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Trading Volume in Dealer Markets

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Abstract

We develop a financial market trading model in the tradition of Glosten and Milgrom (1985) that allows us to incorporate non-trivial volume. We observe that in this model price volatility is positively related to the trading volume and to the absolute value of the net order flow, i.e. the order imbalance. Moreover, higher volume leads to higher order imbalances. These findings are consistent with well-established empirical findings. Our model further predicts that higher trader participation and systematic improvements in the quality of traders' information lead to higher volume, larger order imbalances, lower market depth, shorter duration, and higher price volatility.

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I Introduction

An important empirical regularity is that price volatility is positively related to trading volume and to the order imbalance (i.e. the absolute difference between the volumes of buy and sell orders); see Chordia, Roll, and Subrahmanyam (2002) for a comprehensive list of references or Karpoff (1987) for earlier studies. Volume is generated by trading activity, which is commonly explained by diversification and hedging motives, liquidity needs, or asymmetric information. Order imbalances occur either by chance or because of diverging opinions, which are often the result of heterogenous information.

Focussing on the information motive, we develop a model to study the impact of non-trivial volume on prices. A key feature of our model is the parsimonious formulation of traders' information. It allows us to study how changes in the underlying information environment influence market activity. The economic questions that can then be addressed range from the impact of regulatory changes, such as regulation Fair Disclosure, to the effect of improvements in data processing technology. For instance, do such changes increase or decrease price volatility and do they lead to more or less trading?

Our model combines features of Glosten and Milgrom (1985) and Kyle (1985) and has the following structure. Liquidity is supplied by an uniformed, risk-neutral and competitive market maker (or dealer). Demanders of liquidity either trade for reasons outside the model (e.g., to rebalance their portfolio or inventory), or they have private information about the fundamental value of the security. Specifically, informed traders receive private binary signals of heterogenous precisions, or qualities. There is a continuum of possible qualities, and the quality of trader i's signal is i's private information. Traders first post market orders. The dealer observes the order flow, sets a price that aggregates the information contained in it, and takes positions to clear the market if there is an imbalance between buys and sales. In equilibrium, the price is set so that traders buy (sell) if their private signal is sufficiently encouraging (discouraging).

As in Kyle (1985) or Admati and Pfleiderer (1988), traders post their orders simultaneously, which allows us to analyze the behavior of volume and order imbalances. As in Glosten and Milgrom (1985) traders are restricted to post orders of unit size. Combining unit trades with the continuous information structure allows us to concisely characterize the equilibrium by the marginal buyer's and seller's signal qualities.

Further, our setup allows for random variations in the number of traders and heterogeneity in people's ability to process information, which are essential characteristics of changes in market access or disclosure requirements. Tractably capturing these aspects within existing frameworks, such as Kyle (1985) or Admati and Pfleiderer (1988), is challenging, because the linear structure of the equilibrium in these models relies on the common knowledge of the number of traders and their signal precisions.[1]

In the equilibrium of our model some informed traders choose not to trade because their information is not sufficiently "reliable". We can thus study and quantify the *no-transaction rate*. This rate is loosely related to the time passage between transactions, or duration, a measure of transaction costs that has recently gained attention in the literature (see, for instance, Foucault, Kadan, and Kandel (2005)).

As a first step in our analysis, we verify that our model is consistent with the aforementioned, common empirical findings on the relations among the observables, namely, that price volatility is positively related to both, the trading volume and the order imbalance. Investigating the behavior of the order flow, we find that a higher volume leads to a higher order imbalance. These results lend credibility to our model, showing that it can serve as a conceptual framework for understanding and predicting trader behavior in a dealer market.

The intuition for this first set of results stems from the informational setup. In equilibrium, actions are more likely to be "correct" because signals are informative. Thus the more trades there are, the more "correct" rather than "wrong" decisions there are. This raises the expected order imbalance and leads to a higher price volatility.

A further testable implication of our model is that an increase in the time $t-1$ net order flow increases the expectation of the asset value, hence the expected time t number of traders with favorable information and the expected time t net order flow. In other words, in our model buys generate the expectation of more buys so that there is expected momentum in behavior.

After establishing the core relations among major observable variables, we move on to employ our framework to shed light on how structural shifts in markets affect the major observable variables. We first explore the behavior of prices and volume as the arrival rate of traders increases. Our model predicts that any event resulting in a higher average number of active traders, such as the advent of internet-based trading, will increase volume, the order imbalance and price volatility, and it will lower duration.

Next, we investigate the implications of an improvement in the traders' information quality. Such an improvement can occur, for instance, when a company adopts or a regulator imposes a new disclosure policy that fosters transparency.[2] Our model predicts that, ceteris paribus, stocks of companies with such new policies should exhibit higher

[1] As Kyle (1985) and Admati and Pfleiderer (1988) provide the foundation for a large strand of the dealer market literature, we will comment in detail on the relation of our results to theirs in Sections IV and V to highlight the marginal contribution of our work.

[2] Related to this are many examples of incremental or even dramatic improvements in economy-wide information quality, such as the advent of new data sources or new computing tools that allow faster processing of data. Our model then delivers testable predictions for event studies of such changes.

volume, higher order imbalances, and higher price volatility.

The key feature that admits such a comparative analysis is our informational setup. Focussing on families of quality distributions that are ordered in a first order stochastic dominance sense, we identify a complementarity between the overall and the marginal trader's information: as the aggregate quality increases, traders require signals of higher precision to be willing to trade. Intuitively, a first-order shift in information quality leads to a larger fraction of informed traders who have high quality information. This implies that prices are more informative and the relative disadvantage of the less well informed traders is smaller. At the same time, however, a higher average quality increases adverse selection costs and thus reduces "market depth", so that the price is more sensitive to each trade. A previously marginal trader now faces prices that, given his information, react too strongly to orders and he thus abstains from trading.

At first sight, the impact on volume appears ambiguous: when the marginal buyer has higher quality information, the fraction of people who trade may increase or decrease. Imposing more structure on the quality distribution in the form of a tighter technical monotonicity condition, we show that the probability of an informed trade increases as the overall information quality improves. This effect also leads to an increase in the expected trading volume. The expected order imbalance follows suit: a higher marginal trading quality implies that a smaller fraction of traders take the "wrong" action. Consequently, a larger fraction of traders will trade in the "right" direction, skewing the order imbalance. Combining the effect of the larger price impact of each transaction with the increase in trading volume, we find that price variability also increases. Finally, the mechanism that causes improvements in information quality to increase volume will increase the transaction rate and thus decrease duration.

Our framework admits an alternative way of modeling an improvement in information quality in the market by increasing the probability that any given trader is privately informed. Yet, we show that the equilibrium effects of such an increase are at odds with empirical findings: a higher probability of a trader being uninformed leads to higher volume but *lower* price volatility. Information structure shifts, on the other hand, do yield the empirically observed relation (higher volume and higher price volatility) and thus provide a better interpretation of empirical findings.

The remainder of the paper is organized as follows. Section II outlines our trading model. Section III derives the trading equilibrium. Section IV discusses the evolution of prices and the impact of volume on the order imbalance and price variability. Section V discusses the comparative statics with respect to entry rates and information structures. Section VI concludes. Most proofs are in the appendix.

II The Basic Setup

A Overview of the Trading Process.

Liquidity is supplied by an uniformed, risk-neutral and competitive dealer. Liquidity demanders either have private information about the fundamental value of the security, or they trade for reasons outside the model (e.g. portfolio or inventory rebalancing, diversification). Trade is restricted to unit lots of a single asset.

A random number of traders enter the market simultaneously. Informed traders try to predict the transaction price after receiving their information, and thus determine whether or not it is worthwhile to submit a market order. The dealer observes all orders and then sets a price that reflects her information.[3] All orders clear at a single price; the dealer absorbs the net order flow (the number of buys minus sales) in such a manner that she makes zero expected profits on her position.

Uncertain transaction prices are common in real markets, a few examples being: (a) dealer markets, such as forex and bond-markets (until recently), where liquidity demand precedes liquidity supply so that the clearing price is unknown at the time of the order submission; (b) trades that clear on an upstairs market;[4] (c) trades during the opening session on stock markets; (d) very fast moving markets or markets where the time from placing a market order to its execution is sufficiently long.[5]

B Model Details

Security: There is a single risky asset with a liquidation value V from a set of two potential values $\mathbb{V} = \{\underline{V}, \overline{V}\} \equiv \{0, 1\}$. The two values are equally likely, $\mathsf{Pr}(\overline{V}) = 1/2$, and this prior distribution over $\mathbb{V}$ is common knowledge.[6]

Traders: There is an infinitely large pool of traders, out of which a random number $N_t \geq 0$ of people are drawn in period t according to a Poisson distribution with parameter ν. Each trader can buy or sell *one* round lot (one unit) of the security at prices determined by the dealer, or he can be inactive. As in Glosten and Milgrom

[3] In what follows we will refer to the dealer as female and to traders as male.

[4] There may be prior price indications, but the definitive transaction price is only pinned down when the order is placed. Often large institutional traders instruct their trading department to acquire (or sell) x shares "cheapest" ("highest"). The trading department approaches its floor broker or an upstairs trader, who then helps work the order.

[5] Even with access to tick-to-tick data (for instance as offered by NASDAQ's INET), a retail investor faces a time gap between posting an order and trade execution. Another example is ordering in the immediate aftermarket of an IPO when price uncertainty is large.

[6] As will be clear from the subsequent analysis, the assumption of an equal prior is without loss of generality.

(1985), each trader can trade at most once, immediately upon entering. Traders can post only market orders. The set of possible actions is thus $\{\text{buy}, \text{no trade}, \text{sell}\}$.

Each trader is equipped with private information with probability $\mu \in (0,1)$; if not informed, a trader becomes a noise trader (probability $1-\mu$). The informed traders are risk neutral and rational, and they choose an action to maximize their expected trading profits. Noise traders buy and sell with equal probability; to simplify the exposition, we assume that they always trade. They are not necessarily irrational, but they trade for reasons outside of this model, such as liquidity.[7]

Market and timing: There are $t = 1, \ldots, T$ time periods. At each t, N_t traders enter the market and post their orders simultaneously to the dealer. The latter is risk neutral and competitive; upon observing the order flow, she sets a zero-expected profit market price.

C Information

The structure of the model is common knowledge among all market participants. The identity of a trader and his signal are private information, but everyone can observe the history of trades and transaction prices. "No-trades" by their very nature are unobservable, consequently neither is the total number of traders in the market. The public information H_t at date $t > 1$ is the sequence of numbers of buys and sales together with the realized transaction prices at all dates prior to t: $H_t = ((b_1, s_1, \mathsf{p}_1), \ldots, (b_{t-1}, s_{t-1}, \mathsf{p}_{t-1}))$, where b_τ, s_τ and p_τ are the numbers of buys and sales and the realized transaction price respectively at date $\tau < t$. H_1 refers to the initial history before trades occurred. The numbers of informed and noise traders at time t and qualities of informed traders' information are independent of the past history H_t and the underlying fundamental V.

Dealer. The dealer's information at date t consists of the public history H_t, the number of buy orders b_t and the number of sell orders s_t posted at time t.

Informed Traders' information. We follow most of the GM sequential trading literature and assume that traders receive a binary signal about the true liquidation value V. These signals are private, and they are independently distributed, conditional on the true value V. Specifically, informed trader i is told "with chance q_i, the liquidation

[7] Assuming the presence of noise traders is common practice in the literature on microstructure with asymmetric information to prevent "no-trade" outcomes à la Milgrom-Stokey (1982).

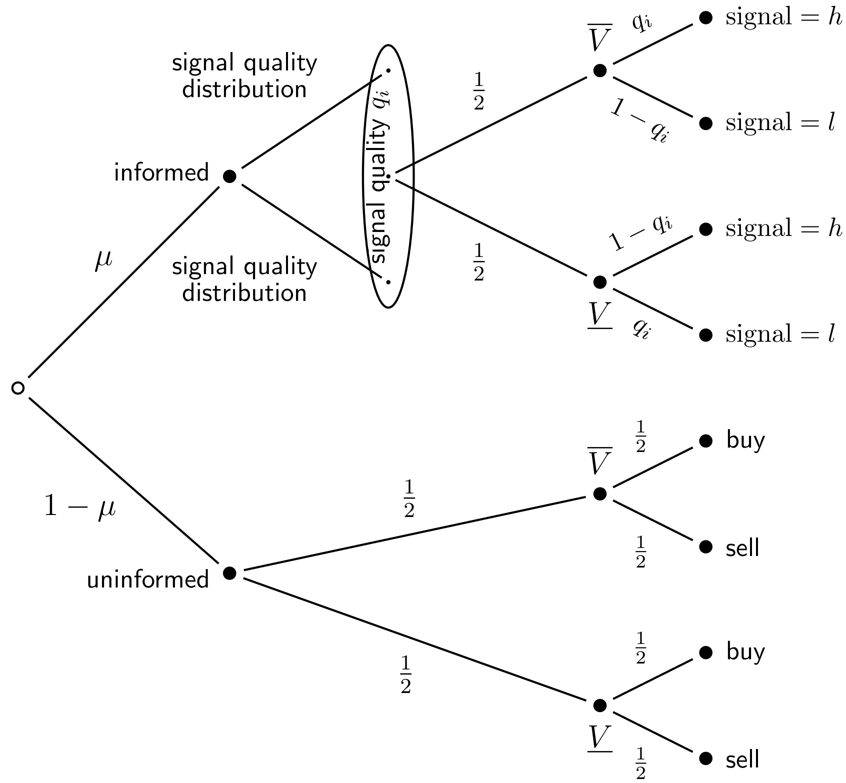

Figure 1: **Signals, Signal Qualities and Noise Trading.** This figure illustrates the mechanics of our signal distribution: first, it is determined whether a trader is informed (probability μ) or noise (probability $1-\mu$). If informed, the signal quality is determined next. The trader receives the "correct" signal with probability q_i and the "wrong" signal with probability $1-q_i$. (The draw of the state V is identical for all agents.) If the trader is a noise trader, then he will buy or sell with equal probability.

value is High/Low (h/l)" where

Pr(signal|true value)	$V=0$	$V=1$
signal = l	q_i	$1-q_i$
signal = h	$1-q_i$	q_i

This q_i is the *signal quality*. In contrast to most of the GM literature, we assume that there is a *continuum* of signal qualities and that q_i is trader i's private information. The distribution of qualities is independent of the asset's true value and can be understood as reflecting, for instance, the distribution of traders' talents to analyze securities. Figure 1 illustrates the distribution of noise and informed traders and the information structure.

In what follows, we will combine the binary signal (h or l) and its quality on [1/2,1] in a single variable on [0,1], namely, the trader's *private belief* that the asset's liquidation value

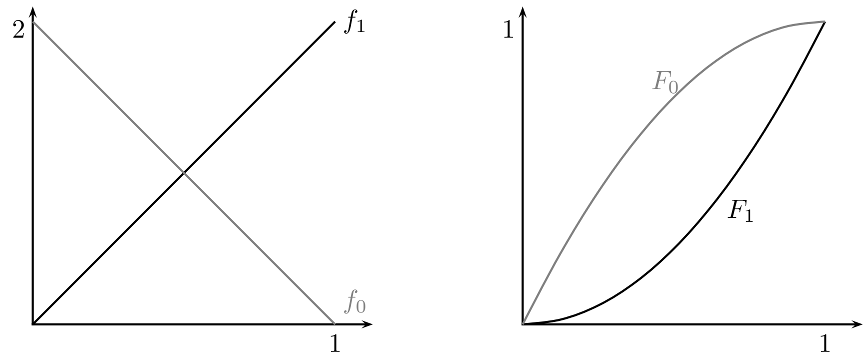

Figure 2: **Plots of Belief Densities and Distributions.** *Left Panel:* The densities of beliefs for an example with uniformly distributed qualities. The densities for beliefs conditional on the true state being 1 and 0 respectively are $f_1(\pi) = 2\pi$ and $f_0(\pi) = 2(1-\pi)$. *Right Panel:* The corresponding conditional distribution functions are $F_1(\pi) = \pi^2$ and $F_0(\pi) = 2\pi - \pi^2$.

is high ($V = 1$). This belief is the trader's posterior on $V = 1$ after he learns his quality and sees his private signal but *before* he observes the public history. A trader's behavior given his private signal and its quality can then be equivalently described in terms of the trader's private belief. This approach allows us to characterize the equilibrium in terms of a continuous scalar variable (as opposed to a vector of traders' private information) and thus simplifies the exposition.

Trader i's private belief is obtained by Bayes Rule and coincides with the signal quality if the signal is h, $\pi_i = \mathsf{Pr}(V = 1|h) = q_i/(q_i + (1 - q_i)) = q_i$. Likewise, $\pi_i = 1 - q_i$ if the signal is l. In what follows we will use the distribution of these private beliefs. Denote the density of beliefs $f_1(\pi)$ if the true state is $V = 1$ and by $f_0(\pi)$ if the true state is $V = 0$. Appendix A fleshes out how these densities are obtained from the underlying distribution of qualities.

Example of private beliefs. Figure 2 depicts an example where the signal quality q is uniformly distributed. The uniform distribution implies that the density of individuals with signals of quality $q \in [1/2, 1]$ is $g(q) = 2q$. In state $V = 1$, private beliefs $\pi \geq 1/2$ are held by traders who receive high (h) signals of quality $q = \pi$, private beliefs $\pi \leq 1/2$ are held by traders who receive low (l) signals of quality $q = 1 - \pi$. Thus, in state $V = 1$, the density of private beliefs π for $\pi \in [1/2, 1]$ is given by $f_1(\pi) = \mathsf{Pr}(h|V = 1, q = \pi)g(q = \pi) = 2\pi$, and for $\pi \in [0, 1/2]$ it is given by $f_1(\pi) = \mathsf{Pr}(l|V = 1, q = 1-\pi)g(q = 1-\pi) = 2\pi$. Similarly, the density conditional on $V = 0$ is $f_0(\pi) = 2(1 - \pi)$. The distributions of private beliefs are then $F_1(\pi) = \pi^2$ and $F_0(\pi) = 2\pi - \pi^2$. Figure 2 further illustrates that signals are informative: recipients in favor of state $V = 0$ are more likely to occur in this state than in state $V = 1$.

D The Trading Equilibrium

Traders who arrive at the same time do not observe each other's actions. Consequently, when posting the order, a trader does not know the transaction price.

The pricing rule: The dealer is competitive and receives zero expected profits. Consequently, given the public history H_t, b_t buy orders and s_t sell orders posted at time t, the price at date t is

$$\mathsf{p}_t = \mathsf{E}[V|b_t, s_t, H_t]. \tag{1}$$

This equilibrium pricing rule is common knowledge.[8]

The informed trader's optimal choice: An informed trader enters the market in period t, receives his private signal and observes history H_t. He submits a buy order if, conditional on his information *and* his own buy-decision, the expected transaction price is at or below his expectation of the asset's liquidation value; likewise for a sell order. He abstains from trading if he expects to make negative trading profits.

The equilibrium concept. We will restrict attention to monotone and symmetric decision rules for traders. Namely, we assume that an informed trader uses a "threshold" rule: he buys if his private belief π_i is at or above the time-t buy threshold $\overline{\pi}_t$, $\pi_i \geq \overline{\pi}_t$, he sells if $\pi_i \leq \underline{\pi}_t \leq \overline{\pi}_t$, and he abstains from trading otherwise. The rules are symmetric in the sense that all informed traders use the same threshold decision rule.

Moreover, we will consider only equilibria in which people have no ex-post regrets. In such equilibria, a trader would not wish to change his trading decision upon observing the order flow. This no-regrets property is commonly attained in other settings. For instance, it is an equilibrium feature of setups where agents submit demand-supply schedules such as a rational expectations equilibrium (e.g. Grossman (1976)). With a regret free equilibrium, the unobservability of the transaction price is benign. Such an equilibrium need not exist, but we show that it does and that it is tractable.[9]

The price set by the dealer given the action profiles for the informed traders is referred to as the equilibrium price.

[8]This equilibrium pricing rule is analogous to Kyle (1985), but there is a subtle difference, as we allow the dealer to see all orders, whereas in Kyle, she observes only the aggregate order flow.

[9]Any regret free equilibrium is also a Perfect Bayesian Equilibrium (PBE), but the converse is not true. Thus the existence of a regret-free equilibrium is not guaranteed by standard game theoretic results. A regret free equilibrium requires no ex-post losses, implying the PBE condition that there are no ex ante expected losses, by the Law of Iterated Expectations.

III Equilibrium Prices and Behavior

A Pricing and Decision Rules

When the asset's liquidation value is high, $V = 1$, a given trader will buy with probability $(1-\mu)/2 + \mu(1 - F_1(\overline{\pi}_t))$, sell with probability $(1-\mu)/2 + \mu F_1(\underline{\pi}_t)$, and not trade with probability $\mu(F_1(\overline{\pi}_t) - F_1(\underline{\pi}_t))$, also accounting for noise traders' actions. To compress notation, for $v \in \{0, 1\}$ we write these probabilities as follows $\beta_{v,t} := \mathsf{Pr}(\text{buy}|V = v, H_t)$, $\sigma_{v,t} := \mathsf{Pr}(\text{sale}|V = v, H_t)$, and $\gamma_{v,t} := \mathsf{Pr}(\text{no trade}|V = v, H_t)$.

The probability that a given trader is informed is independent of other agents' identities and the asset's liquidation value. Moreover, private beliefs are independent conditional on the asset value. Consequently, traders' actions are independent, conditional on the asset value.

The dealer's pricing rule. Suppose the dealer observes b_t buy orders and s_t sell orders. The dealer then revises her belief by Bayes' Rule and sets the zero-expected profit price

$$\text{(2)} \qquad \mathsf{p}_t(b_t, s_t) = \mathsf{E}[V|b_t, s_t, H_t] = \frac{\mathsf{Pr}(b_t, s_t|V = 1, H_t)p_t}{\mathsf{Pr}(b_t, s_t|V = 1, H_t)p_t + \mathsf{Pr}(b_t, s_t|V = 0, H_t)(1 - p_t)},$$

where p_t is the public belief that the state is high after history H_t, $p_t = \mathsf{Pr}(V = 1|H_t)$. To simplify the exposition, we will from now on omit the t subscripts and history H_t when possible without ambiguity. When seeing b buys and s sales, the dealer knows that the total number of traders, N, is at least $b + s$. Summing over all the possible realizations of N,[10]

$$\text{(3)} \qquad \mathsf{Pr}(b, s|V = v) = \frac{e^{-\nu(1-\gamma_v)}(\nu\beta_v)^b(\nu\sigma_v)^s}{b!s!}.$$

Substituting this probability into the pricing rule, we obtain

$$\text{(4)} \qquad \mathsf{p}(b, s) = \frac{e^{-\nu(1-\gamma_1)}\beta_1^b\sigma_1^s p}{e^{-\nu(1-\gamma_1)}\beta_1^b\sigma_1^s p + e^{-\nu(1-\gamma_0)}\beta_0^b\sigma_0^s(1 - p)} \equiv \frac{\ell^M(b, s)p}{\ell^M(b, s)p + (1 - p)},$$

where $\ell^M(b, s)$ denotes the dealer's likelihood ratio of the high to low state, given b buy- and s sell-orders,

$$\text{(5)} \qquad \ell^M(b, s) := e^{\nu(\gamma_1-\gamma_0)}\left(\frac{\beta_1}{\beta_0}\right)^b\left(\frac{\sigma_1}{\sigma_0}\right)^s.$$

[10]The details of the derivation for equation (3) as well as for equation (7) are in Appendix B.1.

The informed trader's trading decision rule. We will focus on the "buy" decision. Consider trader i with private belief $\pi_i = \pi$. He computes his expectation of the asset's value and the transaction price, conditional on his private information. The latter consists of his presence in the market ($N \geq 1$), his private signal and his action. He will buy if $\mathsf{E}[V|\pi, N \geq 1, i \text{ buys}] \geq \mathsf{E}[\mathsf{p}\big|\pi, N \geq 1, i \text{ buys}]$. To simplify the exposition, in what follows we will omit $N \geq 1$ and i's trading decision when writing the informed buyer's expectation. As outlined above, we are searching for an equilibrium such that nobody wishes to change their action upon observing the order flow. This implies, in particular, that the marginal trader with the "buy-threshold" private belief $\overline{\pi}$ is just willing to buy for every realization of buys and sales by others. Consequently, in equilibrium the threshold value $\overline{\pi}$ solves for any $b \geq 1$, $s \geq 0$,[11]

$$\mathsf{E}[V|\overline{\pi}, b, s] = \mathsf{p}(b, s). \tag{6}$$

The Law of Iterated Expectations then implies that the marginal buyer's expectation coincides with this type's expectation of the prices, $\mathsf{E}[V|\overline{\pi}] = \mathsf{E}[\mathsf{p}|\overline{\pi}]$. Next, analogously to the dealer's pricing rule, the marginal trader's expectation is given by

$$\mathsf{E}[V|\overline{\pi}, b, s] = \frac{\ell^I(b, s; \overline{\pi})p}{\ell^I(b, s; \overline{\pi})p + (1-p)}, \tag{7}$$

where $\ell^I(b, s; \overline{\pi})$ is the likelihood ratio of the high to low state, given the informed trader's private belief π, and supposing that he knew that there are $b-1$ buy orders and s sell orders posted by the other agents[12]

$$\ell^I(b, s; \overline{\pi}) = \frac{\overline{\pi}}{1-\overline{\pi}} \frac{\beta_1^{b-1}}{\beta_0^{b-1}} \frac{\sigma_1^s}{\sigma_0^s} \frac{\sum_{h=0}^{\infty} \frac{(\nu\gamma_1)^h}{(h+b+s)h!}}{\sum_{h=0}^{\infty} \frac{(\nu\gamma_0)^h}{(h+b+s)h!}}. \tag{8}$$

B Equilibrium Condition

Using (4) and (7), indifference equation (6) can be rewritten as

$$\frac{\ell^I(b, s; \overline{\pi})p}{\ell^I(b, s; \overline{\pi})p + (1-p)} = \frac{\ell^M(b, s)p}{\ell^M(b, s)p + (1-p)} \text{ for all } (b, s). \tag{9}$$

[11] When the trader buys and there are a total of b buys and s sales, then $b-1$ buys and s sales are posted by the other traders.

[12] This expression is formally derived in Appendix B.2 in Step 1 of the existence proof. The main difference between the dealer's and the informed traders' expectation is their assessment of the number of traders in the economy: an informed trader knows that there is at least one trader (he) and thus he employs a different probability measure compared to the dealer.

Rearranging, in equilibrium $\overline{\pi}$ solves

$$\ell^I(b, s; \overline{\pi}) = \ell^M(b, s) \;\text{ for all } (b, s). \tag{10}$$

Hence, equilibrium decision rules are independent of the public belief about the asset's liquidation value. Moreover, this condition is identical for any trading history and thus the trading threshold is independent of the trading history. The equilibrium condition for the marginal selling type $\underline{\pi}$ is analogous, the difference being that the marginal seller assumes there to be b buy orders and $s - 1$ sell orders by the other traders.

C Equilibrium Existence and Uniqueness

Noise buying and selling occur with the same probability. Suppose the prior p is neutral, $p = 1/2$. Then favourable and unfavourable signals are equally likely in expectation, and assuming an equilibrium exists, intuitively, there must be one with symmetric marginal buying and selling thresholds, $\underline{\pi} = 1 - \overline{\pi}$. Since by (10) the equilibrium decisions rules do not depend on the past trading activity, the symmetry of thresholds extends to any trading history. In Appendix B we show that the symmetric equilibrium exists and that it is the only one.

Symmetry of the thresholds implies that traders require the same signal quality to buy and sell. As the signal quality distribution is independent of the true value, a buy in the high state is as likely as a sale in the low state and vice versa (we show this formally in Appendix A in Lemma 1). Thus a no-trade occurs with equal probability in each state, $\gamma_0 = \gamma_1$. We can then simplify the likelihood ratios $\ell^M(b, s)$ and $\ell^I(b, s; \overline{\pi})$ (equations (5) and (8)) because the effects of no-trades cancel and rewrite the equilibrium conditions $\ell^I(b, s; \overline{\pi}) = \ell^M(b, s)$ and $\ell^I(b, s; \underline{\pi}) = \ell^M(b, s)$ as

$$\overline{\pi} = \frac{\beta_1}{\beta_1 + \beta_0} \quad \text{and} \quad \underline{\pi} = \frac{\sigma_1}{\sigma_1 + \sigma_0}. \tag{11}$$

Since thresholds are symmetric, $\overline{\pi} = 1 - \underline{\pi}$, we have $\underline{\pi} = \frac{\beta_0}{\beta_1 + \beta_0}$. We are now ready to state the existence and uniqueness theorem.

Theorem (Symmetric Equilibrium: Existence and Uniqueness)
There exist unique $(\underline{\pi}, \overline{\pi})$ such that $0 < \underline{\pi} < \overline{\pi} < 1$, any trader with private belief $\pi \geq \overline{\pi}$ buys, any trader with private belief $\pi \leq \underline{\pi}$ sells, any trader with $\pi \in (\underline{\pi}, \overline{\pi})$ does not trade. These thresholds are symmetric, $\overline{\pi} = 1 - \underline{\pi}$, and invariant with respect to time and past order-flows.

The intuition for existence and uniqueness has a parallel in the relation of marginal and average revenues. Suppose that the prior on the high liquidation value, $V = 1$, is 1/2. Then the left hand side of the first expression in (11) is the belief of the *marginal* buyer, the right hand side is the price that a trader must pay if there is only one buy (and no sales). This price accounts for the *average* information content of the buy, conditional on the informed buyer holding a belief at or above $\overline{\pi}$. Loosely speaking, in equilibrium, the *marginal* belief $\overline{\pi}$ must coincide with the *average* belief derived from the observed buying decision.

We will now argue more formally that there exists a unique point where average and marginal coincide and that it is in $(1/2, 1)$. Suppose first that the marginal buyer has a belief 1/2. Since an informed buyer holds a belief at or above 1/2, buying reveals favorable information about the asset. The price that a buyer pays when he is the only trader in the market is then strictly above 1/2, i.e. the average is above the marginal. Now suppose that the marginal buyer has a belief of 1. An informed buy arises with probability 0, and the price for a single buyer is 1/2 so that the average is below the marginal. Continuity yields existence. Analogously to many standard economic problems, average and marginal coincide at an extremum of the average. Since the marginal is strictly increasing there can be at most one such intersection. Hence uniqueness.

Finally, the marginal buyer's and seller's beliefs cannot coincide. Each trade has an effect on the price so that the marginal buyer's belief, $\bar{\pi}$, is strictly above 1/2. Threshold symmetry then implies that $\bar{\pi}$ is strictly above the marginal seller's belief, $\underline{\pi} = 1 - \bar{\pi}$.[13]

IV Price, Volume and Order Imbalance

The five major observable variables in our model are the price, the price-variability, the volume, the net order flow, and the order imbalance. The major primitives of the model are the entry rate, the level of informed trading, and the distribution of information qualities. In this section, we will discuss the relations among the observables for *given* primitives. In the next section we will discuss how *changes* in the primitives affect the observables.

The relations among the observables have been widely analyzed empirically and the predictions of our model are in line with the empirical observations. Some of these predictions are not readily generated by other theoretical models. Moreover, the results in this section are a stepping stone for the novel implications that we derive later on.

Let b_t and s_t denote the numbers of buys and sales respectively in period t. Then

[13] Thresholds $\overline{\pi}$ and $\underline{\pi}$ coincide only when there is no informed trading, $\mu = 0$, which we rule out.

volume is $\mathsf{Vol}_t := b_t + s_t$; the net order flow is $\mathsf{NOF}_t := b_t - s_t$; the order imbalance is the absolute value of the net order flow, $\mathsf{IB}_t := |b_t - s_t|$. As before, p_t is the price at time t and $|\Delta \mathsf{p}_t|$ is the absolute value of the price change from $t-1$ to t. We use $|\Delta \mathsf{p}_t|$ as a measure of price-variability.

We will first analyze how the net order flow affects the price and how the current net order flow affects the expectation of the future net order flow. We then ask how order imbalances and volume affect price changes, how volume affects the order imbalance, and finally how the order imbalance today affects the order imbalance tomorrow. We will not study the relation of past and future prices explicitly. As is common in GM models, the dealer's expectation obeys the Law of Iterated Expectations so that the price process is a martingale.

A Dynamics of Prices and Net Order Flows

Prices in our model incorporate all publicly available past information plus the new information that is revealed by the current order flow and are thus informationally efficient. With competitive pricing, equation (2) holds and thus the past price is a sufficient statistic for the asset's true value: $\mathsf{p}_{t-1}(b_{t-1}, s_{t-1}) = \mathsf{E}[V|b_{t-1}, s_{t-1}, H_{t-1}] = \mathsf{E}[V|H_t] = \mathsf{Pr}(V = 1|H_t) = p_t$. Threshold symmetry and time invariance imply the following law of motion for transaction prices:

Proposition 1 (Price Dynamics)
In equilibrium, transaction price p_t is determined by the number of buy- and sell-orders b_t and s_t respectively at date t and last period's transaction price p_{t-1}. It evolves as follows

$$\mathsf{p}_t = \left(1 + \frac{1-\mathsf{p}_{t-1}}{\mathsf{p}_{t-1}} \left(\frac{\beta_0}{\beta_1}\right)^{\mathsf{NOF}_t}\right)^{-1} \tag{12}$$

Proof: With threshold time invariance, the dealer's pricing rule (4) becomes

$$\mathsf{p}(b_t, s_t|H_t) = \frac{e^{-\nu(1-\gamma_1)}{\beta_1}^{b_t}{\sigma_1}^{s_t}\mathsf{Pr}(V=1|H_t)}{e^{-\nu(1-\gamma_1)}{\beta_1}^{b_t}{\sigma_1}^{s_t}\mathsf{Pr}(V=1|H_t) + e^{-\nu(1-\gamma_0)}{\beta_0}^{b_t}{\sigma_0}^{s_t}(1-\mathsf{Pr}(V=1|H_t))}.$$

The time-t public prior, $\mathsf{Pr}(V = 1|H_t) = p_t$, coincides with the time-$(t-1)$ transaction price p_{t-1}. Threshold symmetry $\overline{\pi} = 1 - \underline{\pi}$ implies $\beta_1 = \sigma_0$, $\beta_0 = \sigma_1$, $\gamma_1 = \gamma_0$. □

Corollary (Prices and Net Order Flows) *For any history, the time-t transaction price p_t strictly increases in the time-t net order flow $\mathsf{NOF}_t = b_t - s_t$.*

The corollary is empirically supported by, among others, Jones, Kaul, and Lipson (1994), Brown, Walsh, and Yuen (1997) or Chordia, Roll, and Subrahmanyam (2002).

Proposition 1 states that the dynamics of prices are linked to the net order flow.[14] It is thus important to understand the behavior of the latter. Threshold symmetry and time-invariance yield the following result:

Proposition 2 ("Buys beget Buys")
The expectation of the future net order flow increases in the current price as follows: the expected time-t net order flow given history H_t is given by

$$\mathsf{E}[\mathsf{NOF}_t|H_t] = (2\mathsf{p}_{t-1} - 1)\nu(\beta_1 - \beta_0).$$

Proposition 2 implies, in particular, that the expectation of the future net order flow increases in the current price. To see why this is true, observe that in the high state, $V = 1$, favourable signals are more likely than unfavourable ones. Consequently, when the high state is more likely than the low state, $p_t > 1/2$, one assigns higher probability to situations with $b_t - s_t > 0$ than to $b_t - s_t < 0$. Since the current price increases in the current net order flow (Proposition 1), a larger past net order flow creates the expectation of a larger future net order flow. Our model thus predicts that buys are more likely to follow buys than sales and vice versa. Empirically this was observed, for instance, by Hasbrouck and Ho (1987) or by Chordia, Roll, and Subrahmanyam (2002).

Proposition 2 further implies that the dealer may accumulate an inventory. This inventory build-up is facilitated by the dealer's risk neutrality. If she were to account for her inventory risk, then she would likely take pro-active steps to reduce her exposure. This would, at least in part, counter the effect that we identify here.

The prediction of the corollary has also been generated in other dealer market models, for instance in Kyle (1985) and Admati and Pfleiderer (1988). Our approach complements the findings from these frameworks by admitting a non-zero expected order flow,[15] and it thus allows us to predict how the past order flow affects the expectation of the future order flow (Proposition 2).

[14] In a degenerate version of our model, in which at most one trader can enter at a time, the corollary is equivalent to Proposition 1 of Glosten and Milgrom (1985). The time-t transaction price p_t becomes the time-t ask price for $(b_t, s_t) = (1, 0)$ and the time-t bid price for $(b_t, s_t) = (0, 1)$. The corollary implies that the ask price is greater and the bid price is smaller than the public expectation, which equals the preceding period's transaction price p_{t-1}.

[15] See, for instance, equation (3.11) or (4.2) in Kyle (1985); the expectation of the net order flows expressed there is zero by the Law of Iterated Expectations; similarly in Admati and Pfleiderer (1988).

B Volume and the Order Imbalance.

While a larger net order flow increases the expectation of the future net order flow, the same need not hold for their *absolute values*, i.e. the order imbalances. Indeed, the first part of the following result shows that in expectation the order imbalance is independent of past trading activity. Next, as signals are informative, an increase in the number of traders yields proportionally more trades in the "right" rather than the "wrong" direction. Hence larger volume should on average lead to larger order imbalances. The second part of the following result confirms this intuition.

Proposition 3 (Order Imbalance and Volume)
(a) *The expected absolute value of the time-t net order flow* $\mathsf{E}[\mathsf{IB}_t \big| H_t]$ *is independent of the transaction history:* $\mathsf{E}[\mathsf{IB}_t \big| H_t] = \mathsf{E}[\mathsf{IB}_{t'} \big| H_{t'}] \;\; \forall t, t'$.
(b) *Conditional on the realized time-t volume, the expected absolute value of the time-t net order flow* $\mathsf{E}[\mathsf{IB}_t \big| \mathsf{Vol}_t, H_t]$ *increases in* Vol_t.

Studies that focus on order imbalances (e.g. Chan and Fong (2000)) implicitly yield (b), and we are not aware of empirical studies that test (a).

The positive relation between volume and order imbalances that we identify here provides insights for the strong empirical performance of the so called Amihud measure. Developed in Amihud (2002), it measures (il-)liquidity by the ratio of the absolute value of the daily dollar return of a stock to its dollar trading volume. At first sight volume seems to be too coarse a measure because, say, 2 million shares bought by or sold to a dealer affect liquidity differently than buy and sell orders of 1 million shares each arriving simultaneously. Proposition 3 predicts, however, that the Amihud measure is correlated with measures that employ the order imbalance — as has been observed empirically.

C Price Variability.

We will now investigate how volume and the order imbalance affect the absolute values of price changes, or price variability, $|\Delta \mathsf{p}_t| := |\mathsf{p}_t - \mathsf{p}_{t-1}|$. Proposition 1 implies that the transaction price increases in the net order flow. Yet, it is not immediate that a larger order imbalance leads to a larger price change because the price is not linear in the net order flow. Suppose that $\mathsf{p}_{t-1} > 1/2$ so that the time-t prior favours the high liquidation value. Then, for a given order imbalance $\mathsf{IB}_t = |b_t - s_t|$, a negative net order flow, $b_t < s_t$, lowers the price by more than a positive net order flow, $b_t > s_t$, would increase it by. In other words, a net order flow that reconfirms the prevailing public opinion leads to smaller price changes than one that is at odds with it. However, for the expected effect we show

Proposition 4 (Price Variability and the Order Imbalance)
The expected absolute value of the price change at time t conditional on the order imbalance, $\mathsf{E}[|\Delta \mathsf{p}_t| \big| \mathsf{IB}_t, H_t]$*, increases in the order imbalance* IB_t*.*

The results of Propositions 3 and 4 show that a larger volume leads to larger order imbalances and that larger order imbalances lead to higher price variability. As a consequence, intuitively, there should be a positive relation between volume and the magnitude of price changes. The following proposition confirms this intuition and shows that in expectation higher volume leads to larger price changes.

Proposition 5 (Price Variability and Volume)
The expectation of the absolute price change conditional on the realized time-t volume $\mathsf{E}[|\Delta \mathsf{p}_t| \big| \mathsf{Vol}_t, H_t]$ *increases in volume* Vol_t*.*

Proposition 5 has been empirically documented in several studies, see, for instance, Gallant, Rossi, and Tauchen (1992) or Karpoff (1987) for earlier references. Chordia, Roll, and Subrahmanyam (2002) and Chan and Fong (2000) additionally highlight the effect of the order imbalance on price variability, providing evidence for both Propositions 4 and 5.

The positive relation between price volatility and volume has also been obtained theoretically in other settings. For instance, Wang (1994) shows this relation in a CARA-Gaussian rational expectations setup. Admati and Pfleiderer (1988) generate a result similar to our Proposition 5 in a setting where the number of traders is endogenously determined by the extent of discretionary noise trading. Specifically, they show that price variability is larger in periods with concentrated trading.

Although rational expectation models with a CARA-Gaussian structure yield valuable insights into the volume-volatility relation, results on order imbalances such as Proposition 4 can only obtain in markets that do not perforce clear, such as intermediated markets. The relation of volume and the order imbalance and that of price-variability and the order imbalance have not been explicitly analyzed in either Kyle (1985) or Admati and Pfleiderer (1988). Further, models in this tradition have an underlying normal structure and for a tractable volume analysis it is important that the expected net order flow is zero.[16] To understand and compute the dynamics of prices, volume, and order imbalances (as studied in Propositions 3 and 4) it may, however, be desirable to have no such restriction. As our framework explicitly admits non-zero expected net order flows (see Proposition 2), our findings in Propositions 3 and 4 are arguably based on a less restrictive premise.

[16]If the equilibrium has this property, then one can employ half-normal distributions to compute the expected volume.

Moreover, higher volume in our model is closely linked to a larger transaction rate, a topic that we discuss below in more detail.

D Duration

The time span between transactions, also called the "duration" (see Engle and Russell (1998) for a formal definition), has recently gained attention in the literature. Lesmond, Ogden, and Trzcinka (1999) use duration to estimate transaction costs. Foucault, Kadan, and Kandel (2005) develop a theoretical model in which liquidity traders account for duration in their choice of submitting market or limit orders.

The simplest cause for a long duration is a low arrival rate of traders. Yet, this view does not consider that the arrival rate of trades is endogenous to the market environment. For instance, high transaction costs may deter people from trading, thus lowering the transaction rate.[17]

In our model the transaction rate is determined by the exogenous Poisson arrival rate ν, the probability that a trader is informed μ, and the trading thresholds $\underline{\pi}$ and $\overline{\pi}$. Under a Poisson process arrival rate, a zero-arrival occurs with positive probability. More importantly, if none of the arriving informed traders hold beliefs that are extreme enough, that is, if $\pi_i \in (\underline{\pi}, \overline{\pi})$ for all informed traders, then there will also be no trading. The probability of no trading in state $V = v$ follows from equation (B-1) in Appendix B.1.[18]

$$\begin{aligned} & \mathsf{Pr}(\text{no trade}|V = v) \\ = \; & \mathsf{Pr}(\text{no arrival}|V = v) + \sum_{j=1}^{\infty} \mathsf{Pr}(j \text{ arrivals}|V = v)\mathsf{Pr}(\forall j : \pi_j \in (\underline{\pi}, \overline{\pi})|V = v) = e^{-\nu(1-\gamma_v)}. \end{aligned}$$

Since $\gamma_1 = \gamma_0$, we have $\mathsf{Pr}(\text{no trade}) = p_t e^{-\nu(1-\gamma_1)} + (1 - p_t)e^{-\nu(1-\gamma_0)} = e^{-\nu(1-\gamma_1)}$, where $\gamma_v = \mu(F_v(\overline{\pi}) - F_v(\underline{\pi}))$. Consequently, the arrival of no-trades is determined solely by the exogenous parameters of the model and the equilibrium behavior, and it does not change with the trading history.

Next, occurrences of the events "trade" and "no-trade" are a Bernoulli process. As

[17]When duration is long, limit orders become implicitly more expensive because it will take longer for them to be filled. In this sense, duration adds to the transaction costs. Lesmond, Ogden, and Trzcinka (1999) discuss this direct impact of duration on transaction costs. In Foucault, Kadan, and Kandel (2005) duration feeds back into transaction costs, the idea being that for longer duration, traders are more inclined to submit market orders (which, by definition, get executed faster than limit orders). Studying the impact of this general relation in a theoretical trading model, they show that longer duration decreases the bid-ask-spread because traders submit more aggressive limit orders. While we do not study the choice between market and limit orders, our framework adds to the discussion by offering an informational angle on endogenous arrival rates.

[18]Equation (B-1) defines the probability that the market marker attaches to b buys and s sales; here we study the special case with $b = s = 0$.

trading in our model occurs at discrete points in time, the time between two periods with transactions is determined by this Bernoulli process. A simple measure for duration in our model is the expected number of future consecutive no-trading periods. The probability of trading is the same for all periods and we can compute

$$\begin{aligned} \mathsf{D}(\nu,\gamma_1) &= 1 \text{ period} \cdot \mathsf{Pr}(1 \text{ period no trade}) + 2 \text{ periods} \cdot \mathsf{Pr}(2 \text{ periods no trade}) + \ldots \\ &= \sum_{t=1}^{\infty} t \cdot \left(e^{-\nu(1-\gamma_1)}\right)^t = \frac{e^{-\nu(1-\gamma_1)}}{(1-e^{-\nu(1-\gamma_1)})^2}. \end{aligned} \tag{13}$$

This measure depends solely on γ_1 and the parameter ν, it is increasing in γ_1 and decreasing in ν. The above discussion implies

Proposition 6 (Duration) *Duration $\mathsf{D}(\nu,\gamma_1)$ is independent of the trading history.*

In models where the underlying noise and information are normally distributed, the number of informed traders must be known (noise trader orders are only considered as an aggregate figure). Moreover, every informed trader in these models trades. Taken together, this effectively fixes the number of transactions, as was pointed out by, for instance, Jones, Kaul, and Lipson (1994). Thus these models are not designed to analyze transaction rates and, relatedly, duration. In our setup, on the other hand, the number of traders is uncertain and the transaction rate is endogenous.

Episodes of no or low trading activity also arise in Easley and O'Hara (1992). These episodes are most likely to obtain when no private information is available and transactions stem exclusively from noise traders. Low trading activity then indicates that trading occurs for non-informational reasons, and it implies low adverse selection costs. We complement Easley and O'Hara (1992) and provide an interpretation of low transaction rates in presence of private information. In our setting, informed traders are always present but they may choose not to trade if their private information is not sufficiently precise. Lower trading activity then corresponds to more accurately informed active traders, and longer duration is associated with higher adverse selection costs and higher transaction costs.

V Changes in Arrival Rates and Information Quality

A Comparative Statics on Arrival Rates

The expected number of traders in our model is governed by the Poisson parameter ν. Intuitively, an increase in ν leads to a higher expected volume, and as a consequence, to a higher absolute value of the order imbalance, larger price changes, and shorter duration.

Proposition 7 (Comparative Statics on the Entry Rate)
As the exogenous entry rate ν increases, so do expected trading volume, $\mathsf{E}[\mathsf{Vol}_t|H_t]$, the expected value of the order imbalance, $\mathsf{E}[\mathsf{IB}_t|H_t]$, and the expected magnitude of the price change, $\mathsf{E}[|\Delta\mathsf{p}_t|\big|H_t]$. Duration $\mathsf{D}(\nu, \gamma_1)$ decreases.

An increase in the arrival rate ν signifies a persistent switch in investor composition, for instance, triggered by a company's inclusion into a major index (so that index-tracking funds must hold the stock) or the advent of internet-based trading. Our model thus predicts, in particular, that any such event should lead to higher price variability, larger volume and lower duration.

In our framework, an increase in ν proportionally increases the (expected) number of both, informed and uninformed traders. Admati and Pfleiderer (1988) perform a related comparative static by increasing the (known) number of informed traders. They find, in their setting with no timing, that such an increase leads to higher trading volume but does not affect the price variability.

B Comparative Statics on Quality Distributions

We will now study how persistent changes in the information quality affect the marginal trading types and through them price informativeness, volatility and volume.

In contrast to setups with normally distributed information, traders in our model need not know the quality of others' information but require only knowledge of the overall distribution of information in the economy. We are thus able to analyze the impact of economy-wide shifts in information processing. Further, traders in our model may abstain from trading, and we can thus study changes in transaction rates. The importance of the transaction rate is highlighted, for instance, by Jones, Kaul, and Lipson (1994), who show that the positive volume-volatility relation is driven by the number of transactions. Chordia, Roll, and Subrahmanyam (2008) argue that turnover has increased over the last decade because of an increase in the frequency of (small) transactions.

Parametrization of Information Quality. Systematic shifts in a quality distribution occur when there is a persistent change in the fraction of traders who are better informed or more capable at processing information. A positive shift can be triggered, for instance, by more extensive analyst coverage for a stock, as this would improve the average trader's information. A stock may also attract a more informed clientele when it gets included in a major index, as major funds will then add it to their portfolios. With such an inclusion, the company often faces additional disclosure requirements, further affecting the distribution of traders' signal qualities. Many of these changes are observable and the predicted impacts can be tested empirically. Additionally, the quality of analysts' earning forecasts can serve as a proxy for the average information quality.

In what follows, we will formally model improvements in information quality by shifts in the underlying distribution in the sense of *first order stochastic dominance* (FOSD). To study these shifts we will employ a family of signal quality distributions that is parameterized by a scalar $\theta \in \Theta$, and we will assume that the signal distributions are twice differentiable with respect to θ; details are in Appendix A.2. Thus when we speak of a *shift in information quality* or an *improvement in information quality* we mean that the "new" quality distribution first-order stochastically dominates the "old" one so that under the new distribution, traders have systematically higher quality information.

Marginal Trading Types, Trade Informativeness and Market Depth. To describe effects of signal quality distributions on empirically observable variables, we must first establish how they affect traders' behavior. The major variable of interest is thus the belief of the marginal buyer and seller.

The informativeness of a transaction is synonymous with its marginal impact on the price. The usual interpretation of the price impact of a trade is that it is inversely related to *market depth*.[19] As can be seen from the price dynamics equation (12), the price impact is inversely related to the ratio of "wrong" to "correct" trading decision probabilities: $\beta_0/\beta_1 = \sigma_1/\sigma_0$. Consequently, the smaller this ratio is, the stronger is the price impact of the marginal order.

Proposition 8 (Trade Informativeness and Price Impact)
As information quality improves,

(*a*) *The buy-threshold private belief* $\overline{\pi}(\theta)$ *increases and the sell-threshold* $\underline{\pi}(\theta)$ *decreases.*

(*b*) *Market depth, measured by* $\beta_0(\overline{\pi}(\theta))/\beta_1(\overline{\pi}(\theta))$, *decreases and each trade has a larger price impact.*

Suppose that a regulatory shift causes traders to be on average better informed and that, for the sake of the argument, the buying and selling thresholds remained unchanged. Then each trade would become more informative and traders would be able to learn more from the price. One might imagine that, as a consequence, the marginal buyers and sellers would require less precise private information. This would cause the selling threshold to increase and the buying threshold to decrease. This argument, however, neglects the dealer's perspective who faces the adverse selection problem and who has to protect herself against well-informed traders. As after the shift traders are systematically

[19]For instance, in models in the tradition of Kyle (1985), the price is a linear function of the net order flow. Market depth is then measured by the inverse of the slope coefficient of the order flow.

better informed, the dealer is more likely to encounter well-informed traders. Thus for a given order flow, she must set prices that are more extreme (she raises prices for positive net order flows and lowers them for negative ones). In equilibrium, the adverse selection effect dominates, and, as a result, traders require more precise information to trade.

Trading Volume, Order Imbalance and Price Variability. As information quality improves, there are two opposing effects. First, there are more traders with high-quality information. For the same threshold, this would increase volume. Second, trading thresholds become more extreme because the dealer has to account for the increased adverse selection problem. The latter effect increases the price impact, and it is thus, effectively, an increase in transaction costs. Ceteris paribus, increased transaction costs would lead to lower volume. In what follows we will provide a sufficient (technical) condition so that the first effect dominates and volume increases.

Volume in our model coincides with the number of transactions. Its expectation is proportional to the probability that a trader who arrives at the market chooses to buy or sell. The expected number of arriving traders is the Poisson parameter ν and traders' decisions are independent conditional on the asset value. The expected number of transactions given this Poisson process can then be obtained by multiplying the exogenous arrival rate with the probability that an arriving trader initiates a transaction in each state (a formal derivation is part of the proof of Proposition 7)

$$\mathsf{E}[\mathsf{Vol}_t] = p_t\nu(\beta_1 + \sigma_1) + (1 - p_t)\nu(\beta_0 + \sigma_0) = \nu(\beta_1 + \sigma_1).$$

The last equality follows from the symmetry of buy- and sell-threshold private beliefs.

For an analytical result, we restrict attention to a class of distributions for which shifts in the quality distribution occur monotonically and not too drastically. Defining

$$\Psi(\pi, \theta) := \int_0^{\pi} \frac{1}{2}[F_1(s|\theta) + F_0(s|\theta)]ds,$$

and denoting partial derivatives by subscripts, we require the distributions to satisfy

$$(\star) \qquad 2\Psi_{\pi,\theta}\Psi + (2\pi - 1)(\Psi_{\pi,\pi}\Psi_\theta - \Psi_{\pi,\theta}\Psi_\pi) < 0.$$

This condition is, for instance, satisfied for the class of quadratic quality distributions that is outlined at the end of Appendix A or for the symmetric Beta distribution.

Proposition 9 (Signal Quality Shifts and Observable Variables)
Assume condition $(\star)$ holds. Then as information quality increases, the expected volume, $\mathsf{E}[\mathsf{Vol}|\theta]$*, the expected order imbalance,* $\mathsf{E}[\mathsf{IB}|\theta]$*, and the expected price variability,* $\mathsf{E}[|\Delta \mathsf{p}_t||\theta]$ *all increase. Duration* $\mathsf{D}(\nu, \gamma_1; \theta)$ *decreases.*

Intuitively, as the threshold quality increases, so does the expected order imbalance — a higher marginal trading quality implies that a smaller fraction of traders take the "wrong" action. Combined with increasing volume, higher imbalances become more likely (as implied by Proposition 3). A similar mechanism applies to price changes and thus price variability. Duration decreases in the transaction probability, $1-\gamma_1 = \beta_1 + \sigma_1$ (expression (13)). The latter is proportional to volume and increases with signal quality.

This proposition is empirically supported by Chordia, Roll, and Subrahmanyam (2008) who argue that "the increase in turnover is associated with greater production of private information".

Our results in Propositions 8 and 9 relate to the theoretical predictions on information quality that have been obtained by varying traders' signal precisions. For instance, Wang (1994) shows in a rational expectations framework that as the precision of information improves, volume increases and the correlation between volume and price variability decreases. In Admati and Pfleiderer (1988), an increase in signal precision is offset by trading activity so that there is no effect on price variability. The price impact (measured by the Kyle-λ) of better quality information is ambiguous.[20]

We contribute to the existing theoretical literature by analyzing how the distributions of economy-wide information and information processing skills affect major observable trading variables. A redistribution of information represents, for instance, a persistent shift in the investorship, which can occur when a stock gets included in a major index. Such an inclusion will typically increase major funds' holdings of the stock and may lead to a relatively higher fraction of well-informed traders. In our framework, a shift in information quality is accompanied by changes in trader participation rates. This feature allows us to derive novel predictions on the impact of signal quality on duration.

C Comparative Statics on Noise Trading

When the fraction of informed traders, μ, increases, then, ceteris paribus, each trade is more likely to be initiated by an informed trader and the adverse selection costs for the dealer increase. She will thus revise prices and informed traders will require a higher quality signal to trade. We then show

Proposition 10 (Comparative Statics for Noise Trading)

As the level of informed trading μ increases
(*a*) *the buy-threshold private belief $\overline{\pi}(\mu)$ increases and the sell-threshold $\underline{\pi}(\mu)$ decreases,*
(*b*) *expected trading volume $\mathsf{E}[\mathsf{Vol}|\mu]$ decreases, and*
(*c*) *duration $\mathsf{D}(\nu, \gamma_1; \mu)$ increases.*

[20]See equation (18) in their paper.

To see why volume declines in the proportion of informed traders, consider the following thought experiment in which a noise trader is substituted with an informed trader. As a noise trader, he would have traded with certainty, whereas as an informed trader, he needs information of sufficient quality to be trading. Moreover, compared to the situation with a lot of noise, he needs better quality information which further reduces the probability of a transaction. These two effects decrease volume and increase duration.

As volume declines, Proposition 3 implies that, ceteris paribus, the expected order imbalance declines. At the same time, each trade is more informative and thus has a higher effect on the price. As the two effects are opposing, the direction of change in price variability is unclear. Numerical simulations reveal that the first effect dominates:[21]

Numerical Observation (Noise and Volatility) *Price variability increases in μ.*

Foster and Viswanathan (1993) show empirically that volume and adverse selection are (weakly) positively correlated, providing support for the thesis that quality improvement shifts (which, by Proposition 9, increase adverse selection) lead to higher volume.

Summary. The above discussion implies that trading volume and volatility are *negatively* related for changes in μ. So while signal quality shifts and increases in informed trading both increase adverse selection costs, they generate opposing implications for the volume-volatility relation. Empirically there is a positive relation between volume and volatility. Thus the implications of shifts in noise trading are at odds with empirical findings, while the implications for information-quality shifts are in line with the data.

VI Conclusion

Combining features of Glosten and Milgrom (1985) and Kyle (1985) in a simple model of a dealer market, we study the dynamic relations among volume, order imbalances and price variability. In financial markets, the number of active traders and the precision of their information are unknown. Incorporating these features into the frameworks that employ normal distributions would render the analysis intractable — in contrast, our setup allows us to study unknown precisions and an unknown number of traders.

We first derive the trading equilibrium and study its properties, in particular, with respect to the dynamic relationships among the major trading variables. We establish that higher volume leads to higher price volatility and higher order imbalances. We then study changes in the information structure to understand the effects of the distribution

[21] Details for the simulations are available upon request from the authors. The distribution class that we employed for the numerical analysis is outlined at the end of Appendix A.

of information. We show that first order stochastic dominance improvements of signal quality lead to higher volume, order imbalances and price variability.

The results from signal quality shifts are consistent with empirical evidence and have appealing economic interpretations. Information quality improves when, for instance, analyst coverage increases, when hedge funds newly invest in a company, when the company adopts a more transparent information release policy, or when regulators require companies to disclose more information. Our results indicate that these measures will lead not only to more efficient prices (because active traders have better information) but also to higher trading volume. Our final comparative static studies changes in the entry rate: as the entry rate increases, so do price variability and volume. Empirically, entry rates are affected, for instance, when a stock is in- or excluded from an index.

In summary, our paper identifies several new channels that affect the volume-volatility relation and thus contributes to our understanding of common empirical findings.

Table 1 summarizes our theoretical findings and empirical predictions. It also either lists existing evidence for the respective result or suggests testing procedures for empirical verification or falsification.

A Appendix: Quality and Belief Distributions

In the main text, we describe trading behavior in terms of the trader's belief π that the asset's value is high. This description is mathematically convenient because we can focus on a scalar variable and not the vector consisting of the signal (h or l) and the signal quality. We now describe how the belief distributions are obtained from the signal quality distributions, and we describe properties of the belief distributions. In the second part of the appendix we discuss the parametrization of signal qualities used in Section V.

A.1 Derivation and Properties of the Belief Distribution

Financial market microstructure models with binary signals and states typically employ a constant, commonly known signal quality $q \in [1/2, 1]$, with $\Pr(S = v|V = v) = q$. Our framework has a continuum of possible qualities with a continuous density function and we will map traders' signals and their qualities into a continuous private belief on $[0, 1]$. The quality parametrization on $[1/2, 1]$ is natural, as a trader who receives a high signal h will update his prior in favor of the high liquidation value, $V = 1$, and a trader who receives a low signal l will update his prior in favor of $V = 0$. We thus use the conventional parametrization on $[1/2, 1]$ in the main text.

However, to characterize the map from traders' signal and qualities into their private beliefs and to derive the distributions of the latter, it is mathematically convenient to normalize the signal quality so that its domain coincides with that of a private belief. We will denote the distribution function of this normalized quality on $[0, 1]$ by G and its density by g, whereas the distribution and density functions of original qualities on $[1/2, 1]$ will be denoted by $\tilde{G}$ and $\tilde{g}$ respectively.

The normalization proceeds as follows. Without loss of generality, we will employ the density function g that is symmetric around 1/2. For $q \in [0, 1/2]$, we then have $g(q) = \tilde{g}(1-q)/2$ and for $q \in [1/2, 1]$, we have $g(q) = \tilde{g}(q)/2$.

Under this specification, signal qualities q and $1-q$ are equally useful for the individual: if someone receives signal h and has quality 1/4, then this signal has "the opposite meaning", i.e. it has the same meaning as receiving signal l with quality 3/4. Signal qualities are assumed to be independent across agents, and independent of the security's liquidation value V.

Beliefs are derived by Bayes Rule, given signals and signal-qualities. Specifically, if a trader is told that his signal quality is q and receives a high signal h then his belief is $q/[q + (1-q)] = q$ (respectively, $1-q$ if he receives a low signal l), because the prior is 1/2. The belief π is thus held by people who receive signal h and quality $q = \pi$ and by those who receive signal l and quality $q = 1-\pi$. Consequently, the density of individuals with belief π is given by $f_1(\pi) = \pi[g(\pi) + g(1-\pi)]$ in state $V = 1$ and analogously by $f_0(\pi) = (1-\pi)[g(\pi) + g(1-\pi)]$ in state $V = 0$. Smith and Sorensen (2008) prove the following property of private beliefs (their Lemma 2):

Lemma 1 (Symmetric beliefs, Smith and Sorensen (2008))
With the above the signal quality structure, private belief distributions satisfy $F_1(\pi) = 1 - F_0(1-\pi)$ *for all* $\pi \in (0, 1)$.

Proof: Since $f_1(\pi) = \pi[g(\pi) + g(1-\pi)]$ and $f_0(\pi) = (1-\pi)[g(\pi) + g(1-\pi)]$, we have $f_1(\pi) = f_0(1-\pi)$. Integrating, $F_1(\pi) = \int_0^\pi f_1(x)dx = \int_0^\pi f_0(1-x)dx = \int_{1-\pi}^1 f_0(x)dx = 1 - F_0(1-\pi)$. □

A direct implication of this lemma is that with symmetric thresholds, $\overline{\pi} = 1 - \underline{\pi}$, a buy in state $V = 1$ is as likely as a sale in state $V = 0$, because

$$\beta_1 = (1-\mu)/2 + \mu(1 - F_1(\overline{\pi})) = (1-\mu)/2 + \mu F_0(1-\overline{\pi}) = (1-\mu)/2 + \mu F_0(\underline{\pi}) = \sigma_0.$$

Similarly, $\beta_0 = \sigma_1$. The belief densities satisfy the monotone likelihood ratio property as

$$\frac{f_1(\pi)}{f_0(\pi)} = \frac{\pi[g(\pi) + g(1-\pi)]}{(1-\pi)[g(\pi) + g(1-\pi)]} = \frac{\pi}{1-\pi}$$

is increasing in π.

One can recover the distribution of qualities on $[1/2, 1]$, denoted by $\tilde{G}$, from G by combining qualities that yield the same beliefs for opposing signals (e.g $q = 1/4$ and signal h is combined with $q = 3/4$ and signal l). With symmetric g, $G(1/2) = 1/2$, and

$$\text{(A-1)} \quad \tilde{G}(q) = \int_{\frac{1}{2}}^{q} g(s)ds + \int_{1-q}^{\frac{1}{2}} g(s)ds = 2\int_{\frac{1}{2}}^{q} g(s)ds = 2G(q) - 2G(1/2) = 2G(q) - 1.$$

A.2 Stochastic Dominance of Belief Distributions.

Section V.B focusses on families of signal quality distributions that are described by a parameter $\theta \in \Theta \subseteq \mathbb{R}$ and that obey first order stochastic dominance with respect to θ. First order stochastic dominance refers to the distribution of qualities $\tilde{G}$ on $[1/2, 1]$. We will now argue that when $\tilde{G}(\cdot|\theta_h)$ *first* order stochastically dominates $\tilde{G}(\cdot|\theta_l)$, then $G(\cdot|\theta_l)$ *second* order stochastically dominates $G(\cdot|\theta_h)$.

Loosely speaking, a first order stochastic dominance shift in $\tilde{G}$ increases the number of high (close to 1) quality signals and decreases the number of low (close to 1/2) quality signals. Symmetry of the quality density function g on $[0, 1]$ then implies that under $G(\cdot|\theta_h)$ there are relatively more signals with qualities close to 0 and 1 and relatively fewer signals with qualities around 1/2. Since g is symmetric around 1/2, the mean quality on $[0, 1]$ is constant and equals 1/2. The above discussion implies, intuitively, that $G(\cdot|\theta_h)$ is a mean preserving spread of $G(\cdot|\theta_l)$.

Formally, suppose that $\tilde{G}(q|\theta_h)$ FOSD $\tilde{G}(q|\theta_l)$ for $q \geq 1/2$, i.e. for $q \in [1/2, 1]$, $\tilde{G}(q|\theta_h) \leq \tilde{G}(q|\theta_l)$. By symmetry of densities, for $q < 1/2$, $G(q|\theta_h) \geq G(q|\theta_l)$ so that

$$\text{(A-2)} \qquad \int_0^x G(s|\theta_h)ds - \int_0^x G(s|\theta_l)ds \geq 0 \quad \forall x \leq 1.$$

In other words, if $\tilde{G}$ FOSD $\tilde{G}'$, then G' SOSD G.

An example for a parametric class of distributions on $[0, 1]$ that obeys second order stochastic dominance is the following class of quadratic quality distributions with density

$$\text{(A-3)} \qquad g(q|\theta) = \theta\left(q - \frac{1}{2}\right)^2 - \frac{\theta}{12} + 1, \quad q \in [0, 1] \text{ and } \theta \in [-6, 12].$$

This class includes the uniform density (for $\theta = 0$); the distribution of qualities G and $\tilde{G}$

on $[1/2, 1]$ can be obtained by integration as outlined in (A-1):

$$\tilde{G}(q|\theta) = 4q + 4\theta q(q-1)(q-1/2)/3 - 1, \quad \tilde{G}_\theta(q|\theta) = 2q(q-1)(q-1/2)/3 < 0 \text{ for } q > 1/2.$$

This class of distributions was used to establish the Numerical Observation in Section V.

B Appendix: Omitted Proofs

B.1 Expectations of Dealer and Informed Traders.

Derivation of Equation (3) in the Dealer's Pricing Rule. Suppose that the dealer observes b buy orders and s sell orders and that people follow thresholds rules so that each trader with belief $\pi \geq \overline{\pi}$ buys and each trader with belief $\pi \leq \underline{\pi}$ sells. The time-t price satisfies $\mathsf{p}(b, s) = \mathsf{E}[V|b, s, H_t]$, which, by equation (2), depends on $\mathsf{Pr}(b, s|H_t, V)$.

No trades are intrinsically unobservable. When seeing b buys and s sales, the dealer knows that the total number of traders, N, is at least $b+s$ — but N may be larger because informed traders with $\pi \in (\underline{\pi}, \overline{\pi})$ choose not to trade. We first determine $\mathsf{Pr}(b, s|H_t, V, N)$ and then sum over all the possible outcomes $N = b + s + h$, where $h = 0, \ldots, \infty$ denotes the number of no trades and N is the number of potential traders. Next, N traders arrive with Poisson probability $\nu^N e^{-\nu}/N!$. Conditional on the asset's true value, and these N people select into buyers, sellers, and those people who do not trade according to a multinomial distribution. For fixed b and s we have

$$\text{(B-1)} \qquad \mathsf{Pr}(b, s|V = v) = \sum_{N=b+s}^{\infty} \mathsf{Pr}(b, s|\ V = v, N)\mathsf{Pr}(N)$$

$$= \sum_{N=b+s}^{\infty} \frac{N!}{b!s!(N-b-s)!}\ {\beta_v}^b {\sigma_v}^s {\gamma_v}^{N-b-s}\ \frac{\nu^N e^{-\nu}}{N!} = \frac{e^{-\nu(1-\gamma_v)}\nu^{b+s}}{b!s!} {\beta_v}^b {\sigma_v}^s.$$

Informed Trader's Expectation. Consider a trader who intends to buy and suppose that the informed trader observes $b-1$ buy orders and s sell orders by others. Then

$$\mathsf{E}[V|\pi, b, s, H_t] \quad = \quad \frac{\pi \mathsf{Pr}(b, s|\pi, H_t, V = 1)p_t}{\pi \mathsf{Pr}(b, s|\pi, H_t, V = 1)p_t + (1-\pi)\mathsf{Pr}(b, s|\pi, H_t, V = 0)(1-p_t)}.$$

As for the dealer, no trades are unobservable. We thus derive the informed trader's expectation for a known number of traders, and then sum over all the possible outcomes $b + s + h \geq 1$, $h = 0, \ldots, \infty$; accounting for the fact that the informed trader knows that there is at least 1 trader. Omitting H_t, for a trader who buys

$$\begin{aligned}
\mathsf{Pr}(b,s|\overline{\pi},V=v) &= \sum_{\mathsf{Vol}=b+s}^{\infty} \mathsf{Pr}(b,s|\overline{\pi},V=1,N=\mathsf{Vol})\mathsf{Pr}(N=\mathsf{Vol}|N\geq 1) \\
&= {\beta_v}^{b-1}{\sigma_v}^{s}\sum_{h=0}^{\infty}\frac{e^{-\nu}}{1-e^{-\nu}}\frac{\nu^{h+b+s}}{(h+b+s)!}\,{\gamma_v}^{h}\,\frac{(h+b+s-1)!}{h!(b-1)!s!} \\
&= \frac{e^{-\nu}}{1-e^{-\nu}}\frac{{\beta_v}^{b-1}{\sigma_v}^{s}\nu^{b-1+s}}{(b-1)!s!}\sum_{h=0}^{\infty}\frac{(\nu\gamma_v)^h}{(h+b+s)h!}.
\end{aligned}$$

B.2 Proof of the Existence and Uniqueness Theorem

The existence theorem states that an equilibrium exists, that the thresholds in this equilibrium are symmetric, $\overline{\pi}=1-\underline{\pi}$, and that it is unique. We first show that in equilibrium the thresholds must be history invariant and symmetric. We then show existence and uniqueness of the indifference thresholds. In the final step, we verify that the posited equilibrium behavior is incentive compatible, i.e. that for traders with beliefs higher than the buying thresholds it is optimal to buy, that for traders with beliefs below the selling threshold it is best to sell, and that all others optimally abstain.

Step 1: History Invariance and Threshold Symmetry. *Assume an equilibrium exists at each time t. Then the indifference thresholds are independent of the trading history and symmetric, $\overline{\pi}=1-\underline{\pi}$ at all dates t.*

Proof: History invariance follows from equation (10).

Inserting the probabilities that we derived at the beginning of this appendix into the second expression in (9), we have that $\ell^I(b,s;\overline{\pi})=\ell^M(b,s)$ is equivalent to

$$\text{(B-2)}\qquad \frac{\beta_1^{b-1}\sigma_1^s\overline{\pi}\sum_{h=0}^{\infty}\frac{(\nu\gamma_1)^h}{(h+b+s)h!}}{\beta_0^{b-1}\sigma_0^s(1-\overline{\pi})\sum_{h=0}^{\infty}\frac{(\nu\gamma_0)^h}{(h+b+s)h!}} = \frac{\beta_1^b\sigma_1^s}{\beta_0^b\sigma_0^s} \quad\Leftrightarrow\quad \frac{\sum_{h=0}^{\infty}\frac{(\nu\gamma_1)^h}{(h+b+s)h!}}{\sum_{h=0}^{\infty}\frac{(\nu\gamma_0)^h}{(h+b+s)h!}} = \frac{\beta_1}{\beta_0}\frac{1-\overline{\pi}}{\overline{\pi}}$$

Since $b+s\geq 1$, each sum in the above expressions converges and the ratios are well-defined. Suppose that $\overline{\pi}\neq 1-\underline{\pi}$. This implies, in particular, that $\gamma_0\neq\gamma_1$. Since $\overline{\pi}$ must solve (B-2) for all (b,s), we must have, in particular,

$$\frac{\sum_{h=0}^{\infty}\frac{(\nu\gamma_1)^h}{(h+1)h!}}{\sum_{h=0}^{\infty}\frac{(\nu\gamma_0)^h}{(h+1)h!}} = \frac{\sum_{h=0}^{\infty}\frac{(\nu\gamma_1)^h}{(h+2)h!}}{\sum_{h=0}^{\infty}\frac{(\nu\gamma_0)^h}{(h+2)h!}} = \frac{\beta_1}{\beta_0}\frac{1-\overline{\pi}}{\overline{\pi}}$$

Using

$$\sum_{h=0}^{\infty}\frac{(\nu\gamma)^h}{(h+1)h!}=\frac{e^{\gamma}-1}{\gamma}\ \text{ and }\ \sum_{h=0}^{\infty}\frac{(\nu\gamma)^h}{(h+2)h!}=\frac{1+e^{\gamma}\gamma-e^{\gamma}}{\gamma^2},$$

$\overline{\pi} \neq 1 - \underline{\pi}$ that solves (B-2) exists only if there exist $\gamma_0 \neq \gamma_1$ on $(0,1)$ such that

$$\varphi(\gamma_1) \equiv \frac{\gamma_1(e^{\gamma_1}-1)}{1+e^{\gamma_1}\gamma_1 - e^{\gamma_1}} = \frac{\gamma_0(e^{\gamma_0}-1)}{1+e^{\gamma_0}\gamma_0 - e^{\gamma_0}} \equiv \varphi(\gamma_0).$$

Function $\varphi(\gamma)$ decreases on $(0,1)$, thus such $\gamma_0 \neq \gamma_1$ do not exist.

Step 2: Existence and Uniqueness. In an equilibrium, any trade is informative, that is indifference thresholds are in (0,1). The argument proceeds by contradiction. Suppose, for instance, that $\underline{\pi} = 0$ and $\overline{\pi} = 1$, that is, no informed trader buys or sells. All trades are then uninformative, and the dealer would set the price to 1/2. This implies that the best response of any informed trader with a private belief below 1/2 is to post a sell-order, and the best response of any informed trader with belief above 1/2 is to post a buy-order, a contradiction. Cases $\overline{\pi} = \underline{\pi} = 0$ and $\overline{\pi} = \underline{\pi} = 1$ are analogous.

Since thresholds are symmetric in equilibrium, we can focus on $\overline{\pi}$. Trade informativeness together with equation (11) on $\overline{\pi}$ imply that $\overline{\pi}$ must be at least 1/2. What remains to show is existence and uniqueness of $\overline{\pi} \in [1/2, 1)$.

Step $2(i)$**:** *We will now show that the first expression in (11) on* $\overline{\pi}$ *can be rewritten as*

$$\text{(B-3)} \qquad \frac{2\overline{\pi}-1}{4}\,\frac{\mu+1}{\mu} - \int_0^{\overline{\pi}} G(s)\ ds = 0,$$

where $G(q)$ *is the cumulative distribution function of the quality distribution.*

In the main text we have shown that if thresholds are symmetric, i.e. if $\overline{\pi} = 1 - \underline{\pi}$, then $\gamma_1 = \gamma_0$ and $\overline{\pi}$ solves $\overline{\pi} = \beta_1/(\beta_1+\beta_0)$. Integrating density $f_1(\pi)$, derived in Appendix A, and using the symmetry of g around 1/2, we have

$$F_1(\pi) = 2\pi G(\pi) - 2\int_0^{\pi} G(s)\ ds \ \Rightarrow \beta_1 = (1-\mu)/2 + \mu\left(1 - 2\overline{\pi}G(\overline{\pi}) + 2\int_0^{\overline{\pi}} G(s)\ ds\right).$$

Expressing $F_0(\pi)$ analogously, $F_1(\pi) + F_0(\pi) = 2G(\pi)$ so that $\beta_1 + \beta_0 = 1 + \mu - 2\mu G(\pi)$. Then $\overline{\pi} = \beta_1/(\beta_1+\beta_0)$ can be rewritten as (B-3).

Step $2(ii)$**:** *We establish that (B-3) has a unique solution* $\overline{\pi} \in (1/2, 1)$.

The left hand side of (B-3) is continuous and strictly increasing in $\overline{\pi}$: its slope is $(1+\mu)/2\mu - G(\overline{\pi}) > 1 - G(\overline{\pi}) > 0$ for all $\mu \in (0,1)$. At $\overline{\pi} = 1/2$, the left hand side of (B-3) is $-\int_0^{1/2} G(s)ds < 0$. At $\overline{\pi} = 1$, it is $(\mu+1)/4\mu - 1/2 > 0$ for all $\mu \in (0,1)$.[22] Consequently, there exists a unique root $\overline{\pi} \in (1/2, 1)$.

Step 3: Monotonic decision rules in equilibrium. We will now argue that the equilibrium behavior depicted above is indeed incentive compatible. Namely, if an

[22]Symmetry of g around 1/2 yields $\int_0^1 G(s)ds = sG(s)ds\big|_0^1 - \int_0^1 sg(s)ds = 1 - 1/2 = 1/2$.

informed trader's has a private belief above the buying threshold, then buying is the optimal action; similarly for selling. Thus for the equilibrium thresholds $\overline{\pi} > \underline{\pi}$ any trader with a private belief $\pi \geq \overline{\pi}$ prefers to post a buy order, any trader with a private belief $\pi \leq \underline{\pi}$ prefers to post a sell order, and any trader with a private belief in $(\underline{\pi}, \overline{\pi})$ chooses to refrain from trading. Specifically, we show the following.

Let $\underline{\pi}, \overline{\pi}$ be the indifference thresholds that satisfy the conditions in (11). Any trader with belief $\pi > \overline{\pi}$ makes a positive profit from buying, any trader with belief $\pi < \overline{\pi}$ makes a loss from buying; any trader with belief $\pi < \underline{\pi}$ makes a positive profit from selling, any trader with belief $\pi > \underline{\pi}$ makes a loss from buying.

Proof: Take $\pi_i > \overline{\pi}$. We shown before (for $\pi_i \equiv \overline{\pi}$) that

$$\mathsf{E}[V|b,s,\pi_i] = \frac{\ell^I(b,s;\pi_i)p}{\ell^I(b,s;\pi_i)p + (1-p)} \quad \text{with} \quad \ell^I(b,s;\pi_i) = \frac{\pi_i}{1-\pi_i} \frac{\beta_1^{b-1}\sigma_1^s \sum_{h=0}^{\infty} \frac{(\nu\gamma_1)^h}{(h+b+s)h!}}{\beta_0^{b-1}\sigma_0^s \sum_{h=0}^{\infty} \frac{(\nu\gamma_0)^h}{(h+b+s)h!}}.$$

Since $\ell^I(b,s;\pi_i)$ increases in π_i and since $\mathsf{E}[V|b,s,\pi_i]$ increases in $\ell^I(b,s;\pi_i)$ we have that the expectation $\mathsf{E}[V|b,s,\pi_i]$ increases in π_i. Since the equilibrium threshold satisfies $\mathsf{E}[V|b,s,\overline{\pi}] = \mathsf{p}(b,s)$ for all b,s, it follows that $\mathsf{E}[V|b,s,\pi_i] > \mathsf{p}(b,s)$ for all b,s. Thus $\mathsf{E}[\mathsf{E}[V|b,s,\pi_i] - \mathsf{p}(b,s)\big|\pi_i] > 0$ for all $\pi_i > \overline{\pi}$. Since $\overline{\pi}$ is common knowledge in equilibrium, any trader with belief $\pi_i > \overline{\pi}$ strictly prefers to buy; analogously any trader with $\pi_i < \overline{\pi}$ has $\mathsf{E}[V|b,s,\pi_i] < \mathsf{p}(b,s)$ for any (b,s). Likewise for the selling case.

B.3 Proof of Proposition 2

As the expectations for b_t and s_t both exist, we write $\mathsf{E}[\mathsf{NOF}_t|H_t] = \mathsf{E}[b_t - s_t|H_t] = \mathsf{E}[b_t|H_t] - \mathsf{E}[s_t|H_t]$. Using equation (3),

$$\begin{aligned}
\mathsf{E}[b_t|H_t] &= \sum_{b=0}^{\infty} b \cdot \sum_{s=0}^{\infty} \mathsf{Pr}(b_t = b, s_t = s|H_t) \\
&= \sum_{b=0}^{\infty} b \cdot \sum_{s=0}^{\infty} \left(\frac{e^{-\nu(\beta_1+\sigma_1)}(\nu\beta_1)^b(\nu\sigma_1)^s}{b!s!} \, \mathsf{p}_{t-1} + \frac{e^{-\nu(\beta_0+\sigma_0)}(\nu\beta_0)^b(\nu\sigma_0)^s}{b!s!} \, (1-\mathsf{p}_{t-1}) \right) \\
&= \nu\beta_1\mathsf{p}_{t-1} + \nu\beta_0(1-\mathsf{p}_{t-1}).
\end{aligned}$$

Likewise, $\mathsf{E}[s_t|H_t] = \nu\sigma_1\mathsf{p}_{t-1} + \nu\sigma_0(1-\mathsf{p}_{t-1})$. With symmetric thresholds, $\sigma_1 = \beta_0$ and $\sigma_0 = \beta_1$, which yield the result.

B.4 Proof of Proposition 3

We first compute the expected absolute value of the order imbalance $\mathsf{IB}_t = |b_t - s_t|$ conditional on the realized number of transactions $b + s = \mathsf{Vol}$. We write more compactly $\mathsf{E}[\mathsf{IB}_t|b_t + s_t = \mathsf{Vol}_t, H_t] =: \mathsf{E}_{\mathsf{Vol}}[\mathsf{IB}_t]$ and compute

$$\mathsf{E}_{\mathsf{Vol}}[\mathsf{IB}_t] = \sum_{b=0}^{\mathsf{Vol}} |2b - \mathsf{Vol}| \mathsf{Pr}(b_t = b, s_t = \mathsf{Vol} - b|H_t).$$

Traders' decisions are independent, conditional on the asset's true liquidation value, and thresholds are time-invariant and symmetric. Using Bayes' Rule, the above is

$$\text{(B-4)} \quad \mathsf{E}_{\mathsf{Vol}}[\mathsf{IB}_t] = \sum_{b=0}^{\mathsf{Vol}} |(2b - \mathsf{Vol})| \left(p_{t-1}\rho^b(1-\rho)^{\mathsf{Vol}-b} + (1-p_{t-1})\rho^{\mathsf{Vol}-b}(1-\rho)^b\right) \binom{\mathsf{Vol}}{b},$$

where $\rho = \beta_1/(\beta_0 + \beta_1)$. Rearranging, we obtain for $\mathsf{Vol} = 2k$ and $\mathsf{Vol} = 2k + 1$

$$\text{(B-5)} \quad \mathsf{E}_{2k}[\mathsf{IB}_t] = \sum_{b=0}^{k}(2k - 2b)\left(\rho^b(1-\rho)^{2k-b} + \rho^{2k-b}(1-\rho)^b\right)\binom{2k}{b},$$

$$\text{(B-6)} \quad \mathsf{E}_{2k+1}[\mathsf{IB}_t] = \sum_{b=0}^{k}(2k + 1 - 2b)\left(\rho^b(1-\rho)^{2k+1-b} + \rho^{2k+1-b}(1-\rho)^b\right)\binom{2k+1}{b}.$$

Proof of (a)**:** The probability of there being Vol trades at time t is given by

$$\begin{aligned}
&\mathsf{Pr}(b_t + s_t = \mathsf{Vol}|V = v, H_t) \\
= &\sum_{N=\mathsf{Vol}}^{\infty} \mathsf{Pr}(N \text{ traders}) \sum_{b=0}^{\mathsf{Vol}} \mathsf{Pr}(N - \mathsf{Vol} \text{ no trades}, b \text{ buys}, \mathsf{Vol} - b \text{ sales}|V = v, H_t) \\
= &\sum_{N=\mathsf{Vol}}^{\infty} \frac{\nu^N e^{-\nu}}{N!} \sum_{b=0}^{\mathsf{Vol}} \frac{N!\ {\gamma_v}^{N-\mathsf{Vol}}{\beta_v}^b{\sigma_v}^{\mathsf{Vol}-b}}{(N - \mathsf{Vol})!b!(\mathsf{Vol} - b)!} = \frac{\nu^{\mathsf{Vol}}(\sigma_v + \beta_v)^{\mathsf{Vol}}}{\mathsf{Vol}!}\ e^{-\nu(1-\gamma_v)}
\end{aligned}$$

With symmetric thresholds the unconditional probability is given by

$$\text{(B-7)} \quad \mathsf{Pr}(b_t + s_t = \mathsf{Vol}|H_t) = \frac{\nu^{\mathsf{Vol}}(\beta_0 + \beta_1)^{\mathsf{Vol}}}{\mathsf{Vol}!}\ e^{-\nu(\beta_0+\beta_1)}.$$

It is independent of the trading history, hence, using (B-5) and (B-6), so is $\mathsf{E}[\mathsf{IB}_t|H_t]$.
Proof of (b)**:** We need to show that $\mathsf{E}[|b - s||N = 2k + 1] - \mathsf{E}[|b - s||N = 2k] > 0$. For

$\mathsf{Vol} \geq b \geq 1$ we use $\binom{2k+1}{b} = \binom{2k}{b} + \binom{2k}{b-1}$ and $\binom{2k+1}{0} = \binom{2k}{0} = 1$ to rewrite (B-6) as

$$\begin{aligned}
\mathsf{E}_{2k+1}[\mathsf{IB}_t] &= \sum_{b=0}^{k}(2k+1-2b)\left(\rho^b(1-\rho)^{2k+1-b} + \rho^{2k+1-b}(1-\rho)^b\right)\binom{2k}{b} \\
&\quad + \sum_{b=0}^{k-1}(2k-2b)\left(\rho^{b+1}(1-\rho)^{2k-b} + \rho^{2k-b}(1-\rho)^{b+1}\right)\binom{2k}{b} \\
&\quad - \sum_{b=0}^{k-1}\left(\rho^{b+1}(1-\rho)^{2k-b} + \rho^{2k-b}(1-\rho)^{b+1}\right)\binom{2k}{b}
\end{aligned}$$

Recognizing that the second sum in the above expression can be rewritten as a sum from 0 to k, and rearranging further, we obtain

$$\begin{aligned}
\mathsf{E}_{2k+1}[\mathsf{IB}_t] - \mathsf{E}_{2k}[\mathsf{IB}_t] &= (\rho^k(1-\rho)^{k+1} + \rho^{k+1}(1-\rho)^k)\binom{2k}{k} \\
&\quad + (2\rho-1)\sum_{b=0}^{k-1}\rho^b(1-\rho)^{2k-b}\left(\left(\frac{\rho}{1-\rho}\right)^{2k-2b} - 1\right)\binom{2k}{b}.
\end{aligned}$$

Since $1/2 < \rho < 1$, this last expression is positive.

B.5 Proof of Proposition 4

We divide the expectation into the sum of two terms, according to the sign of the order imbalance. To simplify notation, we will assume $\nu = 1$, the proof does not rely on this assumption. We will first compute

$$\mathsf{E}^+[|\Delta\mathsf{p}_t||\mathsf{IB}_t] \equiv \mathsf{E}[|\Delta\mathsf{p}_t|||b_t - s_t| = \mathsf{IB}_t, b_t - s_t > 0, H_t]\ \mathsf{Pr}(b_t - s_t > 0||b_t - s_t| = \mathsf{IB}_t, H_t).$$

Omitting subscripts on IB, the probability $\mathsf{Pr}(|b_t - s_t| = \mathsf{IB}|H_t)$ is independent of the history and is given by

$$\mathsf{Pr}(|b_t - s_t| = \mathsf{IB}|H_t) = \sum_{s=0}^{\infty}(\beta_1^s\beta_0^{s+\mathsf{IB}} + \beta_1^{s+\mathsf{IB}}\beta_0^s)\frac{1}{s!(s+\mathsf{IB})!} = \mathcal{B}(\mathsf{IB}, 2\sqrt{\beta_1\beta_0})\frac{\beta_1^{\mathsf{IB}} + \beta_0^{\mathsf{IB}}}{(\sqrt{\beta_1\beta_0})^{\mathsf{IB}}},$$

where $\mathcal{B}(\cdot,\cdot)$ is the Bessel function of the first order. When $b_t > s_t$, the absolute value of the price change is $\mathsf{p}_t - \mathsf{p}_{t-1}$. Using the transaction price law of motion, after some algebraic simplification, we obtain

$$\mathsf{E}^+[|\Delta\mathsf{p}_t||\mathsf{IB}] = \frac{1}{\mathsf{Pr}(|b_t-s_t|=\mathsf{IB}|H_t)}\sum_{s=0}^{\infty}(p_t\beta_1^{s+\mathsf{IB}}\beta_0^s - p_t(p_t\beta_1^{s+\mathsf{IB}}\beta_0^s+(1-p_t)\beta_1^s\beta_0^{s+\mathsf{IB}}))\frac{1}{s!(s+\mathsf{IB})!}$$

A similar expression can be derived for $b_t < s_t$. Since $\mathsf{p}_t - \mathsf{p}_{t-1} = 0$ when $b_t = s_t$, the expected absolute value of the price change is given by $\mathsf{E}^+[|\Delta\mathsf{p}_t||\mathsf{IB}] + \mathsf{E}^-[|\Delta\mathsf{p}_t||\mathsf{IB}]$ and can be simplified to

$$\begin{aligned}\mathsf{E}^+[|\Delta\mathsf{p}_t||\mathsf{IB}] + \mathsf{E}^-[|\Delta\mathsf{p}_t||\mathsf{IB}] &= \frac{2p_t(1-p_t)\sum_{s=0}^{\infty}(\beta_1^{s+\mathsf{IB}}\beta_0^s-\beta_1^s\beta_0^{s+\mathsf{IB}})\frac{1}{s!(s+\mathsf{IB})!}}{\mathsf{Pr}(|b_t-s_t|=\mathsf{IB}|H_t)}\\ &= 2p_t(1-p_t)\,\frac{\beta_1^{\mathsf{IB}}-\beta_0^{\mathsf{IB}}}{\beta_1^{\mathsf{IB}}+\beta_0^{\mathsf{IB}}} = 2p_t(1-p_t)\,\frac{(\beta_1/\beta_0)^{\mathsf{IB}}-1}{\beta_1/\beta_0)^{\mathsf{IB}}+1}.\end{aligned}$$

Since $\beta_1/\beta_0 > 1$, the above expression increases in IB.

B.6 Proof of Proposition 5

We first compute the expected absolute value of the price change conditional on the realized number of transactions $b + s = \mathsf{Vol}$; we use $\Delta\mathsf{p}_t(\cdot)$ to express how the price change depends on order imbalance IB.

$$\mathsf{E}_{\mathsf{Vol}}[|\Delta\mathsf{p}_t|] = \sum_{b=0}^{\mathsf{Vol}}|\Delta\mathsf{p}_t(2b-\mathsf{Vol})|\mathsf{Pr}(b_t=b, s_t=\mathsf{Vol}-b|H_t).$$

Traders' decisions are independent, conditional on the asset's true liquidation value, and thresholds are time-invariant and symmetric. Using Bayes' Rule,

$$\mathsf{E}_{\mathsf{Vol}}[|\Delta\mathsf{p}_t|] = \sum_{b=0}^{\mathsf{Vol}}|\Delta\mathsf{p}_t(2b-\mathsf{Vol})|\left(p_{t-1}\rho^b(1-\rho)^{\mathsf{Vol}-b}+(1-p_{t-1})\rho^{\mathsf{Vol}-b}(1-\rho)^b\right)\binom{\mathsf{Vol}}{b}.$$

Using $\mathsf{Vol} = 2k$ and $\mathsf{Vol} = 2k+1$, we need to show that $\mathsf{E}_{2k+1}[|\Delta\mathsf{p}_t|] - \mathsf{E}_{2k}[|\Delta\mathsf{p}_t|] > 0$. This is true for $k = 0$, and so in what follows we assume $k \geq 1$. Rewrite $\mathsf{E}_{2k+1}[|\Delta\mathsf{p}_t|]$ as

$$\begin{aligned}&\sum_{b=0}^{k}|\Delta\mathsf{p}_t(2b-2k-1)|\left(p_t\rho^b(1-\rho)^{2k+1-b}+(1-p_t)\rho^{2k+1-b}(1-\rho)^b\right)\binom{2k+1}{b}\\ &+\sum_{b=k+1}^{2k+1}|\Delta\mathsf{p}_t(2b-2k-1)|\left(p_t\rho^b(1-\rho)^{2k+1-b}+(1-p_t)\rho^{2k+1-b}(1-\rho)^b\right)\binom{2k+1}{b}.\end{aligned}$$

The second sum can be expressed as a sum where s runs from 0 to k, with $s = 2k + 1 - b$. Since b and s are the numbers of buys and sales respectively, $\sum_{s=0}^{k} = \sum_{b_t > s_t}$ and $\mathsf{E}_{2k+1}[|\Delta \mathsf{p}_t|]$ can be written as

$$\mathsf{E}_{2k+1}[|\Delta \mathsf{p}_t|] = \sum_{b_t < s_t} (\cdot) + \sum_{b_t > s_t} (\cdot) \equiv \mathsf{E}^-_{2k+1}[|\Delta \mathsf{p}_t|] + \mathsf{E}^+_{2k+1}[|\Delta \mathsf{p}_t|],$$

where $\mathsf{E}^-_{2k+1}[|\Delta \mathsf{p}_t|]$ denotes the sum over negative net order flows, $b_t < s_t$, and $\mathsf{E}^+_{2k+1}[|\Delta \mathsf{p}_t|]$ denotes the sum over positive net order flows, $b_t > s_t$.

Likewise, using $|\Delta \mathsf{p}_t(0)| = 0$, we can write $\mathsf{E}_{2k}[|\Delta \mathsf{p}_t|] = \mathsf{E}^-_{2k}[|\Delta \mathsf{p}_t|] + \mathsf{E}^+_{2k}[|\Delta \mathsf{p}_t|]$. We then regroup the sums according to the sign of the net order flow and show that

$$\begin{aligned}
\mathsf{E}^+_{2k+1}[|\Delta \mathsf{p}_t|] - \mathsf{E}^+_{2k}[|\Delta \mathsf{p}_t|] &= |\Delta \mathsf{p}_t(1)|(p_{t-1}\rho(1-\rho)^k + (1-p_{t-1})\rho^k(1-\rho))\binom{2k}{k} \\
\mathsf{E}^-_{2k+1}[|\Delta \mathsf{p}_t|] - \mathsf{E}^-_{2k}[|\Delta \mathsf{p}_t|] &= |\Delta \mathsf{p}_t(-1)|(p_{t-1}\rho^k(1-\rho) + (1-p_{t-1})\rho(1-\rho)^k)\binom{2k}{k}
\end{aligned}$$

We start with the sums over the negative net order flow. Using $\binom{2k+1}{b} = \binom{2k}{b} + \binom{2k}{b-1}$, and $\binom{2k+1}{0} = \binom{2k}{0} = 1$, we rewrite $\mathsf{E}^-_{2k+1}[|\Delta \mathsf{p}_t|]$ as

$$\begin{aligned}
&\sum_{b=0}^{k} |\Delta \mathsf{p}_t(2b-2k-1)| \left(p_{t-1}\rho^b(1-\rho)^{2k+1-b} + (1-p_{t-1})\rho^{2k+1-b}(1-\rho)^b\right)\binom{2k}{b} \\
&+\sum_{b=0}^{k-1} |\Delta \mathsf{p}_t(2b-2k+1)| \left(p_{t-1}\rho^{b+1}(1-\rho)^{2k-b} + (1-p_{t-1})\rho^{2k-b}(1-\rho)^{b+1}\right)\binom{2k}{b}
\end{aligned}$$

An analogous expression obtains for $\mathsf{E}^-_{2k}[|\Delta \mathsf{p}_t|]$. Explicitly writing out $|\Delta \mathsf{p}_t|$ using equation (12) and noting that $\rho/(1-\rho) = \beta_1/\beta_0$, we obtain for all $0 \leq b \leq k-1$:

$$\begin{aligned}
&|\Delta \mathsf{p}_t(2b-2k+1)| - |\Delta \mathsf{p}_t(2b-2k)|) \left(p_{t-1}\rho^{b+1}(1-\rho)^{2k-b} + (1-p_{t-1})\rho^{2k-b}(1-\rho)^{b+1}\right) \\
&= (|\Delta \mathsf{p}_t(2b-2k)| - |\Delta \mathsf{p}_t(2b-2k-1)|) \left(p_{t-1}\rho^{b}(1-\rho)^{2k+1-b} + (1-p_{t-1})\rho^{2k+1-b}(1-\rho)^{b}\right)
\end{aligned}$$

Hence, subtracting the sums, $\mathsf{E}^-_{2k+1}[|\Delta \mathsf{p}_t|] - \mathsf{E}^-_{2k}[|\Delta \mathsf{p}_t|]$, all terms for $b < k$ cancel and the only remaining term is the one for $b = k$ of $\mathsf{E}^+_{2k+1}[|\Delta \mathsf{p}_t|]$. This term is positive as all the arguments are non-negative and some are strictly positive. The sums over the positive net order flows, $\mathsf{E}^+_{2k}[|\Delta \mathsf{p}_t|]$ and $\mathsf{E}^+_{2k+1}[|\Delta \mathsf{p}_t|]$ can be rewritten analogously.

B.7 Proof of Proposition 7

The probability of there being Vol trades at time t is independent of the history and is given by equation (B-7). The expected trading volume is thus $\mathsf{E}[b_t + s_t|H_t] = \nu(\beta_0 + \beta_1)$. Further, the family of distributions of $b_t + s_t|H_t$ increases in ν in the sense of the first order stochastic dominance. To see this, observe that

$$\frac{d}{dx}\sum_{\mathsf{Vol}=0}^{k}\frac{x^{\mathsf{Vol}}}{\mathsf{Vol}!}\,e^{-x} = -\frac{x^k e^{-x}}{k!} < 0,$$

where the equality follows by exchanging the order of summation and differentiation. Propositions 3 and 5 imply that the conditional expectations of the absolute values of the order imbalance and price changes increase in realized volume. An increase in ν leads to a first order stochastic dominance shift in the distribution of volume, which in turns implies an increase in $\mathsf{E}[|b_t - s_t||H_t]$ and $\mathsf{E}[|\Delta \mathsf{p}_t||H_t]$. Duration $\mathsf{D}(\nu, \gamma_1)$ decreases because $e^{-\nu(1-\gamma_1)} = x$ decreases in ν and $\frac{\partial}{\partial x}\frac{x}{(1-x)^2} > 0$. □

B.8 Proof of Proposition 8

(a) From the proof of the existence theorem we know that $\pi = \overline{\pi}$ solves

$$\text{(B-8)} \qquad \frac{2\pi - 1}{4}\,\frac{\mu + 1}{\mu} = \int_0^{\pi} G(s|\theta)\,ds,$$

with the left-hand side intersecting the right-hand side from below at $\overline{\pi} > 1/2$. Take $\tilde{\theta} \geq \theta$. From Appendix A, if the quality distribution $\tilde{G}(\cdot|\theta)$ on $[1/2, 1]$ increases in θ in the sense first order stochastic dominance, then

$$\int_0^{\pi} G(s|\theta)\,ds \leq \int_0^{\pi} G(s|\tilde{\theta})\,ds.$$

Since the left-hand side and the right-hand side of (B-8) are strictly monotonic in π, their intersection for θ is to the left of their intersection for $\tilde{\theta}$.
(b) Equation (11) can be rewritten as $(1 - \overline{\pi}(\theta))/\overline{\pi}(\theta) = \beta_0(\overline{\pi}(\theta))/\beta_1(\overline{\pi}(\theta))$. By part ($a$), $\overline{\pi}(\theta)$ increases in θ, and thus $\beta_0(\overline{\pi}(\theta))/\beta_1(\overline{\pi}(\theta))$ decreases.

B.9 Proof of Proposition 9

(i) ***Expected volume*** *increases.* Expected volume is given by $\nu(\beta_1 + \beta_0)$. From the proof of the existence theorem, $\beta_1(\theta) + \beta_0(\theta) = 1 + \mu - 2\mu G(\overline{\pi}|\theta)$. Thus we need to show that $G(\overline{\pi}(\tilde{\theta})|\tilde{\theta}) < G(\overline{\pi}(\theta)|\theta)$ for $\theta < \tilde{\theta}$. Let $\tilde{\theta} = \theta + \Delta\theta$ with $\Delta\theta$ small. We use subscripts θ and

π for partial derivatives. A first order Taylor approximation of G yields

$$G(\overline{\pi}(\theta+\Delta\theta),\theta+\Delta\theta)=G(\overline{\pi}(\theta),\theta)+G_\theta(\overline{\pi}(\theta),\theta)\Delta\theta+G_\pi(\overline{\pi}(\theta),\theta)\overline{\pi}'(\theta)\Delta\theta.$$

We thus have to show that

$$\text{(B-9)} \qquad G_\theta(\overline{\pi}(\theta),\theta)+G_\pi(\overline{\pi}(\theta),\theta)\cdot\overline{\pi}'(\theta)<0.$$

By Appendix A, when the quality distribution $\tilde{G}(\cdot|\theta)$ on $[1/2,1]$ increases in θ in the sense first order stochastic dominance, then it holds that $G_\theta<0$ for $\pi>1/2$. Yet $G_\pi = g(\cdot|\theta)\geq 0$ (for it is a density) and $\overline{\pi}'>0$ (by Proposition 8).

The change in the threshold, $\overline{\pi}'(\theta)$, is obtained from the equilibrium condition. From the proof of the existence theorem we know that the equilibrium $\overline{\pi}(\theta)$ solves

$$\text{(B-10)} \qquad (2\overline{\pi}(\theta)-1)/4\times(1+\mu)/\mu=\int_0^{\overline{\pi}(\theta)}G(s|\theta)ds\equiv\Psi(\overline{\pi}(\theta),\theta).$$

Since (B-10) holds for all θ as an identity, we can differentiate it with respect to θ to obtain

$$\overline{\pi}'(\theta)=\frac{\Psi_\theta(\overline{\pi}(\theta),\theta)}{\frac{1+\mu}{2\mu}-\Psi_\pi(\overline{\pi}(\theta),\theta)}.$$

Substituting all this back into the LHS of (B-9) yields

$$G_\theta(\overline{\pi}(\theta),\theta)+G_\pi(\overline{\pi}(\theta),\theta)\frac{\Psi_\theta(\overline{\pi}(\theta),\theta)}{\frac{1+\mu}{2\mu}-\Psi_\pi(\overline{\pi}(\theta),\theta)}<0.$$

The size of the last term depends on μ and we will now eliminate it to establish a relation that only depends on the primitive quality distribution. Reformulating (B-10),

$$\frac{1+\mu}{2\mu}=\frac{2\Psi(\overline{\pi}(\theta),\theta)}{2\overline{\pi}(\theta)-1}.$$

Substituting back into the LHS of (B-9) and expressing it in terms of Ψ yields

$$\Psi_{\pi,\theta}(\overline{\pi}(\theta),\theta)+\Psi_{\pi,\pi}(\overline{\pi}(\theta),\theta)\frac{(2\overline{\pi}(\theta)-1)\Psi_\theta(\overline{\pi}(\theta),\theta)}{2\Psi_\theta(\overline{\pi}(\theta),\theta)-(2\overline{\pi}(\theta)-1)\Psi_\pi(\overline{\pi}(\theta),\theta)}<0.$$

As the denominator of the second term is positive, the above is equivalent to ($\star$).

(ii) *The* ***expected order imbalance*** *increases.*

Claim (I): *Given condition ($\star$), the distribution of volume increases in θ the sense of first order stochastic dominance.*

Claim (II): *The distribution of the order imbalance conditional on volume increases in θ in the sense of of first order stochastic dominance.*

Using these claims, the proof obtains as follows. By Proposition 3, $\mathsf{E}[\mathsf{IB}|b_t + s_t = \mathsf{Vol}, \theta]$ increases in Vol. Let $\theta_h > \theta_l$. Then

$$\begin{aligned}
\mathsf{E}[\mathsf{IB}|\theta_h] &= \sum_{\mathsf{Vol}=0}^{\infty} \mathsf{E}[\mathsf{IB}|b_t + s_t = \mathsf{Vol}, \theta_h]\mathsf{Pr}(\mathsf{Vol}|\theta_h) \\
&\geq \sum_{\mathsf{Vol}=0}^{\infty} \mathsf{E}[\mathsf{IB}|b_t + s_t = \mathsf{Vol}, \theta_l]\mathsf{Pr}(\mathsf{Vol}|\theta_h) \\
&\geq \sum_{\mathsf{Vol}=0}^{\infty} \mathsf{E}[\mathsf{IB}|b_t + s_t = \mathsf{Vol}, \theta_l]\mathsf{Pr}(\mathsf{Vol}|\theta_l) = \mathsf{E}[\mathsf{IB}|\theta_l].
\end{aligned}$$

The first inequality follows from Claim (I), and the second from Claim (II).

Proof of **Claim** (I): Define $\alpha(\theta) := \nu(\beta_1 + \beta_0)$. By Proposition 8, $\alpha'(\theta) > 0$. Using equation (B-7), the cumulative distribution of volume is

$$\mathsf{Pr}(\mathsf{Vol} \leq k) = e^{-\alpha}\left(1 + \alpha + \frac{1}{2}\,\alpha^2 + \ldots + \frac{1}{k!}\,\alpha^k\right).$$

The first order effect of a change in θ on this cdf is

$$\frac{\partial}{\partial\theta}\,\mathsf{Pr}(\mathsf{Vol} \leq k) = -\frac{1}{k!}\,e^{-\alpha}\alpha^k\alpha'(\theta) < 0.$$

Since $\mathsf{Pr}(\mathsf{Vol} \leq k) \searrow$ in θ for all k, we have FOSD.

Proof of **Claim** (II): Consider function $\rho = \beta_1/(\beta_1 + \beta_0)$ as defined in the proof of Proposition 3. By Proposition 8, $\rho'(\theta) > 0$. Now suppose that $\mathsf{Vol} = 2k$, i.e. volume is even (the case for odd volume, $\mathsf{Vol} = 2k + 1$ is analogous). To prove that for fixed Vol, the distribution of the order imbalance satisfies FOSD, we need to show that

$$\mathsf{Pr}(\mathsf{IB} \leq 2i|\mathsf{Vol} = 2k) \searrow \quad \text{in } \theta \quad \text{for all } \ 0 \leq i < k.$$

The relevant probabilities for the order imbalance have been derived in the proof of Proposition 3. Suppose that $i = 0$. Then $\mathsf{Pr}(\mathsf{IB} = 0|\mathsf{Vol} = 2k) = \rho^k(1-\rho)^k\frac{(2k)!}{k!k!}$. Since $\rho'(\theta) > 0$ and $\rho \geq 1/2$, the derivative of the above is negative:

$$\frac{\partial}{\partial\rho}\rho^k(1-\rho)^k\frac{(2k)!}{k!k!} = \frac{(2k)!}{k!k!}\cdot k\rho^{k-1}(1-\rho)^{k-1}\,(1-2\rho) < 0.$$

Next, consider the case with $0 < i < k$:

$$\mathsf{Pr}(\mathsf{IB} \leq 2i|\mathsf{Vol}=2k)=\rho^k(1-\rho)^k\frac{(2k)!}{k!k!}+\sum_{j=1}^{i}\left(\rho^{k+j}(1-\rho)^{k-j}+\rho^{k-j}(1-\rho)^{k+j}\right)\frac{(2k)!}{(k+j)!(k-j)!}.$$

Computing the derivative and rearranging, we obtain

$$\frac{\partial}{\partial\rho}\mathsf{Pr}(\mathsf{IB} \leq 2i|\mathsf{Vol}=2k) = \frac{(2k)!}{(k+i)!(k-i-1)!}\rho^{k-i-1}(1-\rho)^{k+i}\left(1-\left(\frac{\rho}{1-\rho}\right)^{2i+1}\right) < 0,$$

where the last step follows as $\rho > 1/2$. Finally observe that $\mathsf{Pr}(\mathsf{IB} \leq 2i|\mathsf{Vol} = 2k) = 1$ for $i = k$. Thus $\mathsf{Pr}(\mathsf{IB} \leq 2i|\mathsf{Vol} = 2k) \searrow$ in ρ and since $\rho \searrow$ in θ, we have that $\mathsf{Pr}(\mathsf{IB} \leq 2i|\mathsf{Vol} = 2k)$ increases in θ in the sense of first order stochastic dominance.

(iii) *The* ***expected price variability*** *increases.* As the proof of Proposition 5 shows, the expected price variability, $\mathsf{E}[|\Delta \mathsf{p}_t||\theta]$ employs the same probabilities as the expected order imbalance, $\mathsf{E}[\mathsf{IB}|\theta]$ for fixed volume. As the distribution of volume satisfies FOSD, the result is analogous to part (ii).

(iv) ***Duration*** *decreases.* As $\beta_v + \sigma_v + \gamma_v = 1$, when $\beta_v + \sigma_v$ increases γ_v decreases.

B.10 Proof of Proposition 10 (Noise Trading)

(a) As μ increases, the left-hand side of (B-3), which implicitly defines $\overline{\pi}$, shifts to the right and thus the root $\overline{\pi}$ shifts to the right, too.

(b) As argued in the Proof of Proposition 7, expected trading volume is proportional to $\beta_1 + \beta_0$. Using Step 2(i) of the proof of the existence theorem,

$$\beta_1 + \beta_0 = 1 + \mu - 2\mu G(\overline{\pi}) = 1 + \mu \underbrace{(1 - 2G(\overline{\pi}))}_{<0 \text{ as } G(\overline{\pi})>1/2}.$$

Since $\overline{\pi}$ increases in μ, $d/(d\mu)G(\overline{\pi}(\mu)) > 0$. Thus $d/(d\mu)(\beta_1(\mu) + \beta_0(\mu)) < 0$ and trading volume declines.

(c) As in Step (iv) in the proof of Proposition 9, duration increases.

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Result	**Variable Shift**	**Observable Reaction**	**Existing evidence or suggested empirical proxies for unobservables**
Corollary 1	net order flow $\nearrow$	price $\nearrow$	Jones, Kaul, and Lipson (1994), Brown, Walsh, and Yuen (1997), Chordia, Roll, and Subrahmanyam (2002)
Proposition 2	past net order flow > 0	average future net order flow > 0	Hasbrouck and Ho (1987), Chordia, Roll, and Subrahmanyam (2002)
Proposition 3 (a)	past order imbalances $\nearrow$	future order imbalances independent	N/A
Proposition 3 (b)	volume $\nearrow$	order imbalance $\nearrow$	Chordia, Roll, and Subrahmanyam (2002), Chan and Fong (2000)
Proposition 4	order imbalance $\nearrow$	average absolute price change $\nearrow$	Chordia, Roll, and Subrahmanyam (2002), Chan and Fong (2000)
Proposition 5	volume $\nearrow$	average absolute price change $\nearrow$	Chordia, Roll, and Subrahmanyam (2002), Chan and Fong (2000), Gallant, Rossi, and Tauchen (1992)
Proposition 7	entry rate $\nearrow$	average volume $\nearrow$ average order imbalance $\nearrow$ average price changes $\nearrow$ duration $\searrow$	causes for changes in entry rate: index inclusion, international market opening financial deregulation, cross-listing
Propositions 8 & 9	information quality $\nearrow$	price impact $\nearrow$ average volume $\nearrow$ average order imbalance $\nearrow$ average price changes $\nearrow$ duration $\searrow$	Chordia, Roll, and Subrahmanyam (2008), causes for changes in information quality: new disclosure rules, changes in analyst coverage, changes in information technology, changes in ownership structure (hedge funds rather than mutual funds), development of a new analysis tool, improvements in analysts' earnings forecasts
Proposition 10	informed trading $\nearrow$	average volume $\searrow$ duration $\nearrow$ (numerically) price changes $\nearrow$	causes for changes in the proportion of informed trading: changes in ownership structure (hedge funds rather than retail), market changes that trigger more retail participation or day trading

Table 1: **Empirical Predictions of the Model**