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Efficient Dynamic Coordination with Individual Learning

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Abstract

We study how the presence of multiple participation opportunities coupled with individual learning about payoffs affects the ability of agents to coordinate efficiently in global coordination games. Two players face the option to invest irreversibly in a project in one of many rounds. The project succeeds if some underlying state variable θ is positive and both players invest, possibly asynchronously. In each round they receive informative private signals about θ, and asymptotically learn the true value of θ. Players choose in each period whether to invest or to wait for more precise information about θ.

We show that with sufficiently many rounds, both players invest with arbitrarily high probability whenever investment is socially efficient, and delays in investment disappear when signals are precise. This result stands in sharp contrast to the usual static global game outcome in which players coordinate on the risk-dominant action. We provide a foundation for these results in terms of higher order beliefs.

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1 Introduction

Coordination problems arise in a wide variety of economic situations. A typical example is of a setting where the successful implementation of some socially beneficial project depends on whether enough agents participate. Such settings may lead to coordination failure, which arises when a given group of agents fails to participate in the project despite the fact that it is in their collective interest to do so.

The traditional theoretical analysis of coordination problems, where payoffs are typically assumed to be commonly known, has been plagued by the existence of multiple equilibria. For a given payoff rule, there exists at least one equilibrium with coordination failure, and one without. Such analysis is unable, therefore, to quantify the extent and relevance of coordination failure, since it is not possible to assign probabilities across equilibria.

The recent literature on global games (Carlsson and van Damme [2], Morris and Shin [17]) has made substantial progress in resolving the problem of multiplicity in the analysis of coordination problems. This literature has identified an important class of coordination games, in which underlying payoffs are observed with small amounts of idiosyncratic noise, where the multiplicity of equilibria is eliminated. The "refinement" thus achieved allows us to quantify the extent of coordination failure, and indeed coordination failures do occur in the unique equilibrium of the canonical global game. Whether coordination failure arises depends on the payoffs of the underlying complete information game. Roughly speaking, agents are only able to coordinate on some risky action in the unique equilibrium of a global game if that action is *risk dominant*, i.e., it is optimal for each agent to choose that action in the underlying complete information game even when there is only a "low" probability that his fellow players will choose that action.[1] This can only happen if the benefits that arise from the action conditional upon success are high relative to the cost of undertaking it. Thus, the global games literature has negative implications for the ability of agents to coordinate on socially beneficial actions: only projects that involve "little strategic risk" will be implemented in equilibrium. In all other cases, coordination failure will arise.

In this paper we evaluate how the incidence of coordination failure in global games is affected by the presence of multiple opportunities to participate between which players individually learn about the fundamental. The canonical global game requires that all agents choose their actions simultaneously. To what extent would the incidence of coordination failure change if we allowed for some asynchronicity in the actions of potential participants

[1] More precisely, equilibria of the underlying complete information game survive in the induced global game only if they are p-dominant (Morris, Rob, and Shin [16]) for "low" p. Exactly how low p must be depends on the structure of the game. In two player games, p-dominant equilibria for $p < \frac{1}{2}$ survive. See Kajii and Morris [14] for a generalization of this idea.

in a coordination problem?

To be specific, consider a setting in which the success of a socially beneficial investment project depends on the total number of agents who invest over the course of T distinct periods. Two players choose at which period (if any) to invest irreversibly, while observing noisy private signals about the underlying state variable (θ). At each period t, the information structure is that of a canonical global game. We assume that agents privately learn the fundamental θ asymptotically: if the number of periods gets arbitrarily large, each agent's cumulative individual information becomes arbitrarily precise. The project succeeds if the fundamental is good and each player invests in some period. Note that it is *not* necessary for success that both players invest in the same period. This is, therefore, an *asynchronous* investment game. The choice between early vs late investment is driven by a trade-off: early investment generates higher payoffs if the project succeeds, while late investors have more accurate private information about payoffs. As in a standard global game, we assume that there exist values of θ that make investment dominant ($\theta \geq 1$) or dominated ($\theta \leq 0$). To fix ideas, imagine that $0 < \theta < 1$ and the payoffs are such that investing is *not* risk-dominant. This means that if agents had to choose their actions simultaneously in some period, say T, and thus play a static global game, then, in the limit as noise vanishes, coordinated investment could not be supported as an equilibrium outcome, and coordination failure arises. To what extent will the possibility of choosing actions asynchronously affect the incidence of coordination failure? We report the following results:

1. Coordination failure almost never arises in a sufficiently long asynchronous investment game.

   For any $\underline{\theta} > 0$ and $\varepsilon > 0$ there exists some $\underline{T}$ such that for any $T \geq \underline{T}$, investment succeeds with probability at least $1 - \varepsilon$ in the asynchronous game with T periods whenever $\theta > \underline{\theta}$. Thus, in the limit as $T \to \infty$, the project succeeds whenever $\theta > 0$. In addition, as noise in observation vanishes (i.e., $\sigma_t \to 0$ for all t) there is also no delay in investment: players successfully coordinate on implementing the project immediately, thus achieving the social optimum.

2. The forces driving our results can be cleanly characterized in terms of higher order beliefs in the asynchronous coordination game.

   Building on the standard belief operators of Monderer and Samet [15], we construct an *asynchronous p-belief* operator which is suitable for characterizing behaviour in our asynchronous investment game. Using this operator, we show that by choosing sufficiently long asynchronous investment games, it is possible to generate adequate levels of generalized approximate common knowledge (i.e., generalized common p-belief for arbitrarily high p) in order to support asynchronous coordination. The generalization lies in allowing the required beliefs to be attained at different times.

If *synchronous* participation at the last round T was necessary for the success of the project, then players invest only if, at T, they commonly p-believe in the standard sense of Monderer and Samet [15], for sufficiently high p, that the fundamental allows success. It is now well understood (see, for example, Morris and Shin [17]) that, however small the private errors are, the global games information structure does not generate common p-belief for $p > \frac{1}{2}$, and thus coordination fails whenever investment is not risk-dominant. In our setting, players do not have to participate synchronously at T, but both players must participate eventually by period T for investment to succeed. In such a situation, only a relaxed version of common beliefs is necessary for coordination. Fix a probability of success $p \in (0, 1)$ sufficient to induce players to invest in period t. Both players will invest by period T, if they both believe with probability p by period T that the fundamental is good, they both believe with probability p by period T that they both believe with probability p by period T that the fundamental is good...etc. We refer to such an event as asynchronous common p-belief of event $\theta > 0$. This variation of standard common belief turns out not to be very demanding in our setting.

To obtain some intuition for why asynchronous approximate common knowledge is attained in long games, consider a game with infinitely many rounds in which each player asymptotically privately learns the fundamental. Then if the fundamental lies in some open set G, all players will eventually p-believe G almost surely for any $p < 1$. This makes event G p-evident in an asynchronous sense, which in turn implies asynchronous common p-belief of G. The shortcoming of this line of argument is that it relies on the assumption that the fundamental is asymptotically perfectly revealed to players. It is thus not clear whether the argument extends to long but finite games in which some information about the fundamental remains uncovered. Existing results on static global games show that there is an important discontinuity in the structure of standard common beliefs as information about the fundamental becomes infinitely precise. We find that such a discontinuity does *not* arise when the infinite asynchronous game is approximated by a sequence of finite asynchronous games. Approximate asynchronous common knowledge *is* attained even if small uncertainty about the fundamental remains.

Our results suggest that allowing asynchronicity and private learning in coordination problems may substantially reduce the extent of coordination failure in global games. In addition to being of theoretical interest, our results are potentially widely applicable. For example, consider the problem of foreign direct investment (FDI) into a newly liberalizing emerging market. Payoffs from FDI depend on whether the emerging economy "takes off", which in turn depends on the amount of FDI. Thus, this is a coordination problem. In addition, it may not matter precisely that all FDI takes place at the same time, but simply that it occurs during the first several months to several years of the liberalization programme. It is not uncommon for liberalization to be accompanied by government subsidies to early investors. Yet, it is also likely that late investors will have better information about the state of the underlying emerging markets. Thus, the class of stylized games outlined above represents trade-offs that are not dissimilar to those outlined in this applied context. The FDI example is not unique. Indeed, it may be reasonable to argue that several of the applications studied to date using global games (e.g., currency crises, bank runs, financial contagion etc.[2]) may well have an element of asynchronicity to them.

The rest of the paper is organized as follows. In section 2 we outline the model. Section 3 states our main result, while section 4 explains the efficiency result in terms of asynchronous common p-belief. Section 5 discusses the role of our main assumptions, considers potential extensions, and concludes. Before proceeding to the main model, we first outline the related literature.

1.1 Literature Review

Our analysis originated in the work of Dasgupta [5]. Dasgupta outlines conditions under which the provision of the option to delay combined with private learning improves the ability of agents to coordinate efficiently in two-stage global games. We use elements of Dasgupta's modeling framework for an analysis of a different but related question. We do not compare coordination outcomes in different finite games; rather, we show that efficient coordination arises in the limit when players have many opportunities to act. Another example of a dynamic global game with private learning can be found in Heidhues and Melissas [12].

The current analysis bears a general connection to models of information dynamics in multi-stage global games (e.g., Chamley [3], Angeletos, Hellwig and Pavan [1]). In contrast to our work, papers in this strand of the literature focus on learning from endogenously generated *public* signals, and focus on the robustness of equilibrium uniqueness in global games. We restrict attention to pure private information settings with a unique long-run outcome, and focus on characterizing the (lack of) incidence of coordination failure.

Our explanation of the efficiency result in terms of higher order beliefs builds on the work of Monderer and Samet [15] and Morris and Shin [19]. The analysis is also related to the work of Cripps, Ely, Mailath, and Samuelson [4]. These authors delineate general condi-

[2] See Morris and Shin [18]for currency crises, Goldstein and Pauzner [9] for bank runs, and Dasgupta [6] and Golstein and Pauzner [10] for financial contagion.

tions under which agents asymptotically attain approximate common knowledge via private learning. The analysis of Cripps *et al* has important implications for long-run outcomes in situations which can be divided into two distinct phases: agents learn privately in the first phase, and attempt to coordinate *synchronously* in the second. We study situations in which those two phases are merged together: players attempt to coordinate *asynchronously* while they privately learn about payoffs. Both papers study whether private learning leads to approximate common knowledge. However, *different* concepts of approximate common knowledge are relevant for ensuring successful coordination in synchronous and asynchronous coordination games, because the payoff-relevant events differ in these two types of games. Cripps *et al* study standard common beliefs as defined in Monderer and Samet [15], while we study an asynchronous form of common beliefs. The two concepts turn out to have very different properties. In our model, private learning fails to deliver common knowledge in the standard sense as studied by Cripps *et al*, but succeeds in delivering asynchronous common beliefs. This explains why coordination failure arises in the synchronous coordination game but does not arise in our asynchronous game.[3]

Gale [8] provides an elegant analysis of the extent of inefficient delay in dynamic coordination games with complete information. He shows that inefficient delay can be eliminated when the period length becomes very small. While our main result implies efficient delay-free coordination in the limit as private signals become accurate, and thus bears a resemblance to Gale's, the situations considered and the arguments offered are very different. Gale's result builds on backward induction based on the observability of past actions in a game of perfect information, while we consider an asymmetric information setting in which players do not observe each others' actions. Hörner [13] studies a model similar to that of Gale [8], and finds that patient players coordinate efficiently when they receive a single noisy signal of payoffs prior to the play of the game.

2 Model

Two players $i \in \{1, 2\}$ play a joint investment game Γ_T, with $T \in \mathbb{N}$. The game consists of T rounds, all of which may take place within a finite, possibly short time window. In each round $t \in \{1, \ldots, T\}$, each player chooses one of the two actions $a_t^i \in \{0, 1\}$; we interpret Action 1 as "invest", and Action 0 as "wait". Each player may invest in at most one round. Investment is irreversible.[4] The payoffs depend on the action profiles and the value of a fundamental parameter $\theta \in \mathbb{R}$ describing the characteristics of the project. The fundamental

[3] In an earlier paper Ely [7] informally discusses the notion of asynchronous common belief, but only to contrast it to the standard common belief which is the relevant concept for the problems he considers.

[4] We discuss the role of irreversibility and other robustness issues in the conclusion.

θ is drawn before the first round according to an improper uniform distribution on $\mathbb{R}$, and remains fixed over all rounds.

The players do not observe the true value of the fundamental θ; instead, they receive private noisy signals of the value of θ in every round. Specifically, each player i receives a signal $\widetilde{x}^{(i,t)} = \theta + \tilde{\sigma}_t \varepsilon^{(i,t)}$ in round t, where the errors $\varepsilon^{(i,t)}$ are drawn from $N(0,1)$ and are independent across players and rounds. The standard errors $\tilde{\sigma}_t$ are strictly positive for all t, and the sequence $(\tilde{\sigma})_{t=0}^{\infty}$ is fixed throughout independent of the value of T. Player i does not observe the choices of player $-i$ before the end of the game.

Players form their beliefs in each period about the true value of the fundamental through Bayesian updating given their received signals. The resulting beliefs over θ conditional on a sequence $(\widetilde{x}^{(i,t)})_{t'=1}^{t}$ are distributed as $N(x^{(i,t)}, \sigma_t^2)$, where

$$x^{(i,t)} = \frac{\sum_{t'=1}^{t} \widetilde{x}^{(i,t)} \frac{1}{\tilde{\sigma}_{t'}^2}}{\sum_{t'=1}^{t} \frac{1}{\tilde{\sigma}_{t'}^2}},$$

and

$$\frac{1}{\sigma_t^2} = \sum_{t'=1}^{t} \frac{1}{\tilde{\sigma}_{t'}^2}.$$

We will refer to $x^{(i,t)}$ as the *cumulative signal*, and to σ_t as the *cumulative standard error.*

We assume that players asymptotically *privately* learn the true fundamental; that is,

$$\lim_{t\to\infty} \sigma_t = 0. \tag{1}$$

Note that, since each standard error $\tilde{\sigma}_t$ is strictly positive, each cumulative standard error σ_t is also strictly positive. Thus even though players learn the true fundamental in the limit over all periods, some uncertainty remains in each round.

The success of the project is determined at the end of the game, based on the fundamental θ and the actions of the players:

- For $\theta \leq 0$, the project fails regardless of the players' actions.
- For $\theta \geq 1$, the project succeeds regardless of the players' actions.
- For $0 < \theta < 1$, the project succeeds if and only if both players invest by round T, *possibly asynchronously.*

Each player's payoff in the game depends on whether and in which round the player invested, and whether the project succeeded. The payoffs are

- 0 if the player never invests,

- $\delta^t b$ if the player invests in round t and the project succeeds, and
- $-\delta^t c$ if the player invests in round t and the project fails,

where the parameters b and c are both strictly positive, and $\delta \in (0,1)$.

The payoffs in this game are consistent with a wide variety of applied settings. For example, they can be easily understood in the context of the FDI example discussed in the introduction. Future payoffs from FDI are positive only if enough foreign firms participate, and the state of the domestic economy (θ) is not too weak. Net benefits from successful FDI participation decline for later participants due to declining subsidies from the emerging market government. Net costs in the event of failure decline for late participants as well, due to a smaller lock-in period for valuable resources.

We now proceed to analyze this game, and show the crucial role of asynchronicity and asymptotic learning in eliminating coordination failure.

3 Analysis

The payoffs outlined above imply two simple properties of the best response correspondence, which we describe below in Lemmas 1 and 2. Our main results, in turn, can be fully stated in terms of these two properties.

Lemma 1 *There exists some $\underline{p} \in (0,1)$ such that, in any round t, waiting is the unique best response for any type that believes the project will succeed with probability less than $\underline{p}$.*

Proof. The payoff to investing immediately is

$$\delta^t \left(pb + (1-p)(-c) \right).$$

The minimum value to waiting is 0. Thus, $\underline{p}$ is defined by

$$\delta^t \left(\underline{p}b + (1-\underline{p})(-c) \right) = 0,$$

or equivalently,

$$\underline{p} = \frac{c}{b+c},$$

as needed. ■

Lemma 2 *There exists some $\overline{p} \in (0,1)$ such that, in any round t, investing is the unique best response for any type that believes the project will succeed will probability greater than $\overline{p}$.*

Proof. The payoff to investing immediately is

$$\delta^t \left(pb + (1-p)(-c) \right).$$

The maximum value to waiting is $\delta^{t+1}b$. Thus, $\overline{p}$ is defined by

$$\delta^t \left(\overline{p}b + (1-\overline{p})(-c) \right) = \delta^{t+1}b$$

or equivalently,

$$\overline{p} = \frac{\delta b + c}{b+c},$$

as needed. ■

We note that $1 > \overline{p} > \underline{p} > 0$. Finally, we observe that the existence of $\overline{p} < 1$ implies that however great the amount future information, any player will choose to invest immediately if she is sufficiently optimistic.

In what follows, we simplify notation by treating $\underline{p}$ and $\overline{p}$ directly as parameters of the game. We are now in a position to state our main results, which demonstrate the stark difference between synchronous and asynchronous coordination games. We begin with the benchmark synchronous case.

3.1 The failure of coordination in the synchronous game

As a benchmark to compare our results to the existing literature on static global games, consider the following static, synchronous version of the game Γ_T. In the synchronous version, which we label by Γ_T^S, for $0 < \theta < 1$, the project succeeds if and only if both players invest synchronously at round T. All other features remain unchanged. We show that for any $\theta < 1$ coordinated investment fails with arbitrarily high probability as long as T is big enough, whenever $\underline{p} > \frac{1}{2}$.

Proposition 1 *Fix any $\underline{p} \in \left(\frac{1}{2}, 1\right)$. For any $\overline{\theta} < 1$ and $\varepsilon > 0$ there exists some $\underline{T}$ such that for any $T \geq \underline{T}$, the project fails with probability at least $1 - \varepsilon$ in Γ_T^S whenever $\theta \leq \overline{\theta}$.*

This result is a consequence of results from the extant literature on static global games (see Morris and Shin [17]), and so we only discuss the argument informally. The game played at round T is a canonical static global game with signals $x^{(1,T)}$ and $x^{(2,T)}$ with precision σ_T. The unique equilibrium of this game is characterized by a threshold, $x^{*,T}$, such that players invest if and only if their signals satisfy $x^{(i,T)} \geq x^{*,T}$. A player observing the threshold signal $x^{(i,T)} = x^{*,T}$ assigns probability $\frac{1}{2} < \underline{p}$ to the event that her opponent received a signal above $x^{*,T}$. This is a consequence of the Laplacian beliefs property of the global games information structure discussed in Morris and Shin [17]. Unless the threshold player assigns probability almost $\underline{p}$ to the event $\theta \geq 1$, she strictly prefers to wait. Thus, the distance of the indifference point $x^{*,T}$ from 1 must be on the order of σ_T, and hence, as T becomes large and σ_T small, $x^{*,T}$ approaches 1.

In order for coordination failure to occur in the synchronous game, it is not essential that agents can invest only at round T. In fact, a similar result would hold in an alternative benchmark game where agents are free to choose in which round to invest, but the project succeeds only if they both end up investing in the same round.

In sharp contrast to these synchronous settings, we now show that coordination almost never fails in the asynchronous game for $\theta > 0$.

3.2 The success of coordination in the asynchronous game

The following proposition establishes that, in the game with many rounds, both players are likely to invest whenever the fundamental allows for success of the project ($\theta > 0$).

Proposition 2 *Fix any $\overline{p} \in (0, 1)$. Suppose that both players play serially interim undominated strategies. For any $\underline{\theta} > 0$ and $\varepsilon > 0$ there exists some $\underline{T}$ such that for any $T \geq \underline{T}$, the project succeeds with probability at least $1 - \varepsilon$ in Γ_T whenever $\theta > \underline{\theta}$.*

Our central result follows from two core lemmas (3 and 4). The proof that follows consists of three steps: First, we briefly outline some notation. Second, we state the two main lemmas. Finally, we argue that the main result follows from the lemmas.

Proof. Fix $q \in (\overline{p}, 1)$. Denote the event that player i q-believes event E at t by $B_q^{(i,t)}(E)$; that is, $B_q^{(i,t)}(E) = \left\{x^{(i,t)} \mid \Pr(E|x^{(i,t)}) \geq q\right\}$. Denote by $l_t^{\theta^*,q}(\theta)$ the probability that the player has q-believed that $\theta > \theta^*$ at least once up to and including round t:

$$l_t^{\theta^*,q}(\theta) = \Pr\left(\bigcup_{t'=1,\ldots,t} B_q^{(i,t)}(\theta \geq \theta^*) \middle| \theta \right).$$

Let $l^{\theta^*,q}(\theta)$ denote $\lim_{t\to\infty} l_t^{\theta^*,q}(\theta)$. This limit exists because $l_t^{\theta^*,q}(\theta)$ is non-decreasing in t.

The following lemma demonstrates one important consequence of our assumption that players asymptotically learn the true state. Such asymptotic learning guarantees that, when the true state is θ^*, each player will, at least once during the course of an arbitrarily long game, believe with arbitrarily high probability that the true state is not below θ^*.

Lemma 3 *For all $0 < q < 1$ and all $\theta^* \in \mathbb{R}$: $l^{\theta^*,q}(\theta^*) = 1$.*

The formal proof is provided in the appendix. The main idea of the proof of Lemma 3 is the following: conditional on θ^*, the probability that a player q-believes $\theta \geq \theta^*$ is $1 - q$ in each round, but with the complication that the posterior probabilities $p^{(i,t)} = \Pr(\theta \geq \theta^* | x^{(i,t)})$ are correlated across rounds. We will show, roughly, that beliefs across sufficiently distant rounds t and t' are approximately independent. The intuition for this is that if the amount of information that a player receives between t and t' is large relative to what she knew at t, then the information at t has only a negligible impact at t'. For long games, we can choose a long subsequence of rounds such that all rounds in the subsequence are sufficiently distant. Hence the probability of q-believing $\theta \geq \theta^*$ in at least one of these rounds approaches one as the number of rounds grows large.

Our next result shows that, in a sufficiently long (but finite) game, whenever a given player believes with probability strictly bigger than $\overline{p}$ that the fundamentals are such that investment succeeds if both players invest, then that player will invest immediately.

Lemma 4 *Suppose that both players play serially interim undominated strategies. For any $\underline{\theta} > 0$ and $q \in (\overline{p}, 1)$, there exists some $\underline{T}$ such that for any $T > \underline{T}$, player i invests in round t of game Γ_T if she believes at t with probability at least q that $\theta \geq \underline{\theta}$.*

Proof of Lemma 4. Let θ^{**} be the infimum of those $\underline{\theta}$ for which the statement holds. We must show that $\theta^{**} = 0$. We proceed by a contagion argument. The statement clearly holds for $\underline{\theta} \geq 1$. The proof consists of showing that if the statement holds for all $\underline{\theta} > \theta^*$ for some $\theta^* > 0$, then there exists $\varepsilon > 0$ such that it holds for all $\underline{\theta} > \theta^* - \varepsilon$. Thus we must have $\theta^{**} = 0$, for otherwise taking $\theta^* = \theta^{**}$ would give a contradiction.

Assume that the statement is true for all $\underline{\theta} > \theta^*$ for some $\theta^* > 0$. Then there exists some T'' such that in any game Γ_T with $T > T''$, player $-i$ invests at t if she $q-$believes that $\theta > \theta^*$. Thus, whenever $\bigcup_{t'=1,\ldots,T} B_q^{(-i,t')}(\theta > \theta^*)$ is true, player $-i$ will invest at some t in the game Γ_T. Fix some $r \in \left(\frac{\overline{p}}{q}, 1\right)$. Lemma 3 implies that there exists T' such that

$$l_{T'}^{\theta^*,q}(\theta^*) > r.$$

The function $l_{T'}^{\theta^*,q}(\cdot)$ is continuous. Therefore, there exists some $\varepsilon \in (0, \theta^*)$ such that

$$l_{T'}^{\theta^*,q}(\theta^* - \varepsilon) \geq r.$$

Since $l_T^{\theta^*,q}(\theta)$ is non-decreasing in T and θ, we have

$$l_T^{\theta^*,q}(\theta) > r$$

for all $T > T'$ and $\theta > \theta^* - \varepsilon$.

Now consider a game Γ_T with $T > \max(T'', T')$. Suppose player i q-believes at t that $\theta > \theta^* - \varepsilon$. Since $T > T'$, $\theta > \theta^* - \varepsilon$ implies that $l_{-i,T}^{\theta^*,q}(\theta) \geq r$. Since $T > T''$, by hypothesis, player $-i$ invests at t if she $q-$believes that $\theta > \theta^*$ at t. Thus, conditional on $\theta > \theta^* - \varepsilon$ the probability that player $-i$ invests is no less than r. Therefore, at t, player i attaches probability at least rq to the event that the project succeeds. Since $rq > \overline{p}$, this implies that player i invests at t. ■

Our main result follows immediately from Lemmas 3 and 4. Fix $\underline{\theta} > 0$ and $\varepsilon > 0$. By Lemma 4, there exists some T' such that each player invests in the game Γ_T with $T > T'$ if she q-believes that $\theta \geq \underline{\theta}$. By Lemma 3, there exists some T'' such that for $T > T''$, when the fundamental is at least $\underline{\theta}$, the probability that both players q-believe that $\theta \geq \underline{\theta}$ in some round in Γ_T is greater than $1 - \varepsilon$. Taking $\underline{T} = \max\{T', T''\}$ gives the result. ■

Thus, in sharp contrast to the synchronous case, coordination failure arises with vanishing probability in the asynchronous case as the number of rounds grows large. In addition, if, as in the synchronous case, we let observation noise vanish, we get the even stronger implication that there is no delay in successful coordination. This is a corollary of Proposition 2. To make this idea precise, consider a family of sequences $(\sigma\sigma_t)_{t=1}^{\infty}$, where $\sigma > 0$ is a scaling factor, and $(\sigma_t)_{t=1}^{\infty}$ is some fixed sequence with strictly positive members converging to 0. We will denote by $\Gamma_T(\sigma)$ game with T rounds and noise parameters $(\sigma\sigma_t)_{t=1}^{T}$.

Corollary 1 *For any $\underline{\theta} > 0$ and $\varepsilon > 0$ there exists some $\overline{\sigma} > 0$ and $\underline{T}$ such that for any $\sigma < \overline{\sigma}$, in any equilibrium of $\Gamma_T(\sigma)$ with $T > \underline{T}$, both players invest in round 1 with probability at least $1 - \varepsilon$ whenever $\theta \geq \underline{\theta}$.*

What explains the stark difference in outcomes in the synchronous and asynchronous coordination games? One instructive way to interpret this difference arises out of characterizing the higher order beliefs of players in these games. We turn to such a characterization in the next section.

4 Higher Order Beliefs

It is well-known that the coordination failure arising in static global games can be explained by the lack of approximate common knowledge. The finding that coordination failure does not arise in our asynchronous global game indicates that some aspects of higher order beliefs differ between synchronous and asynchronous global games. The current section is devoted to examining this difference.

First, we introduce notation for payoff-relevant sets of fundamentals: "good fundamentals," $G = (0, +\infty)$, and "upper dominance fundamentals" $U = [1, +\infty)$. If $\theta \in G$, the project may succeed. If $\theta \in U$, the project must succeed.

4.1 The synchronous case

We first *informally* review the well-known result for the static global game. Consider the simple static game obtained when the dynamic game described in Section 2 has only one round; that is, when $T = 1$. Correspondingly, for the remainder of this subsection we do not superscript or subscript variables by t. The following discussion is based on Morris and Shin [17].

Let $B^i_p(E)$ denote the set of i's types that assign probability at least p to the event E; for types $B^i_p(E)$ we say that i p-believes E. Let $B_p(E)$ denote the profiles at which both players p-believe E.

To simplify the exposition, assume (for this subsection only) that investment of both players is necessary for the project to succeed whenever $\theta > 0$, so that there is no upper dominance region. Then the best response of each player is to invest if and only if she $\underline{p}$-believes both that the fundamental θ is in G, and that the other player invests. Therefore, investment is rationalizable only for types of player i that $\underline{p}$-believe the following list:

- G,
- that player $-i$ $\underline{p}$-believes G,
- that player $-i$ $\underline{p}$-believes that player i $\underline{p}$-believes G,
- etc.

Hence both players will invest only on the intersection

$$\bigcap_{k \geq 1} \left[B_{\underline{p}}\right]^k (G),$$

which is denoted by $C_{\underline{p}}(G)$ and called common $\underline{p}$-belief of G.

However, common $\underline{p}$-belief of G is difficult to achieve in static global games. Suppose $\underline{p} > \frac{1}{2}$. In that case, player i $\underline{p}$-believes G only if $x^i \geq x^{(1)} = 0 + \sigma F^{-1}(\underline{p})$. But for common $\underline{p}$-belief of G, player i must also believe that the opponent's signal exceeds $x^{(1)}$. This belief occurs only if $x^i \geq x^{(2)} = x^{(1)} + \sqrt{2}\sigma F^{-1}(\underline{p})$. Continuing to higher orders of beliefs, we get conditions $x^i \geq x^{(k)}$ where $x^{(k)} = x^{(k-1)} + \sqrt{2}\sigma F^{-1}(\underline{p})$ for all $k > 1$. Since $F^{-1}(\underline{p}) > 0$ for $\underline{p} > \frac{1}{2}$, the sequence $x^{(k)}$ diverges to ∞, and hence there is no state at which G is common $\underline{p}$-belief. Note that this argument holds for arbitrarily small σ.

If we take a snapshot of our dynamic game at any round t, the information structure is identical to that of the static global game with $\sigma = \sigma_t$. Hence the common $\underline{p}$-belief in the above static sense is *not* achieved in any of the rounds of the dynamic game. This explains the coordination failure described in Proposition 1 — the game studied there is essentially a static game with $\sigma = \sigma_T$, and the fact that σ_T decreases in T is irrelevant as long as $\sigma_T > 0$.

4.2 The asynchronous case

The discussion so far indicates that the difference between the asynchronous and synchronous games does not lie in the ability to generate standard common p-belief. In this respect private learning does not help. The difference must lie in the relevant concept of common beliefs which characterize the set of types for which investment is rationalizable. The less restrictive conditions under which the project succeeds in the asynchronous game lead to less demanding belief operators and to a concept of common belief which is satisfied at a large set of states.

4.2.1 Definitions

In what follows, for convenience, we refer to the beliefs and actions of player i at date t as the beliefs and actions of *agent* (i,t). Let Θ denote the set of possible fundamentals, and $X^{(i,t)}$ the set of types of agent (i,t) for $i \in \mathcal{I} = \{1,2\}$ and $t \in \{1,\ldots,T\}$.[5] The set of states is the product $\Theta \times_{i,t} X^{(i,t)}$.

We now define relevant events:

- A Θ-event F_Θ is a subset of Θ. Such events describe the fundamental, θ.
- An (i,t)-event $F^{(i,t)}$ is a subset of $X^{(i,t)}$. Such events describe the type of agent (i,t).
- An i-event $F^i = \times_{t\leq T} F^{(i,t)}$ is a list of (i,t)-events, with each member of the list describing the type of agent (i,t).
- A *compound* event $F = F_\Theta \times \left(\times_{i\in\mathcal{I}} F^i\right)$ is a list containing (i,t)-events for each $i \in \{1,2\}$ and $t \in \{1,\ldots,T\}$, together with one Θ-event.

We will abuse notation by identifying each Θ-event F_Θ with the compound event $F_\Theta \times \left(\times_{i,t} X^{t,i}\right)$, each product $\times_{i\in\mathcal{I}} F^i$ with the compound event $\Theta \times \left(\times_{i\in\mathcal{I}} F^i\right)$, and so on.

We say that an i-event F^i is *eventually true* (or *holds eventually*) if $\cup_{t\leq T} F^{(i,t)}$ is true, that is, if the true state lies in $\cup_{t\leq T} F^{(i,t)}$.

For each player i we define an operator $\alpha^{T,i}(\cdot)$ that maps each compound event $F = F_\Theta \times \left(\times_{(j,t)} F^{(j,t)}\right)$ to

$$\alpha^{T,i}(F) \equiv F_\Theta \cap \left(\bigcap_{j\in\mathcal{I}\setminus\{i\}} \left(\bigcup_{t\leq T} F^{(j,t)} \right) \right). \tag{2}$$

The operator $\alpha^{T,i}(\cdot)$ has a useful interpretation. Suppose that the project succeeds only if the fundamental lies in F_Θ and all players invest by round T. Suppose that each agent (j,t)

[5] In our simple setup $\Theta = X^{(i,t)} = \mathbb{R}$ for all (i,t).

invests only on the event $F^{(j,t)}$. Then $\alpha^{T,i}(F)$ is the event that the project succeeds by round T, conditional on player i investing. Hence $\alpha^{T,i}(F)$ is the payoff-relevant event for player i.

Next we define relevant belief operators:

- The belief operator $A_p^{T,(i,t)}(\cdot)$ of agent (i,t) maps each compound event F to the set of types of (i,t) that assign probability at least p to $\alpha^{T,i}(F)$; that is,
$$A_p^{T,(i,t)}(F) = B_p^{(i,t)}\left(\alpha^{T,i}(F)\right).$$
We refer to $A_p^{T,(i,t)}(F)$ by saying that agent (i,t) asynchronously p-believes F. Note that $A_p^{T,(i,t)}(F)$ is an (i,t)-event.

- The belief operator $A_p^{T,i}(\cdot)$ of player i maps each compound event to a list of (i,t)-events, with each members describing, for some t, the types of (i,t) that asynchronously p-believe F; that is,
$$A_p^{T,i}(F) \equiv \times_{t \leq T} A_p^{T,(i,t)}(F).$$

- The belief operator $A_p^T(\cdot)$ maps each compound event to a list of (i,t)-events, with each member describing, for some (i,t), the types of (i,t) that asynchronously p-believe F; that is,
$$A_p^T(F) \equiv \times_{i \in \mathcal{I}} A_p^{T,i}(F).$$

- The asynchronous common belief operator $D_p^T(\cdot)$ is defined (on compound events F) by $D_p^T(F) \equiv \bigcap_{k=1}^{\infty} \left[A_p^T\right]^k(F)$.

The interpretation of asynchronous common belief $D_p^T(G)$ of a compound event G resembles the interpretation of the usual static concept of common belief. The event $D_p^T(G)$ is a list of events, with each member describing for some agent (i,t) the types of (i,t) that p-believe the following list:

- G,
- that player $-i$ eventually p-believes G,
- that player $-i$ eventually p-believes that player i eventually p-believes G,
- etc.

The interpretation of asynchronous common belief differs from the interpretation of standard common belief on page 13 only in the insertion of the qualifier "eventually". We now proceed to utilize this concept of asynchronous common belief to delineate the set of types for which investment is rationalizable in the asynchronous game.

4.2.2 Rationalizability

In the first step, we formulate a sufficient condition for rationalizability of investment in the dynamic *asynchronous* game in terms of rationalizability in a related *simultaneous move* game. This will allow us to use the results on higher order beliefs in simultaneous move games from Morris and Shin [19]. We refer to this associated game as the *characteristic game*, and define it as follows:

Definition 1 *The* ***characteristic game*** $\tilde{\Gamma}^T$ *is a simultaneous move game with $2T$ players denoted by (i,t) for $i \in \{1,2\}$ and $t \in \{1,\dots,T\}$. The information structure generates the same joint distribution of fundamentals and signals as in the asynchronous game: the fundamental θ is drawn according to an improper uniform distribution on $\mathbb{R}$, and each player (i,t) observes a signal $x^{(i,t)} \sim N(\theta, \sigma_t^2)$. Each player chooses an action from $\{0,1\}$, which we interpret as Not-Invest and Invest respectively. We say that the project succeeds either if $\theta \geq 1$, or if $\theta > 0$ and for each $i \in \{1,2\}$, at least one of the players $\{(i,1),\dots,(i,T)\}$ invests. The payoff for player (i,t) for not investing is 0, and for investing is $\tilde{b}$ if the project succeeds, and $-\tilde{c}$ if the project does not succeed. The parameters $\tilde{b}$ and $\tilde{c}$ satisfy*

$$\underline{p}(\tilde{c},\tilde{b}) = \overline{p}(c,b).^6$$

Note that players in the characteristic game are analogues of agents in the asynchronous game. However, we continue to refer to player (i,t) in the characteristic game as agent (i,t), and the collection $(i,t)_t$ as player i, as would be appropriate in the asynchronous game.

The asynchronous and characteristic games have the same number of agents. In the characteristic game, investment is a best response for agent (i,t) *if and only if* she $\overline{p}(c,b)$-believes that project succeeds. In the asynchronous game, investment is a best response for agent (i,t) *if* she $\overline{p}(c,b)$-believes that project succeeds. Hence, rationalizability of investment in the characteristic game is a sufficient condition for rationalizability of investment in the dynamic game. We proceed to characterize the rationalizability of investment in the characteristic game.

One technical complication we face is that our game is of *common* and not *private* values. Players are sure of their own payoff parameters in private value games, and hence they suffer only from strategic uncertainty; this makes common beliefs directly applicable.[7] In common

[6] Note that we require the probability of success that was *sufficient* for immediate investment in the asynchronous game to be *necessary* in the characteristic game. It must then be the case that payoffs in the characteristic game are less favorable to investing than in the original asynchronous game. It is easy to check that $\tilde{b} = b - \delta b$ and $\tilde{c} = c + \delta b$ works. Finally, note that since the characteristic game is static, $\underline{p}(\tilde{c},\tilde{b}) = \overline{p}(c,b)$ is necessary and sufficient for investment.

[7] This is the reason why most of the higher order beliefs literature deals with private value games. In our case the common value setup is dictated by the motive of private learning in our model.

value games, players suffer also from uncertainty over the fundamental, which requires a slight modification in the relevant belief operators. We introduce these modified operators below and use them to characterize the set of types for which investment is rationalizable. The introduction of the modified operators is only a technical step; later on, we identify sufficient conditions for rationalizability of investment in terms of the unmodified operators defined in Section 4.2.1 above, and thereafter the modified ones will not be needed.

Define the following operators:

- $R_p^{T,(i,t)}(F) \equiv A_p^{T,(i,t)}((F \cap G) \cup U)$.
- $R_p^{T,i}(F) \equiv \times_{t \leq T} R_p^{T,(i,t)}(F)$.
- $R_p^T(F) \equiv \times_{i \in \mathcal{I}} R_p^{T,i}(F)$.
- $Q_p^T(F) \equiv \bigcap_{k=1}^{\infty} \left[R_p^T\right]^k (F)$.

The motivation for the operator $R_p^{T,(i,t)}(\cdot)$ is as follows: suppose agents $(-i,t)$ invest at types $F^{(-i,t)}$, and consider the compound event $F = \Theta \times \left(\times_{i,t} F^{(-i,t)}\right)$. Then $A_p^{T,(i,t)}\left(F^{-i} \cap G\right) \cup U$ is the event that (i,t) asynchronously p-believes that the project succeeds if she invests, since success occurs when the fundamental is good and all players eventually invest $(F \cap G)$, or when the fundamental is in the upper dominance region (U).

Proposition 3 ***(Morris and Shin [19])*** *Investment is rationalizable in the characteristic game at type $x^{(i,t)}$ if and only if $x^{(i,t)}$ is an element of*

$$R_{\overline{p}}^{T,(i,t)}\left(Q_{\overline{p}}^T(G)\right).$$

Proof. See Morris and Shin [19]. ■

To obtain some intuition for Proposition 3, consider iterated deletion of actions which are never best responses. After the first round of deletion, investment survives for types $A_{\overline{p}}^{T,(i,t)}(G) = R_{\overline{p}}^{T,(i,t)}(G)$. After the second round, investment survives for types

$$A_{\overline{p}}^{T,(i,t)}\left(\left(R_{\overline{p}}^T(G) \cap G\right) \cup U\right) = R_{\overline{p}}^{T,(i,t)}\left(R_{\overline{p}}^T(G)\right).$$

After the third round, investment survives for types

$$A_{\overline{p}}^{T,(i,t)}\left(\left(\left[R_{\overline{p}}^T\right]^2(G) \cap G\right) \cup U\right) = R_{\overline{p}}^{T,(i,t)}\left(\left[R_{\overline{p}}^T\right]^2(G)\right),$$

and so on.

The following lemma specifies sufficient conditions for the events $R_p^T(F)$ and $Q_p^T(F)$ to occur in terms of the events $A_q^T(F)$ and $D_q^T(F)$ for sufficiently high q.

Lemma 5 *Suppose* $q \geq \frac{p+1}{2}$ *and* $A_q^{T,(i,t)}(F) \subseteq A_q^{T,(i,t)}(G)$*. Then*

$$\begin{aligned} &\textit{(i)} \quad A_q^{T,(i,t)}(F) \subseteq R_p^{T,(i,t)}(F), \\ \textit{and} \quad &\textit{(ii)} \quad D_q^T(F) \subseteq Q_p^T(F). \end{aligned}$$

Proof. Since $A_q^{T,(i,t)}(F) \subseteq A_q^{T,(i,t)}(G)$, we have

$$A_q^{T,(i,t)}(F) \subseteq A_q^{T,(i,t)}(F) \cap A_q^{T,(i,t)}(G).$$

The right-hand side of this last expression is contained in $A_{2q-1}^{T,(i,t)}(F \cap G)$, which is in turn contained in $R_{2q-1}^{T,(i,t)}(F)$. Since $q \geq \frac{p+1}{2}$, we have $2q-1 \geq p$, and hence $R_{2q-1}^{T,(i,t)}(F) \subseteq R_p^{T,(i,t)}(F)$. This proves part (i).

We now use part (i) to prove part (ii) of the lemma. The event $D_q^T(F)$ is contained in $A_q^{T,(i,t)}(F)$. By part (i), $D_q^T(F)$ is contained in $R_p^{T,(i,t)}(F)$. Since this containment holds for all (i,t), the event $D_q^T(F)$ is contained in $R_p^T(F)$. Furthermore, $D_q^T(F)$ is contained in $A_q^{T,(i,t)}\left(D_q^T(F)\right)$, and hence also in $A_q^{T,(i,t)}\left(R_p^T(F)\right)$. Applying part (i) again gives containment in $R_p^T\left(R_p^T(F)\right) = [R_p^T]^2(F)$. Continuing in this fashion, we obtain containment in $[R_p^T]^k(F)$ for any order k. ■

We are now ready to state sufficient conditions for rationalizability of investment in terms of the operators $A_q^{T,(i,t)}(\cdot)$ and $D_q^T(\cdot)$.

Proposition 4 *Investment is rationalizable in the characteristic game for types of agent* (i,t) *in*

$$A_q^{T,(i,t)}\left(D_q^T\left(G\right)\right) \tag{3}$$

for $q \geq \frac{\overline{p}+1}{2}$*.*

Proof. A sufficient condition for the event $R_p^{T,(i,t)}\left(Q_p^T\left(G\right)\right)$ to occur is for $A_q^{T,(i,t)}\left(D_q^T\left(G\right)\right)$ to occur with $q \geq \frac{p+1}{2}$. To see this, note first that $A_q^{T,(i,t)}\left(D_q^T(G)\right) \subseteq A_q^{T,(i,t)}(G)$, so the conditions of Lemma 5 are satisfied with $F = D_q^T(G)$. Hence we have

$$A_q^{T,(i,t)}\left(D_q^T\left(G\right)\right) \subseteq R_p^{T,(i,t)}\left(D_q^T\left(G\right)\right) \subseteq R_p^{T,(i,t)}\left(Q_p^T\left(G\right)\right),$$

where the second containment follows from part (ii) of Lemma 5 with $F = G$. By Proposition 3, investment is rationalizable for types of agent (i,t) in $R_{\overline{p}}^{T,(i,t)}\left(Q_{\overline{p}}^T\left(G\right)\right)$. ■

4.2.3 Characterization of asynchronous beliefs

Section 4.2.2 established sufficient conditions for rationalizability of investment in terms of asynchronous common beliefs. In this section, we show that asynchronous common belief is easily attained in sufficiently long games.

Following Monderer and Samet [15], we say that E is an *asynchronous p-evident event* (for T rounds) if $E \subseteq A_p^T(E)$. The following proposition restates a result due to Monderer and Samet [15], but in the asynchronous setting.

Proposition 5 *A state ω lies in $D_p^T(F)$ if and only if there exists an asynchronous p-evident event E containing ω such that $E \subseteq A_p^T(F)$.*

Proof. See Monderer and Samet [15], Proposition 3. ■

We use the characterization of asynchronous common beliefs from Proposition 5 to prove the next result.

Proposition 6 *For all $r > q$, there exists some $\underline{T}$ such that for all $T \geq \underline{T}$,*

$$A_r^{T,(i,t)}(G) \subseteq A_q^{T,(i,t)}\left(D_q^T(G)\right).$$

Proof. Let $\alpha^T(F) \equiv \bigcap_i \alpha^{T,i}(F)$. Recalling the definition of $\alpha^{T,i}(\cdot)$ from page 14, $\alpha^T(F)$ may be interpreted as the event that $F^{(i,t)}$ is eventually true for each player i and F_Θ holds.[8]

For $\theta^* = 0$, Lemma 3 states that for all $r < 1$ and all $\theta \in G$,

$$\lim_{T\to\infty} \Pr\left(\alpha^T\left(A_r^T(G)\right)\middle|\,\theta\right) = 1.$$

Denoting $\Pr\left(\alpha^T\left(A_r^T(G)\right)\middle|\,\theta = 0\right)$ by s_T, we have

$$A_r^{T,(i,t)}(G) \subseteq A_{r\cdot s_T}^{T,(i,t)}\left(A_r^T(G)\right)$$

because any type of agent (i,t) that assigns probability r to G assigns probability at least $r \cdot s_T$ to the event that G holds and the opponent eventually r-believes G.

Since $r > q$ and $s_T \to 1$ as $T \to \infty$, the product $r \cdot s_T$ exceeds q for sufficiently large T. Hence we have

$$A_r^{T,(i,t)}(G) \subseteq A_q^{T,(i,t)}\left(A_r^T(G)\right), \tag{4}$$

and, since this holds for all agents (i,t), $A_r^T(G)$ is an asynchronous q-evident event.

By Proposition 5, the event $A_r^T(G)$ must be contained in $D_q^T(G)$. Combining this with (4) gives

$$A_r^{T,(i,t)}(G) \subseteq A_q^{T,(i,t)}\left(A_r^T(G)\right) \subseteq A_q^{T,(i,t)}\left(D_q^T(G)\right),$$

as needed. ■

Proposition 6 indicates that the sufficient conditions for rationalizability of investment given above are not demanding when T is large. All that is needed is *first-order* r-belief of G with $r > \frac{\overline{p}+1}{2}$, which is achieved for signals exceeding $F^{-1}(r)\sigma_t$. Investment is therefore rationalizable for all positive signals except in a small neighborhood of 0 of size on the order of σ_t.

[8] We apply the operator $\alpha^T(\cdot)$ only to compound events for which $F_\Theta = \Theta$, and hence F_Θ holds trivially.

5 Conclusion

Static coordination games represent a useful abstraction for studying coordination problems in the real world. However, the associated requirement of synchronicity in participation may be a strong restriction: the outcomes generated in such models may not be good representations for real-world coordination problems where agents are able to participate at different points of time and can learn about payoffs while deciding when to participate. We illustrate the radical difference between synchronous and asynchronous coordination problems within the framework of global games. In canonical synchronous (one-shot) global games, the risk-dominant equilibrium of the underlying complete information game is selected. Thus, coordination failure is endemic in static global games: there exist a wide class of payoffs for which players fail to efficiently coordinate in the unique equilibrium of the canonical global game despite the fact that it is in their collective interest to do so. At the other extreme, we introduce a class of enriched asynchronous global games where agents have an infinity of opportunities to participate, while they asymptotically and privately learn the true payoffs. In such games, we show that equilibrium play ensures Pareto dominant outcomes. Coordination failure is eliminated.

In conclusion, it is useful to discuss the role of our major assumptions, and to consider potential extensions. First, it is clear that though our analysis considers only two players, our results extend immediately to any finite number of players. Our core results, described in Section 3, are phrased in terms of $\underline{p}$ and $\overline{p}$, which do not depend on the number of players. In explaining our results in terms of higher order beliefs in Section 4, we have used definitions and notation that do not rely on the specific number of players (as long as the set is finite). However, phrasing the core analysis in terms of two-player games facilitates a clean comparison to the static game in terms of risk dominance vs. Pareto dominance.

Second, irreversibility plays an important role in our analysis, and, more generally, in the analysis of dynamic coordination games. The tendency towards efficiency in our model is related to the fact that we chose the efficient rather than the inefficient action to be irreversible. This assumption is natural in the context of many applications, including the leading example of foreign direct investment which we used to motivate our stylized model. However, in other applications, alternative assumptions may be more appropriate. Had we chosen differently, that is, had we assumed that the project succeeds only if all players choose to invest in *all* rounds, the project would always fail except in the upper dominance region. The coordination outcome in dynamic coordination games is, therefore, sensitive to the details of the dynamic setup. A deeper understanding of dynamic coordination problems may pinpoint detailed changes in the design of coordination processes that could help to avoid coordination failures. Our results provide a benchmark for such design exercises.

Finally, while it is useful as a benchmark exercise to study the extreme cases in which players learn nothing or everything during the play of the game, or when investment is fully reversible vs irreversible, from an applied perspective it is of greatest interest to learn about intermediate cases, i.e., about finite-rounds asynchronous global games with individual learning during which players learn something but not everything. These intermediate cases remain interesting problems for future research.

6 Appendix

Proof of Lemma 3. For any infinite sequence $\tau = (t_1, t_2, \ldots)$, let

$$l^{\theta^*,q}(\theta^*;\tau) = \Pr\left(\bigcup_{t'=t_1,t_2,\ldots} B_q^{(i,t')}(\theta \geq \theta^*) \middle| \theta^* \right).$$

We have $l^{\theta^*,q}(\theta^*;\tau) \leq l^{\theta^*,q}(\theta^*)$, so it suffices to show that, for any $\varepsilon > 0$, there exists some sequence τ for which $l^{\theta^*,q}(\theta^*;\tau) \geq 1-\varepsilon$.

Given $\varepsilon > 0$, let $\underline{x} < \theta^*$ be such that $\Pr(x^{(i,1)} < \underline{x}|\theta^*) < \varepsilon(1-\varepsilon)$. Let $t_1 = 1$ and choose each subsequent t_k so that

$$\Pr(x^{(i,t_k)} < \underline{x}|\theta^*) < \varepsilon^k(1-\varepsilon).$$

For this sequence $t_1, t_2, \ldots$, we have

$$\Pr(x^{(i,t_k)} < \underline{x} \text{ for some } k|\theta^*) < \varepsilon.$$

Thus it suffices to show that, as long as $x^{(i,t_k)} \geq \underline{x}$ for all k, there almost surely exists some period t_k in this sequence at which the player q-believes that $\theta \geq \theta^*$. By the Borel-Cantelli Lemma, it suffices to show that for some $\delta > 0$ and some subsequence τ of this sequence, the player q-believes that $\theta \geq \theta^*$ with independent probability δ in each period in τ.

Player i q-believes in period t that $\theta \geq \theta^*$ as long as $\frac{x^{(i,t)}-\theta^*}{\sigma_t} > \Phi^{-1}(q)$, where $\Phi(\cdot)$ denotes the standard normal distribution function. Given that $x^{(i,t_k)} \geq \underline{x}$, for $t > t_k$, we have

$$\begin{aligned} x^{(i,t)} &= \frac{\sigma_t^2}{\sigma_{t_k}^2} x^{(i,t_k)} + \sigma_t^2 \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2} \\ &\geq \frac{\sigma_t^2}{\sigma_{t_k}^2} \underline{x} + \sigma_t^2 \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2}. \end{aligned}$$

Hence $\frac{x^{(i,t)}-\theta^*}{\sigma_t} > \Phi^{-1}(q)$ whenever

$$\frac{\frac{\sigma_t^2}{\sigma_{t_k}^2}\underline{x} + \sigma_t^2 \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2} - \theta^*}{\sigma_t} > \Phi^{-1}(q). \tag{5}$$

Since

$$\frac{1}{\sigma_t^2} = \frac{1}{\sigma_{t_k}^2} + \sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}, \tag{6}$$

we have $\frac{1}{\sigma_t} = \frac{\sigma_t}{\sigma_{t_k}^2} + \sigma_t \sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}$, and the left-hand side of Inequality (5) may be rewritten as

$$\begin{aligned}
\frac{\frac{\sigma_t^2}{\sigma_{t_k}^2}\underline{x} + \sigma_t^2 \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2} - \theta^*}{\sigma_t} &= \frac{\sigma_t}{\sigma_{t_k}^2}\underline{x} + \sigma_t \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2} - \left(\frac{\sigma_t}{\sigma_{t_k}^2} + \sigma_t \sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2} \right) \theta^* \\
&= \frac{\sigma_t}{\sigma_{t_k}^2}\left(\underline{x} - \theta^*\right) + \sigma_t \sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2} \left(\frac{\sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2}}{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} - \theta^* \right) \\
&= \frac{\sigma_t}{\sigma_{t_k}^2}(\underline{x} - \theta^*) \\
&\quad + \left(\sigma_t \sqrt{\frac{1}{\sigma_t^2} - \frac{1}{\sigma_{t_k}^2}} \right) \left(\sqrt{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} \left(\frac{\sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2}}{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} - \theta^* \right) \right),
\end{aligned}$$

where the last equality follows again from Equation (6). Inequality (5) is therefore equivalent to

$$\frac{\sigma_t}{\sigma_{t_k}^2}(\underline{x} - \theta^*) + \left(\sigma_t \sqrt{\frac{1}{\sigma_t^2} - \frac{1}{\sigma_{t_k}^2}} \right) \left(\sqrt{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} \left(\frac{\sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2}}{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} - \theta^* \right) \right) > \Phi^{-1}(q).$$

The first term on the left-hand side of this inequality tends to zero as t grows large. The second term is a product of two factors, the first of which tends to one as t grows large, and the second of which is a standard normal random variable independent of the realizations of all signals up to period t_k. Therefore, for small enough $\delta > 0$, the inequality holds with probability at least δ when t is sufficiently large given δ, q, $\underline{x}$, and t_k. We may therefore construct the desired subsequence τ by selecting sufficiently distant elements of the sequence $t_1, t_2, \ldots$. ■

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