University of Toronto
Department of Economics

Working Paper 343

Directed Search for Equilibrium Wage-Tenure Contracts

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October 27, 2008

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August 2008
(forthcoming in Econometrica)

Abstract

I construct a theoretical framework in which firms offer wage-tenure contracts to direct the search by risk-averse workers. All workers can search, on or off the job. I characterize an equilibrium and prove its existence. The equilibrium generates a non-degenerate, continuous distribution of employed workers over the values of contracts, despite that all matches are identical and workers observe all offers. A striking property is that the equilibrium is block recursive; that is, individuals' optimal decisions and optimal contracts are independent of the distribution of workers. This property makes the equilibrium analysis tractable. Consistent with stylized facts, the equilibrium predicts that (i) wages increase with tenure, (ii) job-to-job transitions decrease with tenure and wages, and (iii) wage mobility is limited in the sense that the lower the worker's wage, the lower the future wage a worker will move to in the next job transition. Moreover, block recursivity implies that changes in the unemployment benefit and the minimum wage have no effect on an employed worker's job-to-job transitions and contracts.

JEL classifications: E24, C78, J6.
Keywords: Directed search; On-the-job search; Wage-tenure contracts.

* I thank a co-editor and three anonymous referees for providing thoughtful suggestions that led to significant improvements of the paper. I also thank Giuseppe Moscarini, Guido Menzio, and Alok Kumar for serving as discussants of the paper. Useful comments have also been received from the participants of the workshops and conferences at University of Pennsylvania (2006), Rochester (2006), Yale (2006), University of British Columbia, Calgary, Pittsburgh, the Microfoundations of Markets with Frictions (UQAM, 2007), Research on Money and Markets (Toronto, 2007), the NBER Summer Institute (Cambridge, 2007), Canadian Macro Study Group Meeting (Ottawa, 2007), Cornell-Penn State Macro Conference (2008), and the Econometric Society Winter Meeting (2008). Financial support from the Bank of Canada Fellowship and from the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged. The opinion expressed in the paper is my own and does not reflect the view of the Bank of Canada. All errors are mine.

1. Introduction

Search on the job is prevalent and generates large job-to-job transitions. On average, 2.6 percent of employed workers in the U.S. change employers each month, and nearly two-fifths of new jobs represent employer changes (Fallick and Fleischman, 2004). This large flow of workers between jobs exhibits three stylized patterns. First, the longer the tenure that a worker has on his current job, the less likely he will quit for another job (Farber, 1999). Second, controlling for individual heterogeneity, wage is a key determinant of mobility: a worker with a higher wage is less likely to quit for another job (Topel and Ward, 1992). Third, wage mobility is limited in the following sense: controlling for individual characteristics, most of the transitions take place between adjacent quintiles of wages at the lower end of the wage distribution and probabilities of staying in a quintile are higher at the higher quintiles (Buchinsky and Hunt, 1999).

To explain these facts, I construct a theoretical framework to integrate wage-tenure contracts and on-the-job search. In the model, firms enter the labor market competitively and offer wage-tenure contracts. Workers are risk averse and identical, all of whom can search for jobs. Search is directed in the sense that, when making search decisions, agents take into account that a higher offer yields a lower matching rate for an applicant and a higher matching rate for a firm. Firms can commit to the contracts but workers cannot commit to staying with a firm. I characterize an equilibrium, prove its existence, and explore its properties.

The framework provides consistent explanations for the above facts. First, wages increase with tenure, and job-to-job transitions decrease with tenure and wages. Making wages increase continuously with tenure is the optimal way for a firm to backload wages when workers are risk averse. As wages rise with tenure, a worker is less likely to quit because the probability of finding higher wages elsewhere falls. Second, directed search strengthens the negative dependence of job-to-job transitions on wages. As an optimal tradeoff between offers and matching rates, workers with low wages choose to search for relatively low offers. Because low offers are relatively easier to obtain, low-wage workers make job transitions with higher probabilities than high-wage workers. Third, and similarly, directed search generates limited wage mobility. By climbing up the wage ladder gradually, workers maximize the expected gain from search in each job transition.

An equilibrium has a non-degenerate, continuous distribution of wages or values, despite the assumptions that all matches are identical and all workers observe all offers. On-the-job search generates a wage ladder among identical workers by creating dispersion among workers' histories of search outcomes. Wage-tenure contracts fill in the gap between any two rungs of the ladder by increasing wages continuously with tenure.

In addition to explaining the stylized facts, this paper formalizes and explores a key property of an equilibrium with directed search, called "block recursivity". That is, individuals' decisions and equilibrium contracts are independent of the distribution of workers over wages, although the distribution affects aggregate statistics. In general, the non-degenerate distribution can serve as a state variable in individuals' decisions. By eliminating this role of the distribution, block recursivity makes the equilibrium analysis tractable. Block recursivity arises from directed search, because the optimal tradeoff between offers and matching rates implies that workers at different wages choose to apply for different offers. With such endogenous separation, the workers who apply for a particular offer cares about only the matching rate at that offer, but not the distribution of workers over other offers. In turn, the matching rate at each offer is determined by free entry of firms independently of the distribution of workers. Besides tractability, block recursivity has a novel policy implication – changes in the unemployment benefit and the minimum wage have no effect on an employed worker's job-to-job transitions.

This paper is closely related to Burdett and Coles (2003, *BC* henceforth). Both papers predict that on-the-job search induces firms to backload wages, thus making wages increase and quit rates decrease with tenure. As a main difference, BC assume that search is undirected as workers exogenously receive offers.[1] With undirected search, wage mobility is not limited as in the data, because all workers have the same probability of obtaining any particular offer regardless of their current wages. Moreover, to sustain a non-degenerate distribution of wages among identical matches, BC assume that every worker observes at most one offer before applying. In contrast, I assume that all workers observe all offers, which makes wage dispersion more robust and on-the-job search more potent for explaining worker turnover. In addition, a different apparatus is required to establish existence of an equilibrium with directed search.

I model directed search as in Moen (1997) and Acemoglu and Shimer (1999a,b). To the literature of directed search, the main contributions here are to incorporate wage-tenure contracts and on-the-job search, and to formally establish existence of an equilibrium.[2] Moen and Rosen (2004) examine on-the-job search with contracts, but their assumption that on-the-job search is entirely driven by changes in productivity eliminates the main issues and theoretical challenges that I face here. Delacroix and Shi (2006) examine directed search on the job with identical workers, but they assume that firms offer constant wages, rather than wage-tenure contracts.

In this paper, all matches are identical and the productivity of a match is public information. Although heterogeneity, private information and learning about productivity are important for wage dynamics and turnover in reality, as modeled by Jovanovic (1979), Harris and Holmstrom (1982), and Moscarini (2005), abstracting from them enables me to focus on search. Most of the proofs are omitted in this paper but are available as supplementary materials.

2. The Model Environment

Consider a labor market that lasts forever in continuous time. There is a unit measure of homogeneous, risk-averse workers whose utility function in each period is $u(w)$, where w is income. The utility function has the standard properties: $0 < u'(w) < \infty$ and $-\infty < u''(w) < 0$ for all $w \in (0, \infty)$, and $u'(0) = \infty$. Workers cannot borrow against their future income. An employed worker produces a flow of output, $y > 0$, and an unemployed worker enjoys the unemployment benefit, $b > 0$. A worker dies at a Poisson rate $\delta \in (0, \infty)$, and is replaced with a newborn who is unemployed. Death is the only exogenous separation. Firms are identical and risk-neutral. Jobs enter the market competitively: a firm can post a vacancy at a flow cost $k > 0$, and can

[1] BC extend the wage-posting model of Burdett and Mortensen (1998). Undirected search is also the feature of another class of search models, pioneered by Diamond (1982), Mortensen (1982), and Pissarides (1990).

[2] Peters (1984, 1991) and Montgomery (1991) are two of the earliest formulations of directed search. Other examples of directed search models include Julien, et al. (2000), Burdett, et al. (2001), Shi (2001, 2002), Coles and Eeckhout (2003), and Galenianos and Kircher (2005).

treat different jobs independently. Firms announce wage-tenure contracts to recruit. A contract is denoted as $W = \{\tilde{w}(t)\}_{t=0}^{\infty}$, which specifies the wage at each tenure length, t, conditional on that the worker stays with the firm. Firms are assumed to commit to the contracts, but workers can quit a job at any time. In particular, a firm cannot respond to the employee's outside offers. Normalize the production cost to 0. Firms and workers discount future with the same rate of time preference $\rho \geq 0$. Denote $r = \rho + \delta$ as the effective discount rate.[3]

Throughout this paper, t denotes tenure rather than the calendar time. Denote $V(t)$ as the value of a contract at t, i.e., the worker's lifetime expected utility generated by the remaining contract at t. The value of a contract at $t = 0$ is called *an offer* and denoted as $x = V(0)$. Denote an unemployed worker's "tenure" as $t = \varnothing$, the unemployment benefit as $b = \tilde{w}(\varnothing)$, and the value of unemployment as $V_u = V(\varnothing)$. All offers are bounded in $[\underline{V}, \bar{V}]$, where

$$\bar{V} = u(\bar{w})/r, \quad \underline{V} = u(b)/r. \tag{2.1}$$

$\bar{w}$ is the highest wage given later by (3.8), $\bar{V}$ the lifetime utility of a worker who is employed at $\bar{w}$ permanently until death, and $\underline{V}$ the lifetime utility of a worker who stays in unemployment forever. I use the phrases "all x" and "all V" to mean all values in $\left[\underline{V}, \bar{V}\right]$.

All workers, employed or unemployed, can search. There is a continuum of submarkets indexed by the offer x. Each submarket x has a tightness, $\theta(x)$, which is the ratio of applicants to vacancies in that submarket. The total number of matches in submarket x is given by a linearly homogeneous matching function, $\mathcal{M}(N(x), N(x)/\theta(x))$, where $N(x)$ is the number of applicants in the submarket. In submarket x, a vacancy is filled at the Poisson rate $q(x) \equiv \mathcal{M}(\theta(x), 1)$, and an applicant obtains an offer at the rate $p(x) \equiv \mathcal{M}(1, 1/\theta(x))$. I refer to $q(.)$ as the *hiring rate function*, and $p(.)$ as the *employment rate function*. In an equilibrium, $q(x)$ is increasing and $p(x)$ decreasing in x. Thus, search is directed in the sense that agents face a tradeoff between offers and matching rates when choosing which submarket to enter. Since the matching rates act as hedonic prices in each submarket, search is competitive.[4]

Although the function $\mathcal{M}$ is exogenous, the functions $q(.)$, $p(.)$ and $\theta(.)$ are equilibrium objects. Following Moen and Rosen (2004), I eliminate θ from the above expressions for p and q to express $p(x) = M(q(x))$. Because the function $M(q)$ inherits all essential properties of the function, $\mathcal{M}$, I will take $M(.)$ as a primitive of the model and refer to it as the *matching function*.

Focus on stationary equilibria where the set of offered contracts and the functions, $q(x)$ and $p(x)$, are time invariant. Moreover, I focus on an equilibrium in which $p(.)$ satisfies:

$$\left.\begin{array}{l} \text{(i) } p\left(\bar{V}\right) = 0; \quad \text{(ii) } p(x) \text{ is bounded, continuous and concave for all } x; \\ \text{(iii) } p(x) \text{ is strictly decreasing and continuously differentiable for all } x < \bar{V}. \end{array}\right\} \tag{2.2}$$

[3]The assumptions on the contracts and the separation process are the same as in BC (2003). The main difference of my model from BC is that search is directed here. Also, I do not impose BC's assumption $u(0) = -\infty$. For a model in which firms can counter outside offers, see Postel-Vinay and Robin (2002).

[4]Moen (1997) and Acemoglu and Shimer (1999a,b) formulate this competitive process of directed search. An alternative is to formulate the process as a strategic game, e.g., Peters (1991), Burdett et al. (2001), and Julien et al. (2000). The strategic formulation endogenizes the matching function, $\mathcal{M}$, but the function converges to a linearly homogeneous function when the number of participants in the market goes to infinity. Moreover, Acemoglu and Shimer (1999b) relax the assumption that each applicant observes all offers, and Galenianos and Kircher (2005) allow each applicant to send two or more applications at once.

I will first characterize individuals' decisions under any arbitrary p function that satisfies (2.2) and then verify, in Theorem 4.1, that an equilibrium satisfying (2.2) exists indeed.

3. Workers' and Firms' Optimal Decisions

3.1. A Worker's Optimal Search Decision

Refer to a worker's value, V, as the worker's *state* or *type*. If the worker searches in submarket x, he obtains the offer x at rate $p(x)$, which yields the gain, $(x - V)$. The expected gain from search in submarket x is $p(x)(x - V)$. The optimal search decision, x, solves:

$$S(V(t)) \equiv \max_{x \in [V(t), \bar{V}]} p(x)\,(x - V)\,. \tag{3.1}$$

Denote the solution as $x = F(V)$. I prove the following lemma in Appendix A:

Lemma 3.1. *Assume (2.2). Then, $F(\bar{V}) = \bar{V}$. For all $V < \bar{V}$, the following results hold: (i) $F(V)$ is interior, strictly increasing in V, and satisfies:*

$$V = F(V) + \frac{p(F(V))}{p'(F(V))}; \tag{3.2}$$

(ii) $F(V)$ is unique for each V, and continuous in V; (iii) $S(V)$ is differentiable, with $S'(V) = -p(F(V)) < 0$; (iv) $F(V_2) - F(V_1) \le (V_2 - V_1)/2$ for all $V_2 \ge V_1$; (v) If $p''(.)$ exists, then $F'(V)$ and $S''(V)$ exist, with $0 < F'(V) \le 1/2$.

The following properties are noteworthy. First, $F(V)$ is unique for each V. For a worker at the state V, offers higher than $F(V)$ have too low employment rates to be optimal, while offers lower than $F(V)$ have too low values. Only the offer $F(V)$ provides the optimal tradeoff between the value and the employment rate. Second, $F(V)$ is strictly increasing in V. That is, the higher a worker's state, the higher the offer for which the worker will apply. Thus, the applicants separate themselves according to their states. This endogenous separation arises because an applicant's current job is a backup for him when he fails to obtain the applied job. The higher this backup value is, the more the worker can afford to "gamble" on the application and, hence, the higher the offer for which he will apply. Third, the expected gain from search, $S(V)$, and the actual gain in percentage, $(F - V)/V$, diminish as V increases. Moreover, $S''(V) > 0$; i.e., the expected gain from search diminishes at a lower rate as V increases.

Endogenous separation of the applicants is a common result in directed search models (see Acemoglu and Shimer, 1999a, Shi, 2001, Moen and Rosen, 2004, and Delacroix and Shi, 2006). However, it is not a result in undirected search models (e.g., BC, and Burdett and Mortensen, 1998). With undirected search, workers receive offers randomly and exogenously, and so there is no counterpart to the search decision in (3.1). In section 5, I will contrast my model with undirected search models on worker turnover and wage mobility.

3.2. Value Functions of Workers and Firms

Denote $\dot{f} = df/dt$ for any variable f. Consider first an employed worker whose tenure is $t \geq 0$. From the analysis above, the worker searches for the offer $x(t) = F(V(t))$. At rate $p(F(V(t)))$, he gets the offer and quits the current job. If the worker does not get the offer, he stays in the current contract whose value increases by $\dot{V}$. Taking into account time discounting and the event of death, the value for the worker obeys:

$$\rho V(t) = u(\tilde{w}(t)) + \dot{V}(t) + p(F(V(t)))\left[F(V(t)) - V(t)\right] - \delta V(t).$$

Using $S(V)$ defined by (3.1), and the effective discount rate $r = \rho + \delta$, I can rewrite:

$$\dot{V}(t) = rV(t) - u(\tilde{w}(t)) - S(V(t)). \tag{3.3}$$

Because I focus on stationary equilibria, the change, $\dot{V}$, is entirely caused by changes in wages with tenure. If wages are constant, then $\tilde{w}(t) = \tilde{w}$ and $\dot{V}(t) = 0$ for all t.[5] In particular, because the unemployment benefit is constant, $\dot{V}_u = 0$, and V_u obeys:

$$0 = rV_u - u(b) - S(V_u). \tag{3.4}$$

Since $S(V_u) > 0$, it is clear that $V_u > \underline{V}$, where $\underline{V}$ is defined in (2.1).

Now consider the value of a firm whose worker has a contract with a remaining value, $V(t)$. Let $\tilde{J}(t)$ denote this firm's value. Similar to (3.3), I can derive

$$d\tilde{J}(t)/dt = \left[r + p(F(V(t)))\right]\tilde{J}(t) - y + \tilde{w}(t). \tag{3.5}$$

Note that $\tilde{J}(t)$ is bounded above and below for all t. For any arbitrary $t_a \in [0, t]$, define

$$\gamma(t, t_a) \equiv \exp\left[-\int_{t_a}^{t}\left[r + p(F(V(\tau)))\right]d\tau\right]. \tag{3.6}$$

Since $\lim_{t\to\infty}\gamma(t, t_a) = 0$, integrating (3.5) yields:

$$\tilde{J}(t_a) = \int_{t_a}^{\infty}\left[y - \tilde{w}(t)\right]\gamma(t, t_a)dt. \tag{3.7}$$

3.3. Optimal Recruiting Decisions and Contracts

A firm's recruiting decision contains two parts. The first is to choose an offer x to maximize the expected value of recruiting, $q(x)\tilde{J}(0)$, taking the function $q(.)$ as given. The second part is to choose a contract to deliver the value x and to maximize $\tilde{J}(0)$.

For the first part, I will later show that there is a continuum of optimal offers, denoted as $\mathcal{V} = \left[v_1, \bar{V}\right]$, where $v_1 \equiv F(V_u)$. A high offer increases the chance of filling the vacancy but yields lower profit ex post. A low offer yields higher ex post profit, but reduces the chance of filling the

[5] Although a worker can quit the job to become unemployed, it is not optimal to do so in an equilibrium, because optimal contracts provide higher values in employment than in unemployment. Also, because an employed worker never returns to unemployment, the worker has no incentive to save provided that wages increase with tenure.

vacancy. A firm is indifferent among the offers in $\mathcal{V}$, because they all yield the same expected value of recruiting. That is, $q(x)\tilde{J}(0) = k$ for all $x \in \mathcal{V}$, where k is the vacancy cost.

The highest offer, $\bar{V}$, is delivered by the highest wage, $\bar{w}$. To derive $\bar{w}$, note that $p(F(\bar{V})) = p(\bar{V}) = 0$. Since (3.1) implies $S(\bar{V}) = 0$ and (2.1) implies $\bar{V} = u(\bar{w})/r$, (3.3) implies $\dot{V} = 0$ at $\bar{V}$. Similarly, (3.5) implies that the value of a firm that employs a worker at $\bar{w}$ is $\underline{J} = (y - \bar{w})/r$. Because $q(\bar{V})\underline{J} = k$, then $\bar{w} = y - rk/q(\bar{V})$. Let $\bar{q} \in (0, \infty)$ be the upper bound on q, discussed further in Assumption 1. Then, $q(\bar{V}) = \bar{q}$: if $q(\bar{V}) < \bar{q}$, offering a constant wage slightly above $\bar{w}$ would yield a higher expected value to the firm.[6] Therefore,

$$\bar{w} = y - rk/\bar{q} \quad (< y), \qquad \underline{J} = k/\bar{q} \quad (> 0). \tag{3.8}$$

For the contracting part of a firm's decisions, the optimal contract, $\{\tilde{w}(t)\}_{t=0}^{\infty}$, solves:

$$(\mathcal{P}) \ \max \ \tilde{J}(0), \text{ subject to } V(0) = x.$$

This problem differs from that in BC (2003) in two aspects. First, BC assume $u(0) = -\infty$ to prove $\tilde{w}(t) > 0$ and $d\tilde{w}(t)/dt > 0$ for all t. This assumption is not necessary in the current model, because employed and unemployed workers face the same employment rate function. Second, the quit rate of a worker employed at V is $p(F(V))$ here, but it is $\lambda[1 - Q(V)]$ in BC, where $Q(.)$ is the distribution of offers and λ a constant. Despite these differences, the following lemma is similar to the results in BC:

Lemma 3.2. *Assume (2.2). Optimal contracts have the following features: (i)* $0 < \tilde{w}(t) \le \bar{w}$ *for all* $t > 0$*; (ii)* $d\tilde{w}(t)/dt > 0$ *for all* $t < \infty$, $\tilde{w}(t) \nearrow \bar{w}$ *as* $t \to \infty$*, and*

$$\frac{d\tilde{w}(t)}{dt} = \frac{[u'(\tilde{w}(t))]^2}{u''(\tilde{w}(t))}\tilde{J}(t)\left[\frac{dp(F(V(t)))}{dV}\right], \ \text{all } t; \tag{3.9}$$

(iii) $\dot{V}(t) > 0$ *and* $d\tilde{J}(t)/dt < 0$ *for all* $t < \infty$*, with* $V(t) \nearrow \bar{V}$ *and* $\tilde{J}(t) \searrow \underline{J}$ *as* $t \to \infty$*. Moreover,*

$$d\tilde{J}(t)/dt = -\frac{\dot{V}(t)}{u'(\tilde{w}(t))}, \quad \text{all } t. \tag{3.10}$$

Optimal contracts have several properties. First, wages are continuous and increasing in tenure for all finite tenure lengths. This property is generated by firms' incentive to backload wages and workers' risk aversion. Because a worker cannot commit to a job, a firm backloads wages to entice the worker to stay. A rising wage profile is less costly to the firm than a constant profile that promises the same value to the worker: as wages rise with tenure, it is more difficult for the worker to find a better offer elsewhere, and so the worker's quit rate falls. However, if workers are risk neutral, one optimal way to backload wages is to offer a very low wage initially, with promised wage jumps in the future (see Stevens, 2004). Risk aversion makes such jumps suboptimal. Thus, wages increase continuously in tenure in optimal contracts.

[6] Suppose that $q(\bar{V}) = \bar{q} - a$ for some $a > 0$. In this case, $\bar{w} = y - rk/(\bar{q} - a)$. A firm that deviates from $\bar{w}$ to $\bar{w} + \varepsilon$, with $\varepsilon > 0$, attracts all of the workers who are employed at $\bar{w}$, because the deviating firm is the only one that offers a wage higher than $\bar{w}$. Thus, $q(\hat{V}) = \bar{q}$, where $\hat{V} = u(\bar{w} + \varepsilon)/r$. The deviating firm's expected value of recruiting is $(y - \bar{w} - \varepsilon)\bar{q}/r$, that exceeds k for sufficiently small $\varepsilon > 0$. A contradiction.

Second, wages and values are *strictly* increasing in tenure for all finite tenure lengths. To explain this result, suppose that an optimal contract has a constant segment of wages. This segment should be put at the beginning of the contract in order to increase the room for backloading wages. Moreover, the constant segment must be at the constrained level, 0; otherwise, the firm can increase its expected value by reducing the initial wage and shortening the constant segment. However, such a contract has a strictly lower value than the value of unemployment, because an unemployed worker enjoys a positive benefit and faces the same job opportunities as an employed worker does. Thus, a contract with such a constant segment of wages will not be accepted.

Third, optimal contracts induce efficient sharing of the value between a firm and its worker, in the sense described by (3.10). To elaborate, note that $-d\tilde{J}/dt$ is the marginal cost to the firm of increasing wages, while $\dot{V}/u'(\tilde{w})$ is the marginal benefit to the worker of the wage increase, measured in the same unit as profit. Thus, (3.10) requires that a wage increase should have the same marginal cost to the firm as the marginal benefit to the worker.

Fourth, all optimal contracts are sections of a baseline contract. The *baseline contract*, denoted as $\{\tilde{w}_b(t)\}_{t=0}^{\infty}$, is an optimal contract where $\tilde{w}_b(0)$ is the lowest wage in an equilibrium. The entire set of optimal contracts can be constructed as follows:

$$\left\{\{\tilde{w}(t)\}_{t=0}^{\infty} : \tilde{w}(t) = \tilde{w}_b(t + t_a),\ t_a \in [0, \infty), \text{ for all } t\right\}.$$

That is, the "tail" of the baseline contract from any arbitrary tenure t_a onward is an optimal contract by itself when offered at the beginning of the match. This property is an implication of the principle of dynamic optimality.[7]

With the above property, it suffices to examine only the baseline contract. From now on, I suppress the subscript b on the baseline contract. In particular, $V(t)$ denotes the value of the baseline contract for a worker at tenure t. Note that the set of equilibrium offers across contracts at any given time can be obtained alternatively by tracing out the baseline contract over tenure. That is, $\mathcal{V} = \{x : x = V(t), \text{ all } t \geq 0\}$.

4. Equilibrium and Block Recursivity

I will use V instead of t as the variable in various functions. To do so, define

$$T(V(t)) = t, \quad w(V) = \tilde{w}(T(V)), \quad J(V) = \tilde{J}(T(V)). \tag{4.1}$$

$T(V)$ is the inverse function of $V(t)$, and records the length of time for the value to increase from the lowest equilibrium offer, v_1, to V. A contract of a value V starts with the wage, $w(V)$, and generates a present value, $J(V)$, to a firm. Refer to $w(V)$ as the *wage function.* Since $T'(V(t)) = 1/\dot{V}(t)$, then $d\tilde{J}(t)/dt = J'(V(t))\dot{V}(t)$, and (3.10) becomes:

$$J'(V) = -\frac{1}{u'(w(V))}, \quad \text{all } V < \bar{V}. \tag{4.2}$$

[7]If the property does not hold for some tenure $t_a > 0$, then there is another contract, $\{\hat{w}(t)\}_{t=0}^{\infty}$, that yields a higher value to the firm than the contract, $\{\tilde{w}(t)\}_{t=0}^{\infty}$, where $\tilde{w}(t) = \tilde{w}_b(t + t_a)$ for all t. Replace the tail of the baseline contract from tenure t_a onward by letting $\hat{w}_b(t + t_a) = \hat{w}(t)$ for all t. The new baseline contract yields a higher value to the firm than the original baseline contract, contradicting the optimality of the latter.

4.1. Definition of the Equilibrium and Block Recursivity

An *equilibrium* consists of a set of offers, $\mathcal{V} = \{V(t) : t \geq 0\}$, a hiring rate function, $q(.)$, an employment function, $p(.)$, an application strategy, $F(.)$, a value function, $J(.)$, a wage function, $w(.)$, a distribution of employed workers over values, $G(.)$, and a fraction of employed workers, n, that satisfy the following requirements: (i) Given $p(.)$, $F(V)$ solves (3.1); (ii) Given $F(.)$ and $p(.)$, each offer $x \in \mathcal{V}$ is delivered by a contract that solves ($\mathcal{P}$), and the resulting value function of the firm is $J(x)$; (iii) Zero expected profit of recruiting: $q(x)J(x) = k$ for all $x \in [\underline{V}, \bar{V}]$, and $q(x)J(x) < k$ otherwise, where $q(x) = M^{-1}(p(x))$; (iv) G and n are stationary.[8]

Most elements of this definition are self-explanatory, except (iii). Requirement (iii) asks expected profit of recruiting to be zero for all $x \in [\underline{V}, \bar{V}]$, and it implies that an equilibrium indeed has meaningful tradeoffs between offers and matching rates in all submarkets. Given $J(.)$, the requirement yields the hiring rate function as $q(x) = k/J(x)$, and the employment rate function as $p(x) = M(k/J(x))$. Because $J(x)$ decreases in x (see (4.2)), the hiring rate is increasing and the employment rate decreasing in the offer, as I have used in previous sections. Note that requirement (iii) is imposed not just on equilibrium offers in $\mathcal{V}$, but on all offers in $[\underline{V}, \bar{V}]$. Because the lowest equilibrium offer is $v_1 = F(V_u) > V_u > \underline{V}$, $\mathcal{V}$ is a strict subset of $[\underline{V}, \bar{V}]$. Thus, requirement (iii) restricts the beliefs out of the equilibrium. By completing the markets, this restriction refines the set of equilibria and has been commonly used in directed search models, e.g., Moen (1997), Acemoglu and Shimer (1999b) and Delacroix and Shi (2006).[9]

A striking property of an equilibrium is *block recursivity*: Although the distribution of workers over wages or values depends on the aggregation of individuals' decisions, parts (i) – (iii) above are self-contained and independent of the distribution. Thus, the distribution plays *no role* in individuals' decisions, optimal contracts, the equilibrium functions, $p(.)$ and $q(.)$, and employed workers' job-to-job transitions. The reason for this independence is that directed search separates the applicants into different submarkets and, in each submarket, free entry of firms determines the number of vacancies independently of the distributions of workers in other submarkets. As a result, the matching rate functions, $p(.)$ and $q(.)$, are independent of the distributions.

To elaborate, consider the fixed-point problem formed by (i) – (iii) in the above definition. Given $q(.)$, the matching function yields the employment rate function, $p(.)$. Knowing $p(.)$ is sufficient for the workers to choose the optimal target, $F(.)$. The functions, $p(V)$ and $F(V)$, determine the quit rate of a worker at each V. For a firm, the worker's quit rate summarizes all the effects of competition on the firm's expected stream of profits. Thus, given the quit rate, the firm can calculate the expected value delivered by any wage contract, and hence can choose the contract optimally. This optimal choice determines the wage function, $w(.)$, and the firm's

[8] The model can be extended to allow for a sunk cost of creating a vacancy, C, in addition to the flow cost, k. Let R be the expected value of a vacancy, measured after a firm has incurred C. Then, R and the optimal offer solve: $\rho R = -k + \max_x \{q(x)[J(x) - R]\}$. Free entry of vacancies requires $R = C$. In this economy, (iii) in the equilibrium definition is modified as $q(x)[J(x) - C] \leq k + \rho C$, with equality for all $x \in [\underline{V}, \bar{V}]$. An equilibrium is well defined if either $k > 0$ or $\rho > 0$. Only when $k = \rho = 0$ is there no finite R that satisfies $R = C$.

[9] To see why there can be missing markets in general, suppose that all agents believe that no one will participate in submarket x. With such beliefs, no firm will post a vacancy and no worker will search in submarket x. Thus, the beliefs that submarket x will be missing is self-fulfilling. This outcome of a missing market may not be robust to a trembling-hand event that exogenously puts some firms in submarket x.

value function, $J(.)$. Finally, the free-entry condition ties the loop by determining the hiring rate function $q(.)$ that, in an equilibrium, must be the same as the one with which the process started. The distributions of offers and workers do not appear in this process.

Note that an equilibrium is block recursive even if there is exogenous separation into unemployment, which I have assumed away. With such separation, the value for a worker can still be determined given the function, $p(.)$, without any reference to the distributions.[10]

Block recursivity relies critically on endogenous separation of workers, which is an implication of directed search. Not surprisingly, undirected search models (e.g., Burdett and Mortensen, 1998, and BC, 2003) do not have this property. With undirected search, a worker's quit rate is a function of the distribution of offers because the worker can receive an offer anywhere in the distribution of offers, and a firm's hiring rate is a function of the distribution of workers because all workers whose current values are less than the firm's offer will accept the offer. Thus, the distributions of offers and workers affect individuals' decisions and contracts in undirected search models. In turn, these decisions and contracts affect the flows of workers that determine the distribution of workers. The two-way dependence and the dimensionality of the distribution make an equilibrium analysis complicated in undirected search models. Block recursivity simplifies an equilibrium drastically.

4.2. Existence of an Equilibrium

This subsection determines the equilibrium functions, $p(.)$, $q(.)$, $w(.)$, $F(.)$ and $J(.)$. I refer to existence of these functions as existence of an equilibrium, although an equilibrium also involves the distribution of workers that will be determined in section 6 later.

The following procedure formalizes the fixed-point problem discussed above. It is more convenient to develop a mapping on the wage function, $w(V)$, than on $q(.)$. Start with an arbitrary function, $w(.)$. First, integrating (4.2), and using $J(\bar{V}) = \underline{J} = k/\bar{q}$ (see (3.8)), I get:

$$J_w(V) = k/\bar{q} + \int_V^{\bar{V}} \frac{1}{u'(w(z))} dz. \tag{4.3}$$

The subscript w on J, and on (q, p, F, S) below, indicates the dependence on the initial function w. Second, the zero-profit condition yields: $q_w(V) = k/J_w(V)$. Since $p = M(q)$, then

$$p_w(V) = M\left(\frac{k}{J_w(V)}\right). \tag{4.4}$$

Third, with $p_w(V)$, the solution to (3.1) yields a worker's optimal search as $F_w(V)$, and the expected gain from search as $S_w(V)$. (3.3) yields $\dot{V}_w$, and (3.5) yields $dJ_w(V(t))/dt$.

Fourth, I combine (3.10) with (3.5) and (3.3). Recall that optimal contracts require $\dot{V}(t) > 0$ for all $t < \infty$ (see Lemma 3.2). However, $\dot{V}_w(t)$ constructed from an arbitrary w may not necessarily be positive. To ensure that every step of the equilibrium mapping satisfies $\dot{V}_w \geq 0$, I

[10] If the number of firms is fixed, rather than being determined by free entry, the expected value of recruiting is endogenous and depends on the distribution of workers. Even in this case, the distribution plays only a limited role because it affects individuals' decisions and the functions $p(.)$ and $q(.)$ entirely through a one-dimensional object, i.e., the expected value of recruiting.

modify (3.10) as $d\tilde{J}_w(t)/dt = -\max\{0, \dot{V}_w\}/u'(\tilde{w})$. Substituting $\dot{V}_w$ from (3.3), and $d\tilde{J}_w(t)/dt$ from (3.5), into (3.10), I get $w(V) = \psi w(V)$, where the mapping ψ is defined as follows:

$$\psi w(V) \equiv y - [r + p_w(F_w(V))] J_w(V) - \frac{\max\{0, rV - S_w(V) - u(w(V))\}}{u'(w(V))}. \tag{4.5}$$

The equilibrium wage function is a fixed point of ψ. With this fixed point, the first three steps above recover $q(.)$, $p(.)$, $J(.)$, $F(.)$ and $S(.)$ in an equilibrium. Clearly, all these functions are independent of the distribution of workers.

To characterize the fixed point of ψ, define:

$$\begin{aligned} \Omega = \; & \{w(V) : w(V) \in [\underline{w}, \bar{w}] \text{ for all } V;\; w(\bar{V}) = \bar{w}; \\ & \text{and } w(V) \text{ is continuous and (weakly) increasing}\}, \end{aligned} \tag{4.6}$$

$$\Omega' = \{w \in \Omega : w(V) \text{ is strictly increasing for all } V < \bar{V}\}. \tag{4.7}$$

The equilibrium wage function must lie in Ω' (see Lemma 3.2). In addition, I must verify that the equilibrium wage function induces a function $p(.)$, through (4.4), that indeed satisfies (2.2). To this end, I impose the following assumption on the matching function $M(q)$:

Assumption 1. *(i) $M(q)$ is continuous and $q(V) \in [\underline{q}, \bar{q}]$ for all V, where $\underline{q}$ will be specified in (4.8) and $\bar{q} < \infty$; (ii) $M'(q) < 0$ and $M(\bar{q}) = 0$; (iii) $M(q)$ is twice differentiable for all $q \in [\underline{q}, \bar{q}]$, where $|M'| \leq m_1$ and $|M''| \leq m_2$ for some finite constants m_1 and m_2; (iv) $qM''(q) + 2M'(q) \leq 0$.*

Part (i) is a regularity condition. In particular, the upper bound on q is imposed to apply a fixed-point theorem on bounded and continuous functions. Part (ii) captures the intuitive feature that if it is easy for a firm to fill a vacancy, it must be difficult for a worker to obtain a job. In the extreme case where a firm can fill a vacancy at the maximum rate, the employment rate is 0. Part (iii) simplifies the proof of existence significantly. By restricting convexity of $M(q)$, part (iv) helps establishing concavity of $p(.)$, which is stated in (2.2) and used to ensure uniqueness of each worker's optimal search decision. Assumption 1 is satisfied by the so-called telegraph matching function, $\mathcal{M}(\theta, 1) = \bar{q}\theta/(1+\theta)$, which implies $M(q) = \bar{q} - q$.[11]

Next, I specify the following bounds on various functions. Define:

$$\underline{J} \equiv k/\bar{q}, \quad \bar{J} \equiv J_{\bar{w}}(\underline{V}), \quad \underline{q} \equiv k/\bar{J}, \quad \bar{p} \equiv M(\underline{q}), \quad \bar{S} \equiv S_{\bar{w}}(\underline{V}). \tag{4.8}$$

Because $J_w(V)$, $p_w(V)$ and $S_w(V)$ are decreasing in V, and $q_w(V)$ increasing in V, then

$$J_w(V) \in [\underline{J}, \bar{J}], \; q_w(V) \in [\underline{q}, \bar{q}], \; p_w(V) \in [0, \bar{p}], \; S_w(V) \in [0, \bar{S}], \text{ all } w \in \Omega, \text{ all } V.$$

Choose the lower bound on wages, $\underline{w}$, to be a strictly positive number sufficiently close to 0.

[11] As another example, consider the Cobb-Douglas matching function, which has $\mathcal{M}(\theta, 1) = \theta^\alpha$, where $\alpha \in (0, 1)$. This function implies $p = \hat{M}(q) \equiv q^{(\alpha-1)/\alpha}$. Let $\bar{q}$ be a sufficiently large but finite constant, and let $M(q) = \hat{M}(q) - \hat{M}(\bar{q})$. Then $M(q)$ satisfies Assumption 1 iff $\alpha \geq 1/2$.

Assumption 2. *Assume that b, $\underline{V}$ and $\underline{w}$ satisfy:*

$$(0<)\ b<\bar{w}=y-rk/\bar{q}, \tag{4.9}$$

$$y-\left[r+p_{\bar{w}}\left(F_{\bar{w}}\left(\underline{V}\right)\right)\right]\bar{J}\geq\underline{w}+\frac{\left[u\left(b\right)-S_{\underline{w}}\left(\underline{V}\right)-u\left(\underline{w}\right)\right]}{u'\left(\underline{w}\right)}, \tag{4.10}$$

$$1+\frac{u''\left(w\right)}{\left[u'\left(w\right)\right]^{2}}\left[u\left(\bar{w}\right)-u\left(w\right)\right]\geq 0,\ \textit{all}\ w\in\left[\underline{w},\bar{w}\right]. \tag{4.11}$$

Note that all elements in the above assumption can be derived exclusively from exogenous objects of the model. (4.9) is necessary for there to be any worker employed. (4.10) is sufficient for $\psi w(V)\geq\underline{w}$ for all V, and (4.11) sufficient for ψ to map increasing functions into increasing functions. There is a non-empty region of parameter values that satisfy all of these conditions.[12]

The following theorem establishes existence of an equilibrium that indeed satisfies (2.2) (see Appendix B for a proof):

Theorem 4.1. *Maintain Assumptions 1 and 2. The mapping ψ has a fixed point $w^*\in\Omega'$. Moreover, the equilibrium has the following properties: (i) $J_{w^*}(V)$ is strictly positive, bounded in $\left[\underline{J},\bar{J}\right]$, strictly decreasing, strictly concave, and continuously differentiable for all V, with $J_{w^*}\left(\bar{V}\right)=\underline{J}$; (ii) $p_{w^*}\left(V\right)$ has all the properties in (2.2) and is strictly concave for all $V<\bar{V}$; (iii) $\dot{V}_{w^*}>0$ and $dJ_{w^*}\left(V\left(t\right)\right)/dt<0$ for all $V<\bar{V}$.*

Remark 1. *Although I have focused on an equilibrium that satisfies (2.2), all equilibria must have a strictly decreasing $p\left(.\right)$. If $p\left(V_2\right)\geq p\left(V_1\right)$ for some $V_2>V_1$, then $q\left(V_2\right)\leq q\left(V_1\right)$. In this case, no worker would apply to V_1, no firm would recruit at V_2, and so V_1 and V_2 could not both be equilibrium offers. Similarly, $p\left(.\right)$ must be continuous in all equilibria. In contrast, not all equilibria necessarily have a concave and differentiable $p\left(.\right)$. However, it is natural to focus on equilibria with a concave and differentiable $p\left(.\right)$. Concavity of $p\left(.\right)$ is useful for ensuring that each worker's optimal search decision is unique, and differentiability of $p\left(.\right)$ allows me to characterize this optimal decision with the first-order condition.*

I will suppress the asterisk on w^* and the subscript w^* on the functions J, p, q, F and S. Moreover, I will focus on a wage function, $w\left(V\right)$, that is differentiable.

5. Job Transitions, Wage Mobility and Policy Analysis

A typical worker in this model experiences continuous wage increases when he stays with a job and discrete jumps in wages when he transits to another job. For example, consider a worker in

[12] (4.9) can be easily satisfied. By choosing $\underline{w}$ sufficiently close to 0, and using the assumption $u'\left(0\right)\rightarrow\infty$, I can ensure (4.10) if $\left[r+p_{\bar{w}}\left(F_{\bar{w}}\left(\underline{V}\right)\right)\right]\bar{J}<y$. Because the left-hand side of this inequality is a decreasing function of $\underline{V}$, the inequality puts a lower bound on $\underline{V}$. This lower bound is smaller than $\bar{V}$, because $\left[r+p_{\bar{w}}\left(F_{\bar{w}}\left(\bar{V}\right)\right)\right]J_{\bar{w}}\left(\bar{V}\right)=r\underline{J}<y$. Using the definition of $\underline{V}$, I can translate this lower bound on $\underline{V}$ into a lower bound on b, which is smaller than $\bar{w}$. Hence, there are values of b that satisfy both (4.9) and (4.10). Finally, there are utility functions that satisfy (4.11). For example, the utility function with constant relative risk aversion satisfies (4.11) if the relative risk aversion is lower than a critical level.

unemployment. The worker's value is V_u and he applies for the offer $v_1 = F(V_u)$. If he obtains the offer, the value jumps to v_1, and the target of his next search is $v_2 = F(v_1)$. If the worker obtains the next offer, his value jumps to v_2. If the worker fails to obtain the offer v_2, the value for the worker increases continuously according to the contract. In both cases, the worker revises the target of search according to $F(.)$. This process continues to increase the worker's value toward $\bar{V}$ asymptotically until the worker dies.

The above process has the following predictions that are consistent with the stylized facts described in the introduction. First, wages and values strictly increase with tenure, as shown by $w'(V) > 0$ and $\dot{V}(t) > 0$. Second, the rate at which a worker quits a job for a better offer strictly decreases with tenure and wages, as shown by the result that $p(F(V))$ strictly decreases in V. The cause for this feature is directed search, rather than the fact that a low-wage worker has more wage levels to which he can transit to than a high-wage worker does. With directed search, low-wage workers optimally choose to search for relatively low offers that are easier to get, and so they make job transitions with higher probabilities than high-wage workers do. Third, wage mobility is limited endogenously, because the workers at a wage $w(V)$ optimally choose to search only for the contract that starts at the wage $w(F(V))$. The lower a worker's current wage, the lower the future wage he will move to in the next job transition.

Now consider two policies: an increase in the unemployment benefit, b, and a minimum-wage requirement, $w \geq w_{\min}$. For the minimum wage to be non-trivial, assume that $w(v_1) < w_{\min}$, where $w(v_1)$ is the lowest equilibrium wage in the absence of the minimum wage. The following corollary summarizes the effects of these policies (the proof is straightforward and omitted):

Corollary 5.1. *Changes in b and $w_{\min}$ do not affect the functions $w(.)$, $F(.)$, $p(.)$, $q(.)$, and $J(.)$. Hence, they do not affect an employed worker's transitions or contracts, conditional on the worker's current wage. However, they affect the distribution of workers and increases the lowest offer in an equilibrium, v_1. Moreover, an increase in b increases the value for unemployed workers, V_u, and reduces the measure of employed workers, n. An increase in $w_{\min}$ reduces n and V_u.*

To see more clearly the effects of the policies, suppose that the policies increase v_1 to $\hat{v}_1$. The offers in $[v_1, \hat{v}_1)$ are no longer equilibrium offers, but the new baseline contract is the tail of the original baseline contract that starts at $\hat{v}_1$. Since the latter is an equilibrium contract prior to the policy change, the set of equilibrium contracts after the policy change is a subset of the original set of equilibrium contracts. Conditional on a worker's current value (or wage), the worker's optimal application, the wage-tenure contract and the worker's transition rate to another job are all independent of the two policies. The reason for this independence is block recursivity of an equilibrium with directed search. Because the fixed-point problem that determines q, p, F, J and w involves only employed workers and not unemployed workers, its solution does not depend on policies that affect only unemployed workers.

The policies do affect aggregate activities in the current model, by affecting (v_1, V_u) and the distribution of workers. These effects, stated in Corollary 5.1, are intuitive. For example, a higher unemployment benefit reduces employment, because it makes unemployed workers "picky" about offers. Note that an increase in the minimum wage reduces the value for unemployed workers, despite that it raises the target value of an unemployed worker's search. The explanation is that the original target value, v_1, provides the best tradeoff for an unemployed applicant between the offer and the employment rate. By raising the target value, the minimum wage reduces an unemployed worker's transition rate into employment by so much that it cannot be adequately compensated by the rise in the target value.

Let me contrast the results in this section with those in BC (2003). Modeling search as an undirected process, BC has also shown that wages increase, and quit rates fall, with tenure. However, their model does not generate limited wage mobility; instead, even a worker at the bottom of the wage distribution can immediately transit to the top of the distribution. Moreover, because their model does not have block recursivity, the two policies above affect contracts and individuals' transitions through the distribution of workers. In particular, an increase in the unemployment benefit in that model increases the equilibrium distribution of offers, the job-to-job transition rate, and the slope of the wage-tenure contracts.[13]

6. Equilibrium Distribution of Workers

Let G be the cumulative distribution function of employed workers over $\mathcal{V} = \left[v_1, \bar{V}\right]$, and g the corresponding density function.[14] For any arbitrary $V \in \mathcal{V}$ and a small interval of time, dt, let me examine the flows in and out of the group of workers who are employed at values less than or equal to V. The measure of this group is $nG(V)$. The only inflow is unemployed workers who find matches at v_1, which is $(1-n)\,p(v_1)dt$. There are three outflows. First, death generates an outflow, $\delta n G(V)\,dt$. Second, the contracts increase the values for the workers in $(V - \dot{V}dt,\, V]$ above V, the flow of which is $n[G(V) - G(V - \dot{V}dt)]$. Third, some workers in the group quit for offers higher than V. These quitters are currently employed in $(F^{-1}(V), V]$ if $F^{-1}(V) \geq v_1$, and in $(v_1, V]$ if $F^{-1}(V) < v_1$. Thus, quitting generates the following outflow:

$$(dt)\, n \int_{\max\{v_1, F^{-1}(V)\}}^{V} p\left(F(z)\right) dG(z)\,.$$

Equating the inflows to the sum of outflows, and taking the limit $dt \downarrow 0$, I obtain:

$$\lim_{dt \downarrow 0} \frac{G(V) - G(V - \dot{V}dt)}{dt} = \frac{1-n}{n} p(v_1) - \delta G(V) - \int_{\max\{v_1, F^{-1}(V)\}}^{V} p(F(z))dG(z). \tag{6.1}$$

Theorem 6.1. *Denote $v_j = F^{(j)}(v_0),\ \ j = 1, 2, ...,$ where $F^{(0)}(v_0) = v_0 \equiv V_u$ and $F^{(j)}(v_0) = F(F^{(j-1)}(v_0))$. Then, $G(V)$ is continuous for all V, with $G(v_1) = 0$. The density function, $g(V)$, is continuous for all V, and differentiable except for $V = v_2$. Moreover,*

$$n = p(v_1)\,/\left[\delta + p(v_1)\right], \tag{6.2}$$

[13] Both the current model and BC (2003) assume that there is no exogenous separation into unemployment. If such exogenous separation is introduced, the two policies will affect equilibrium contracts and employed workers' transitions in the current model, because the value of unemployment will appear in the equation that determines the value for employed workers. Even in this extension of the current model, it is still true that the policies do not affect contracts and worker transitions through the distribution of workers.

[14] The distribution of employed workers over wages can be deduced as $G_w(w(V)) = G(V)$, with a density function $g_w(w(V)) = g(V)/w'(V)$.

$$g(V)\dot{V} = \delta\left[1 - G(V)\right] - \int_{\max\{v_1, F^{-1}(V)\}}^{V} p(F(z))dG(z). \tag{6.3}$$

With the function $T(V)$ in (4.1), define:

$$\Gamma(z_2, z_1) = \exp\left[-\int_{T(z_1)}^{T(z_2)} \left[\delta + p\left(F\left(V\left(t\right)\right)\right)\right] dt\right], \; z_1, z_2 \geq v_1. \tag{6.4}$$

Add a subscript j to $g(V)$ for $V \in [v_j, v_{j+1})$. g can be recursively solved piece-wise as follows:

$$g_1(V)\dot{V} = \delta\Gamma\left(V, v_1\right), \tag{6.5}$$

$$g_j(V)\dot{V} - g_j(v_j)\dot{v}_j\Gamma(V, v_j) = \int_{v_j}^{V} \Gamma(V, z)p(z)g_{j-1}(F^{-1}(z))dF^{-1}(z), \tag{6.6}$$

where (6.6) holds for $j \geq 2$. Moreover, $g_j(v_j) = \lim_{V \uparrow v_j} g_{j-1}(V)$ for all j.

The above theorem documents several features. First, the equilibrium distribution of employed workers is non-degenerate and continuous, despite that all matches are identical and search is directed. Both on-the-job search and wage-tenure contracts are important for this dispersion of values. If on-the-job search were prohibited, only one value, v_1, would be offered in an equilibrium, as in most models of directed search with homogeneous matches. On-the-job search produces jumps in values, and hence a non-degenerate distribution of values. However, without wage-tenure contracts, on-the-job search alone would only produce a wage ladder formed by the set, $\{v_1, v_2, ..., \bar{V}\}$, as in Delacroix and Shi (2006). Wage-tenure contracts provide continuous increases in the values to fill in the gaps between any two levels in this discrete set.

Second, there is no mass point anywhere in the support of the distribution. It is particularly remarkable that there is no build-up of workers at v_1. Although all unemployed workers only apply for v_1, all workers at v_1 move out of v_1 in any arbitrarily short length of time, as a result of quits, death, or wage increases in the contracts. Moreover, the density function is differentiable except at $V = v_2$. It is not differentiable at v_2 because offers above v_2 receive applications from employed workers but offers below v_2 do not.[15]

Finally, more workers are employed at low values than at high values, because the job-to-job transition rate decreases sharply in the target value. In particular, as V approaches $\bar{V}$, the employment rate declines to 0, which requires the measure of recruiting firms per applicant to approach zero. Thus, the density function $g(V)$ can be decreasing for V close to $\bar{V}$.

7. Conclusion

I have constructed a theoretical framework in which firms offer wage-tenure contracts to direct the search by risk-averse workers. All workers can search, on or off the job. I have characterized an equilibrium and proved its existence. The equilibrium generates a non-degenerate, continuous

[15] Similarly, the density function of offers is discontinuous, because a mass of firms recruit at v_1 but no firm recruits at $V \in (v_1, v_2)$. To eliminate non-differentiability of g at v_2 and discontinuity of the offer density, an earlier version of this paper assumes that b is distributed in an interval whose upper bound is equal to $\bar{w}$.

distribution of employed workers over the values of contracts, despite that all matches are identical and workers observe all offers. A striking property is that the equilibrium is block recursive; that is, individuals' optimal decisions and optimal contracts are independent of the distribution of workers. This property makes the equilibrium analysis tractable. Consistent with stylized facts, the equilibrium predicts that (i) wages increase with tenure, (ii) job-to-job transitions decrease with tenure and wages, and (iii) wage mobility is limited in the sense that the lower the worker's wage, the lower the future wage a worker will move to in the next job transition. Moreover, block recursivity implies that changes in the unemployment benefit and the minimum wage have no effect on an employed worker's job-to-job transitions and contracts.

The theoretical framework is tractable for a wide range of applications and extensions, because of block recursivity. In particular, Menzio and Shi (2008) incorporate aggregate and match-specific shocks into the model to examine dynamics and business cycles with on-the-job search.

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Appendix

A. Proof of Lemma 3.1

The result $F(\bar{V}) = \bar{V}$ is evident. Let $V < \bar{V}$, and denote $K(x,V) = p(x)(x - V)$. Let V_1 and V_2 be two arbitrary values with $V_1 < V_2 < \bar{V}$, and denote $F_i = F(V_i)$, where $i = 1, 2$.

For part (i), because $p(.)$ is bounded and continuous, $K(x,V)$ is bounded and continuous. Thus, the maximization problem in (3.1) has a solution. Since $K(x,V) > 0$ for all $x \in \left(V, \bar{V}\right)$, and $K(V,V) = 0 = K(\bar{V}, V)$, the solutions are interior. Interior solutions and differentiability of $p(.)$ imply that $F(V)$ is given by the first-order condition, (3.2). Take two distinct values, V_2 and V_1. They must generate different values for the right-hand side of (3.2). Thus, $F(V_1) \cap F(V_2) = \varnothing$ for all $V_2 \neq V_1$. This result implies that $K(F_i, V_i) > K(F_j, V_i)$ for $j \neq i$. I have:

$$0 > [K(F_2, V_1) - K(F_1, V_1)] + [K(F_1, V_2) - K(F_2, V_2)] = [p(F_2) - p(F_1)](V_2 - V_1).$$

Thus, $p(F_2) < p(F_1)$. Because $p(.)$ is strictly decreasing, $F(V_2) > F(V_1)$.

For part (ii), I show that $K(x,V)$ is strictly concave in x for all $x \in (V, \bar{V})$. Let x_1 and x_2 be two arbitrary values with $x_2 > x_1 > V$. Let $x_\alpha = \alpha x_1 + (1-\alpha)x_2$, where $\alpha \in (0,1)$. Then,

$$\begin{aligned} K(x_\alpha, V) &\geq [\alpha p(x_1) + (1-\alpha)p(x_2)]\,[\alpha(x_1 - V) + (1-\alpha)(x_2 - V)] \\ &= \alpha K(x_1, V) + (1-\alpha)K(x_2, V) + \alpha(1-\alpha)[p(x_1) - p(x_2)][x_2 - x_1] \\ &> \alpha K(x_1, V) + (1-\alpha)K(x_2, V). \end{aligned}$$

The first inequality comes from concavity of p, and the last from strictly decreasing $p(.)$. Thus, $K(x,V)$ is strictly concave in x, and $F(V)$ is unique. Uniqueness implies that $F(V)$ is continuous in V, by the Theorem of the Maximum (see Stokey and Lucas, 1989, p62).

For part (iii), note that $K(F_1, V_1) > K(F_2, V_1)$ and $K(F_2, V_2) > K(F_1, V_2)$. Then,

$$S(V_2) - S(V_1) > K(F_1, V_2) - K(F_1, V_1) = -p(F_1)(V_2 - V_1);$$

$$S(V_2) - S(V_1) < K(F_2, V_2) - K(F_2, V_1) = -p(F_2)(V_2 - V_1).$$

Divide the two inequalities by $(V_2 - V_1)$ and take the limit $V_2 \to V_1$. Because $F(.)$ is continuous, the limit shows that $S'(V_1) = -p(F_1)$. Since V_1 is arbitrary, part (iii) holds for all V.

For part (iv), because p is decreasing and concave, $p(F_1) \geq p(F_2) - p'(F_1)(F_2 - F_1)$. Substituting this inequality into (3.2) yields:

$$V_2 - V_1 \geq 2(F_2 - F_1) + p(F_2)\frac{p'(F_1) - p'(F_2)}{p'(F_1)\,p'(F_2)} \geq 2(F_2 - F_1).$$

This implies $F_2 - F_1 \leq (V_2 - V_1)/2$, and so F is Lipschitz.

For part (v), if p is twice differentiable, then differentiating (3.2) generates $F'(V)$. Part (iv) implies $F'(V) \leq 1/2$. Hence, $S''(V) = -p'(F(V))F'(V)$. **QED**

B. Proof of Theorem 4.1

Consider the sets, Ω and Ω', defined by (4.6) and (4.7) respectively. It can be verified that Ω is non-empty, closed, bounded and convex. Lemma B.1 below shows that properties (i) and (ii) in Theorem 4.1 are satisfied not only by J_{w^*} and p_{w^*}, but also by J_w and p_w that are constructed through (4.3) and (4.4) with any arbitrary $w \in \Omega$. Thus, (2.2) and parts (i) - (iv) of Lemma

3.1 hold in every iteration of the mapping ψ (defined by (4.5)), not just with the fixed point. In particular, $F_w(V)$ is strictly increasing and satisfies (3.2) in every iteration. Lemma B.2 below, whose proof uses (4.11), describes additional properties that will be used in the proofs of Lemmas B.3, B.4 and B.5. The latter three lemmas establish that the mapping ψ satisfies the conditions of the Schauder fixed-point theorem (see Stokey and Lucas, 1989, p520). Therefore, ψ has a fixed point in Ω, denoted as w^*. Lemma B.3 then implies $w^*(V) = (\psi w^*)(V) \in \Omega'$.

Finally, I show that $\dot{V}_{w^*} > 0$ for all $V < \bar{V}$, as in (iii) of Theorem 4.1. Once this is done, (4.5) implies $dJ_{w^*}(V(t))/dt = -\max\{0, \dot{V}^*_w(V)\}/u'(w^*(V)) < 0$ for all $V < \bar{V}$. Suppose that $\dot{V}_{w^*} \leq 0$ for some $V_1 < \bar{V}$, contrary to the theorem. In this case, (4.5) implies: $w^*(V_1) = y - [r + p_{w^*}(F_{w^*}(V_1))] J_{w^*}(V_1)$, and so

$$\dot{V}_{w^*} = rV_1 - S_{w^*}(V_1) - u(y - [r + p_{w^*}(F_{w^*}(V_1))] J_{w^*}(V_1)).$$

With the properties of J_{w^*} and p_{w^*} in Lemma B.1, the right-hand side of the above equation is strictly decreasing in V_1, and equal to 0 at $V_1 = \bar{V}$. Thus, $\dot{V}_{w^*} > 0$ at V_1. A contradiction. This completes the proof of Theorem 4.1. **QED**

The proofs of Lemmas B.1, B.2, B.3, and B.4 below are omitted but can be found in the supplementary material.

Lemma B.1. *For any $w \in \Omega$, $J_w(V)$ and $p_w(V)$, defined by (4.3) and (4.4), have the following properties: (i) They are bounded, with $J_w(V) \in [\underline{J}, \bar{J}]$, $p_w(V) \in [0, \bar{p}]$, $J_w(\bar{V}) = \underline{J}$, and $p_w(\bar{V}) = 0$, where $\underline{J}$, $\bar{J}$ and $\bar{p}$ are defined in (4.8); (ii) They are strictly decreasing, continuously differentiable, and concave for all V; (iii) If $w \in \Omega'$, then $J_w(V)$ and $p_w(V)$ are strictly concave.*

Lemma B.2. *Let $w_1, w_2, w \in \Omega$. (i) $p_w(F_w(V))$ is increasing in w in the sense that if $w_2(V) \geq w_1(V)$ for all V, then $p_{w_2}(F_{w_2}(V)) \geq p_{w_1}(F_{w_1}(V))$ for all V; (ii) For all $V_2 \geq V_1$,*

$$\frac{u(w(V_2)) - u(w(V_1))}{u'(w(V_1))} \geq \Delta \geq \frac{[rV_1 - S_w(V_1)] - [rV_2 - S_w(V_2)]}{u'(w(V_1))}, \tag{B.1}$$

where Δ is defined as follows:

$$\begin{aligned} \Delta = \; & \tfrac{1}{u'(w(V_1))} \max\{0, rV_1 - S_w(V_1) - u(w(V_1))\} \\ & - \tfrac{1}{u'(w(V_2))} \max\{0, rV_2 - S_w(V_2) - u(w(V_2))\}. \end{aligned} \tag{B.2}$$

Lemma B.3. *$\psi : \Omega \to \Omega' \subset \Omega$.*

Lemma B.4. *ψ is Lipschitz continuous in the sup norm.*

Lemma B.5. *With an arbitrary $w \in \Omega$, define $\psi^0 w = w$ and $\psi^{j+1} w = \psi(\psi^j w)$ for $j = 0, 1, 2, \ldots$. The family of functions, $\{\psi^j w\}_{j=0}^{\infty}$, is equicontinuous.*

Proof. Take an arbitrary $w \in \Omega$ and construct the family, $\{\psi^j w\}_{j=0}^{\infty}$. The family is equicontinuous if it satisfies the following requirement (see Stokey and Lucas, 1989, p520): For any given $\varepsilon > 0$, there exists $a > 0$ such that, for all V_1 and V_2,

$$|V_2 - V_1| < a \Longrightarrow \left|\psi^j w(V_2) - \psi^j w(V_1)\right| < \varepsilon, \quad \text{all } j. \tag{B.3}$$

I use the following procedure to establish (B.3). In the entire procedure, fix w as an arbitrary function in Ω and $\varepsilon > 0$ as an arbitrary number. First, because w is continuous, and the domain of w, $[\underline{w}, \bar{w}]$, is bounded and closed, w is uniformly continuous. Thus, there exists $a_0 > 0$ such that, for all V_1 and V_2, $|V_2 - V_1| < a_0 \Longrightarrow |w(V_2) - w(V_1)| < \varepsilon$. Second, I show that there exists $a_1 > 0$ such that, for all V_1 and V_2, if $|w(V_2) - w(V_1)| < \varepsilon$, then

$$|V_2 - V_1| < a_1 \Longrightarrow |\psi w(V_2) - \psi w(V_1)| < \varepsilon. \tag{B.4}$$

Let $a = \min\{a_0, a_1\}$. Then, w and ψw both satisfy (B.3). Third, replacing w with ψw, and a_0 with a, the above two steps yield that $\psi^2 w$ satisfies (B.3). Repeating this process but *fixing a at the level just defined*, it is easy to see that $\psi^j w$ satisfies (B.3) for all j.

Only (B.4) needs a proof. Take arbitrary V_1 and V_2 that satisfy: $|w(V_2) - w(V_1)| < \varepsilon$. As before, shorten the notation $f(V_i)$ to f_i, where f includes the functions w, J_w, F_w, p_w, S_w and ψw. Without loss of generality, assume $V_2 \geq V_1$. Since $w(V)$ and $\psi w(V)$ are increasing functions, then $w_2 \geq w_1$ and $\psi w_2 \geq \psi w_1$. With the first inequality in (B.1), I have:

$$\begin{aligned} (0 \leq)\ \ \psi w_2 - \psi w_1 \ &\leq [r + p_w(F_{w1})](J_{w1} - J_{w2}) \\ &\quad + J_{w2}[p_w(F_{w1}) - p_w(F_{w2})] + [u(w_2) - u(w_1)]/u'(w_1). \end{aligned} \tag{B.5}$$

Examine the three terms on the right-hand side in turn. Using (4.3), I obtain $0 \leq J_{w1} - J_{w2} \leq (V_2 - V_1)/u'(\bar{w})$. For the difference in $p_w(F)$, recall that $F_{w2} - F_{w1} \leq (V_2 - V_1)/2$ (see Lemma 3.1). Also, because $q_w \leq \bar{q}$, I can verify that $\left|\frac{dM(k/J_w)}{dJ_w}\right| \leq B_1 \equiv m_1 \bar{q}^2/k$, where m_1 is specified in Assumption 1. Thus,

$$\begin{aligned} 0 \ &\leq p_w(F_{w1}) - p_w(F_{w2}) = M\left(\tfrac{k}{J_w(F_{w1})}\right) - M\left(\tfrac{k}{J_w(F_{w2})}\right) \\ &\leq B_1[J_w(F_{w1}) - J_w(F_{w2})] \leq B_1[F_{w2} - F_{w1}]/u'(\bar{w}) \leq B_1(V_2 - V_1)/[2u'(\bar{w})]. \end{aligned} \tag{B.6}$$

To examine the last term in (B.5), define $L(w) \equiv u(w) - u(w_1) - u'(w_1)(w - w_1) + \frac{\mu_1}{2}(w - w_1)^2$, where $\mu_1 \equiv \min_{w \in [\underline{w}, \bar{w}]} |u''(w)| > 0$. Because L is concave, and $L'(w_1) = 0$, $L(w)$ is maximized at $w = w_1$, and so $L(w_2) \leq L(w_1) = 0$. Since $w_1 \geq \underline{w}$, I get:

$$(0 \leq) \quad \frac{u(w_2) - u(w_1)}{u'(w_1)} \leq w_2 - w_1 - \frac{\mu_1}{2u'(\underline{w})}(w_2 - w_1)^2. \tag{B.7}$$

The RHS of (B.7) is maximized at $w_2 - w_1 = [u'(\underline{w})/\mu_1]^{1/2}$. Recall that $w_2 - w_1 < \varepsilon$. If $\varepsilon \leq [u'(\underline{w})/\mu_1]^{1/2}$, the RHS of (B.7) is increasing in $(w_2 - w_1)$, and so it is strictly smaller than the value at $w_2 - w_1 = \varepsilon$, which is $\left[\varepsilon - \frac{\mu_1}{2u'(\underline{w})}\varepsilon^2\right]$. If $\varepsilon > [u'(\underline{w})/\mu_1]^{1/2}$, then RHS(B.7) $< \frac{1}{2}[u'(\underline{w})/\mu_1]^{1/2} < \varepsilon/2$. In both cases, I have:

$$(0 \leq) \quad \frac{u(w_2) - u(w_1)}{u'(w_1)} < \varepsilon \max\left\{\frac{1}{2}, 1 - \frac{\mu_1 \varepsilon}{2u'(\underline{w})}\right\} = \varepsilon - \varepsilon \min\left\{\frac{1}{2}, \frac{\mu_1 \varepsilon}{2u'(\underline{w})}\right\}. \tag{B.8}$$

Substitute the above bounds on the terms on the RHS of (B.5). Noting that $p_w(F_w) \leq \bar{p}$ and $J_w \leq \bar{J}$, where $\bar{p}$ and $\bar{J}$ are defined in (4.8), I obtain:

$$(0 \leq) \quad \psi w_2 - \psi w_1 < A_3(V_2 - V_1) + \varepsilon - \varepsilon \min\left\{\frac{1}{2}, \frac{\mu_1 \varepsilon}{2u'(\underline{w})}\right\}, \tag{B.9}$$

where $A_3 \in (0, \infty)$ is defined as $A_3 \equiv \left(r + \bar{p} + \frac{\bar{J}B_1}{2}\right)/u'(\bar{w})$. A sufficient condition for $\psi w_2 - \psi w_1 < \varepsilon$ is that RHS(B.9) $\leq \varepsilon$. This condition can be expressed as $0 \leq V_2 - V_1 \leq a_1$, where $a_1 \equiv \frac{\varepsilon}{A_3}\min\left\{\frac{1}{2}, \frac{\mu_1 \varepsilon}{2u'(\underline{w})}\right\}$. Because $A_3 \in (0, \infty)$, $\underline{w} > 0$, and $\mu_1 \in (0, \infty)$, then $a_1 > 0$, and (B.4) holds. Moreover, a_1 is independent of a_0, given ε. This completes the proof of Lemma B.5. **QED**

Directed Search for Equilibrium Wage-Tenure Contracts,
Supplementary Material: Supplementary Appendix
Shouyong Shi
August 2008

Abstract. This supplementary appendix provides the proofs of Lemmas 3.2, B.1, B.2, B.3, and B.4, and Theorem 6.1. Lemmas B.1, B.2, B.3, and B.4 are used in the proof of Theorem 4.1 in the paper. All cross references to equations and sections use the numbering in the paper.

C. Supplementary Appendix

C.1. Proof of Lemma 3.2

Consider the firm's optimization problem, $(\mathcal{P})$. The state variable is V that obeys (3.3). Treat γ, defined in (3.6), as an auxiliary state variable whose law of motion is:

$$\frac{d}{dt}\gamma\left(t,t_a\right) = -\left[r + p(F(V(t)))\right]\gamma(t,t_a). \tag{C.1}$$

Denote the shadow price of V as Λ_V, and of γ as Λ_γ. Then, the Hamiltonian of $(\mathcal{P})$ is:

$$\mathcal{H}(t) = (y-\tilde{w})\,\gamma(t,0) + \Lambda_V\left[rV - S(V) - u(\tilde{w})\right] - \Lambda_\gamma\left[r + p\left(F\left(V\right)\right)\right]\gamma(t,0),$$

where I have suppressed the dependence of the variables on t, except that of γ. Denote $\Lambda_c(t) = \Lambda_V(t)/\gamma(t,0)$, where the subscript c indicates the "current value". The optimality conditions of $\tilde{w}$, V and γ are as follows:

$$-u'(\tilde{w})\Lambda_c - 1 \leq 0 \text{ and } \tilde{w} \geq 0, \text{ with complementary slackness;} \tag{C.2}$$

$$\dot{\Lambda}_c = \Lambda_\gamma dp\left(F\left(V\right)\right)/dV; \tag{C.3}$$

$$\dot{\Lambda}_\gamma = -\left(y-\tilde{w}\right) + \Lambda_\gamma\left[r + p\left(F\left(V\right)\right)\right]. \tag{C.4}$$

To derive (C.3), I have used the fact that $S'(V) = -p\left(F\left(V\right)\right)$ (see Lemma 3.1).

Using (C.1), I can rewrite (C.4) as $\frac{d}{dt}\left[\gamma(t,0)\Lambda_\gamma(t)\right] = -\left[y-\tilde{w}\left(t\right)\right]\gamma(t,0)$. Integrating this equation under the transversality condition, $\lim_{t\to\infty}\gamma(t,0)\Lambda_\gamma(t) = 0$, I get $\Lambda_\gamma(t) = \tilde{J}(t)$ for all t, where $\tilde{J}(.)$ is given by (3.7). Substituting $\Lambda_\gamma = \tilde{J}$ into (C.3) and the Hamiltonian yields:

$$\dot{\Lambda}_c = \tilde{J}\,\frac{dp\left(F\left(V\right)\right)}{dV}, \tag{C.5}$$

$$\mathcal{H}(t) = \gamma(t,0)\left[-d\tilde{J}\left(t\right)/dt + \Lambda_c\left(t\right)\dot{V}\left(t\right)\right]. \tag{C.6}$$

Because $p\left(F\left(V\right)\right)$ strictly decreases in V for all $t<\infty$, $\dot{\Lambda}_c(t) < 0$ for all $t < \infty$.

Define t_0 by $\Lambda_c\left(t_0\right) = 0$. There is at most one such t_0, because $\dot{\Lambda}_c < 0$. Moreover, $\Lambda_c(t) > 0$ for all $t < t_0$, and $\Lambda_c(t) < 0$ for all $t > t_0$. For all $t \leq t_0$, $-u'\left(\tilde{w}\left(t\right)\right)\Lambda_c(t) \leq 0$, in which case (C.2) implies $\tilde{w}(t) = 0$. For all $t > t_0$, the assumption $u'(0) = \infty$ ensures $\tilde{w}\left(t\right) > 0$: if $\tilde{w}(t) = 0$, then $-u'\left(\tilde{w}\left(t\right)\right)\Lambda_c\left(t\right) - 1 = \infty > 0$, which contradicts (C.2).

The remainder of the proof establishes a sequence of results. First, $d\tilde{w}\left(t\right)/dt > 0$ for all $t > t_0$. Suppose, to the contrary, that $d\tilde{w}\left(t\right)/dt \leq 0$ at $t = t_1$ for some $t_1 \in (t_0,\infty)$. Because $\dot{\Lambda}_c < 0$, then

$$\frac{d}{dt}\left[-u'\left(\tilde{w}\left(t\right)\right)\Lambda_c\left(t\right)\right] > -u''\left(\tilde{w}\left(t\right)\right)\Lambda_c\left(t\right)\frac{d\tilde{w}\left(t\right)}{dt}, \quad \text{all } t<\infty.$$

Because $d\tilde{w}(t)/dt \leq 0$ at $t = t_1$ and $\Lambda_c(t) < 0$ for $t > t_0$, the derivative above on the RHS is strictly positive for t near t_1. As a result, there exists $\varepsilon > 0$ such that $-u'(\tilde{w}(t))\Lambda_c(t) > -u'(\tilde{w}(t_1))\Lambda_c(t_1) = 1$ for $t \in (t_1, t_1 + \varepsilon]$, where the equality follows from (C.2) and $\tilde{w}(t_1) > 0$. This result contradicts (C.2). Thus, I have shown that the wage path has the following form:

$$\begin{cases} \tilde{w}(t) = 0, & \text{for } t < t_0; \\ \tilde{w}(t) > 0 \text{ and } d\tilde{w}(t)/dt > 0, & \text{for } t \in (t_0, \infty). \end{cases} \tag{C.7}$$

Because $\tilde{w}(t)$ is bounded for all t, and increasing, then $\tilde{w}(t) \nearrow \bar{w}$ as $t \to \infty$.

Second, $\mathcal{H}(t) = 0$ for all t. Differentiating (C.6) with respect to t and substituting (C.5) yields:

$$\frac{d\mathcal{H}(t)}{dt} = -\gamma(t,0)\left[1 + u'(\tilde{w}(t))\Lambda_c(t)\right]\frac{d\tilde{w}(t)}{dt} = 0,$$

where the second equality uses the results that $d\tilde{w}(t)/dt = 0$ for $t < t_0$, and $1 + u'(\tilde{w}(t))\Lambda_c(t) = 0$ for $t \geq t_0$. Because $\lim_{t\to\infty}\mathcal{H}(t) = 0$, then $\mathcal{H}(t) = 0$ for all t, which can be rewritten as

$$d\tilde{J}(t)/dt = \Lambda_c(t)\dot{V}(t), \quad \text{all } t. \tag{C.8}$$

Third, $\dot{V}(t) > 0$ for all $t < \infty$, and $\tilde{J}(t)$ is maximized at $t = t_0$. Suppose, to the contrary, that $\dot{V}(t_1) \leq 0$ for some $t_1 < \infty$. If $t_1 > t_0$, then $d\tilde{w}(t)/dt > 0$ for all $t \in [t_1, \infty)$ (see (C.7)). Differentiating (3.3) yields:

$$\frac{d\dot{V}(t)}{dt} = [r + p(F(V(t)))]\dot{V}(t) - u'(\tilde{w}(t))\frac{d\tilde{w}(t)}{dt}.$$

$\dot{V}(t_1) \leq 0$ implies $d\dot{V}(t)/dt < 0$ at $t = t_1$. By induction, $d\dot{V}(t)/dt < 0$ for all $t \in [t_1, \infty)$. Thus, $V(t)$ strictly decreases toward $\bar{V}$ as t increases from t_1 to ∞, contradicting the fact that $V(t) \leq \bar{V}$ for all $t < \infty$. If $t_1 \leq t_0$, then $\tilde{w}(t_1) = 0$ by (C.7), and so (3.3) implies: $rV(t_1) - S(V(t_1)) \leq u(0)$. This result and (3.4) yield:

$$rV_u - S(V_u) - u(b) - [rV(t_1) - S(V(t_1))] + u(0) \geq 0.$$

Because $S'(V) < 0$, the left-hand side of the equation is strictly decreasing in $V(t_1)$. Because the left-hand side is negative at $V(t_1) = V_u$, then $V(t_1) < V_u$. In this case, the worker will quit into unemployment, which will be suboptimal to the firm. A contradiction.

Recall that $\Lambda_c(t) > 0$ for all $t < t_0$, and $\Lambda_c(t) < 0$ for all $t > t_0$. (C.8) and $\dot{V} > 0$ imply that $d\tilde{J}(t)/dt > 0$ for all $t < t_0$, and $d\tilde{J}(t)/dt < 0$ for all $t > t_0$. That is, $\tilde{J}(t)$ is maximized at $t = t_0$.

Fourth, $t_0 \leq 0$; thus, $\tilde{w}(t) > 0$ for all $t > 0$, and $d\tilde{w}(t)/dt > 0$ for all $t < \infty$ (see (C.7)). Suppose $t_0 > 0$, to the contrary. Then, $\tilde{J}(t_0) > \tilde{J}(0)$ by the previous result. Let $\{\tilde{w}(t)\}_{t=0}^{\infty}$ be the optimal contract that generates $\tilde{J}(0)$ to the firm. Consider an alternative contract, $\{\hat{w}(t)\}_{t=0}^{\infty}$, where $\hat{w}(t) = \tilde{w}(t + t_0)$ for all t. This alternative contract is feasible and generates a higher value to the firm, $\tilde{J}(t_0)$, than the optimal contract. A contradiction.

Finally, (3.9) and (3.10) hold. Because $\tilde{w}(t) > 0$ for all t, then $\Lambda_c(t) = -1/u'(\tilde{w}(t))$ for all t. Differentiating this equation with respect to t, and substituting (C.5), I get (3.9). Substituting Λ_c into (C.8) yields (3.10). Because $\dot{V}(t) > 0$, and $\tilde{w}(t) > 0$, for all $t < \infty$, then $d\tilde{J}(t)/dt < 0$ for all $t < \infty$. **QED**

C.2. Proof of Lemma B.1

Let $w \in \Omega$. Part (i) of the lemma was established in the analysis immediately following (4.8). It is easy to verify from (4.3) that $J_w(V)$ is strictly decreasing and continuously differentiable, with $J_w'(V) = -1/u'(w(V)) < 0$. Since $w(V)$ is increasing, then $J_w'(V)$ is decreasing, and so $J_w(V)$ is (weakly) concave. Because $q_w(V) = k/J_w(V)$ and $p_w(V) = M(q_w(V))$, I have:

$$p_w'(V) = \frac{M'(q_w(V))[q_w(V)]^2}{u'(w(V))k} < 0,$$

where $M'(q) < 0$ by Assumption 1. This shows that $p_w(V)$ is strictly decreasing and continuously differentiable. Moreover, parts (iii) and (iv) of Assumption 1 imply that $\left[M'(q)q^2\right]$ is decreasing in q. Because $q_w(V)$ is increasing in V, $M'(q_w(V))[q_w(V)]^2$ is decreasing in V. Because $1/u'(w(V))$ is increasing in V, and $M' < 0$, then $p_w'(V)$ is decreasing. That is, $p_w(V)$ is (weakly) concave, and so part (ii) of the Lemma holds.

If $w \in \Omega'$, i.e., if $w(V)$ is strictly increasing for all $V < \bar{V}$, then it is straightforward to strengthen the argument for part (ii) to show that $J_w(V)$ and $p_w(V)$ are strictly concave, as stated in part (iii). **QED**

C.3. Proof of Lemma B.2

To prove part (i) of the lemma, pick two arbitrary functions $w_1, w_2 \in \Omega$, with $w_2(V) \geq w_1(V)$ for all V. Simplify the notation J_{w_i} to J_i, F_{w_i} to F_i and p_{w_i} to p_i, where $i = 1, 2$. Because $w_2(V) \geq w_1(V)$ for all V, (4.3) implies $J_2(V) \geq J_1(V)$, and the assumption $M' < 0$ implies $p_2(V) \geq p_1(V)$, for all V. Suppose, contrary to part (i) of the Lemma, that $p_1(F_1(V)) > p_2(F_2(V))$ for some V. Let $q_i = k/J_i(F_i(V))$, $i = 1, 2$. Because $p_i(F_i(V)) = M(q_i)$, and $M(q)$ is strictly decreasing in q, the supposition implies $q_1 < q_2$, and hence $J_1(F_1(V)) > J_2(F_2(V))$. Monotonicity of J_w in w implies $J_2(F_2(V)) \geq J_1(F_2(V))$. In this case, $J_1(F_1(V)) > J_1(F_2(V))$, and so $F_1(V) < F_2(V)$. With these results, I can derive:

$$\begin{aligned}
0 &< p_1(F_1(V)) - p_2(F_2(V)) \\
&= p_2'(F_2(V))[F_2(V) - V] - p_1'(F_1(V))[F_1(V) - V] \\
&< [p_2'(F_2(V)) - p_1'(F_1(V))][F_1(V) - V] \\
&= \tfrac{F_1(V)-V}{k}\left[\tfrac{M'(q_2)(q_2)^2}{u'(w_2(F_2(V)))} - \tfrac{M'(q_1)(q_1)^2}{u'(w_1(F_1(V)))}\right] \\
&\leq \tfrac{F_1(V)-V}{u'(w_1(F_1(V)))k}\left[M'(q_2)(q_2)^2 - M'(q_1)(q_1)^2\right].
\end{aligned}$$

The first inequality comes from the supposition, the first equality from (3.2), the second inequality from $F_2(V) > F_1(V)$ and $p_2'(F_2) < 0$, the second equality from computing $p_i'(F_i)$, and the last inequality from $M'(q_2) < 0$ and $w_2(F_2(V)) \geq w_1(F_2(V)) \geq w_1(F_1(V))$. Parts (iii) and (iv) of Assumption 1 imply that $M'(q)q^2$ is decreasing in q. Because $q_2 > q_1$, as shown above, the expression in the last line above is non-positive. A contradiction.

To prove part (ii) of the lemma, let $w \in \Omega$, and $V_2 \geq V_1$. Note that $w(V_2) \geq w(V_1)$, because $w \in \Omega$. Moreover, because $[rV - S_w(V)]$ is strictly increasing in V, $rV_2 - S_w(V_2) \geq rV_1 - S_w(V_1)$. Hence, the following inequality holds:

$$\Delta \leq \Delta_1 \equiv \frac{1}{u'(w(V_1))}\left[\begin{array}{l} \max\{0, rV_1 - S_w(V_1) - u(w(V_1))\} \\ -\max\{0, rV_1 - S_w(V_1) - u(w(V_2))\} \end{array}\right].$$

Consider all possible cases: (a) $rV_1 - S_w(V_1) \geq u(w(V_2))$; (b) $rV_1 - S_w(V_1) \leq u(w(V_1))$; and (c) $u(w(V_1)) < rV_1 - S_w(V_1) < u(w(V_2))$. In each case, it can be verified that

$$\Delta_1 \leq \frac{u(w(V_2)) - u(w(V_1))}{u'(w(V_1))}.$$

Thus, the first inequality in (B.1) holds.

To establish the second inequality in (B.1), I first show that

$$\max\{0, rV_2 - S_w(V_2) - u(w(V_2))\} \leq \frac{u'(w(V_2))}{u'(w(V_1))}\max\{0, rV_2 - S_w(V_2) - u(w(V_1))\}.$$

This inequality is evident when $rV_2 - S_w(V_2) \leq u(w(V_2))$, because the left-hand side is 0 in that case. If $rV_2 - S_w(V_2) > u(w(V_2))$, the above inequality becomes:

$$\frac{rV_2 - S_w(V_2) - u(w(V_2))}{u'(w(V_2))} \leq \frac{rV_2 - S_w(V_2) - u(w(V_1))}{u'(w(V_1))}.$$

Because $[rV - S_w(V)]$ is strictly increasing in V, $rV_2 - S_w(V_2) \leq r\bar{V} - S_w(\bar{V}) = u(\bar{w})$. In this case, (4.11) implies that $[rV - S_w(V) - u(w)]/u'(w)$ is decreasing in w, for any given V and $S_w(V)$. Since $w(V_2) \geq w(V_1)$, the above inequality holds.

Using the above result, I obtain:

$$\Delta \geq \frac{1}{u'(w(V_1))}\left[\begin{array}{c} \max\{0, rV_1 - S_w(V_1) - u(w(V_1))\} \\ -\max\{0, rV_2 - S_w(V_2) - u(w(V_1))\} \end{array}\right].$$

Consider all of the possible cases: (a) $u(w(V_1)) \geq rV_2 - S_w(V_2)$; (b) $u(w(V_1)) \leq rV_1 - S_w(V_1)$; and (c) $rV_1 - S_w(V_1) < u(w(V_1)) < rV_2 - S_w(V_2)$. In each case, it is straightforward to deduce the second inequality in (B.1) from the above relation. **QED**

C.4. Proof of Lemma B.3

Let $w \in \Omega$, and consider the function $\psi w(V)$. With Lemma B.1, $\psi w(V)$ is a continuous and bounded function of V. Next, I prove that $\psi w(V)$ is an increasing function. To do so, let V_1 and V_2 be arbitrary values in $[\underline{V}, \bar{V}]$, with $V_2 \geq V_1$. Simplify the notation $f(V_i)$ to f_i, where f includes the functions w, J_w, F_w, S_w and ψw. I show that $\psi w_2 \geq \psi w_1$. To do so, use the second inequality in (B.1) to obtain:

$$\begin{aligned} \psi w_2 - \psi w_1 &\geq [r + p_w(F_{w1})] J_{w1} - [r + p_w(F_{w2})] J_{w2} \\ &\quad + [rV_1 - S_{w1} - (rV_2 - S_{w2})]/u'(w_1) \\ &= [r + p_w(F_{w1})](J_{w1} - J_{w2}) + J_{w2}[p_w(F_{w1}) - p_w(F_{w2})] \\ &\quad + [rV_1 - S_{w1} - (rV_2 - S_{w2})]/u'(w_1). \end{aligned}$$

Because $[rV - S_w(V)]' = r + p_w(F_w)$, and $[r\bar{V} - S_w(\bar{V})] = u(\bar{w})$, then

$$rV - S_w(V) = u(\bar{w}) - \int_V^{\bar{V}} [r + p_w(F_w(z))]\, dz.$$

Using this result, and expressing $J_w(V)$ as in (4.3), I get:

$$\begin{aligned} &[r + p_w(F_{w1})](J_{w1} - J_{w2}) + [rV_1 - S_{w1} - (rV_2 - S_{w2})]/u'(w_1) \\ = &\int_{V_1}^{V_2} \left[\tfrac{r + p_w(F_{w1})}{u'(w(z))} - \tfrac{r + p_w(F_w(z))}{u'(w_1)}\right] dz \geq 0. \end{aligned} \tag{C.9}$$

The inequality follows from $p_w(F_{w1}) \geq p_w(F_w(z))$, and $u'(w(z)) \leq u'(w_1)$, for all $z \in [V_1, V_2]$. Because $p_w(F)$ is a decreasing function of F, I have established:

$$\psi w_2 - \psi w_1 \geq J_{w2}\left[p_w(F_{w1}) - p_w(F_{w2})\right] \geq 0. \tag{C.10}$$

Now I verify $\psi w(V) \in [\underline{w}, \bar{w}]$ for all V, with $\psi w\left(\bar{V}\right) = \bar{w}$. Because $w\left(\bar{V}\right) = \bar{w}$, $J_w\left(\bar{V}\right) = k/\bar{q}$, and $p_w\left(\bar{V}\right) = 0$, it is clear that $\psi w\left(\bar{V}\right) = \bar{w}$. Since $\psi w(V)$ is increasing, $\psi w(V) \leq \psi w\left(\bar{V}\right) = \bar{w}$ for all V. Similarly, $\psi w(V) \geq \underline{w}$ for all V if and only if $\psi w(\underline{V}) \geq \underline{w}$. To establish the latter inequality, note that $w(\underline{V}) \geq \underline{w}$, because $w \in \Omega$. Using (4.11) and the fact that $r\underline{V} = u(b)$, I have:

$$\frac{1}{u'(w(\underline{V}))}\left[r\underline{V} - S_w(\underline{V}) - u(w(\underline{V}))\right] \leq \frac{1}{u'(\underline{w})}\left[u(b) - S_w(\underline{V}) - u(\underline{w})\right].$$

The right-hand side of the inequality is non-negative, because $\underline{w}$ is set to be small. Thus,

$$\begin{aligned}\psi w(\underline{V}) &\geq y - \left[r + p_w(F_w(\underline{V}))\right] J_w(\underline{V}) - \tfrac{1}{u'(\underline{w})}\left[u(b) - S_w(\underline{V}) - u(\underline{w})\right] \\ &\geq y - \left[r + p_{\bar{w}}(F_{\bar{w}}(\underline{V}))\right] \bar{J} - \tfrac{1}{u'(\underline{w})}\left[u(b) - S_{\underline{w}}(\underline{V}) - u(\underline{w})\right].\end{aligned}$$

The first inequality comes from the preceding result. The second inequality uses part (i) of Lemma B.2, the upper bound on J (defined in (4.8)), and the fact that $S_w(V)$ is increasing in w for any given V. With the above result, (4.10) implies $\psi w(\underline{V}) \geq \underline{w}$. Therefore, ψ maps functions in Ω into functions in Ω.

Finally, if $V_2 > V_1$, the inequalities in (C.9) and (C.10) are strict, because $F_w(V)$ is strictly increasing and $p_w(F_w(V))$ is strictly decreasing in V for all $V < \bar{V}$ (see Lemma B.2). In this case, $\psi w \in \Omega' \subset \Omega$. This completes the proof of Lemma B.3. **QED**

C.5. Proof of Lemma B.4

I prove that the following inequality holds for all $w_1, w_2 \in \Omega$, and all V:

$$|\psi w_2(V) - \psi w_1(V)| \leq A\left\|w_2 - w_1\right\|, \tag{C.11}$$

where the norm is the sup norm and A is a finite constant. Once this is done, Lipschitz continuity of ψ is evident from the following inequality:

$$\left\|\psi w_2 - \psi w_1\right\| = \max_V |\psi w_2(V) - \psi w_1(V)| \leq A\left\|w_2 - w_1\right\|.$$

To show (C.11), take two arbitrary functions, $w_1, w_2 \in \Omega$, and fix V at an arbitrary value in $[\underline{V}, \bar{V}]$. Without loss of generality, assume $\psi w_2(V) \geq \psi w_1(V)$ for this given V. Since V is fixed, I suppress it from the functions if this does not cause confusion. Also, shorten the subscript w_i on J, p, F, and S to i, where $i = 1, 2$. I have:

$$0 \leq \psi w_2(V) - \psi w_1(V) = \left[r + p_1(F_1)\right](J_1 - J_2) + J_2\left[p_1(F_1) - p_2(F_2)\right] + \Delta_2,$$

where

$$\Delta_2 = \max\left\{0, \frac{rV - S_1 - u(w_1)}{u'(w_1)}\right\} - \max\left\{0, \frac{rV - S_2 - u(w_2)}{u'(w_2)}\right\}.$$

To proceed, note that the following inequalities hold for all a_1 and a_2:

$$\max\{0, a_1\} - \max\{0, a_2\} \leq \max\{0, a_1 - a_2\} \leq |a_1 - a_2|.$$

Using these results, it is easy to verify that

$$\Delta_2 \leq \left| \frac{rV - S_1 - u(w_1)}{u'(w_1)} - \frac{rV - S_1 - u(w_2)}{u'(w_2)} \right| + \frac{|S_2 - S_1|}{u'(w_2)}.$$

Denote the first term on the right-hand side above as Δ_3. Define:

$$\mu_1 = \min_{w \in [\underline{w}, \bar{w}]} \left| u''(w) \right|, \quad \mu_2 = \max_{w \in [\underline{w}, \bar{w}]} \left| u''(w) \right|. \tag{C.12}$$

μ_1 and μ_2 are positive and finite. Because $(rV - S_1)$ is strictly increasing in V, $rV - S_1 \leq u(\bar{w})$. Also, concavity of u implies: $u(\bar{w}) \leq u(w) + u'(w)(\bar{w} - w)$. Then,

$$\left| \frac{d}{dw} \left(\frac{rV - S_1 - u(w)}{u'(w)} \right) \right| \leq 1 + \frac{\mu_2}{u'(\bar{w})} (\bar{w} - \underline{w}) \equiv A_1, \tag{C.13}$$

Hence,

$$\Delta_3 \leq A_1 |w_2 - w_1|, \quad \Delta_2 \leq A_1 |w_2 - w_1| + |S_2 - S_1| / u'(\bar{w}).$$

Substituting these results into the earlier expression for $[\psi w_2(V) - \psi w_1(V)]$, and using the bounds in (4.8), I obtain:

$$\begin{aligned} 0 \leq \psi w_2(V) - \psi w_1(V) & \leq (r + \bar{p}) |J_1 - J_2| + \bar{J} |p_2(F_2) - p_1(F_1)| \\ & \quad + |S_2 - S_1| / u'(\bar{w}) + A_1 |w_2 - w_1|. \end{aligned} \tag{C.14}$$

Let me examine the first three terms on the right-hand side above. With μ_2 defined in (C.12), the following inequality holds for all w_1, $w_2 \in [\underline{w}, \bar{w}]$:

$$\left| \frac{1}{u'(w_1(z))} - \frac{1}{u'(w_2(z))} \right| \leq A_2 \|w_2 - w_1\|, \text{ where } A_2 \equiv \frac{\mu_2}{[u'(\bar{w})]^2}. \tag{C.15}$$

Using this result and (4.3), I have:

$$|J_1 - J_2| \leq \int_{\underline{V}}^{\bar{V}} \left| \frac{1}{u'(w_1(z))} - \frac{1}{u'(w_2(z))} \right| dz \leq A_2 \left(\bar{V} - \underline{V} \right) \|w_2 - w_1\|. \tag{C.16}$$

To put a bound on the difference, $|p_2(F_2) - p_1(F_1)|$, define:

$$B_1 \equiv m_1 \bar{q}^2 / k, \quad B_2 \equiv (\bar{q} m_2 + 2m_1) \bar{q}^3 / k^2, \tag{C.17}$$

where m_1 and m_2 are the bounds specified in Assumption 1. Clearly, B_1 and B_2 are finite. Because $k/J_w = q_w \leq \bar{q}$, it is straightforward to verify that

$$\left| \frac{dM(k/J_w)}{dJ_w} \right| \leq B_1, \quad \left| \frac{d^2 M(k/J_w)}{dJ_w^2} \right| \leq B_2. \tag{C.18}$$

Using these bounds, (C.15) and (C.16), I can derive the following results for all $z \in \left[\underline{V}, \bar{V}\right]$:

$$|p_2(z) - p_1(z)| \leq B_1 |J_2(z) - J_1(z)| \leq B_1 A_2 \left(\bar{V} - \underline{V} \right) \|w_2 - w_1\|; \tag{C.19}$$

$$\begin{aligned} |p_2'(z) - p_1'(z)| & = \left| \tfrac{1}{u'(w_2(z))} \tfrac{d}{dJ_2} M\left(\tfrac{k}{J_2(z)}\right) - \tfrac{1}{u'(w_1(z))} \tfrac{d}{dJ_1} M\left(\tfrac{k}{J_1(z)}\right) \right| \\ & \leq B_1 \left| \tfrac{1}{u'(w_2(z))} - \tfrac{1}{u'(w_1(z))} \right| + \tfrac{1}{u'(w_2(z))} \left| \tfrac{d}{dJ_2} M\left(\tfrac{k}{J_2(z)}\right) - \tfrac{d}{dJ_1} M\left(\tfrac{k}{J_1(z)}\right) \right| \\ & \leq B_1 A_2 \|w_2 - w_1\| + \tfrac{B_2}{u'(\bar{w})} |J_2(z) - J_1(z)| \\ & \leq \left[B_1 + \tfrac{B_2}{u'(\bar{w})} \left(\bar{V} - \underline{V} \right) \right] A_2 \|w_2 - w_1\|. \end{aligned} \tag{C.20}$$

Now, examine the difference, $|p_2(F_2) - p_1(F_1)|$. Assume $F_2 \geq F_1$, without loss of generality. (If $F_2 \leq F_1$, switch the roles of F_1 and F_2 in the proof, and the resulting bound is the same.) In the case where $p_2(F_2) < p_1(F_1)$, I have the following inequalities:

$$0 < p_1(F_1) - p_2(F_2) = -p_1'(F_1)(F_1 - V) + p_2'(F_2)(F_2 - V) \leq (F_1 - V)\left[p_2'(F_1) - p_1'(F_1)\right].$$

The equality follows from (3.2), and the last inequality from the fact that $p_2'(F)(F-V)$ is decreasing in F. Because $0 \leq F_1 - V \leq \bar{V} - \underline{V}$, the above result and (C.20) imply:

$$|p_2(F_2) - p_1(F_1)| \leq \left[B_1 + \frac{B_2}{u'(\bar{w})}\left(\bar{V} - \underline{V}\right)\right] A_2\left(\bar{V} - \underline{V}\right)\|w_2 - w_1\|. \tag{C.21}$$

In the case where $p_2(F_2) \geq p_1(F_1)$, the following inequalities hold:

$$0 \leq p_2(F_2) - p_1(F_1) \leq p_2(F_1) - p_1(F_1) \leq B_1 A_2\left(\bar{V} - \underline{V}\right)\|w_1 - w_2\|.$$

The second inequality comes from the fact that p is decreasing, and the last inequality from (C.19). Thus, (C.21) holds in this case too.

Next, turn to the difference, $|S_2 - S_1|$. Because S_1 is the maximum of $p_1(F)(F-V)$ over F, then $S_1 \geq p_1(F_2)(F_2 - V)$. Using the inequality and (C.19), I have:

$$\begin{aligned} S_2 - S_1 &\leq p_2(F_2)(F_2 - V) - p_1(F_2)(F_2 - V) \\ &= (F_2 - V)\left[p_2(F_2) - p_1(F_2)\right] \leq B_1 A_2\left(\bar{V} - \underline{V}\right)^2\|w_2 - w_1\|. \end{aligned}$$

Similarly, using the inequality, $S_2 \geq p_2(F_1)(F_1 - V)$, I can show that $(S_1 - S_2)$ is bounded by the same upper bound as above. Hence,

$$|S_2 - S_1| \leq B_1 A_2\left(\bar{V} - \underline{V}\right)^2\|w_2 - w_1\|. \tag{C.22}$$

Assembling (C.16), (C.21) and (C.22) into (C.14), I obtain (C.11), where A is given as

$$A = A_1 + A_2\left(\bar{V} - \underline{V}\right)\left\{(r + \bar{p}) + \left[B_1\bar{J} + \frac{B_1 + B_2\bar{J}}{u'(\bar{w})}\left(\bar{V} - \underline{V}\right)\right]\right\}.$$

Clearly, A is finite. Moreover, A is independent of the particular functions w_1 and w_2 with which the functions $(J_i, q_i, p_i, F_i, S_i)$ are constructed. **QED**

C.6. Proof of Theorem 6.1

First, I derive (6.2). Set $V = \bar{V}$ in (6.1). Because $\dot{V} = 0$ at $V = \bar{V}$, the left-hand side of (6.1) is equal to 0 at $V = \bar{V}$. Moreover, the integral in (6.1) is equal to zero, because $F^{-1}(\bar{V}) = \bar{V}$. Thus, at $V = \bar{V}$, (6.1) yields (6.2).

Second, I show that G is continuous; i.e., G does not have any mass point. Suppose, to the contrary, that G has a mass $a > 0$ at some value $V \in [v_1, \bar{V}]$. Then, $G(V) - G(V - \dot{V}dt) \geq a$ for all $dt > 0$, and so the left-hand side of (6.1) is equal to ∞. This is a contradiction, because the right-hand side of (6.1) is bounded.

Third, to establish (6.3) and continuity of g, denote the left-hand side derivative of G as $g(V_-)$. The left-hand side of (6.1) is equal to $g(V_-)\dot{V}$. Because G, F, F^{-1} and $p(.)$ are continuous, the right-hand side of (6.1) is continuous in V. Thus, $g(V_-)\dot{V}$ must be continuous. Because $\dot{V}$ is continuous, g must be continuous. Then, I can express the left-hand side of (6.1) as $g(V)\dot{V}$. After substituting $p(v_1)$ from (6.2), (6.1) becomes (6.3).

Fourth, g is continuously differentiable for all $V \neq v_2$. To see this, note that F, F^{-1} and $p(.)$ are continuously differentiable. Since g is continuous, G is continuously differentiable, and so the right-hand side of (6.3) is continuously differentiable for all $V \neq v_2$. Thus, the left-hand side of the equation, $g(V)\dot{V}$, must be continuously differentiable for all $V \neq v_2$. Because $\dot{V}$ is continuously differentiable, $g(V)$ is continuously differentiable for all $V \neq v_2$.

Fifth, I derive (6.5). For $V \in (v_1, v_2)$, $F^{-1}(V) < v_1$, and so (6.3) becomes:

$$g_1(V)\dot{V} = \delta\left[1 - G_1(V)\right] - \int_{v_1}^{V} p(F(z))g_1(z)dz. \tag{C.23}$$

Note that $T'(V) = 1/\dot{V}$ from (4.1). Differentiating the function Γ in (6.4) yields:

$$d\Gamma(V, v_1)/dV = -\left[\delta + p(F(V))\right]\Gamma(V, v_1)/\dot{V}. \tag{C.24}$$

With (C.24) and (C.23), it is straightforward to verify:

$$\frac{d}{dV}\left[\frac{\dot{V}g_1(V)}{\Gamma(V, v_1)}\right] = 0. \tag{C.25}$$

Recall that $G_1(v_1) = 0$, because $G(V)$ is continuous for all V. Taking the limit $V \downarrow v_1$ in (C.23) leads to $g_1(v_1)\dot{v}_1 = \delta$. With this initial condition, integrating (C.25) from v_1 to V yields (6.5). Since g is continuous, taking the limit $V \uparrow v_2$ in (6.5) gives $g(v_2)$.

Finally, I derive (6.6) by examining the case $V \in [v_j, v_{j+1})$, where $j \geq 2$. In this case, $F^{-1}(V) \geq v_1$, and so (6.3) becomes:

$$g_j(V)\dot{V} = \delta\left[1 - G(V)\right] - \int_{F^{-1}(V)}^{v_j} p(F(z))g_{j-1}(z)\,dz - \int_{v_j}^{V} p(F(z))g_j(z)\,dz. \tag{C.26}$$

I have separated the two groups of applicants who obtain jobs with values above V: one coming from $(F^{-1}(V), v_j]$ and the other from $[v_j, V]$. With (C.26) and (C.24), I can derive:

$$\frac{d}{dV}\left[\frac{\dot{V}g_j(V)}{\Gamma(V, v_1)}\right] = \frac{p(V)}{\Gamma(V, v_1)}g_{j-1}\left(F^{-1}(V)\right)\frac{dF^{-1}(V)}{dV}. \tag{C.27}$$

Integrating this equation from v_j to V yields (6.6). Because g is continuous, then $g_j(v_j) = \lim_{V \uparrow v_j} g_{j-1}(V)$, all j. This completes the proof of Theorem 6.1. **QED**