex, and strictly convex in μ if $V(.)$ is strictly convex.*

Proof of Lemma A.2:

Assume that V is convex. Take arbitrary μ_a and μ_b in M, with $\mu_a > \mu_b$. Let γ be an arbitrary number in $(0,1)$, and define $\mu_\gamma = \gamma\mu_a + (1-\gamma)\mu_b$. We first prove that $\mu_\gamma V(\phi(\mu_\gamma)) \leq \gamma\mu_a V(\phi(\mu_a)) + (1-\gamma)\mu_b V(\phi(\mu_b))$, which establishes convexity of $\mu V(\phi(\mu))$. Shorten the notation $\phi(\mu_i)$ to ϕ_i, where $i \in \{a, b, \gamma\}$. Denote $\kappa = (\phi_\gamma - \phi_b)/(\phi_a - \phi_b)$. Clearly, $\kappa \in [0,1]$, and $\phi_\gamma = \kappa\phi_a + (1-\kappa)\phi_b$. Moreover, since $\mu\phi(\mu)$ is a linear function of μ, we can verify that $\kappa\mu_\gamma = \gamma\mu_a$ and $(1-\kappa)\mu_\gamma = (1-\gamma)\mu_b$. Thus,

$$\begin{aligned}&\mu_\gamma V(\phi_\gamma) = \mu_\gamma V(\kappa\phi_a + (1-\kappa)\phi_b)\\ &\leq \mu_\gamma\,[\kappa V(\phi_a) + (1-\kappa)V(\phi_b)] = \gamma\mu_a V(\phi_a) + (1-\gamma)\mu_b V(\phi_b).\end{aligned}$$

The inequality comes from convexity of V and the fact that $\mu_\gamma > 0$. The last equality comes from the facts that $\kappa\mu_\gamma = \gamma\mu_a$ and $(1-\kappa)\mu_\gamma = (1-\gamma)\mu_b$. If V is strictly convex, then the above inequality is strict, in which case $\mu V(\phi(\mu))$ is strictly convex.

Note that the function $(1+z\mu)H(-z,\mu)$ is also linear in μ for any given z. A similar proof as the above establishes that this function is convex if V is convex, and strictly convex if V is strictly convex. *Q.E.D.*

B. Proof of Theorem 4.1

First, we prove that $\hat{R}(z,\mu)$ is strictly supermodular. Take arbitrary z_a, $z_b \in -X$ and arbitrary μ_a, $\mu_b \in M$, with $z_a > z_b$ and $\mu_a > \mu_b$. Denote:

$$D = \left[\hat{R}(z_a,\mu_a) - \hat{R}(z_a,\mu_b)\right] - \left[\hat{R}(z_b,\mu_a) - \hat{R}(z_b,\mu_b)\right].$$

We need to show $D > 0$. Temporarily denote $\phi_j = \phi(\mu_j)$, $H_{ij} = H(-z_i,\mu_j)$ and $V_{ij} = V(H_{ij})$, where $i,j \in \{a,b\}$. Computing D, we have:

$$D = D_1 - \frac{\delta}{A}[V(\phi_a) - V(\phi_b)](z_a - z_b),$$

where

$$D_1 = (z_a + \mu_a^{-1})V_{aa} - (z_b + \mu_a^{-1})V_{ba} - (z_a + \mu_b^{-1})V_{ab} + (z_b + \mu_b^{-1})V_{bb}.$$

Denote $\tilde{H} = \min\{H_{ba}, H_{ab}\}$. Because $H(-z,\mu)$ is a strictly increasing function of z and μ for all $\mu \in (a_L, a_H)$, then $H_{aa} > \tilde{H} \geq H_{bb}$. Because V is convex, we have:

$$\min\left\{\frac{V_{aa} - V_{ba}}{H_{aa} - H_{ba}}, \frac{V_{aa} - V_{ab}}{H_{aa} - H_{ab}}\right\} \geq \frac{V_{aa} - V(\tilde{H})}{H_{aa} - \tilde{H}} \geq \frac{V_{aa} - V_{bb}}{H_{aa} - H_{bb}}.$$

Substituting V_{ba}, V_{ab} and V_{bb} from these inequalities, and substituting H, we have:

$$\begin{aligned} D_1 &\geq \tfrac{V_{aa} - V(\tilde{H})}{H_{aa} - \tilde{H}} \Big\{ \tfrac{1}{\mu_a}\left[(1 + z_a\mu_a)H_{aa} - (1 + z_b\mu_a)H_{ba}\right] \\ &\qquad\qquad - \tfrac{1}{\mu_b}\left[(1 + z_a\mu_b)H_{ab} - (1 + z_b\mu_b)H_{bb}\right]\Big\} \\ &= \tfrac{V_{aa} - V(\tilde{H})}{H_{aa} - \tilde{H}}(z_a - z_b)(\phi_a - \phi_b). \end{aligned}$$

Thus, $D > 0$ if

$$\frac{\delta}{A} < \left[\frac{V_{aa} - V(\tilde{H})}{H_{aa} - \tilde{H}}\right] \Bigg/ \left[\frac{V(\phi_a) - V(\phi_b)}{\phi_a - \phi_b}\right].$$

Because V is convex, then

$$\frac{V_{aa} - V(\tilde{H})}{H_{aa} - \tilde{H}} \geq V'(a_L^+); \quad \frac{V(\phi_a) - V(\phi_b)}{\phi_a - \phi_b} \leq V'(a_H^-),$$

where $V'(\mu^+)$ is the right derivative, and $V'(\mu^-)$ the left derivative, of V at μ. Hence, a sufficient condition for $D > 0$ is $\delta/A < V'(a_L^+)/V'(a_H^-)$, which is implied by Assumption 3 (see Lemma A.1).

Thus, the function $\hat{R}(z,\mu)$ is strictly supermodular. Because $-X$ is a lattice, the monotone selection theorem in Topkis (1998, Theorem 2.8.4, p. 79) implies that every selection from $Z(\mu)$ is increasing. As a result, every selection $g(\mu)$ from $G(\mu)$ is decreasing, and $w(\mu) = W(g(\mu))$ is increasing.

Finally, we establish that the five statements (i) - (v) in Theorem 4.1 are equivalent.

(i) $\Longleftrightarrow$ (ii): Optimal learning has the following standard property (see Nyarko, 1994, Proposition 4.1): The value function is strictly convex in beliefs if and only if there do not exist μ_a and μ_b in M, with $\mu_a > \mu_b$, and a choice z_0 such that $z_0 \in Z(\mu)$ for all $\mu \in [\mu_b, \mu_a]$. Since $z(\mu)$ is an increasing function, as proven above, the standard property implies that V is strictly convex if and only if every selection $z(\mu)$ is strictly increasing for all μ.

(ii) $\Longrightarrow$ (iii): Suppose $\{-a_H^{-1}\} \in Z(\mu_a)$ for some $\mu_a > a_L$ so that (iii) is violated. Because every selection $z(\mu)$ is increasing, $Z(\mu)$ contains only the singleton $\{-a_H^{-1}\}$ for all $\mu < \mu_a$. In this case, (ii) does not hold for $\mu \leq \mu_a$. Note that since $z(\mu) < 0$ by Theorem 3.1, the result $\{-a_H^{-1}\} \notin Z(\mu)$ implies that $Z(\mu)$ is interior.

(iii) $\Longrightarrow$ (iv): This follows from $a_H > a_L$.

(iv) $\Longleftrightarrow$ (v): See part (ii) of Lemma A.1 in Appendix A.

(iv) $\Longrightarrow$ (i): We prove that a violation of (i) implies that $\{-a_H^{-1}\} \in Z(a_H)$, which violates (iv). Suppose that V is not strictly convex. Proposition 4.1 in Nyarko (1994) implies that there exist μ_a and μ_b in M, with $\mu_a > \mu_b$, and a choice z_0 such that $z_0 \in Z(\mu)$ and $V(\mu)$ is linear for all $\mu \in [\mu_b, \mu_a]$. Since $\mu_a > \mu_b$, let $\mu_b > a_L$ and $\mu_a < a_H$ without loss of generality. (If μ_a or μ_b is at the boundary, we can find μ'_a and μ'_b, with $\mu_a > \mu'_a > \mu'_b > \mu_b$.) We deduce that $V(\mu)$ is linear for all $\mu \in [\phi(\mu_b), \phi(\mu_a)]$: If $V(\mu)$ were strictly convex in any subinterval of $[\phi(\mu_b), \phi(\mu_a)]$, Lemma A.2 above would imply that $R(-z_0, \mu)$ is strictly convex μ in some subinterval of $[\mu_b, \mu_a]$. Similarly, $V(\mu)$ is linear for all $\mu \in [H_b, H_a]$, where H_i denotes $H(-z_0, \mu_i)$ for $i \in \{a, b\}$. Denote the slope of V as $V'(\phi_b)$ for $\mu \in [\phi(\mu_b), \phi(\mu_a)]$ and $V'(H_b)$ for $\mu \in [H_b, H_b]$. For all $\mu \in [\mu_b, \mu_a]$, we have

$$\begin{aligned}\hat{R}(z,\mu) = \; & -\tfrac{zW(-z)}{(1-\sigma)A} - \tfrac{\delta z}{A}\{V(\phi(\mu_b)) + V'(\phi_b)[\phi(\mu) - \phi(\mu_b)]\} \\ & + \tfrac{1}{\mu}(1 + z\mu)\{V(H_b) + V'(H_b)[H(-z,\mu) - H_b]\}.\end{aligned}$$

Because $(1 + z\mu)H(-z, \mu)$ is linear in z, the last two terms in the above expression are linear in z. In this case, part (iii) in (3.11) implies that $\hat{R}(z,\mu)$ is strictly concave in z and twice continuously differentiable in z and μ for all $\mu \in [\mu_b, \mu_a]$. Thus, the optimal choice $z(\mu)$ is unique. By the supposition, this optimal choice is $z(\mu) = z_0$ for all $\mu \in [\mu_b, \mu_a]$.

Using these results and the fact that $z_0 < 0$ (see Theorem 3.1), we conclude that that z_0 satisfies the complementary slackness condition, $\hat{R}_1(z_0, \mu) \leq 0$ and $z_0 \geq -1/a_H$. Moreover, in this case, strict supermodularity of $\hat{R}$ implies $\hat{R}_{12}(z, \mu) > 0$ and strictly concavity of $\hat{R}$ in z implies $\hat{R}_{11}(z, \mu) < 0$ for all $\mu \in [\mu_b, \mu_a]$. If $z_0 > -1/a_H$, then $\hat{R}_1(z_0, \mu) = 0$, which implies $dz_0/d\mu = -\hat{R}_{12}/\hat{R}_{11} > 0$. This contradicts the supposition that z_0 is constant for all $\mu \in [\mu_b, \mu_a]$. Thus, $z_0 = -1/a_H$.

Repeat the above argument for all $\mu \in [\phi^i(\mu_b), \phi^i(\mu_a)]$, where $\phi^i(\mu) = \phi(\phi^{i-1}(\mu))$ and $i = 1, 2, ...$ For such μ, V is linear and $Z(\mu)$ is the singleton $\{-a_H^{-1}\}$.

Take an arbitrary $\mu_c \in (\mu_b, \mu_a)$. Since $Z(\phi^i(\mu_c)) = \{-a_H^{-1}\}$ for all positive integers i, then $\lim_{i\to\infty} Z(\phi^i(\mu_c)) = \{-a_H^{-1}\}$. From the definition of $\phi(\mu)$, it is clear that $\phi(a_H) = a_H$, $\phi(a_L) = a_L$, and $\phi(\mu) > \mu$ for all $\mu \in (a_L, a_H)$. Thus, $\lim_{i\to\infty} \phi^i(\mu) = a_H$ for every $\mu \in (a_L, a_H)$ and, particularly, for $\mu = \mu_c$. Because Z is upper hemi-continuous, we conclude that $\{-a_H^{-1}\} \in Z(a_H)$. *Q.E.D.*

B.1. The relationship between single-crossing and supermodularity

The original objective function of a worker's maximization problem is:

$$R(-z, \mu) = -\frac{z\mu}{A}\left[\frac{W(-z)}{1-\sigma} + \delta V(\phi(\mu))\right] + (z\mu + 1)\, V(H(-z, \mu)).$$

The transformed function is $\hat{R}(z, \mu) = \frac{1}{\mu} R(-z, \mu)$. Consider arbitrary z_a, $z_b \in -X$ and arbitrary μ_a, $\mu_b \in M$, with $z_a > z_b$ and $\mu_a > \mu_b$. The original function R has *strict single crossing* in (z, μ) if

$$R(-z_a, \mu_b) \geq R(-z_b, \mu_b) \Longrightarrow R(-z_a, \mu_a) > R(-z_b, \mu_a).$$

The transformed function $\hat{R}$ is *strictly supermodular* in (z, μ) if

$$D \equiv [\hat{R}(z_a, \mu_a) - \hat{R}(z_b, \mu_a)] - [\hat{R}(z_a, \mu_b) - \hat{R}(z_b, \mu_b)] > 0.$$

Claim 1. *Strict supermodularity of $\hat{R}\,(z, \mu)$ is sufficient but not necessary for strict single crossing of $R\,(-z, \mu)$.*

Proof. Consider arbitrary z_a, z_b, μ_a and μ_b in the above definitions. Substituting $\hat{R} = R/\mu$ into the definition of D and re-arranging terms, we have:

$$R(-z_a, \mu_a) - R(-z_b, \mu_a) = \mu_a D + \frac{\mu_a}{\mu_b}[R(-z_a, \mu_b) - R(-z_b, \mu_b)].$$

Suppose that $\hat{R}(z,\mu)$ is strictly supermodular in (z,μ), i.e., $D > 0$. If $R(-z_a,\mu_b) \geq R(-z_b,\mu_b)$, then the above equation clearly implies $R(-z_a,\mu_a) > R(-z_b,\mu_a)$, which establishes strict single crossing of R. The converse is not true. If R has strict single crossing, then $\frac{\mu_a}{\mu_b}[R(-z_a,\mu_b) - R(-z_b,\mu_b)] \geq 0$ implies $[R(-z_a,\mu_a) - R(-z_a,\mu_a)] > 0$. However, the first difference can be greater than the second difference, in which case $D < 0$. *Q.E.D.*

In practice, verifying strict supermodularity of $\hat{R}$ is the operational way to verify strict single crossing of R, although the former is not necessary for the latter. To see why, suppose that $R(-z_a,\mu_b) \geq R(-z_b,\mu_b)$. To verify that R has strict single crossing, we need to verify that $R(-z_a,\mu_a) - R(-z_b,\mu_a) > 0$. The latter difference involves the term $[z_a W(-z_a) - z_b W(-z_b)]$. For arbitrary (z_a,z_b), this term is unrelated to (μ_a,μ_b), and it can be either positive or negative (because the function $zW(-z)$ is not monotone). The operational way to verify strict single crossing of R is to use the hypothesis in strict single crossing to substitute the above term. That is, rewrite the hypothesis, $R(-z_a,\mu_b) \geq R(-z_b,\mu_b)$, as

$$\begin{aligned}&\tfrac{z_a W(-z_a) - z_b W(-z_b)}{A(1-\sigma)} \\ \leq\ &\tfrac{\delta V(\phi(\mu_b))}{A}[z_a - z_b] + \left[\left(z_a + \tfrac{1}{\mu_b}\right)V(H(-z_a,\mu_b)) - (z_b + \tfrac{1}{\mu_b})V(H(-z_b,\mu_b))\right].\end{aligned}$$

Using this inequality to substitute $[z_a W(-z_a) - z_b W(-z_b)]$, we obtain:

$$R(-z_a,\mu_a) - R(-z_b,\mu_a) \geq \mu_a D.$$

Working on the above inequality to verify $R(-z_a,\mu_a) > R(-z_b,\mu_a)$ amounts to proving the sufficient condition, $D > 0$, which is strict supermodularity of $\hat{R}$.

C. Proof of Theorem 5.1

Fix $\mu \in (a_L,a_H)$ and use the notation $h(\mu) = H(-z(\mu),\mu)$.

Part (i): Because $H(-z,\mu)$ is increasing in z, $H(-z^+(\mu),\mu) = h^+(\mu)$ and $H(-z^-(\mu),\mu) = h^-(\mu)$. Note that the convex function V has left and right derivatives. Because $W(-z)$ is continuous, V is continuous and convex, and H is continuously differentiable, then

$$\begin{aligned}\hat{R}_1(z^+(\mu),\mu) = \ &\tfrac{z(\mu)W'(-z(\mu)) - W(-z(\mu))}{A(1-\sigma)} - \tfrac{\delta}{A}V(\phi(\mu)) \\ &+ V(h(\mu)) - (\mu^{-1} + z(\mu))\,V'(h^+(\mu))H_1(-z(\mu),\mu).\end{aligned}$$

$\hat{R}_1(z^-(\mu),\mu)$ is given similarly with $h^-(\mu)$ replacing $h^+(\mu)$. Recall that H_1 denotes the derivative of $H(-z,\mu)$ with respect to $-z$, rather than to z. Since $H_1 < 0$ and V is convex, we can deduce that $\hat{R}_1(z^+(\mu),\mu) \geq \hat{R}_1(z^-(\mu),\mu)$. However, because $z(\mu)$ is optimal,

$\hat{R}_1(z^+(\mu),\mu) \le 0 \le \hat{R}_1(z^-(\mu),\mu)$. It must be true that $\hat{R}_1(z^-(\mu),\mu) = \hat{R}_1(z^+(\mu),\mu) = 0$, which requires that $V'(h^-(\mu)) = V'(h^+(\mu)) = V'(h(\mu))$.

Part (ii): Let $\{\mu_i\}$ be a sequence with $\mu_i \to \mu$ and $\mu_i \ge \mu_{i+1} \ge \mu$ for all i. Because $\bar{z}(\mu)$ is an increasing function, $\{\bar{z}(\mu_i)\}$ is a decreasing sequence, and $\bar{z}(\mu_i) \ge \bar{z}(\mu)$ for all i. Thus, $\bar{z}(\mu_i) \downarrow z_c$ for some $z_c \ge \bar{z}(\mu)$. On the other hand, the Theorem of the Maximum implies that the correspondence $Z(\mu)$ is upper hemi-continuous (uhc) (see Stokey et al., 1989, p. 62). Because $\mu_i \to \mu$, and $\bar{z}(\mu_i) \in Z(\mu_i)$ for each i, uhc of Z implies that there is a subsequence of $\{\bar{z}(\mu_i)\}$ that converges to an element in $Z(\mu)$. This element must be z_c, because all convergent subsequences of a convergent sequence must have the same limit. Thus, $z_c \in Z(\mu)$, and so $z_c \le \max Z(\mu) = \bar{z}(\mu)$. Therefore, $\bar{z}(\mu_i) \downarrow z_c = \bar{z}(\mu)$, which shows that $\bar{z}(\mu)$ is right-continuous. Similarly, by examining the sequence $\{\mu_i\}$ with $\mu_i \to \mu$ and $\mu \ge \mu_{i+1} \ge \mu_i$ for all i, we can show that $\underline{z}$ is left-continuous.

Part (iii): Let μ_a be another arbitrary value in the interior of (a_L, a_H). Because $\bar{z}(\mu)$ maximizes $R(-z,\mu)$ for each given μ, then

$$\begin{aligned}(1+r)V(\mu_a) &= b + (1-\sigma)R(-\bar{z}(\mu_a),\mu_a) \ge b + (1-\sigma)R(-\bar{z}(\mu),\mu_a)\\ (1+r)V(\mu) &= b + (1-\sigma)R(-\bar{z}(\mu),\mu) \ge b + (1-\sigma)R(-\bar{z}(\mu_a),\mu).\end{aligned}$$

For $\mu_a > \mu$, we have:

$$\frac{R(-\bar{z}(\mu),\mu_a) - R(-\bar{z}(\mu),\mu)}{(1+r)(\mu_a-\mu)} \le \frac{V(\mu_a) - V(\mu)}{(1-\sigma)(\mu_a-\mu)} \le \frac{R\left(-\bar{z}(\mu_a),\mu_a\right) - R(-\bar{z}(\mu_a),\mu)}{(1+r)(\mu_a-\mu)}.$$

Take the limit $\mu_a \downarrow \mu$. Under (4.2), $V'(H(-\bar{z}(\mu_a),\mu_a))$ exists for each μ_a (see part (i)). Because $\bar{z}(\mu_a)$ is right-continuous, $\lim_{\mu_a\downarrow\mu} \bar{z}(\mu_a) = \bar{z}(\mu)$. Thus, all three ratios above converge to the same limit, $\frac{1}{1-\sigma}V'(\mu^+) = \frac{1}{1+r}R_2(-\bar{z}(\mu),\mu^+)$, where

$$\begin{aligned}R_2(-\bar{z}(\mu),\mu^+) = \;& \bar{z}(\mu)\left[-\tfrac{W(-\bar{z}(\mu))}{(1-\sigma)A} - \tfrac{\delta}{A}V(\phi(\mu)) + V(H(-\bar{z}(\mu),\mu))\right]\\ & -\tfrac{\mu\bar{z}(\mu)\delta}{A}V'(\phi^+(\mu))\phi'(\mu) + \left[\mu\bar{z}(\mu)+1\right]V'(H(-\bar{z}(\mu),\mu))H_2(-\bar{z}(\mu),\mu).\end{aligned}$$

Now conduct the above exercise with $\underline{z}$ replacing $\bar{z}$. For $\mu_a < \mu$, we have:

$$\frac{R\left(-\underline{z}(\mu_a),\mu_a\right) - R(-\underline{z}(\mu_a),\mu)}{\mu_a-\mu} \le \frac{(1+r)[V(\mu_a) - V(\mu)]}{(1-\sigma)(\mu_a-\mu)} \le \frac{R(-\underline{z}(\mu),\mu_a) - R(-\underline{z}(\mu),\mu)}{\mu_a-\mu}.$$

Taking the limit $\mu_a \uparrow \mu$, and using left-continuity of $\underline{z}(\mu_a)$, we have $(1+r)V'(\mu^-) = (1-\sigma)R_2(-\underline{z}(\mu),\mu^-)$.

Part (iv): Convexity of V implies that $V'(\mu^+) \ge V'(\mu^-)$. To find the conditions for V to be differentiable at μ, use the definition $R(-z,\mu) = \mu\hat{R}(z,\mu)$ to compute:

$$R_2(-z(\mu),\mu) = \hat{R}(z(\mu),\mu) + \mu\hat{R}_2(z(\mu),\mu).$$

Note the following features. First, because $\hat{R}(z,\mu)$ is strictly supermodular, $\hat{R}_2(\bar{z}(\mu),\mu) \geq \hat{R}_2(\underline{z}(\mu),\mu)$, where the inequality is strict if and only if $\bar{z}(\mu) > \underline{z}(\mu)$. Second, μ^+ appears in the expression for $R_2(-z(\mu),\mu^+)$ only through the term $V'(\phi^+(\mu))$, and μ^- appears in the expression for $R_2(-z(\mu),\mu^-)$ only through the term $V'(\phi^-(\mu))$. Since V is strictly convex (and $\underline{z},\bar{z} < 0$), we have $R_2(-z(\mu),\mu^+) \geq R_2(-z(\mu),\mu^-)$ for all $z(\mu) \in Z(\mu)$, where the inequality is strict if and only if $V'(\phi^+(\mu)) > V'(\phi^-(\mu))$. Third, $\hat{R}(\bar{z}(\mu),\mu) = \hat{R}(\underline{z}(\mu),\mu)$, since both $\bar{z}(\mu)$ and $\underline{z}(\mu)$ maximize $\hat{R}(z,\mu)$. These features imply:

$$\begin{aligned} R_2(-\bar{z}(\mu),\mu^+) &\geq R_2(-\bar{z}(\mu),\mu^-) \\ &\geq \hat{R}(\underline{z}(\mu),\mu^-) + \mu\hat{R}_2(\underline{z}(\mu),\mu^-) = R_2(-\underline{z}(\mu),\mu^-). \end{aligned}$$

The first inequality comes from strict convexity of V, and it is strict if and only if $V'(\phi^+(\mu)) > V'(\phi^-(\mu))$. The second inequality comes from strict supermodularity of $\hat{R}(z,\mu)$, and it is strict if and only if $\bar{z}(\mu) > \underline{z}(\mu)$. Therefore, $V'(\mu^+) = V'(\mu^-)$ if and only if $V'(\phi(\mu))$ exists and $\bar{z}(\mu) = \underline{z}(\mu)$.

Part (v): Assume that $V'(\mu_a)$ exists for a particular (interior) μ_a, such as $\mu_a = h(\mu)$ for any arbitrary interior μ. By part (iv), $z(\mu_a)$ is unique and $V'(\phi(\mu_a))$ exists. Recall that $V'(h(\mu_a))$ always exists, by part (i). Since V is now differentiable at all posterior beliefs reached from μ_a under the optimal choice, we can take each of these subsequent nodes and repeat the argument. This shows that the optimal choice is unique and the value function is differentiable at all nodes on the tree generated from μ_a in the equilibrium. *Q.E.D.*