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Communication Can Destroy Common Learning

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Abstract

We show by example that communication can generate a failure of common knowledge acquisition. In the absence of communication, agents acquire approximate common knowledge of some parameter, but with communication they do not.

1 Introduction

The significance of common knowledge in determining equilibrium outcomes of games has become well established since the seminal work of Lewis (1969). In settings where players acquire information over time, an important question is whether (approximate) common knowledge of certain events will eventually be attained. An interesting recent paper by Cripps, Ely, Mailath, and Samuelson (2008, henceforth CEMS) identifies conditions under which a parameter becomes common knowledge if agents privately learn the value of the parameter over time. They refer to common knowledge acquisition as "common learning."

In addition to private learning, economic agents frequently acquire information through communication. Intuitively, one might think that introducing communication could only help to achieve common learning since it improves the information agents have about each others' knowledge and beliefs. This intuition is false. We show by example that communication can cause common learning to fail. Our example exhibits common learning of an underlying parameter if players do not communicate, but when communication is introduced according to a particular protocol, common learning does not occur. Moreover, the failure of common learning is profound; approximate common knowledge of the parameter fails uniformly across all periods in every state.

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The example is as follows. Two agents, 1 and 2, independently observe the value of some underlying parameter at stochastic times. If the agents do not communicate, then the value of the parameter becomes approximate common knowledge since each agent eventually assigns high probability to the other agent having observed the parameter.

In addition to direct observation of the parameter, agents communicate according to the following variant of the Rubinstein (1989) email game. When Agent 1 observes the value of the parameter, she sends a message to Agent 2, which is privately received following some stochastic delay. Upon receipt, Agent 2 sends a confirmation message to Agent 1, which is again subject to stochastic delay. Agent 1 in turn sends a confirmation to Agent 2, and so on. There is no other communication. All communication is truthful and consists only of each agent (partially) reporting her own information to the other agent.

Under this communication protocol, common learning of the parameter fails (for some delay distributions). With communication, if Agent 2 has not received the first message from Agent 1, it is no longer true that she assigns high probability to Agent 1 having observed the parameter, even after many periods. Although the unconditional probability that Agent 1 has observed the parameter becomes high, the probability conditional on the first message not having been received is bounded away from 1. Agent 2 therefore faces second-order uncertainty, that is, uncertainty about Agent 1's beliefs about the parameter, until she receives the first message. Since Agent 1 is uncertain of the time at which this message is received, she faces third-order uncertainty until she receives Agent 1's confirmation. Continuing in this fashion, some higher order uncertainty persists regardless of how many messages have been delivered.

The Rubinstein (1989) email game showed that communication can have a double-edged effect on common knowledge acquisition. In the email game, Agent 1 observes a parameter, sends a message informing Agent 2 of the parameter, Agent 2 sends a confirmation message, and so on. Communication terminates at each step with some small fixed probability. On the one hand, communication enhances knowledge acquisition; without it, Agent 2 never learns the value of the parameter. Furthermore, as discussed by Rubinstein, if communication is restricted to a fixed number of messages, beliefs approach common knowledge with high probability as the likelihood of delivery failure vanishes. On the other hand, when the number of messages is unbounded, approximate common knowledge of the parameter is never acquired. Our example differs from the original email game in two significant respects. First, common knowledge is attained without communication, and thus communication only hinders common learning. Second, in our example, every finite order of interactive knowledge of the parameter is eventually acquired with probability 1, whereas in the email game knowledge is (almost surely) acquired only to some finite order.

Our example fits into the framework of CEMS when there is no communication. CEMS assume that each agent learns about the underlying parameter through an infinite sequence of signals that are i.i.d. across time conditional on the parameter. They prove that if the signal spaces are finite then individual learning of the parameter implies common learning. The addition of communication in our example can be viewed as a relaxation of the i.i.d. assumption. Communication naturally generates dependence in signal profiles across time since any informative message received by an agent depends on the information possessed by the sender at the time the message was sent.

2 The Example

The example is closely based on the framework of CEMS with the addition of a specific form of communication. Two agents, 1 and 2, learn about a parameter θ in periods $t = 0, 1, \ldots$. The parameter θ is drawn before period 0 from the set $\Theta = \{\theta_1, \theta_2\}$ according to the prior distribution $p(\theta_1) = p(\theta_2) = 1/2$, and remains fixed over time. In each period t, each agent i receives a signal $z_t^i \in \mathcal{Z}^i = \{\theta_1, \theta_2, u\}$. Conditional on θ, these signals are i.i.d. across time and agents. Signals are generated with probabilities $\Pr(z_t^i = \theta_k \mid \theta_k) = \lambda$ and $\Pr(z_t^i = u \mid \theta_k) = 1 - \lambda$ for each $k = 1, 2$ and some fixed $\lambda \in (0, 1)$. Note that after receiving signal $z_t^i = \theta_k$, Agent i knows that the parameter is $\theta = \theta_k$. If $z_s^i = \theta$ for some $s \leq t$, we will say that Agent i has observed θ by t. Also note that the signal u carries no information about the value of θ, and hence, absent communication, agents beliefs about θ remain equal to their prior beliefs until they observe θ.

Our main purpose is to understand whether approximate common knowledge of the parameter θ will eventually be acquired by the two agents. Accordingly, following CEMS, we say that Θ is *commonly learned* if for each $\theta \in \Theta$ and $q \in (0, 1)$, there exists some T such that for all $t > T$,

$$\Pr\left(\theta \text{ is common } q\text{-belief at } t \mid \theta\right) > q,$$

where common q-belief is as defined by Monderer and Samet (1989).

It easy to show that in the absence of communication, Θ is commonly learned in this setting.[1] Consider the event F that $\theta = \theta_k$ and both agents have observed θ by time t. At any state in F, each agent assigns probability $1-(1-\lambda)^{t+1}$ at time t to the other agent having observed θ. Conditional on θ_k, the event F occurs with probability $\left(1-(1-\lambda)^{t+1}\right)^2$. Choosing T large enough so that $q < \left(1-(1-\lambda)^{T+1}\right)^2$, θ_k is common q-belief on the event F, which occurs with probability greater than q conditional on θ_k.

We now enrich the example by adding communication according to the following protocol. In each period t, each agent i privately observes a message $m_t^i \in \mathcal{M}^i = \{c, s\}$, representing "confirmation" and "silence" respectively. The messages m_t^i are determined by the following stochastic process. As soon as Agent 1 first observes θ in some period t_0, she sends the message c to Agent 2. This message is received by Agent 2 at some date $t_1 > t_0$ according to the distribution described below. At time t_1, Agent 2 sends a message c which is received by Agent 1 at some time $t_2 > t_1$. At time t_2, Agent 1 again sends a message c received by Agent 2 at time $t_3 > t_2$, and so on. In every period $t \neq t_k$ for k odd, Agent 2 receives the message s, and similarly Agent 1 receives the message s in every period $t \neq t_k$ for $k \geq 2$ even.[2]

The distribution of delivery times is determined as follows. With probability 1/2, there is *odd delay*, otherwise there is *even delay*. With odd delay, each message c from Agent 1 is received by Agent 2 with stochastic delay according to a geometric distribution with parameter $\delta \in (\lambda, 1)$; that is, given t_k with k even, $t_{k+1} - t_k$ is geometrically distributed on the set $\{1, 2, \ldots\}$ with parameter δ. Each message c from Agent 2 is received by Agent 1 exactly one period later; that is, $t_{k+1} - t_k = 1$ for all odd k. Even delay is identical to odd delay except with the roles of the two agents reversed.

Letting $\mathcal{M} = \mathcal{M}^1 \times \mathcal{M}^2$ and $\mathcal{Z} = \mathcal{Z}^1 \times \mathcal{Z}^2$, the set of states is given by $\Theta \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$. The information of Agent i at time t is captured by the natural projection of $\Theta \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$ onto $(\mathcal{Z}^i)^{t+1} \times (\mathcal{M}^i)^{t+1}$. We will write $h_t^i(\omega) \in (\mathcal{Z}^i)^{t+1} \times (\mathcal{M}^i)^{t+1}$ for the private history of Agent i at time t. We abuse notation by writing θ for the event $\{\theta\} \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$.

As above, we will write $t_0(\omega)$, or simply t_0 when the state is clear, for the time at which Agent 1 first observes the parameter. For $k \geq 1$, we will write $t_k(\omega)$, or simply t_k, for the time at which

[1] This result also follows immediately from either Proposition 2 or Proposition 3 of CEMS.

[2] The main result would be unchanged if the agents similarly exchanged messages beginning with Agent 2's observation of θ. We focus on the asymmetric version to keep the notation and analysis simple.

the kth confirmation message is received. Formally, let $t_0 = \min\{t \mid z_t^1 = \theta\}$ and for $k \geq 1$, define t_k recursively by $t_k = \min\{t > t_{k-1} \mid m_t^i = c \text{ for } i = 1 \text{ or } 2\}$.

The following result indicates that communication can destroy common learning.

Proposition 1. *In the example with communication, for each $\theta \in \Theta$, there exists some $q \in (0,1)$ such that θ is not common q-belief at any t in any state of the world. In particular, common learning does not occur.*

For integers $k, t \geq 0$, let

$$M_t^k = \{\omega : t_k(\omega) \leq t \text{ and } t_{k+1}(\omega) > t\}.$$

Thus M_t^k consists of those states in which, by time t, Agent 1 has observed θ and exactly k confirmation messages have been received. Similarly, let M_t^{-1} denote the event that, by time t, Agent 1 has not observed θ; formally,

$$M_t^{-1} = \{\omega : t_0(\omega) > t\}.$$

Lemma 1. *There exists some $\overline{p} > 0$ such that, for each $t \geq 0$ and $k = 0, \ldots, t$, given any $\omega \in M_t^k$, $\Pr\left(M_t^{k-1} \mid h_t^i(\omega)\right) \geq \overline{p}$ for some $i \in \{1, 2\}$.*

For $k \geq 1$, Lemma 1 states that if exactly k confirmation messages have been received by t then one of the agents assigns probability at least $\overline{p}$ to only $k-1$ confirmation messages having been received. For $k = 0$, the lemma states that if Agent 2 has not received a confirmation message by t then she assigns probability at least $\overline{p}$ to Agent 1 not having observed θ. The proof of Lemma 1 is in the appendix.

Proof of Proposition 1. Choose any $q \in \left(\frac{1}{2}, 1\right)$ such that $1 - q < \overline{p}$, with $\overline{p}$ as in Lemma 1. Suppose for contradiction that θ is common q-belief at time t in some state ω. By the characterization of Monderer and Samet (1989), there exists an event F containing ω such that, at time t, F is q-evident and both agents q-believe θ on F.

We will show that F contains a state in M_t^{-1}, i.e. one in which Agent 1 has not observed θ by time t. In such a state, Agent 1 assigns probability $1/2$ to the event θ' for $\theta' \neq \theta$. Since $q > 1/2$, these beliefs violate the hypothesis that both agents q-believe θ on F at time t, giving the desired contradiction.

Let $k^* = \min\left\{k \mid F \cap M_t^k \neq \emptyset\right\}$ and choose some $\omega' \in F \cap M_t^{k^*}$. We will show that $k^* = -1$. Suppose for contradiction that $k^* \geq 0$. By Lemma 1, for some i, Agent i assigns probability at least $\overline{p}$ to the event $M_t^{k^*-1}$ at the private history $h_t^i(\omega')$. Writing $B_p^i(E)$ for the event that Agent i p-believes the event E at time t, we have

$$\omega' \in B_{\overline{p}}^i\left(M_t^{k^*-1}\right).$$

Since F is q-evident at time t, we also have

$$\omega' \in B_q^i(F).$$

By the choice of q, $\overline{p}+q > 1$ and hence $M_t^{k^*-1} \cap F \neq \emptyset$, contradicting the definition of k^*. Therefore, $k^* = -1$. $\square$

3 Discussion

The key to the example is that the possibility of delay generates persistent higher order uncertainty regarding whether Agent 1 has observed the parameter. This feature does not arise if, instead of delay, each message fails to be delivered with some positive probability (as in the original email game). In this case, common learning turns out to occur because if Agent 2 does not receive the first message from Agent 1, then after many periods Agent 2 assigns high probability to the event that Agent 1 observed the parameter but her message was not delivered. Similarly, a simpler alternative to our example would be to suppose that each agent's messages can be delayed in each round of communication.[3] However, common learning occurs under this alternative formulation. One can show that conditional on not having received a confirmation of the last message she sent, an agent's belief that her last message has been received tends to 1 over time. This feature suffices to generate common learning since, for any $q \in (0,1)$, the event that the message was received eventually

[3] Morris (2001) studies this form of communication in a finite horizon continuous time setting without private learning. Common learning again depends on whether the receipt of the first message becomes common knowledge, which fails more easily with a finite horizon.

becomes q-evident. In our example, the agents' uncertainty about the delay distributions prevents this convergence of beliefs.

For common learning to fail in our example, it was necessary to assume that $\delta > \lambda$, so that delays in communication tend to be shorter than delays in agents' observation of θ. Otherwise, if after many periods Agent 2 has not received the first message from Agent 1, then she assigns high probability to Agent 1 having observed θ but the message having been delayed. Paradoxically, lowering δ can rescue common learning even though doing so makes communication worse in the sense that message delays tend to be longer.

Questions about the influence of communication on common knowledge acquisition are related to a larger literature on the emergence of consensus with communication. A consensus is said to emerge about an event E if all agents eventually have the same belief about E. Heifetz (1996) showed that, as suggested by Parikh and Krasucki (1990), consensus can emerge in dynamic settings without ever becoming common knowledge. Koessler (2001) proved that, although consensus may emerge, full common knowledge of an event is never attained under any noisy and non-public communication protocol unless the event was common knowledge initially. We diverge from this literature by combining communication with the individual learning of CEMS. Consensus about θ almost surely emerges in our example with or without communication. Unlike the previous literature, however, common learning of θ fails only *with* communication.

It is easy to construct examples in which communication enables common learning, that is, in which common learning occurs with communication but fails without it. This would be the case, for instance, if only one agent privately learns the parameter, and communication consists of that agent publicly announcing each of her signals. That communication can also cause common learning to fail raises interesting questions about the role of communication in common knowledge acquisition and the conditions under which it enhances or hinders common learning. We plan to pursue these questions in future research.

A Appendix

Proof of Lemma 1. Let O and E denote the events that there is odd or even delay respectively, that is, let $O = \{\omega \mid t_{k+1}(\omega) - t_k(\omega) = 1 \text{ for all } k \text{ odd}\}$, and $E = \{\omega \mid t_{k+1}(\omega) - t_k(\omega) = 1 \text{ for all } k \text{ even}\}$.

We begin by calculating each agent's beliefs over O and E after any finite history, beginning with Agent 2.

Fix $t > 0$. Since O and E are equally likely *ex ante*,

$$\Pr(E \mid t_1 = t) = \frac{\Pr(t_1 = t \mid E)}{\Pr(t_1 = t \mid E) + \Pr(t_1 = t \mid O)}. \tag{1}$$

We have

$$\Pr(t_1 = t \mid E) = \lambda(1-\lambda)^{t-1} \tag{2}$$

and

$$\begin{aligned}\Pr(t_1 = t \mid O) &= \sum_{s=0}^{t-1} \lambda(1-\lambda)^s \delta(1-\delta)^{t-s-1} \\ &= \delta\lambda \frac{(1-\lambda)^t - (1-\delta)^t}{\delta - \lambda}.\end{aligned} \tag{3}$$

Substituting equations (2) and (3) into equation (1) gives

$$\Pr(E \mid t_1 = t) = \left(1 + \frac{\delta(1-\lambda)}{\delta-\lambda}\left(1 - \left(\frac{1-\delta}{1-\lambda}\right)^t\right)\right)^{-1}. \tag{4}$$

Since $\delta > \lambda$ by construction, this last expression is decreasing in t and approaches $\frac{\delta-\lambda}{2\delta-\delta\lambda-\lambda} > 0$ as t tends to infinity.

Note that, in any state ω, Agent 2 assigns probability $\Pr(E \mid t_1(\omega) = t)$ to E at any time $t' \geq t_1(\omega)$ since the distribution of all subsequent messages received by Agent 2 is independent of O or E. Similarly, Agent 1 assigns probability $\frac{1}{2}$ to O after any finite history. Let σ_1 denote Agent's 1 belief in O and σ_2 denote Agent's 2 belief in E, suppressing from the notation the dependence of σ_2 on the history.

Next we compute Agent i's belief that her most recent message has been received. For any $k \geq 1$, consider the event M_t^k that k confirmation messages have been received by time t. Let i be the sender of the kth confirmation message, that is $i = 1$ for k odd and $i = 2$ for k even. Fix any state $\omega \in M_t^k$, and let $d = t - t_{k-1}$ be the length of time that has passed since Agent i sent her last confirmation message. Note that $d > 0$. We have (given σ_i)

$$\Pr\left(M_t^{k-1} \mid h_t^i(\omega)\right) = \frac{\sigma_i(1-\delta)^d}{(1-\sigma_i)(1-\delta)^{d-1} + \sigma_i\delta(1-\delta)^{d-1} + \sigma_i(1-\delta)^d} = \sigma_i(1-\delta). \tag{5}$$

Since σ_i is bounded away from 0, there exists some $\overline{p}_1 > 0$ such that

$$\Pr\left(M_t^{k-1} \mid h_t^i(\omega)\right) \geq \overline{p}_1.$$

Finally, consider Agent 2's belief of whether Agent 1 has observed θ if she has not yet received a confirmation message at time $t \geq 0$. For any $\omega \in M_t^0$, we have

$$\begin{aligned} \Pr\left(M_t^{-1} \mid h_t^2(\omega)\right) &= \frac{(1-\lambda)^{t+1}}{(1-\lambda)^{t+1} + \lambda(1-\lambda)^t + \frac{1}{2}\sum_{s=0}^{t-1}\lambda(1-\lambda)^s(1-\delta)^{t-s}} \\ &= \frac{\delta-\lambda}{1 - \frac{1-\delta}{1-\lambda} + \frac{1}{2}\lambda\frac{1-\delta}{1-\lambda}\left(1 - \left(\frac{1-\delta}{1-\lambda}\right)^t\right)}, \end{aligned} \tag{6}$$

which is decreasing in t and approaches $\frac{2(\delta-\lambda)(1-\lambda)}{2\delta-\lambda-\delta\lambda} > 0$ as t tends to infinity.

Taking $\overline{p} = \min\left\{\overline{p}_1, \frac{2(\delta-\lambda)(1-\lambda)}{2\delta-\lambda-\delta\lambda}\right\}$ gives the result. □

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