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Comment: Identification of a Simple Dynamic Discrete Game under Rationalizability

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Abstract

This paper studies the identification power of *rationalizability* in a simple dynamic discrete game model. The paper extends to dynamic games some of the results in Aradillas-Lopez and Tamer (2007). The most commonly used equilibrium concept in empirical applications of dynamic games is Markov Perfect Equilibrium (MPE). I study the identification of structural parameters when we replace the MPE assumption with weaker conditions such as rational behavior or rationalizability. I present identification results for a simple dynamic game of market entry-exit with two players. Under the assumption of level-2 rationalizability (i.e., players are rational and they know that they are rational), exclusion restrictions and large-support conditions on the exogenous explanatory variables are sufficient for point-identification of all the structural parameters. Though the model is fully parametric, the key identifying assumptions are nonparametric in nature and it seems that these identification results might be extended to a semiparametric version of the model.

Keywords: Identification, Empirical dynamic discrete games, rational behavior, rationalizability.

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*Comment on the paper *"The Identification Power of Equilibrium in Games,"* by Andres Aradillas-Lopez and Elie Tamer, invited paper for the annual conference of the *Journal of Business and Economics Statistics.* Salt Lake City. July 2007.

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1 Introduction

Structural econometric models of individual or firm behavior typically assume that agents are rational in the sense that they maximize expected payoffs given their subjective beliefs about uncertain events. Empirical applications of game theoretic models have used stronger assumptions than rationality. Most studies that have estimated games have used the Nash equilibrium concept, or some of its refinements, to explain agents' strategic behavior. The Nash equilibrium (NE) concept is based on assumptions on players' knowledge and beliefs which are more restrictive than rationality. Though there is not a set of necessary conditions to generate the NE outcome, the set of sufficient conditions includes the assumption that players' actions are common knowledge.[1] This assumption may be too restrictive or unrealistic in some applications. Therefore, it seems relevant to study whether we can identify the parameters of empirical games under weaker conditions than NE. For instance, we would like to know if rationality (together with mutual knowledge of payoffs) is sufficient for identification. It is also relevant to study the identification power of other assumptions on players' knowledge which are stronger than rationality but weaker than NE, such as common knowledge rationality: i.e., everybody knows that players are rational; everybody knows that everybody knows that players are rational, etc. The solution concepts *iterated strict dominance* and *rationalizability* are closely related to the assumption of common knowledge rationality (see chapter 2 in Fudenberg and Tirole, 1991).

The paper by Andres Aradillas-Lopez and Elie Tamer (2007) is the first study that deals with these interesting identification issues. The authors study the identification power of rational behavior and rationalizability in three classes of static games which have received significant attention in empirical applications: binary choice games, with complete and with incomplete information, and auction games under the independent private values paradigm. Their paper contributes to the literature on identification of incomplete econometric mod-

[1] Aumann and Brandenburger (1995) have derived sufficient conditions on players' knowledge and beliefs to generate the NE as an outcome of a game. They show that mutual knowledge of the payoff functions and of rationality, and common knowledge of the conjectures (actions), imply that the conjectures form a NE.

els, i.e., models which do not provide unique predictions on the distribution of endogenous variables (see also Tamer, 2003, and Haile and Tamer, 2003). Aradillas-Lopez and Tamer's paper shows that standard exclusion restrictions and large-support conditions are sufficient to identify structural parameters despite the non-uniqueness of the model predictions. Note that, though structural parameters can be point-identified, when we use the estimated model to perform counterfactual experiments we have that players' behavior under the counterfactual scenario is not point-identified. This problem also appears in models with multiple equilibria. However, a nice feature of Aradillas-Lopez and Tamer's *rationalizability approach* is that, at least for the class of models that they consider, it is very simple to obtain bounds for the model the predictions on players' behavior.

The main purpose of this paper is to study the identification power of rational behavior and rationalizability in a class of empirical games that has not been analyzed in Aradillas-Lopez and Tamer's paper: dynamic discrete games. Dynamic discrete games are of interest in economic applications where agents interact over several periods of time and make decisions that affect both current and future payoffs. The most commonly used equilibrium concept in applications of dynamic games is Markov Perfect Equilibrium (MPE). As in the case of NE, the concept of MPE is based on the assumption that players' strategy functions are common knowledge. Here I study the identification of structural parameters when we replace the MPE assumption with weaker conditions such as rational behavior or rationalizability.

I present identification results for a simple dynamic game of market entry-exit with two players. For this simple model the results are similar to the ones in Aradillas-Lopez and Tamer's paper for static games of incomplete information. Under the assumption of level-2 rationalizability (i.e., players are rational and they know that they are rational), exclusion restrictions and large-support conditions on the exogenous explanatory variables are sufficient for point-identification of all the structural parameters. Though the model is fully parametric, the key identifying assumptions are nonparametric in nature and it seems that these identification results might be extended to a semiparametric version of the model.

2 Dynamic discrete games

2.1 Model and assumptions

There are two firms which decide whether to operate or not in a market. I use the indexes $i \in \{1, 2\}$ and $j \in \{1, 2\}$ to represent a firm and its opponent, respectively. Time is discrete an indexed by t. Let $Y_{it} \in \{0, 1\}$ be the indicator of the event "firm i is active in the market at period t". Every period t the two firms decide simultaneously whether to be active in the market or not. A firm makes this decision to maximize its expected and discounted profits $E_t \left(\sum_{s=0}^{\infty} \delta^s \Pi_{i,t+s} \right)$, where $\delta \in (0, 1)$ is the discount factor and Π_{it} is the firm's profit at period t. The decision to be active in the market has implications not only on a firm's current profits but also on its expected future profits. More specifically, there is an entry cost that should be paid only if a currently active firm was not active at previous period ($Y_{it} = 1$ and $Y_{it-1} = 0$). Therefore, the lagged entry decision Y_{it-1} affects current profits and the model is dynamic. The one-period profit function is:

$$\Pi_{it} = \begin{cases} Z_i \eta_i + \gamma_i Y_{it-1} + \alpha_i Y_{jt} - \varepsilon_{it} & if \quad Y_{it} = 1 \\ 0 & if \quad Y_{it} = 0 \end{cases} \tag{1}$$

Y_{jt} represents the opponent's decision. η_i, γ_i and α_i are parameters. The parameter γ_i represents firm i's entry cost. The parameter $\alpha_i \leq 0$ captures the competitive effect. The variable ε_{it} is private information of firm i. For the sake of simplicity, I assume that the exogenous market and firm characteristics in Z_1 and Z_2 are constant over time. The vector of parameters $\theta \equiv \{\eta_i, \gamma_i, \alpha_i : i = 1, 2\}$ and the vector of variables $Z \equiv (Z_1, Z_2)$ are common knowledge. The variables ε_{1t} and ε_{2t} are independent of Z, independent of each other, and independently and identically distributed over time. Their distribution functions, H_1 and H_2, are absolutely continuous and strictly increasing with respect to the Lebesgue measure on $\mathbb{R}$.

2.2 Rational behavior and rationalizability

The literature on estimation of dynamic discrete games has used the concept of Markov Perfect Equilibrium (MPE). This type of equilibrium assumes that: (1) players' strategy functions depend only on payoff relevant state variables and they are constant over time (Markov assumption); (2) players are forward looking, maximize expected intertemporal payoffs and know their own strategy functions; and (3) players' strategies are common knowledge. The concept of rational behavior that I consider here maintains the assumptions (1) and (2) of Markov strategy functions and forward looking behavior. I relax condition (3). In our game, the payoff relevant state variables for player i are $\{X_t, \varepsilon_{it}\}$ where $X_t \equiv (Z_1, Z_2, Y_{1t-1}, Y_{2t-1})$. Let $\sigma_i(X_t, \varepsilon_{it})$ be a strategy function for player i. Given a strategy function $\sigma_i(X_t, \varepsilon_{it})$, we can define a conditional choice probability (CCP) function $P_i(X_t) \equiv \int I\{\sigma_i(X_t, \varepsilon_{it}) = 1\} dH_i(\varepsilon_{it})$, where $I\{.\}$ is the indicator function. It will be convenient to represent players' behavior using CCPs. Player i's beliefs about the expected behavior of his opponent can be represented as a CCP function $P_j(X_t)$.

A strategy function $\sigma_i(X_t, \varepsilon_{it})$ is *rational* if for every possible value of (X_t, ε_{it}) the action $\sigma_i(X_t, \varepsilon_{it})$ maximizes player i's expected and discounted payoffs given player i's belief about his opponent's strategy. More formally, $\sigma_i(X_t, \varepsilon_{it})$ is a *rational strategy function* if there is a CCP function $P_j(X_t)$ such that for any possible state (X_t, ε_{it}):

$$\sigma_i(X_t, \varepsilon_{it}) = \begin{cases} 1 & \text{if } \varepsilon_{it} \leq v_i^{P_j}(X_t) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where the function $v_i^{P_j}(X_t)$ represents the expected value of firm i if it is active today minus its value if it is not active today given that: (1) current state is X_t; (2) firm i behaves optimally in the future; and (3) firm i believes that its opponent's CCP function is P_j. We denote $v_i^{P_j}$ as the *differential value function.* For given P_j, the function $v_i^{P_j}$ is implicitly defined as the unique solution of a contraction mapping. I do not present here the details of the fixed point mapping that defines $v_i^{P_j}$. Assumption M and Lemmas 1 and 2, below, present several properties of the function $v_i^{P_j}$ which are used to prove the identification results. According to this definition of rational strategy, we say that a CCP function P_i is rational if there is a CCP function P_j for the opponent such that for every value X:

$$P_i(X) = H_i\left(v_i^{P_j}(X)\right) \tag{3}$$

Assumption *M*, below, establishes a monotonicity property of the differential value function in this game: if the opponent increases his probability of being active at some state X, then the differential value $v_i^{P_j}$ declines at any state. This monotonicity of the differential value function with respect to P_j implies that the empirical implications of rationalizability can be represented in terms of bounds on players' choice probabilities. $\alpha_i \leq 0$ is a necessary condition for Assumption M to hold, but it is not sufficient.

ASSUMPTION M: Let P_j^A and P_j^B be two CCP functions such that $P_j^A(X) \geq P_j^B(X)$ for any value of X. Then, $v_i^{P_j^A}(X) \leq v_i^{P_j^B}(X)$ for any value of X.

Assumption M has several implications. Let use $\{P_j = 1\}$ to denote the CCP function with $P_j(X) = 1$ for every value of X. And let's use $\{P_j = 0\}$ to denote the CCP function with $P_j(X) = 0$ for every value of X. Assumption M implies that for any CCP P_j and any value of X:

$$v_i^{\{P_j=1\}}(X) \quad \leq \quad v_i^{P_j}(X) \quad \leq \quad v_i^{\{P_j=0\}}(X) \tag{4}$$

If player i beliefs that his opponent will be active in the market with probability one under any possible state, then this belief generates the lowest differential value and the lowest probability of being active for player i. Similarly, if player i believes that he is a monopolist without threat of entry, then this belief generates the largest differential value and the highest probability of being active for player i. Therefore, if we do not know player i's beliefs, we can say that P_i is consistent with rational behavior if for every value X:

$$H_i\left(v_i^{\{P_j=1\}}(X)\right) \leq P_i(X) \leq H_i\left(v_i^{\{P_j=0\}}(X)\right) \tag{5}$$

A CCP function P_i is rationalizable of level 2 if there is a probability function P_j, which represents player i's belief about player $j's$ behavior, such that P_j is consistent with player

j's rational behavior, and P_i maximizes firm $i's$ expected value given his belief P_j. More formally, P_i is rationalizable of level 2 if there is a P_j such that for every X:

$$\begin{cases} H_j\left(v_j^{\{P_i=1\}}(X)\right) \leq P_j(X) \leq H_j\left(v_j^{\{P_i=0\}}(X)\right) \\ \text{and} \\ P_i(X) = H_i\left(v_i^{P_j}(X)\right) \end{cases} \tag{6}$$

We can use Assumption M to represent the restrictions of level-2 rationalizability as bounds on players' CCPs. Let's use $P_j^{L,1}$ and $P_j^{U,1}$ to denote the lower and upper bounds, respectively, on player j's CCPs given level-1 rationality: i.e., $P_j^{L,1}(X) \equiv H_j\left(v_j^{\{P_i=1\}}(X)\right)$ and $P_j^{U,1}(X) \equiv H_j\left(v_j^{\{P_i=0\}}(X)\right)$. Assumption M implies that for any P_j that satisfies player j's level-1 rationalizability and for any value of X:

$$v_i^{P_j^{U,1}}(X) \;\leq\; v_i^{P_j}(X) \;\leq\; v_i^{P_j^{L,1}}(X) \tag{7}$$

Therefore, if we do not know player i's beliefs, we can say that a strategy P_i is consistent with rationalizability of level-2 iff:

$$H_i\left(v_i^{P_j^{U,1}}(X)\right) \leq P_i(X) \leq H_i\left(v_i^{P_j^{L,1}}(X)\right) \tag{8}$$

It is straightforward to extend this result to level-k rationalizability. Under level-k rationalizability a strategy P_i should be between a lower bound $P_i^{L,k}$ and an upper bound $P_i^{U,k}$ where these bounds can be obtained recursively as follows. For any $k \geq 1$:

$$\begin{aligned} P_i^{L,k}(X) &= H_i\left(v_i^{P_j^{U,k-1}}(X)\right) \\ P_i^{U,k}(X) &= H_i\left(v_i^{P_j^{L,k-1}}(X)\right) \end{aligned} \tag{9}$$

with $P_j^{L,0} = 0$ and $P_j^{U,0} = 1$.

2.3 Identification under rationalizability

Suppose that we have a random sample of independent markets at some period t. For each market in the sample we observe a realization of the variables $\{Y_{1t}, Y_{2t}, Y_{1t-1}, Y_{2t-1}, Z_1, Z_2\}$.

Notice that the variables Z_1 and Z_2 do not vary over time but they have sample variability because they vary across markets. We are interested in using this sample to estimate the vector of structural parameters $\theta \equiv \{\eta_i, \gamma_i, \alpha_i : i = 1, 2\}$.

Let X_i be the vector of exogenous and predetermined explanatory variables (Z_i, Y_{it-1}). And define $\beta_i \equiv (\eta_i, \gamma_i)$ such that $X_i\beta_i \equiv Z_i\eta_i + \gamma_i Y_{it-1}$. Also, define $X \equiv (X_1, X_2)$ and let S_X be the support of X. Consider the following conditions:

(C1) The variance-covariance matrixes $V(X_1)$ and $V(X_2)$ have full rank.

(C2) For any player i, there is a variable $X_{i\ell} \subset X_i$ such that $\beta_{i\ell} \neq 0$ and conditional on any value of the other variables in X , that we represent as $(X_{i(-\ell)}, X_j)$, the random variable $\{X_{i\ell}|(X_{i(-\ell)}, X_j)\}$ has unbounded support.

(C3) For any player i, $\alpha_i \leq 0$ and Assumption M holds.

These assumptions are standard exclusion restrictions and large-support conditions on X. The following Lemmas are used in the proofs of the identification results.

LEMMA 1: For any P_j, the function $v_i^{P_j}(X)$ is strictly increasing in $X_i\beta_i$. Furthermore, $\lim_{X_i\beta_i \to +\infty} v_i^{P_j}(X) = +\infty$ and $\lim_{X_i\beta_i \to -\infty} v_i^{P_j}(X) = -\infty$.

LEMMA 2: For any P_j, the function $v_i^{P_j}(X)$ is strictly increasing in α_i.

Let $P_i^0(X)$ be the true conditional probability function $\Pr(Y_{it} = 1|X_t = X)$ in the population. And let θ^0 be the true value of θ in the population. Level-k rationalizability implies the following restrictions on choice probabilities: for $i \in \{1, 2\}$ and any $X \in S_X$

$$P_i^{L,k}\left(X, \theta^0\right) \leq P_i^0(X) \leq P_i^{U,k}\left(X, \theta^0\right) \tag{10}$$

To prove point identification of θ^0 we should show that for any vector $\theta \neq \theta^0$ there are values of $X \in S_X$ for which the previous inequality does not hold: i.e., either $P_i^{L,k}(X, \theta) > P_i^0(X)$ or $P_i^{U,k}(X, \theta) < P_i^0(X)$.

THEOREM 1 (Point identification under level-1 rationalizability). Suppose that conditions (C1)-(C3) hold and players are level-1 rational. Then, the parameters $\beta_{1\ell}^0$ and $\beta_{2\ell}^0$ are point-identified. The rest of the parameters may not be point-identified.

PROOF: Suppose that θ is such that $\beta_{i\ell} \neq \beta^0_{i\ell}$. By Assumption *M,* Lemma 1 and conditions (C1) and (C2), for any value of $(X_{i(-\ell)}, X_j)$ we can always find a finite value of $X_{i\ell}$, that we denote by $X^*_{i\ell}$, such that $v_i^{\{P_j=1\}}(X^*_{i\ell}, X_{i(-\ell)}, X_j; \theta) = v_i^{\{P_j=0\}}(X^*_{i\ell}, X_{i(-\ell)}, X_j; \theta^0)$. If $\beta_{i\ell} > \beta^0_{i\ell}$, then for values of X with $X_{i\ell} > X^*_{i\ell}$ we have that by construction:

$$P_i^{L,1}(X, \theta) = H_i\left(v_i^{\{P_j=1\}}(X, \theta)\right) > H_i\left(v_i^{\{P_j=0\}}(X, \theta^0)\right) \geq P_i^0(X)$$

Similarly, if $\beta_{i\ell} < \beta^0_{i\ell}$, then for values of X with $X_{i\ell} < X^*_{i\ell}$ we have that $P_i^{L,1}(X, \theta) = H_i\left(v_i^{\{P_j=1\}}(X, \theta)\right) > H_i\left(v_i^{\{P_j=0\}}(X, \theta^0)\right) \geq P_i^0(X)$. Q.E.D. ■

THEOREM 2 (Point identification under level-2 rationalizability). Suppose that conditions (C1)-(C3) hold and that players are rational and they know that they are rational. Then, all the structural parameters in θ^0 are point-identified.

PROOF: The proof goes through three cases which cover all the possible values of θ with $\theta \neq \theta^0$.

Case (i): Suppose that θ is such that $\beta_{i\ell} \neq \beta^0_{i\ell}$. By Assumption *M,* Lemma 1 and conditions (C1) and (C2), for any value of $(X_{i(-\ell)}, X_j)$ we can always find a finite value of $X_{i\ell}$, that we denote by $X^*_{i\ell}$, such that $v_i^{P_j^{U,1}}(X^*_{i\ell}, X_{i(-\ell)}, X_j; \theta) = v_i^{P_j^{L,1}}(X^*_{i\ell}, X_{i(-\ell)}, X_j; \theta^0)$. If $\beta_{i\ell} > \beta^0_{i\ell}$, then for values of X with $X_{i\ell} > X^*_{i\ell}$ we have that by construction:

$$P_i^{L,2}(X, \theta) = H_i\left(v_i^{P_j^{U,1}}(X, \theta)\right) > H_i\left(v_i^{P_j^{L,1}}(X, \theta^0)\right) = P_i^{U,2}(X, \theta^0) \geq P_i^0(X)$$

Similarly, if $\beta_{i\ell} < \beta^0_{i\ell}$, then for values of X with $X_{i\ell} < X^*_{i\ell}$ we have that $P_i^{L,2}(X, \theta) > P_i^{U,2}(X; \theta^0) \geq P_i^0(X)$.

Case (ii): Suppose that θ is such that $(\beta_{1\ell} = \beta^0_{1\ell})$ and $(\beta_{2\ell} = \beta^0_{2\ell})$ but $\beta_{i(-\ell)} \neq \beta^0_{i(-\ell)}$. By condition (C1), there should be some value of $X_{i(-\ell)}$ for which $X_{i(-\ell)}\left(\beta_{i(-\ell)} - \beta^0_{i(-\ell)}\right) > 0$. Given this value of $X_{i(-\ell)}$, by condition (C2) we can find a value of $X_{j\ell}$ (either large or small enough, depending on the sign of $\beta^0_{j\ell}$) such that both $P_j^{U,1}(X, \theta)$ (i.e., $H_j\left(v_j^{\{P_i=0\}}(X, \theta)\right)$) and $P_j^{L,1}(X, \theta^0)$ (i.e., $H_j\left(v_j^{\{P_i=1\}}(X, \theta^0)\right)$) and arbitrarily close to 1. For this value of X, we have that:

$$P_i^{L,2}(X,\theta) \simeq H_i\left(v_i^{\{P_j=1\}}(X,\theta)\right) > H_i\left(v_i^{\{P_j=1\}}(X,\theta^0)\right) \simeq P_i^{U,2}(X,\theta^0) \geq P_i^0(X)$$

where the inequality $H_i\left(v_i^{\{P_j=1\}}(X,\theta)\right) > H_i\left(v_i^{\{P_j=1\}}(X,\theta^0)\right)$ holds because $X_{i(-\ell)}\beta_{i(-\ell)} > X_{i(-\ell)}\beta^0_{i(-\ell)}$ and Lemma 1.

Case (iii): Suppose that θ is such that $\beta_1 = \beta_1^0$ and $\beta_2 = \beta_2^0$ but $\alpha_i \neq \alpha_i^0$. Suppose that $\alpha_i > \alpha_i^0$. By condition (C2) we can make $X_j\beta_j^0$ large enough such that $P_j^{L,1}(X,\theta^0) \simeq 1$ and $P_j^{U,1}(X,\theta) \simeq 1$. Then, given that $\alpha_i > \alpha_i^0$, by Lemma 2 we have that:

$$P_i^{L,2}(X,\theta) \simeq H_i\left(v_i^{\{P_j=1\}}(X,\theta)\right) > H_i\left(v_i^{\{P_j=1\}}(X,\theta^0)\right) \simeq P_i^{U,2}(X,\theta^0) \geq P_i^0(X)$$

Suppose that $\alpha_i < \alpha_i^0$. By condition (C2) we can make $X_j\beta_j^0$ large enough such that $P_j^{L,1}(X,\theta) \simeq 1$ and $P_j^{U,1}(X,\theta^0) \simeq 1$. Then, given that $\alpha_i < \alpha_i^0$, by Lemma 2 we have that:

$$P_i^{U,2}(X,\theta) \simeq H_i\left(v_i^{\{P_j=1\}}(X,\theta)\right) < H_i\left(v_i^{\{P_j=1\}}(X,\theta^0)\right) \simeq P_i^{L,2}(X,\theta^0) \leq P_i^0(X) \qquad \text{Q.E.D.} \blacksquare$$

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