April 15, 2004

Revised: July 13, 2005

# DIVISIBLE MONEY IN AN ECONOMY WITH VILLAGES

Miquel Faig*

University of Toronto

*I am grateful of the comments received from Neil Wallace, Randy Wright, Gabriele Camera, Tao Zhu, and Shouyong Shi as well as those received from participants of the Summer North American Meetings of the Econometric Society and the Summer Workshop at the Cleveland Fed in 2004. I am also grateful of the financial support from SSHRC of Canada. The usual disclaimer applies. Author's address: 140 St. George Street, Suite 707, Toronto, Canada, M5S 3G6. E-mail: mfaig@chass.utoronto.ca. Phone: (416) 978-0308. Fax: (416) 978-5519.

**Abstract**

This paper provides a tractable search model with divisible money that encompasses the two frameworks currently used in the literature. Individuals belong to many villages. Inside a village, individuals know each other so financial contracts are feasible. Money is essential to facilitate trade across villages. When financial markets inside a village are complete, the model generalizes the framework advanced by Lagos and Wright (2005) without having to assume quasi-linear preferences. Likewise, complete financial markets in each village substitutes for the representative household in the framework advanced by Shi (1997). The paper describes sets of financial arrangements that complete the markets inside the villages. In general, these financial arrangements include a combination of credit and insurance. However, if individuals choose period by period the trading role they play outside their village, then under some parametric restrictions either a lottery or a risk-free bond market are sufficient.

Keywords: monetary search, divisible money.

JEL: E40.

# 1 Introduction

Monetary search models have provided rich insights on the foundations of money, and they have become the dominant paradigm in this field of economics. To facilitate tractability, early monetary search models made strong assumptions on the properties of money (indivisibility and limited storage capacity). These strong assumptions prevented the study of many interesting issues such as inflation. Thanks to the work of Shi (1997) and Lagos and Wright (2005), we have now two distinct frameworks that yield tractable monetary search models with divisible money. Both frameworks use a trick to obtain a tractable distribution of money balances. In the case of Shi, the trick is the assumption that individuals belong to large households. In the case of Lagos and Wright, the trick is the assumption that utility is linear on a good traded in a competitive market. The present paper introduces a framework that encompasses those advanced by Shi (1997), and Lagos and Wright (2005).

In the model of this paper, individuals belong to villages.[1] Each village contains a large number of individuals, but it is only a small part of the global economy. In a village, individuals are not altruistic as in a household, but they know their neighbors. Therefore, financial contracts such as insurance and credit are feasible among individuals of the same village. Despite the existence of financial contracts inside the village, money is still essential to facilitate trade with anonymous individuals from other villages. This model captures in a simple fashion that in our daily economic interactions sometimes we deal with well identified and easy to trace individuals and sometimes we deal with relative strangers that can easily disappear from our lives.

If the set of financial markets inside the villages is complete (all individuals share the same marginal rates of substitution for all commodities traded there), then the financial deals inside villages substitute for the representative household in Shi (1997) to attain

[1] Jin and Temzelides (2004) advance also a random search model with villages to generate equilibria where money is used in some exchanges while credit is used in some others. However, they do not attempt to use villages to make tractable the divisibility of money. In their model, both money and goods are indivisible, and credit is a gift giving equilibrium with trigger strategies.

a simple distribution of money holdings. The idea that the large household construct substitutes for financial markets goes, at least, as far back as Lucas (1990). The present contribution takes seriously Lucas' idea and fleshes out the mechanisms that arise in the village to exhaust the gains from trade among villagers. There are several advantages to design these mechanisms explicitly. Typically, the mechanisms are financial contracts which are interesting on themselves. Moreover, by designing the financial contracts needed to complete the market eliminates the ambiguity on the objectives of buyers and sellers when they interact in a trade meeting. Finally, using numerical methods, one can in principle relax the assumptions that are required to give rise to a complete set of financial markets inside the village to find out how relevant they are to a particular issue.

At the same time, the complete set of financial markets inside the villages allows for more general preferences than those assumed by Lagos and Wright (2005). In Lagos and Wright, during the day individuals trade in a frictionless competitive market a good that yields constant marginal utility (quasi-linear preferences), at night they trade anonymously in frictional search markets. In the framework of this paper, we can add that during the day individuals are able to trade financial instruments with their fellow villagers, but at night these financial markets are closed because individuals are away from their village. With quasi-linear preferences, the financial markets inside the villages are complete even without any financial trades. As shown by Lagos and Wright (2005), during the day the unbounded trades on the good with a constant marginal utility equalizes the marginal rates of substitution for any pair of commodities among all people. In turn, this implies a simple distribution of money holdings. With the village structure, quasi-linear preferences are not a necessary assumption for this result. Financial arrangements can substitute for the trades on the good that yields linear utility to obtain similar outcomes.

Relaxing the quasi-linearity of preferences is important for several reasons. Quasi-linear preferences imply risk neutrality. Moreover, their usefulness in delivering a tractable distribution of money holdings rest on the absence of liquidity constraints during the day. Therefore, resting on these preferences to support tractability rules out most of the issues dealt in financial economics. Also, quasi-linear preferences rule out wealth effects on all goods except for the one that yields constant marginal utility. This hinders the study of many interesting issues in macroeconomics where wealth effects are important.

The main conclusions of the paper can be summarized as follows. As economists trained in the basic theorems of welfare economics would expect, each village acts as a well defined representative household when the set of financial markets inside the villages is complete. What is more surprising is the type of instruments that in some instances are sufficient to complete these financial markets. If the individuals choose endogenously their trading role (buyer-seller) in the frictional night markets, then, under some parametric restrictions, either a lottery or a risk-free bond are sufficient instruments to exhaust the gains from financial trades inside the village. In general, the complete set of markets includes the insurance of risks on trading opportunities.

The rest of the paper is organized as follows. The basic model of this paper is analyzed in Section 2. This model uses credit and insurance to complete the financial markets inside the villages. Section 3 provides the conditions for a simple lottery to substitute credit and insurance as the instrument for achieving market completeness inside the villages. Section 4 discusses the robustness of the main results of the paper and concludes. An Appendix collects the most technical parts of the proofs.

# 2 The Model

The economy is composed of a continuum of measure one of individuals. Individuals live in a continuum of measure one of symmetric villages. Inside their village, individuals know each other. Outside their village, individuals can easily hide their true identity, so they are anonymous.

Time is discrete and the horizon is infinite. Each period consists of two subperiods: day and night. During the day, all the individuals can produce and consume a general nondurable good, which is traded competitively inside each village. Also during the day, individuals trade competitively with their fellow villagers a set of financial contracts to be specified below. During the night, individuals trade nondurable goods specific to each village. All individuals can produce the good from their own village. Moreover, they all consume one specific good from another village. The specific good consumed differs randomly across time and across villagers. This environment gives rise to potential gains from trading outside the village of origin. At the same time, it precludes the possibility of long term relationships between buyers and sellers from different villages. Finally, it avoids that one individual can purchase outside goods for other fellow villagers.

At night, goods are traded in search markets where each individual is a price taker as in the competitive equilibrium concept employed by Rocheteau and Wright (2003 and 2005).[2] In the present environment, we can visualize this equilibrium concept as follows. Every night, a market for each specific good (one for each village) opens. In every market there is a Walrasian auctioneer that finds the competitive price that clears the market. All potential traders must search for the appropriate market place of the good they want to buy or sell. If successful, individuals are able to trade this good at the competitive price.

Individuals do not find by accident markets for which they do not search, so each individual must decide during the day if they are going to search for the market that trades the specific good from their own village, in which case the individual is going to be a seller, or for the market that trades the specific good they would like to consume, in which case the individual is going to be a buyer. For short, it is convenient to call the individuals that choose the first option "sellers" and individuals that choose the second option "buyers."

In the symmetric equilibrium we focus on, all markets for each specific good have the same measures of buyers and sellers, and all markets clear at the same price. Therefore, each one of these markets is representative of all the others. The probabilities that a buyer and a seller are able to trade in this representative market are denoted respectively by $\pi^b(\theta)$ and $\pi^s(\theta)$, where $\theta$ is the measure of individuals that choose to be sellers in period $t$.

---

[2] As Rocheteau and Wright acknowledge, this concept is closely related to the one used much earlier by Lucas and Prescott (1974).

(Throughout the paper time subscripts are omitted when they are not strictly necessary.) Both functions $\pi^b$ and $\pi^s$ are continuously differentiable. The function $\pi^b$ is convex and increasing, with terminal conditions: $\pi^b(0) = 0$ and $\pi^b(1) = 1$. The function $\pi^s$ is concave, decreasing, $\pi^s(0) = 1$, and $\pi^s(1) = 0$. In bilateral matching, these functions must also satisfy: $(1-\theta)\pi^b(\theta) = \theta\pi^s(\theta)$. However, this restriction is not necessary for most results, so it will only be imposed as an interesting special case.

Despite the competitive nature of all markets, there is a role for money in this model because at night buyers are anonymous in the markets they trade and there is a lack of double coincidence of wants.[3] Money is an intrinsically useless, perfectly divisible, and storable asset. The money supply grows at a constant gross rate $\gamma : M_{+1} = \gamma M$, where $M$ is the aggregate quantity of money and subscript $+1$ denotes next period. New money is injected via a lump-sum transfer to all individuals at the beginning of each day.

The objective of individuals is to maximize their expected lifetime utility: $E\sum_{t=0}^{\infty}\beta^t U_t$. The discount factor $\beta$ belongs to the interval $(1/2, 1)$.[4] The one period utility is equal to:

$$U = U^d(x, y) + EU^n\left(q^b, q^s\right), \tag{1}$$

where $x$ and $y$ are respectively quantities consumed and produced of the general good, and $q^b$ and $q^s$ are the quantities consumed or produced of specific goods. The expectation in (1) is conditional on the information at the beginning of period $t$. The functions $U^d$ and $U^n$ are continuously differentiable, concave, increasing in $x$ and $q^b$, and decreasing in $y$ and $q^s$. The maximum quantities an individual can produce, $y$ and $q^s$, are bounded. Finally, $U^n(0, 0) = 0$, and the standard Inada conditions for interior solutions apply. The discount and money growth factors obey: $\gamma > \beta$.

A typical period proceeds as follows. During the day, individuals produce, trade, and consume the general good in their village of origin. Also during the day, individuals trade

[3] Levine (1991) provides an early discussion of this point. See also Kocherlakota (1998).

[4] The restriction that $\beta$ is greater than 1/2 implies that the real interest rate is lower than 1. This restriction is important for the results of this section, but it is not for those in the next one.

with their fellow villagers an array of financial securities to be described below and choose the quantity of money to be held overnight. The optimal choices made during the day depend on the choice of being a seller or a buyer at night. Therefore, it is optimal for the individuals to make this choice prior to the other activities that take place during the day. At night, individuals search for one of the markets trading specific goods. Sellers search for the market trading the specific good of their own village. Buyers search for the market trading the specific good they would like to consume. Individuals successful in finding the market they are searching for trade money for goods at the competitive price. As a result of these trades, sellers produce, buyers consume, and money changes hands from buyers to sellers.

One of the objectives of the paper is to investigate alternative sets of financial securities that complete the financial markets inside the villages (equate the intertemporal marginal rates of substitution for the general good across all villagers.) At this point, it is assumed that individuals can trade two type of instruments: credit and insurance. As it will be seen, these instruments are sufficient to complete the financial markets inside the village. Later on, the paper inquires for conditions that make one or both of these types of instruments redundant. In so doing, the paper characterizes alternative informational requirements that support equilibria with simple distributions of money holdings.

The credit instrument individuals can trade inside their village is a risk-free real bond. This bond is a promise to deliver one unit of the general good next period at the price of $(1+r)^{-1}$ general goods today. Individuals demand bonds in this model because in some periods they accumulate wealth (while they are sellers), while other periods they spend their wealth (while they are buyers). Also, individuals may use these bonds to self-insure. The only informational requirement for the viability of this financial instrument is that the issuers of these bonds are known and they can be punished upon default. These bonds cannot be traded across villages because the buyers of these bonds would not know the issuer.

In addition to credit, individuals can buy insurance in their village against the risks on trading opportunities at the night markets. Specifically, buyers can purchase a contract for the delivery of $\mu^b$ general goods next morning contingent upon reaching the targeted search market at night. To implement this contract, buyers must be able to prove that they reached the market. This is clearly accomplished if all other fellow villagers can observe this event. However, it can also be accomplished with less demanding informational requirements. For example, it is sufficient if one can take the proverbial self-portrait holding today's newspaper with the targeted market in the background. Certainly, the requirement that the buyer can provide such as proof is not logically incompatible with being anonymous at night. Likewise, sellers can purchase a contract for the delivery of $\mu^s$ general goods next morning contingent upon failing to meet the market where the specific village good is traded. Again, the seller must be able to prove that such event occurred, which is not incoherent with the anonymity required for money to be essential. For example, since all sellers of the same village go to the same market, it is reasonable to assume that the list of those who reached such a market and the list of those who did not is public knowledge in the village. Even if sellers in that market are known by other sellers, this does not mean that buyers cannot be anonymous, which is what makes money essential. The fair premia to acquire the insurance contracts are respectively $\mu^b\pi^b$ and $\mu^s\left(1-\pi^s\right)$.[5] For notational ease, these premia are assumed to be payable next morning.

The paper focusses on symmetric, stationary, monetary competitive equilibria. In a competitive equilibrium, individuals maximize utility taking as given the sequence of prices of the following items: the general good, the specific goods, the real bonds, and the insurance premia. They also take as given the lump-sum transfers from the government and the aggregate fraction of individuals that choose to be sellers at night. The equilibrium is monetary if money is valued. The equilibrium is stationary if the price of specific goods relative to the price of the general good, $p$, the real quantity of money held by buyers in terms of the general good, $m^b$, and the fraction of individuals that choose to be sellers, $\theta$, are constant over time. The equilibrium is symmetric if these magnitudes are equal across

[5] To avoid cumbersome expressions, the argument $\theta$ is dropped in $\pi^b\left(\theta\right)$ and $\pi^s\left(\theta\right)$.

both villages and markets for specific goods.

In a stationary equilibrium, the nominal prices of the general good and the specific goods must grow at the gross rate $\gamma$. Moreover, the real interest rate must be equal to the subjective discount rate: $r = \beta^{-1} - 1$, otherwise consumption would grow or decline over time. Therefore, to characterize an equilibrium we will proceed as follows. First, we will characterize the behavior of individuals in a stationary environment with a constant inflation rate and a real interest rate equal to the subjective discount rate. Then, we will determine the relative price $p$ and the measure of sellers $\theta$ that clear the markets for general and specific goods.

## 2.1 The Behavior of Individuals

Consider an individual facing fair insurance and a constant vector $(p, \theta, \gamma, r)$ where $r = \beta^{-1} - 1$. Early each day, the individual receives a monetary transfer which real value is constant and equal to $\tau$.

Prior to all trades, the individual chooses day by day the trading role to be performed at night. This is a binary non-convex choice, in which the individual picks the alternative, buyer or seller, that yields the highest utility. As a result, the value function $V$ of the individual at the beginning of each day obeys:

$$V(a) = \max \left\{ V^b(a), V^s(a) \right\}; \tag{2}$$

where $a$ is the wealth (in units of the general good), and $V^b$ and $V^s$ are the value functions conditional on being, respectively, a buyer or a seller during the day.

To characterize the optimal plans of an individual as a buyer and as a seller, it is convenient to start with the conditional optimal demands for money. In the environment considered, the gross real rate of return on bonds is $\beta^{-1}$. Meanwhile, the gross real rate of return on money is $\gamma^{-1}$ (the inverse of the gross inflation rate.) Therefore, the assumption $\gamma > \beta$ implies that bonds earn a higher return than money. As long as $V$ is increasing (it will be), it is never optimal for an individual to demand a dollar today that with certainty will not be spent tonight. Therefore, the demand for money of a seller is zero, while the real demand for money of a buyer is $m^b = pq^b$, where $q^b$ is the quantity of specific goods demanded at night if search is successful.

Conditional on being a buyer, the individual chooses the quantities of the general good to be consumed $x^b$ and produced $y^b$, the real demands for money $m^b$ and bonds $b^b$, and the insurance coverage $\mu^b$. These choices must satisfy the budget constraint: $x^b + m^b + b^b (1+r)^{-1} = y^b + a + \tau$. Using $m^b = pq^b$, this constraint simplifies to:

$$x^b + pq^b + b^b (1+r)^{-1} = y^b + a + \tau. \tag{3}$$

The optimal plan for the individual as a buyer is represented by a vector $(x^b, y^b, q^b, b^b, \mu^b)$ that solves the following maximization program:

$$V^b(a) = \max \left\{ U^d \left(x^b, y^b\right) + \pi^b \left[U^n \left(q^b, 0\right) + \beta V \left(a^{b1}_{+1}\right)\right] + \left(1-\pi^b\right) \beta V \left(a^{b0}_{+1}\right) \right\}, \tag{4}$$

subject to (3). The terms $a^{b1}_{t+1}$ and $a^{b0}_{t+1}$ denote the wealth next morning contingent on succeeding to encounter a trading opportunity (superscript $b1$) or not (superscript $b0$). Therefore,

$$\begin{aligned} a^{b0}_{+1} &= b^b + m^b \gamma^{-1} - \mu^b \pi^b, \text{ and} & (5) \\ a^{b1}_{+1} &= b^b + \mu^b \left(1-\pi^b\right). & (6) \end{aligned}$$

With probability $\pi^b$, the individual is successful searching at night. In this case, the individual consumes $q^b$ at night, and the wealth next period is given by (6). With complementary probability, the individual fails to find the targeted market. As a result, the individual consumes nothing at night, which yields zero utility. In this instance, the wealth next period is (5). Analogously, conditional on being a seller the individual chooses $(x^s, y^s, q^s, b^s, \mu^s)$ to solve:

$$V^s(a) = \max \left\{ U^d \left(x^s, y^s\right) + \pi^s \left[U^n \left(0, q^s\right) + \beta V \left(a^{s1}_{+1}\right)\right] + \left(1-\pi^s\right) \beta V \left(a^{s0}_{+1}\right) \right\}, \tag{7}$$

subject to the budget constraint

$$x^s + b^s (1+r)^{-1} = y^s + a + \tau. \tag{8}$$

The terms $a_{+1}^{b1}$ and $a_{+1}^{s0}$ denote, respectively, the wealth next morning if the individual finds or not a trading opportunity. Therefore,

$$\begin{aligned} a_{+1}^{s0} &= b_t^s + \mu_t^s \pi^s, \text{ and} & (9) \\ a_{+1}^{s1} &= b^s + \gamma^{-1} p q_t^s - \mu^s (1-\pi^s). & (10) \end{aligned}$$

In addition to all constraints specified above, the individual faces a No-Ponzi game condition. That is, there is an endogenous lower bound on next period's wealth to ensure that the individual will be able to repay the amounts borrowed with probability one without reliance to unbounded borrowing:

$$a_{+1} \geq a_{\min} \text{ with probability one.} \tag{11}$$

In general, the wealth at the beginning of next period, $a_{+1}$, is stochastic because it depends not only on the buyer-seller choice but also on the random success of the night's search. The lower bound $a_{\min}$ is equal to minus the present discounted value of the maximum guaranteed income the individual can obtain as a seller.

The optimization program described in equations (3) to (11) can be characterized using standard recursive methods. This is a convenient feature of this model, which is absent with quasi-linear preferences and unbounded demands for the general good. A key step in this characterization is the following properties about the value function:

**Proposition 1:** *The value function $V$ is continuously differentiable, increasing, and concave. Furthermore, $V$ is affine in an interval $[\underline{a}, \overline{a}] \subseteq [a_{\min}, \infty)$ :*

$$V(a) = v_0 + va, \text{ for } a \in [\underline{a}, \overline{a}]; \tag{12}$$

*where $v_0$ and $v$ are coefficients independent of wealth. The interval $[\underline{a}, \overline{a}]$ is absorbing in the sense that optimal behavior implies that if $a \in [\underline{a}, \overline{a}]$, then $a_{+1} \in [\underline{a}, \overline{a}]$ with* *probability one.*

The linear segment of $V$ (12) is due to the daily endogenous choice of trading role by each individual. Intuitively, if an individual is not rich enough to be a buyer forever and not so poor to have to be a seller at perpetuity, then the individual will alternate between being a buyer and a seller. As the individual does so, wealth does not affect the quantities consumed or produced conditional on being a buyer or a seller. Instead, wealth affects how often and how early the individual decides to be a buyer or a seller. Since utility is linear on the times and the timing an individual gets the incremental expected utilities of being a buyer or a seller for one period, the value function is linear.

The property that the interval $[\underline{a}, \overline{a}]$ is absorbing simplifies the characterization of an equilibrium dramatically. As long as all individuals have initial wealth in the interval $[\underline{a}, \overline{a}]$, as it will be assumed from now on, the behavior of all buyers and all sellers is independent from their wealth. Therefore, the distribution of money holdings is easily characterized.

The proof of Proposition 1 is in the Appendix. The crucial step in this proof is presented here. This step uses the Bellman's equation (3) to (11) to define a mapping $T$ from the value function $V$ for period $t+1$ in the right hand sides of (4) and (7) onto the value function for period $t$ in the left hand side of (2), to be denoted for the rest of this proof $TV$. As shown in the next few paragraphs, if $V$ is increasing and concave, and it has the linear segment (12), then, for a particular set of values of the coefficients $v_0$, $v$, $\underline{a}$, and $\overline{a}$, $TV$ is also increasing and concave, and it has an identical linear segment to the one $V$ has. Since $T$ is a contraction mapping (see the Appendix), the unique true value function must have all these properties.

Consider an individual whose value function $V$ for period $t+1$ is increasing and concave with the linear segment (12). Because $V$ is concave, it is an optimal plan at $t$ to fully insure risks on trading opportunities. Hence, it is optimal for the individual to purchase contracts that satisfy: $\mu^b = \gamma^{-1} p q^b$ and $\mu^s = \gamma^{-1} p q^s$. Furthermore, only plans with full insurance are optimal if there is a positive probability that $a_{+1}$ will lie on a strictly concave region of $V$. With full insurance, $a_{+1}$ is not stochastic. Let $a_{+1}^b$ and $a_{+1}^s$ be the respective wealths at $t+1$

that the individual attains as a fully insured buyer or as a fully insured seller. If $a_{+1}^{b}$ lies in the linear interval $[\underline{a}, \overline{a}]$, then (12) and $m^b = pq^b$ imply that the utility at $t$ conditional on being a buyer (4) is equal to:

$$V^b(a) = S^b + \beta v_0 + v\tau + va; \tag{13}$$

where $S^b$ is the expected trade surplus of the individual as a buyer:

$$S^b = \max_{\{x^b, y^b, q^b\}} \left[ U^d\left(x^b, y^b\right) + \pi^b U^n\left(q^b, 0\right) + v\left(y^b - x^b - pq^b \frac{\pi^b + i}{1+i}\right)\right], \tag{14}$$

and $i \equiv (\gamma - \beta)/\beta = (1+r)\gamma - 1$ is the nominal rate of interest earned on bonds. Since $S^b$ is independent from $a$, the value function $V^b(a)$ has a linear segment with the same slope as the one in $V$. Similarly, if $a_{+1}^{s} \in [\underline{a}, \overline{a}]$, the utility at $t$ conditional on being a seller (7) is

$$V^s(a) = S^s + \beta v_0 + v\tau + va; \tag{15}$$

where $S^s$ is the expected trade surplus of the individual as a seller:

$$S^s = \max_{\{x^s, y^s, q^s\}} \left[ U^d\left(x^s, y^s\right) + \pi^s U^n\left(0, q^s\right) + v\left(y^s - x^s + pq^s \frac{\pi^s}{1+i}\right)\right]. \tag{16}$$

Again, $V^s(a)$ has a linear segment with the same slope as $V$.

The optimal plan $\left(x^{*b}, y^{*b}, q^{*b}, x^{*s}, y^{*s}, q^{*s}\right)$ that solves the maximization programs (14) and (16) is characterized by the following set of first order conditions:

$$\begin{aligned} U_1^d\left(x^{*b}, y^{*b}\right) &= U_2^d\left(x^{*b}, y^{*b}\right) = v, \\ U_1^n\left(q^{*b}, 0\right) &= \frac{vp}{1+i}\left(1 + \frac{i}{\pi^b}\right), \\ U_1^d\left(x^{*s}, y^{*s}\right) &= U_2^d\left(x^{*s}, y^{*s}\right) = v, \text{ and} \\ U_2^n\left(0, q^{*s}\right) &= -\frac{vp}{1+i}. \end{aligned} \tag{17}$$

The properties of $U^d$ and $U^n$ ensure that (17) has a unique solution, which is interior. Moreover, the structure of the system of equations (17) implies that

$$y^{*b} = y^{*s} \text{ and } x^{*b} = x^{*s}. \tag{18}$$

Define the optimal net expenditure of a buyer and a seller at $t$ respectively as:

$$z^{*b} = x^{*b} - y^{*b} - \tau + pq^{*b}\frac{\pi^b + i}{1+i}, \text{ and} \tag{19}$$

$$z^{*s} = x^{*s} - y^{*s} - \tau - pq^{*s}\frac{\pi^s}{1+i}. \tag{20}$$

With these definitions, the flow budget constraint for an individual simplifies into $a^j_{+1} = (1+r)(a - z^{*j})$ for $j = b$ and $s$. Let $[\underline{a}^j, \overline{a}^j]$ be the interval of present wealth that leads to $a^j_{+1} \in [\underline{a}, \overline{a}]$ next period for $j = b$ and $s$. That is, $[\underline{a}^b, \overline{a}^b]$ and $[\underline{a}^s, \overline{a}^s]$ are respectively the linear segments in $V^b(a)$ and $V^s(b)$. Using the flow budget constraint, we obtain:

$$\underline{a}^j = \frac{\underline{a}}{1+r} + z^{*j}, \text{ and } \overline{a}^j = \frac{\overline{a}}{1+r} + z^{*j} \quad \text{for } j = b \text{ and } s. \tag{21}$$

Let the values of the coefficients $v_0$, $v$, $\underline{a}$, and $\overline{a}$ be implicitly defined by the following set of equations:

$$\overline{a} = \frac{z^{*b}(1+r)}{r}, \tag{22}$$

$$\underline{a} = \frac{z^{*s}(1+r)}{r}, \tag{23}$$

$$v_0 = \frac{(S^{*s} + v\tau)(1+r)}{r}, \text{ and} \tag{24}$$

$$S^{*b} = S^{*s}, \tag{25}$$

where $S^{*b}$ and $S^{*s}$ are the optimized values of (14) and (16).[6] With these coefficient values, the linear segments of $V^b$ and $V^s$ lie on a common affine function as displayed in Figure 1.

In Figure 1, the value functions $V^s$ and $V^b$ have been displayed with linear intervals with bounds that obey: $\underline{a}^s < \underline{a}^b < \overline{a}^s < \overline{a}^b$. These inequalities are implied by $r \in (0,1)$, (18), (19), (20), and (21) to (23).[7] Moreover, $V^b$ and $V^s$ cross only in the interval that they share a common slope. This property is implied by the following argument. The inequality $z^{*b} > z^{*s}$ implies that $a^b_{+1} < a^s_{+1}$. Using the Envelope Theorem, the derivatives

[6] The existence of a unique solution to this system of equations can be easily established using the following facts: If $v \to 0$, $S^{*b} > S^{*s}$. If $v \to \infty$, $S^{*b} < S^{*s}$. $S^{*b}$ is decreasing with $v$. $S^{*s}$ is increasing with $v$.

[7] If $\beta$ were lower than 0.5, then $r > 1$ and $\underline{a}^b > \overline{a}^s$. As a result, the segment $[\underline{a}, \overline{a}]$ is not absorbing and the proof fails.

of $V^b$ and $V^s$ are respectively proportional to $V'(a^b_{+1})$ and $V'(a^s_{+1})$. Hence, the concavity of $V$ combined with $a^b_{+1} < a^s_{+1}$ implies that $V^b$ cannot be flatter than $V^s$ when evaluated at the same wealth. Therefore, $V^b$ and $V^s$ can only cross once. This crossing must be in the linear interval for the inequalities stated above to hold.

Figure 1 is useful to describe when an individual chooses to be a buyers or a seller. At each level of wealth, the individual picks the trading role that brings maximum utility. Therefore, the individual chooses to be a seller if $a < \underline{a}^b$. The individual is indifferent between being a buyer and a seller if $a \in \left[\underline{a}^b, \overline{a}^s\right]$. Finally, the individual decides to be a buyer if $a > \overline{a}^s$. Given the definitions of $\left[\underline{a}^b, \overline{a}^b\right]$ and $\left[\underline{a}^s, \overline{a}^s\right]$, these choices imply that if $a \in [\underline{a}, \overline{a}] = \left[\underline{a}^s, \overline{a}^b\right]$, then $a_{+1} \in [\underline{a}, \overline{a}]$. Therefore, $[\underline{a}, \overline{a}]$ is absorbing. Moreover, $TV(a)$ is increasing and concave, and it has the same linear segment as $V$. As a result, the true value function must have these properties.

## 2.2 Equilibrium

In a competitive equilibrium, the markets for the general good during the day and the specific goods at night must clear. Therefore, the equilibrium relative price of specific and general goods and the fraction of individuals choosing to be sellers ($p$ and $\theta$) are determined by the following two market clearing conditions:

$$(1-\theta)\, x^{*b} + \theta x^{*s} = (1-\theta)\, y^{*b} + \theta y^{*s}, \text{ and} \tag{26}$$

$$(1-\theta)\, \pi^b(\theta)\, q^{*b} = \theta \pi^s(\theta)\, q^{*s}. \tag{27}$$

The lump-sum transfer must satisfy the government budget constraint:

$$\tau = \left(1-\gamma^{-1}\right)(1-\theta)\, p q^{*b}. \tag{28}$$

Formally, an equilibrium is an optimal plan $\left(x^{*b}, y^{*b}, q^{*b}, x^{*s}, y^{*s}, q^{*s}\right)$, a marginal value of wealth $(v)$, and a triple $(p, \theta, \tau)$ that satisfy (17), the equality of trading surpluses (25), the market clearing conditions (26) and (27), and the government budget constraint (28).

The characterization of an equilibrium is greatly simplified by the fact that (18) and (26) imply $x^{*s} = x^{*b} = y^{*s} = y^{*b} \equiv x^*$. Using (17), these equalities imply that the equilibrium value of $x^*$ is determined by

$$U_1^d(x^*, x^*) = U_2^d(x^*, x^*). \tag{29}$$

Consequently, consumption and production in the general goods market is independent from monetary policy. Using this result, (17) and (25) simplify into

$$\frac{U_1^n\left(q^{*b}, 0\right)}{-U_2^n(0, q^{*s})} = 1 + \frac{i}{\pi^b(\theta)}, \text{ and} \tag{30}$$

$$\frac{U^n\left(q^{*b}, 0\right) - U_1^n\left(q^{*b}, 0\right) q^{*b}}{U^n(0, q^{*s}) - U_2^n(0, q^{*s})\, q^{*s}} = \frac{\pi^s(\theta)}{\pi^b(\theta)}. \tag{31}$$

Hence, the equilibrium values of $q^{*b}, q^{*s}$, and $\theta$ are determined by (28), (30), and (31).

Most of the literature that follows the seminal contributions of Shi (1997) and Lagos and Wright (2005) assume bilateral matching:

$$(1-\theta)\,\pi^b(\theta) = \theta\pi^s(\theta) \tag{32}$$

With bilateral matching, the market clearing condition (27) simplifies to $q^{*s} = q^{*b} \equiv q^*$. Using this equality, equations (30), and (31) become equivalent to the equations that define an equilibrium in Rocheteau and Wright (2003) without having assumed quasi-linear preferences.

## 2.3 The Role of Insurance

In the equilibrium characterized above, all individuals have wealths in the linear interval of the value function. Therefore, individuals, being locally risk neutral, are indifferent as to whether or not they purchase insurance as long as their future respective wealths remain inside the interval $[\underline{a}, \overline{a}]$ with probability one. The role of insurance is to allow individuals to insure against risks that would drive their future wealths into the strictly concave regions of the value function. This role is essential if individuals cannot avoid these type of risks while following optimal strategies. Otherwise, the insurance of trading risks is not an essential financial instrument to complete the markets inside the villages. This subsection studies the conditions that make insurance essential or redundant.

Consider again the optimization problem of an individual whose value function $V$ for period $t+1$ is increasing and concave with the linear segment (12). Define the mapping $T$ as in subsection 2.1 (except that now the individual has no access to insurance), so $TV$ is the value function for period $t$. The absence of insurance only matters for actions that would lead the individual outside the interval $[\underline{a}, \overline{a}]$. Therefore, it only makes a difference in the characterization of the bounds of the linear interval of $TV$.

In the absence of insurance, the optimal saving at $t$ depends not only on the trading role chosen but also on the outcome of the random search at night. Denoting with superscript 1 a successful search and with superscript 0 a failed search, the contingent net expenditures as a seller and as a buyer are respectively:

$$z^{*s1} = x^{*s} - y^{*s} - \tau - \frac{pq^{*s}}{1+i} \text{ and } z^{*s0} = x^{*s} - y^{*s} - \tau, \text{ and} \tag{33}$$

$$z^{*b1} = x^{*b} - y^{*b} - \tau + pq^{*b} \quad \text{and } z^{*b0} = x^{*b} - y^{*b} - \tau + \frac{i}{1+i} pq^{*b}. \tag{34}$$

With these definitions, the flow budget constraint for the individual is $a_{+1}^{jk} = (1+r)\left(a - z^{*jk}\right)$ for $j = b$ and $s$, and $k = 0$ and 1. Let $\left[\underline{a}^b, \overline{a}^b\right]$ be the interval that contains the values of $a$ with optimal strategies that lead to $a_{+1}^{b0}, a_{+1}^{b1} \in [\underline{a}, \overline{a}]$. Graphically, $\left[\underline{a}^b, \overline{a}^b\right]$ is the linear interval of $V^b(a)$. Since $z^{*b1} > z^{*b0}$, the bounds $\underline{a}^b$ and $\overline{a}^b$ are implicitly characterized by the following two equations:

$$\overline{a} = (1+r)\left(\overline{a}^b - z^{*b0}\right) \text{ and } \underline{a} = (1+r)\left(\underline{a}^b - z^{*b1}\right). \tag{35}$$

Similarly, since $z^{*s1} < z^{*s0}$, the bounds of the interval $[\underline{a}^s, \overline{a}^s]$ which contains the values of $a$ with optimal strategies that lead to $a_{+1}^{s0}, a_{+1}^{s1} \in [\underline{a}, \overline{a}]$ are characterized by

$$\overline{a} = (1+r)\left(\overline{a}^s - z^{*s1}\right) \text{ and } \underline{a} = (1+r)\left(\underline{a}^s - z^{*s0}\right). \tag{36}$$

The values of the coefficients $v_0$, $v$, $\underline{a}$, and $\overline{a}$ that are candidates to generate a linear interval in $TV$ identical to that of $V$ are implicitly defined by (24), (25), and the following two equations:

$$\overline{a} = \frac{z^{*b1}(1+r)}{r}, \text{ and} \tag{37}$$

$$\underline{a} = \frac{z^{*s1}(1+r)}{r}. \tag{38}$$

With these coefficient values, the linear segments of $V^b$ and $V^s$ lie on a common affine function as it happened with insurance. However, now two possible cases may arise. In the first case, the intervals $\left[\underline{a}^b, \overline{a}^b\right]$ and $[\underline{a}^s, \overline{a}^s]$ overlap in which case their union constitutes $[\underline{a}, \overline{a}]$ in $TV$. In this case, insurance is redundant because an individual with $a \in [\underline{a}, \overline{a}]$ can always pick an optimal strategy that leads to $a_{+1} \in [\underline{a}, \overline{a}]$ without purchasing insurance. In the second case, the intervals $\left[\underline{a}^b, \overline{a}^b\right]$ and $[\underline{a}^s, \overline{a}^s]$ do not overlap so their union does not generate $[\underline{a}, \overline{a}]$ in $TV$. In this case, the conjecture that the true value function has a linear interval cannot be validated. As a result, insurance is essential in the sense that their existence strictly improves the well being of individuals.

To characterize when the intervals $\left[\underline{a}^b, \overline{a}^b\right]$ and $[\underline{a}^s, \overline{a}^s]$ overlap, note that even without insurance (18) holds, so $z^{*s1} > z^{*b0}$ and $z^{*s0} < z^{*b1}$. These inequalities together with (36) imply $\underline{a}^s < \underline{a}^b$ and $\overline{a}^s < \overline{a}^b$. Therefore, the intervals $\left[\underline{a}^b, \overline{a}^b\right]$ and $[\underline{a}^s, \overline{a}^s]$ overlap if an only if $\underline{a}^b \leq \overline{a}^s$. Using (35) to (38), this condition is equivalent to

$$z^{*b0} + rz^{*s1} \geq z^{*s0} + rz^{*b1}. \tag{39}$$

Finally, using (18) and the definitions (33) and (34), (39) simplifies into condition (40) in the following proposition:

**Proposition 2:** *Consider the equilibrium allocation that solves the system of equations (*25*) to (*28*). The insurance of trading risks is redundant to implement this equilibrium if and only if parameter values are such that the following condition holds:*

$$i \geq \frac{r}{1-r}\left(1 + \frac{q^{*s}}{q^{*b}}\right). \tag{40}$$

With bilateral matching, condition (40) can be further simplified because then $q^{*b} = q^{*s}$.

Using this equality together with $i \equiv (1+r)\gamma - 1$, we obtain the following corollary to Proposition 2:

**Corollary 1:** *With bilateral matching (32), the insurance of trading risks is redundant if and only if the following condition holds:*

$$\gamma \geq \frac{1}{1-r}. \tag{41}$$

The simplicity of Corollary 1 helps to provide intuition on Proposition 2. Consider an equilibrium with a constant money supply $(\gamma = 1 \text{ and } \tau = 0)$. According to Corollary 1 this monetary policy makes insurance essential, so let us see why this is the case. In the equilibrium with complete markets inside the village, individuals produce the quantity of the general good that they consume day after day. In the absence of insurance, an individual may fail to find a trading opportunity at night for an indefinite time. If this happens, any positive holdings of bonds produces interest, so wealth accumulates above any potential upper bound $\overline{a}$. Similarly, if night trading opportunities are not realized, a positive debt increases without bound, so wealth falls below any potential lower bound $\underline{a}$. Consequently, no credit can take place without incurring a positive probability of escaping a potential interval $[\underline{a}, \overline{a}]$.

Conversely, if condition (41) holds, even if a buyer fails to be able to purchase the specific good, the opportunity cost of holding money is sufficiently large to avoid wealth from overtaking the value of $\overline{a}$ that satisfies (37). Similarly, the lump-sum transfers associated with the growth of the money supply are sufficient to guarantee a minimum income to sellers to avoid wealth falling below $\underline{a}$ in (38). Therefore, if (41) holds, there is a set of optimal strategies for the individual that allows wealth to remain in the linear interval forever, so insurance is redundant.

# 3 Financial Contracts versus Lotteries

The linearity of the value function found in the previous section is related to the derivation of quasi-linear preferences in the Rogerson (1988) model of indivisible labor. In both cases, the linearity arises from a binary choice in the space of feasible policies. In the indivisible labor literature, lotteries are commonly used to support equilibrium allocations.[8] This section shows that under certain conditions lotteries can play a similar role here. That is, it shows that a fair lottery can substitute for credit and insurance to attain market completeness inside the villages, even if these lotteries require no memory about personal histories and no observability of trading opportunities.

Consider an environment identical to the one studied in the previous section except that now individuals are anonymous both inside and outside their village. As a result, individuals have no access to credit and insurance contracts. Instead, individuals can play a fair lottery during the day. A lottery ticket delivers the same day $\delta$ goods with probability $\psi$ at the cost $\psi\delta$. As long as $\psi\delta$ is paid, the individual is able to choose both $\delta$ and $\psi$ subject to the constraints $\delta \geq 0$ and $\psi \in [0,1]$.

The optimal plans of a representative individual are characterized using similar recursive methods to those of the previous section. Let $V(a_{+1})$ be the value function at the beginning of period $t+1$ and consider the optimal plans for period $t$. Conditional on being a buyer, the utility of having wealth $\tilde{a}$ after playing the lottery is given by a value function $V^b(\tilde{a})$ that satisfies:

$$V^b(\tilde{a}) = \max_{\{x^b, y^b, q^b\}} \left\{ U^d\left(x^b, y^b\right) + \pi^b\left[U^n\left(q^b, 0\right) + \beta V\left(a_{+1}^{b1}\right)\right] + \left(1-\pi^b\right)\beta V\left(a_{+1}^{b0}\right)\right\}, \quad (42)$$

subject to the flow budget constraints:

$$a_{+1}^{b0} = \left(\tilde{a} + \tau + y^b - x^b\right)\gamma^{-1}, \text{ and} \quad (43)$$

$$a_{+1}^{b1} = \left(\tilde{a} + \tau + y^b - x^b - pq^b\right)\gamma^{-1}. \quad (44)$$

[8] See Rocheteau, Rupert, Shell, and Wright (2004) for a monetary search model that uses the indivisibility of labor to motivate the quasi-linear preferences in a Lagos and Wright (2005) model.

Analogously, conditional on being a seller, the value function $V^s(\tilde{a})$ satisfies:

$$V^s(\tilde{a}) = \max_{\{x^s,y^s,q^s\}} \left\{U^d\left(x^s, y^s\right) + \pi^s\left[U^n\left(0, q^s\right) + \beta V\left(a^{s1}_{+1}\right)\right] + \left(1-\pi^s\right)\beta V\left(a^{s0}_{+1}\right)\right\}, \tag{45}$$

subject to

$$a^{s0}_{+1} = \left(\tilde{a} + \tau + y^s - x^s\right)\gamma^{-1}, \text{ and} \tag{46}$$

$$a^{s1}_{+1} = \left(\tilde{a} + \tau + y^s - x^s + pq^s_t\right)\gamma^{-1}. \tag{47}$$

Since money is the only asset to carry wealth, the gross return of the unspent wealth after the night market is equal to inverse of the inflation factor $\gamma$. Furthermore, since the individual cannot hold negative amounts of wealth, the wealth at the beginning of next period must obey:

$$a_{+1} \geq 0. \tag{48}$$

Depending on the trading role picked and the success or failure in finding a trading opportunity, the wealth $a_{+1}$ is equal to $a^{b1}_{+1}$, $a^{b0}_{+1}$, $a^{s1}_{+1}$, or $a^{b0}_{+1}$.

Each day, after the lottery has been played, the individual chooses the trading role to be played at night. This binary choice implies that the utility of an individual prior to any consumption-production activity but after the lottery has been played is given by:

$$\tilde{V}\left(\tilde{a}\right) = \max\left\{V^b(\tilde{a}), V^s(\tilde{a})\right\}. \tag{49}$$

The individual can randomize initial wealth using the lottery. Therefore, the value function $TV\left(a\right)$ at the beginning of period $t$ must satisfy:

$$TV(a) = \max_{\{\delta,\psi\}}\left\{\psi\tilde{V}\left[a + \delta\left(1-\psi\right)\right] + \left(1-\psi\right)\tilde{V}(a - \psi\delta)\right\}, \tag{50}$$

subject to $\psi \in [0,1]$ and $\delta \geq 0$. The wealth $\tilde{a}$ after the lottery outcome is stochastic. If the individual wins the lottery, $\tilde{a} = a + \delta\left(1-\psi\right)$. Otherwise, $\tilde{a} = a - \psi\delta$. The true value function $V$ must obey the Bellman's equation: $V = TV$.

The following proposition states the conditions for the lottery to complete the markets inside the village. Also, it characterizes the value function in a stationary equilibrium in which this condition is satisfied.

**Proposition 3:** *Let* $(x^{*b}, y^{*b}, q^{*b}, x^{*s}, y^{*s}, q^{*s}, v, p, \theta, \tau)$ *be a monetary equilibrium vector of the model in Section 2, so it solves (*25*) to (*28*). Define*

$$\bar{z}^b = x^{*b} - y^{*b} - \tau + pq^{*b}, \text{ and } \quad \bar{z}^s = x^{*s} - y^{*s} - \tau. \tag{51}$$

*If the following inequalities are satisfied:*

$$p\gamma^{-1} \max\left\{q^{*b}, q^{*s}\right\} \leq \bar{z}^b, \quad \bar{z}^s \leq 0, \tag{52}$$

*then the following statements hold in a monetary equilibrium with lotteries:*

*1 - The lottery supports* $(x^{*b}, y^{*b}, q^{*b}, x^{*s}, y^{*s}, q^{*s}, v, p, \theta, \tau)$ *as an equilibrium, so the markets inside the village are complete without credit and insurance.*

2 - *The value function* $V(a)$ *is continuously differentiable, increasing, and concave. Moreover,* $V(a)$ *is affine in the interval* $[0, \bar{z}^b]$ *with slope* $v$ *and intercept (*24*).*

The linear segment of $V$ comes now from the randomization of wealth at the beginning of a period. This randomization convexifies the binary choice of trading role to be performed at night. Intuitively, inside the linear segment, higher wealth allows the individual to purchase lottery tickets with a higher probability of winning. Therefore, wealth changes the probabilities of being a buyer or a seller, but conditional on a particular trading role individuals consume and produce quantities that are independent from their initial wealth. With expected utility preferences, utility is linear on these probabilities, so $V$ is an affine function of wealth in the interval $[0, \bar{z}^b]$.

The proof of Proposition 3 is in the Appendix. The crucial step of this proof is summarized here and illustrated in Figure 2. Consider an individual whose value function $V$ for period $t+1$ is increasing and concave. Conditional on being a buyer or a seller, the individual is maximizing a concave objective subject to a convex set of feasible policies. Moreover, the set of feasible policies unambiguously expands with $\tilde{a}$. Therefore, the value functions $V^b(\tilde{a})$ and $V^s(\tilde{a})$ are increasing and concave as it is depicted by the thin lines in Figure 2. In general, the value function $\tilde{V}(\tilde{a})$, defined as the maximum of $V^b(\tilde{a})$ and $V^s(\tilde{a})$, is not concave (see Figure 2). However, the individual has an incentive to gamble the initial wealth $a$ to avoid the values of $\tilde{a}$ that lie in the regions where $\tilde{V}$ is not concave. In Figure 2, an individual with $a$ inside the interval $\left[\bar{z}^s, \bar{z}^b\right]$ has an incentive to gamble $a - \bar{z}^s$ of the initial wealth to buy a lottery ticket with payout $\bar{z}^b - \bar{z}^s 0$. As a result of this gamble, the individual attains $\tilde{a} = \bar{z}^b$ with probability $(a - \bar{z}^s) / \left(\bar{z}^b - \bar{z}^s\right)$, and $\tilde{a} = \bar{z}^s$ with complementary probability. The expected utility of the gamble is the straight line tangent to $V^b$ and $V^s$, which lies above $\tilde{V}$. The value function $TV$ for period $t$ (thick line in Figure 2) is the concave hull of $\tilde{V}$. Therefore, $TV$ is increasing and concave. Furthermore, as long as $V^b$ and $V^s$ cross (as it is implied by the premises of Proposition 3), it has a linear segment.

In Proposition 3, we need to impose condition (52) to ensure that the interval of wealth for which the value function is linear is absorbing. With bilateral matching, condition (52) simplifies neatly to a restriction on the rates of growth of the money supply. This restriction is stated in the following corollary.

**Corollary 2:** *With bilateral matching (*32*), in a stationary monetary equilibrium a fair lottery substitutes for credit and insurance of trading risks to complete the markets inside the village if the net rate of growth of the money supply is not negative, so*

$$\gamma \geq 1. \tag{53}$$

As in the previous section, the first order conditions (17) and equilibrium in the goods market (26) imply $x^{*s} = x^{*b} = y^{*s} = y^{*b}$. In addition, bilateral matching implies $q^{*b} = q^{*s}$. Consequently, the equalities in (51) simplify to $\bar{z}^s = -\tau$, and $\bar{z}^b = pq^{*b} - \tau$. These equalities together with the government budget constraint (28), imply that (53) is equivalent to (52).

The intuition why condition (53) is needed for the lottery to achieve market completeness inside the villages is the following. Individuals use the lottery to gamble in such a way that if they win they finish the day with the money balances needed to pay the night purchases of a successful buyer, and if they loose they finish the day with zero money balances. With deflation ($\gamma < 1$) the monetary transfer is negative (it is a lump-sum tax). Therefore, individuals that start the day with zero wealth (successful buyers and frustrated sellers) cannot afford any lottery gamble that delivers at least the wealth necessary to pay the lump-sum tax that finances the deflation while maintaining $x^{*b} = y^{*b}$ or $x^{*s} = y^{*s}$. These individuals have to break one of these equalities to finance the lump-sum tax. In doing so, they break the condition for the markets inside the village to be complete.[9]

# 4 Discussion

The analysis of the preceding sections shows that complete financial markets in each village support an equilibrium with simple distributions of money holdings. In this sense, complete financial markets substitutes for the representative household in the framework advanced by Shi (1997) and for the quasi-linear preferences in the framework advanced by Lagos and Wright (2005). Moreover, with the assumptions made so far, individuals have an absorbing linear segment in their value functions. This linearity is convenient not only to characterize an equilibrium, but also to provide mild conditions under which either a lottery or a risk-free bond are sufficient to achieve market completeness inside a village. This section discusses the generality of these results.

An endogenous buyer-seller choice at the beginning of each day is crucial for the individuals to have a linear segment in their value function. However, the endogenous buyer-seller choice is not needed to have market completeness inside the villages. For example, if exogenously some individuals (buyers) are capable of consuming at night but not producing, while other individuals (sellers) have the complementary abilities, then the insurance of trading risks leads to the equality of the intertemporal marginal rates of substitution for the general good. As a result, an equalitarian distribution of wealth is

[9] This outcome is reminiscent to Bewley's (1983) difficulty of deflationary policies although it does not hinge on the existence or not of a well defined equilibrium.

perpetuated indefinitely. Hence, the distribution of money holdings is easy to characterize. To support an equilibrium in this variation of the model, insurance is always essential. That is, neither credit nor lotteries can substitute for insurance to achieve market completeness inside the villages.

The frameworks introduced by Shi (1997) and Lagos and Wright (2005) are applicable to a wide variety of equilibrium concepts. For example, Rocheteau and Wright (2005) show how the Lagos and Wright framework can be adapted to three alternative ways of determining the terms of trade in the frictional markets that use money as the medium of exchange: generalized Nash bargaining, Walrasian competition, and competitive search. This paper assumes Walrasian competition because it is the simplest and most widely used equilibrium concept in economics. However, the key results summarized in Propositions 1 to 3 are applicable to the other equilibrium concepts used by Rocheteau and Wright (2005) with the following qualifications. First, the concavity of the value function is harder to prove in non-Walrasian environments. In particular, with generalized Nash bargaining the conditions on first principles that ensure the concavity of value functions are quite restrictive. This technical problem is well discussed in Lagos and Wright (2005) and their discussion applies to the framework of this paper. Second, insurance contracts may be more complicated in non-Walrasian environments. Again, generalized Nash bargaining offers special difficulties in this respect because the outcome of bargaining depends on the reservation utilities of potential traders. To calculate these reservation utilities, we have to allow for the possibility that bargaining in a trade meeting breaks down. Once this possibility is introduced, if the value function is strictly concave, individuals have an incentive to insure not only trading opportunities but also the outcome of bargaining. To accomplish this, insurance contracts have to specify the details of bargaining strategies. This raises the issue of how these contracts can be enforced if trades occur in decentralized markets. This difficulty does not appear either with an endogenous buyer-seller choice (individuals are locally risk neutral so they have not incentive to insure the outcome of bargaining) or with competitive search.

## Appendix

**Proof of Proposition 1**

Let $\mathcal{C}(a)$ be the space of bounded and continuous functions $f : [a_{\min}, \infty) \rightarrow R$ with the sup norm. Use the Bellman's equation implied by (3) to (11) to define the mapping $T$ of $\mathcal{C}(a)$ onto itself by substituting $f$ for $V$ in the right hand sides of (4) and (7) and denoting as $Tf(a)$ the left hand side of (2). For a given $a$, the set of feasible policies is non-empty, compact-valued, and continuous. The utility function $U$ is bounded and continuous on the set of feasible policies, and $0 < \beta < 1$. Therefore, Theorem 4.6 in Stokey and Lucas with Prescott (1989) implies that there is a unique fixed point to the mapping $T$, which is the value function $V$.

Let $\mathcal{V}(a)$ be the subset of functions in $\mathcal{C}(a)$ that are increasing, concave, and with the linear segment (12). Consider again the mapping $T$ defined in the previous paragraph. The argument following Proposition 1 in the main text shows that if the coefficients $v_0$, $v$, $\underline{a}$, and $\overline{a}$ have the values specified in (22) to (24) with $z^{*j}$ and $S^{*j}$ for $j = b$ and $s$ consistent with (17), $T$ maps $\mathcal{V}(a)$ onto itself. Therefore, since $T$ is a contraction mapping and $\mathcal{V}(a)$ is closed, $V$ satisfies the properties of Proposition 1 with the coefficients specified by (22) to (24). With the Inada conditions assumed on $U$, the choices of the individual are interior, so $V$ is continuously differentiable■

**Proof of Proposition 3**

Using the mapping $T$ defined by (42) to (50), construct a sequence of value functions $V_0$, $V_1, ...$ that satisfy: $V_n = TV_{n-1}$ for $n = 1, 2, ...$ Since money is the only asset and money holdings cannot be negative the domain of these functions is $[0, \infty)$.

Let $V_0$ be the affine value function $V_0(a) = v_0 + va$, where $v_0$ and $v$ are the values that solve (24) to (28). Conditional on being a buyer or a seller, the value functions that map wealth $\tilde{a}$ after playing the lottery onto utilities are:

$$V^b(\tilde{a}) = \max_{\{x^b, y^b, q^b\}} U^d\left(x^b, y^b\right) + \pi^b U^n\left(q^b, 0\right) + \beta v_0 + \beta v \gamma^{-1}\left(\tilde{a} + y^b + \tau - x^b - \pi^b p q^b\right), \quad (54)$$

subject to

$$\tilde{a} + y^b + \tau \geq x^b + pq^b; \text{ and} \tag{55}$$

$$V^s(\tilde{a}) = \max_{\{x^s, y^s, q^s\}} U^d(x^s, y^s) + \pi^s U^n(0, q^s) + \beta v_0 + \beta v \gamma^{-1}(\tilde{a} + y^s + \tau - x^s + \pi^s p q^s), \tag{56}$$

subject to

$$\tilde{a} + y^s + \tau \geq x^s. \tag{57}$$

The timing of events allows the wealth $\tilde{a}$ to be negative, in which case the individual gambles part of the net earnings made during the day in the lottery. Therefore, the domain of $V^b$ and $V^s$ is $[-\bar{y} - \tau, \infty)$, where $\bar{y}$ is the maximum possible production during the day. In the domain of $V^b$ and $V^s$, the programs (54) to (57) maximize a continuously differentiable, increasing, and concave objective on a non-empty and convex set of policies. Therefore, the solutions to (54) to (57) are unique, and the value functions $V^b$ and $V^s$ are well defined, increasing, and concave. With the Inada conditions assumed on $U$, the non-negativity constraints on $x^b, y^b$, and $q^b$ are not binding, so $V^b$ and $V^s$ are also continuously differentiable.

The conditions that characterize the buyer's maximization program (54)-(55) are:

$$\begin{aligned} U_1(x^b, y^b) &= \lambda + \beta v \gamma^{-1}, \\ U_2(x^b, y^b) &= -(\lambda + \beta v \gamma^{-1}), \text{ and} \\ \pi^b U_1(q^b, 0) &= (\lambda + \beta v \gamma^{-1} \pi^b) p; \end{aligned} \tag{58}$$

where $\lambda$ is the Lagrange multiplier associated with (55). If this multiplier is equal to $v - \beta v \gamma^{-1}$, the first order conditions (58) are identical to (17). So for $\lambda = v - \beta v \gamma^{-1}$, the solution to (58) is $(x^{*b}, y^{*b}, q^{*b})$. With this solution, the constraint (55) holds with equality if $\tilde{a} = \bar{z}^b$ (see [51]). Therefore, for $\tilde{a} = \bar{z}^b$ the unique solution to (54)-(55) is $(x^{*b}, y^{*b}, q^{*b})$. Using the Envelope Theorem, the marginal value of wealth at $\tilde{a} = \bar{z}^b$ is $V_a^b(\bar{z}^b) = \lambda + \beta v \gamma^{-1} = v$. Moreover, the concavity of $V^b$ implies $V_a^b(\tilde{a}) \leq v$ if $\tilde{a} \geq \bar{z}^b$, and $V_a^b(\tilde{a}) \geq v$ if $\tilde{a} \leq \bar{z}^b$. An analogous treatment to the seller's maximization program

(56)-(55) yields the following results. At $\tilde{a} = \bar{z}^s$, the optimal choice is $(x^{*s}, y^{*s}, q^{*s})$. If $\tilde{a} \geq \bar{z}^s$, then $V_a^s(\tilde{a}) \leq v$. If $\tilde{a} \leq \bar{z}^s$, then $V_a^s(\tilde{a}) \geq v$. Since the solution of (54)-(55) for $\tilde{a} = \bar{z}^b$ is $\left(x^{*b}, y^{*b}, q^{*b}\right)$ and the solution of (56)-(55) for $\tilde{a} = \bar{z}^s$ is $(x^{*s}, y^{*s}, q^{*s})$, using (25) and (51), we obtain:

$$V^b\left(\bar{z}^b\right) - V^s\left(\bar{z}^s\right) = v\left(\bar{z}^b - \bar{z}^s\right). \tag{59}$$

Furthermore, at $\bar{z}^s$ the individual replicates the same consumption and production as with credit and insurance, and the relevant portion of next period's value function and wealth are the same. Therefore, $V^s(\bar{z}^s) = v_0 + v\bar{z}^s$.

Figure 2 uses the properties derived in the previous paragraph to display $V^b$ and $V^s$. In Figure 2, $V^b$ and $V^s$ are continuously differentiable, increasing, and concave functions. The slopes of these functions must satisfy: $V_a^b\left(\bar{z}^b\right) = V_a^s(\bar{z}^s) = v$. Furthermore, the linear segment that connects $V^b\left(\bar{z}^b\right)$ and $V^s(\bar{z}^s)$ must also have slope $v$. For a given wealth $\tilde{a}$ after the lottery is played, the individual picks the trading role that brings maximum utility. In Figure 2, the individual chooses to be a buyer for $\tilde{a} > \hat{a}$ and to be a seller for $\tilde{a} < \hat{a}$. The individual is indifferent between the two roles at $\tilde{a} = \hat{a}$. Therefore, utility of the individual after the lottery is given by the function: $\tilde{V}(\tilde{a}) = \max\left\{V^b(\tilde{a}), V^s(\tilde{a})\right\}$.

Consider now the optimal lottery gamble of an individual with initial wealth $a$. As long as the probability $\psi$ is between zero and one, the individual can use the lottery to randomize between any two values of $\tilde{a}$ : $\tilde{a}_0$ and $\tilde{a}_1$. The respective probabilities of these two outcomes are: $(a - \tilde{a}_0)/(\tilde{a}_1 - \tilde{a}_0)$ and $(\tilde{a}_1 - a)/(\tilde{a}_1 - \tilde{a}_0)$. The condition $\psi \in [0,1]$ is equivalent to $a \in [\tilde{a}_0, \tilde{a}_1]$. Graphically, the utility achieved from this gamble is given by the vertical distance from the horizontal axis to the straight segment connecting the utilities achieved with the two possible outcomes of $\tilde{a}$. It is clear from observing Figure 2 that the optimal gamble for an individual with $a \in \left[\bar{z}^s, \bar{z}^b\right]$ is two pick $\tilde{a}_0 = \bar{z}^s$ and $\tilde{a}_1 = \bar{z}^b$. Algebraically, the same result is attained from the first order conditions of problem (50).

The utility of the individual prior playing the lottery is given by $V_1(a) = TV_0(a)$, which graph is the convex-hull of the graph of $\tilde{V}(a)$. The value function $V_1$ is continuously differentiable, increasing, and concave. Furthermore, in the interval comprised between $\bar{z}^s$

and $\bar{z}^b$ the function $V_1$ is affine with the same coefficients $v_0$ and $v$ as $V_0$. The behavior of an individual whose value function for next period is $V_1$ is identical to the behavior just described for an individual whose value function is $V_0$ as long as the next period wealth under the actions described is in the affine interval with slope $v$. Condition (52) ensures that this is the case. That is, if the individual is a failed buyer or a successful seller ends up with real wealth $p\gamma^{-1}q^{*b}$, and $p\gamma^{-1}q^{*s}$ respectively. These values cannot be greater than $\bar{z}^b$. If the individual is a successful buyer or a failed seller ends up with zero real wealth, so $\bar{z}^s$ cannot be greater than zero. For initial wealths outside the interval comprised between $\bar{z}^s$ and $\bar{z}^b$, the concavity of $V_1$ implies that $V_1(a) \leq V_0(a)$.

The same arguments apply to all members of the sequence of value functions $V_0, V_1, ...$ As long as (52) is satisfied, all these functions are increasing and concave with an identical affine interval $\left[0, \bar{z}^b\right]$. The monotonicity of $T$ implies $V_n(a) \leq V_{n-1}(a)$ for all $n = 1, 2, ...$ Moreover, all these functions are bounded below by zero. Therefore, they must converge point-wise to a function $V$, which is a fixed point of $T$. (If $U$ is bounded on the set of feasible policies, $T$ is a contraction and so the convergence must be uniform to the unique fixed point of $T$.) The fixed point $V$ must be increasing and concave with the same affine interval $\left[0, \bar{z}^b\right]$ as all members of the sequence. Since $V$ is concave and $U$ is differentiable, the function $V$ must be continuously differentiable because the Inada conditions on $U$ imply that the consumption and production choices are interior (non-negative). The first statement in Proposition 3 has been proved above given that the optimal policies with terminal value functions $V_0$ and $V$ coincide■

## References


[1] Bewley, Truman, "A Difficulty with the Optimum Quantity of Money," *Econometrica*, 51, 1485-1504, 1983.

[2] Jin, Yi, and Ted Temzelides, "On the Local Interaction of Money and Credit," *Review of Economic Dynamics*, 7, 143-156, 2004.

[3] Kocherlakota, Narayana R., "Money is Memory," *Journal of Economic Theory* 81 (1998), 232-251

[4] Lagos, Ricardo, and Randall Wright, "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113 (2005), 463-484.

[5] Levine, David, "Asset Trading Mechanisms and Expansionary Policy," *Journal of Economic Theory* 54, 148-164, 1991.

[6] Lucas, Robert E., "Liquidity and Interest Rates," *Journal of Economic Theory* 50, 237-264, 1990.

[7] Lucas, Robert E., and Edward C. Prescott, "Equilibrium Search and Unemployment," *Journal of Economic Theory* 7 (1974), 188-209.

[8] Rocheteau, Guillaume, and Randall Wright, "Inflation and Welfare in Models with Trading Frictions," Penn Institute for Economic Research Working Paper 03-032, 2003.

[9] Rocheteau, Guillaume, and Randall Wright, "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium, *Econometrica* 73 (2005), 175-203.

[10] Rocheteau, Guillaume, Peter Rupert, Karl Shell, and Randall Wright (2004), "A Simple Model of Monetary Exchange Based on Nonconvexities and Sunspots," manuscript.

[11] Rogerson, Richard, "Indivisible Labor, Lotteries, and Equilibrium," *Journal of Monetary Economics* 21 (1), 3-16, 1988.

[12] Shi, Shouyong, "A Divisible Search Model of Fiat Money," *Econometrica* 65, 75-102, 1997.

[13] Stokey, Nancy L. and Robert E. Lucas with Edward C. Prescott, *Recursive Methods in Economic Dynamics,* Cambridge, MA. Harvard University Press, 1989.

Figure 1

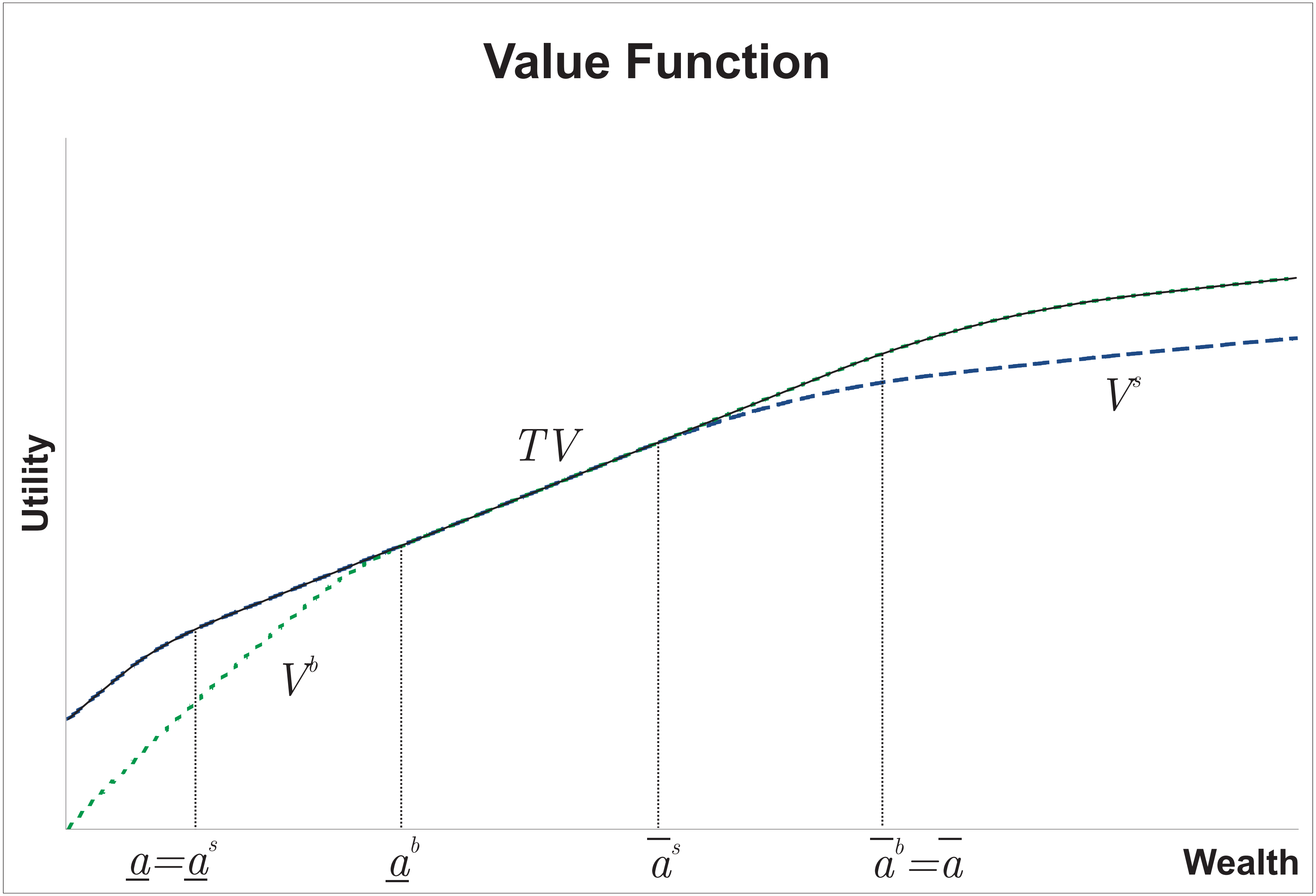
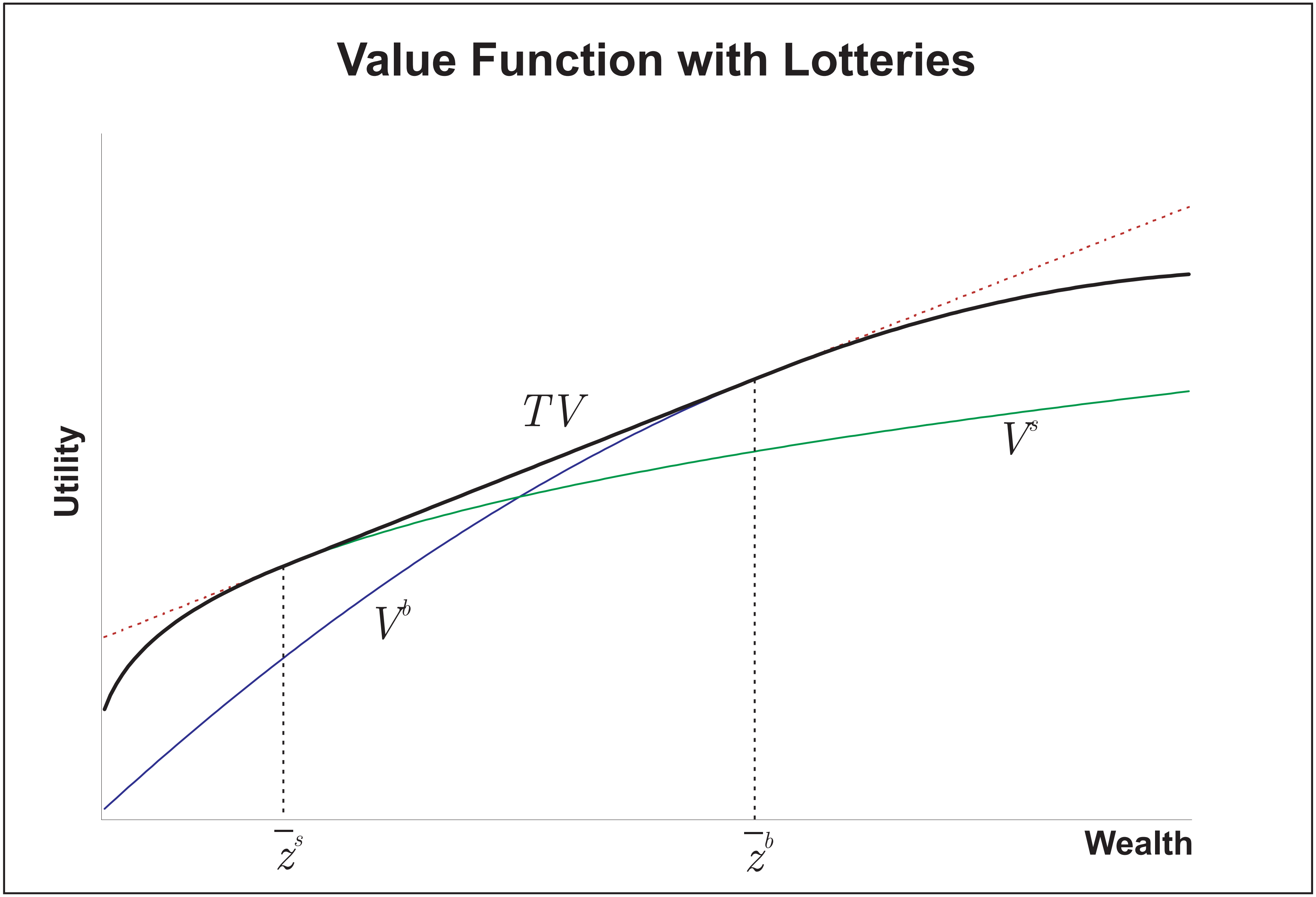

Value Function
Utility
$TV$
$V^b$
$V^s$
$\underline{a}=\underline{a}^s$
$\underline{a}^b$
$\overline{a}^s$
$\overline{a}^b=\overline{a}$
Wealth

Figure 2


Value Function with Lotteries
Utility
Wealth
$TV$
$V^b$
$V^s$
$\bar{z}^s$
$\bar{z}^b$