

# Precautionary Balances and the Velocity of Circulation of Money<sup>†</sup>

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## Abstract

The low velocity of circulation of money implies that households hold more money than they normally spend. This behavior is explained if households face uncertain expenditure needs, so that they have a precautionary motive for holding money. We investigate this motive in a search model where households are subject to preference shocks. The model predicts that the velocity is not only low but also interest elastic. The model closely fits United States data on velocity and interest rates (1892-2004). The empirical analysis reveals a dramatic reduction in precautionary balances towards the end of our sample, which is important for policy issues.

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*Key Words:* Precautionary Balances, Monetary Search, Velocity of Circulation of Money, Money Demand.

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# 1 Introduction

From 1892 to 2004, checkable deposits plus the currency in circulation inside the United States ( $M1^*$ ) was worth on average the GDP for 3 months.<sup>1</sup> Since during this period almost all households received an income at least once a month and quite often once in a fortnight or once a week, a large fraction of  $M1^*$  must have remained unspent in the course of a pay period. Recently, the worth of  $M1^*$  has fallen to be around 1 month of GDP. Therefore, the fraction of unspent  $M1^*$  during a pay period, although probably still positive, must have dropped dramatically relative to historical levels. These observations raise the following two questions.

Why do households hold more money than they normally spend if other assets bear higher interest? A natural answer to this question is that households hold precautionary balances in order to accommodate uncertain expenditure needs, such as a breakdown of the household car or unexpected travel. Households typically hold precautionary balances in order to face these contingencies, even if holding money is costly.

Why have unspent money balances fallen so dramatically in recent years? A natural answer is that improvements in information technology have made possible credit cards, telephone banking, internet banking, and low fees for rebalancing portfolios. Consequently, individuals are now able to face unexpected expenses without holding large precautionary balances.

Our paper shows how a simple search model with precautionary balances explains the historical data on the quantity of money in the United States and draws conclusions for substantive issues in monetary economics.

In this paper, we investigate the precautionary demand for money in a disaggregated model where households are subject to preference shocks. In earlier work, the precautionary demand for money was analyzed by Svensson (1985) in a representative agent model with a cash-in-advance constraint. As in our model, the agent experienced a preference shock after deciding on the demand for money. In an influential paper, Hodrick, Kocherlakota, and Lucas (1991) dismissed the quantitative importance of this precautionary demand in the context of an aggregated model. In their numerical analysis, they found that once the purchases of the representative agent are calibrated to match the smooth aggregate consumption expenditures in the United States, the timing of the

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<sup>1</sup> $M1^*$  differs from the standard  $M1$  by subtracting the currency in circulation outside the United States. See Section 2 for data sources.

preference shock is quantitatively irrelevant. Yet, they acknowledged that, “It is possible that a similar model that seriously treated the aggregation problem could produce reasonable velocity predictions.” We address this issue by building a tractable yet fully disaggregated search model that even in a steady state with constant aggregate consumption, generates large precautionary balances due to individual preference shocks.

We estimate our model using United States data from 1892-2004. Our model closely fits the velocity of circulation of  $M1^*$  and its elasticity with respect to interest rates. Moreover, this empirical implementation reveals that the demand for precautionary balances has dramatically declined in recent years. This decline has had profound consequences for most substantive issues in monetary economics. For example, it has reduced the elasticity of the demand for money, the seigniorage the Fed collects at a given rate of inflation, and the welfare cost of inflation. This last change is dramatic. For most of the past century, raising the nominal interest rate from 0 to 10 per cent used to induce an equivalent reduction of consumption of around 1 per cent. (This figure is similar to the estimates of Lucas, 2000). In contrast, the same increase today induces an equivalent reduction of consumption of only 0.13 per cent.

Since the seminal work of Kiyotaki and Wright (1989), search models have become a common paradigm in the micro-foundations of money. In the earlier versions of these models, money was assumed indivisible to simplify the endogenous distribution of money holdings. Recent contributions have built highly tractable models with divisible money by introducing devices that render the distribution of money holdings degenerate across individuals. Shi (1997) assumed that individuals belong to large symmetric households. More recently, Lagos and Wright (2005) assumed that individuals sometimes trade bilaterally and sometimes in a centralized market, and agents have quasi-linear preferences. Our paper uses a framework related to these earlier contributions,<sup>2</sup> which proves useful in focusing on precautionary money balances by avoiding the existence of both a double layer of decision making and goods traded without holding money for at least one period.

We assume that individuals belong to a large number of villages. Money is essential in facilitating trade across villages because the trading process is anonymous, enforcement is limited, and there is a double coincidence of wants problem. (See Kocherlakota (1998) and Wallace (2001).) However, within a village, financial contracts are viable because fellow villagers know each other. With this village construct we capture that individuals in modern societies sometimes deal with well-known parties with whom financial contracts are viable, while at other times they deal with

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<sup>2</sup>See Faig (2004) for a comparison of the three frameworks.

relative strangers with whom future promises are impractical to enforce. The financial markets inside a village, by allowing individuals to rebalance their portfolios, render the distribution of money holdings tractable.<sup>3</sup>

Our model is in many ways different from most previous search models of monetary exchange. One key difference is that we allow for preference shocks since, as we explain above, these shocks are the foundation for our precautionary demand for money. In addition, we assume that the terms of trade are determined as a result of a competitive search process while most of the literature assumes Nash bargaining (see, however, Rocheteau and Wright (2005)). Finally, we abstract from the uncertainty of meeting a trading partner by assuming that matching is efficient (the short-side of the market is always served).

The concept of competitive search has been widely used in labor economics since the work of Moen (1997) and Shimer (1996), and it is an attractive equilibrium concept for several reasons. First, under competitive search the split of the trading surplus between a buyer and a seller is determined endogenously as a result of competition instead of being determined by an exogenous bargaining weight. Second, competitive search induces efficient search decisions. As shown by Rocheteau and Wright (2005), this implies that the first best is attained under the Friedman rule, while in a search environment the equilibrium is generically inefficient at the Friedman rule either with Nash bargaining or with Walrasian pricing.

The consequences of inflation in our model are as follows. Inflation increases the velocity of circulation of money because it reduces the precautionary balances individuals hold. This reduction is detrimental for welfare because a larger fraction of buyers turn out to be liquidity constrained, so they are forced to purchase inefficiently low quantities of goods. These consequences of inflation are similar to those found in Berentsen and Rocheteau (2002), who introduce idiosyncratic tastes in a search model with divisible money à la Shi (1997). Despite this similarity, the two papers differ both in the focus of analysis and in some important assumptions. Berentsen and Rocheteau center their analysis around the efficiency consequences of the divisibility of money, while we seek to explain the history of the velocity of circulation of money in the United States and draw implications for monetary policy. Also, they assume bargaining to determine the terms of trade (with take-it-or-leave-it offers by buyers), while we assume competitive search. While we conjecture that assuming

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<sup>3</sup>The role of financial markets inside a village replaces the role of the large household assumption in Shi (1997) and dispenses with the assumption that some goods yielding constant marginal utility are traded in a centralized market in Lagos and Wright (2005).

bargaining in our environment would not change our qualitative results, it should increase our estimates of the welfare costs of inflation. As noted by Rocheteau and Wright (2005), increasing inflation above the Friedman rule creates only a second-order loss with competitive search, while it creates a first-order welfare loss with bargaining.<sup>4</sup>

A consequence of precautionary balances is that some individuals end up with unspent money while others are cash constrained. As long as inflation is above the Friedman rule, individuals would gain from readjusting their balances after preference shocks are realized. However, this is not possible in our baseline model because all financial markets close prior to the realization of the preference shocks. Berentsen, Camera, and Waller (2004) consider a related environment where these readjustments take place through banks which take deposits from individuals with excess cash and make loans to cash-constrained individuals. The existence of these banks eliminates precautionary balances and is welfare enhancing. Moreover, this type of banking provides a rationale for a positive inflation rate when enforcement is limited because inflation raises the cost of defaulting on loans.

The paper is organized as follows. Section 2 reviews historical data on the velocity of circulation of money in the United States and documents the facts the paper seeks to explain. Section 3 presents the theoretical model. Section 4 estimates the model using United States data. Section 5 concludes.

## 2 The Data

Figure 1 displays the annual time series of a short-term nominal interest rate and the velocity of circulation of money (ratio of nominal GDP over the quantity of money) in the United States from 1892 to 2004.<sup>5</sup> We plot the velocity for two measures of money: M1 and M1\*. M1 is the standard aggregate reported by the Federal Reserve that includes currency and checkable deposits. M1\* is M1 minus the currency reported by the Federal Reserve as being circulated abroad. Measures of currency abroad are first reported for December 1964, when its importance was minimal. However, since then currency abroad grew to 25 per cent of M1 in 2004. Our analysis concentrates on the velocity of M1\* because it is the most meaningful of our two measures, but we plot the velocity of M1 to facilitate the comparison with prior studies.

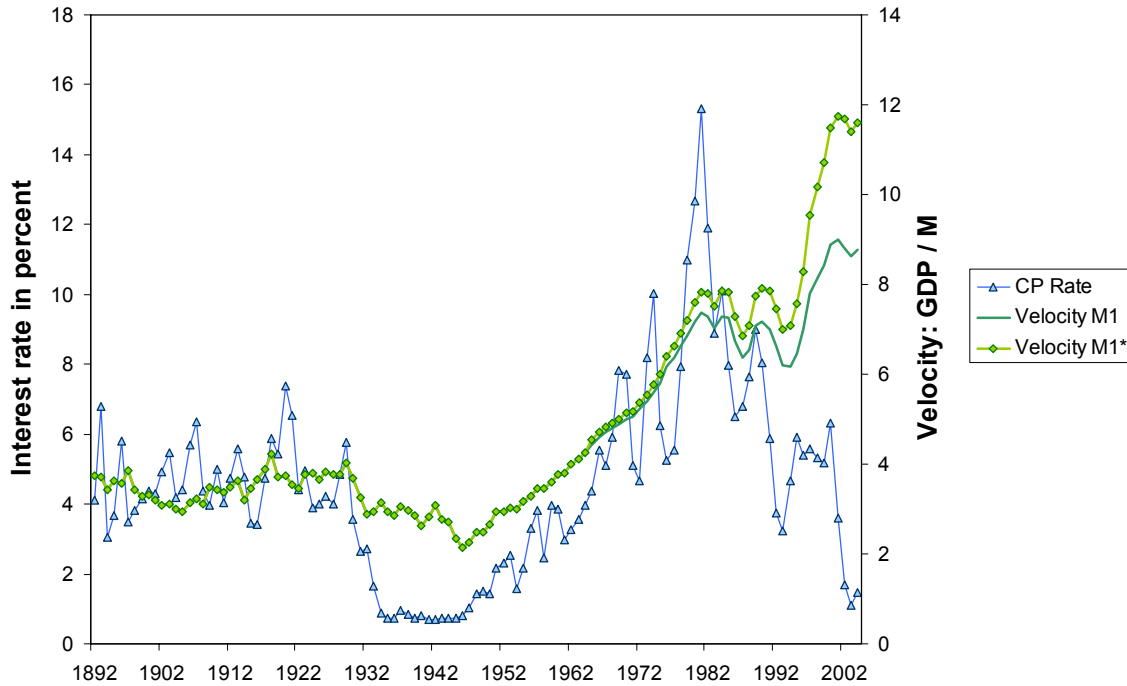
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<sup>4</sup>Although in Berentsen and Rocheteau (2002) the Friedman rule yields the first best, this would not be true in our model because the composition of buyers and sellers is endogenous.

<sup>5</sup>The data plotted in this figure are similar to those analyzed by Lucas (2000) extended backward from 1900 to 1892 and forward from 1994 to 2004. See the Appendix for the sources.

Figure 1

### Velocity of M1 and Interest Rates in the United States



From 1892 to 2004, the velocity of circulation of money was on average low: M1\* circulated on average 4.8 times each year. Over time, velocity has changed widely. Until 1946, velocity seldom reached 4, but since then it has experienced a marked upward trend. This upward trend has accelerated in recent years, which is informative about the factors that may have driven the change. From 1946 to 1982, the upward trend in velocity seems to be due to increases in the nominal rate of interest experienced during that period. However, velocity increased dramatically since 1982, at a time when nominal interest rates have collapsed. Therefore, other elements must have played an important role in the dynamics of velocity. In principle, an upward trend in velocity could be driven by GDP growth if the transactions elasticity of the demand for money were lower than one. However, this hypothesis does not fit the data well. GDP experienced long-term growth for the whole period of analysis, and actually slowed down since the early seventies. In contrast, we observe in Figure 1 that the upward trend in velocity accelerated precisely at the time GDP growth experienced a slowdown. A much better explanation is that the upward trend in velocity is due to the revolution in information technology, which in the last three decades has radically reduced the costs of communication and record keeping.

Besides an upward trend in velocity, Figure 1 also shows a positive correlation between velocity and interest rates. There are many theoretical models that explain this correlation. Two classical examples are the Baumol-Tobin model and the cash-credit goods model of Lucas and Stokey (1987). It is beyond the scope of this paper to discriminate among these various models. Instead, we show that the precautionary demand for money not only provides a rationale for the large holdings of money we observe, but is also consistent with the historical correlation between the velocity of circulation of money and the nominal interest rate in the United States.

### 3 The Model

The economy consists of a measure one of individuals. Individuals live in a large number of symmetric villages. The members of each village are *ex ante* identical. They all produce a perishable good specific to their own village and consume the goods produced in all other villages. Consequently, individuals must trade outside their village to consume.

Time is a discrete, infinite sequence of days. Each morning an individual must choose to be either a buyer or a seller in the goods market that convenes later in the day. Within a village some individuals will be buyers and others will be sellers each day. However, over time individuals will typically alternate between these two roles.

Individuals seek to maximize their expected lifetime utility:

$$E \sum_{t=0}^{\infty} \beta^t \mathcal{U}(\varepsilon_t, q_t^b, q_t^s), \quad (1)$$

where

$$\mathcal{U}(\varepsilon, q^b, q^s) = \varepsilon U(q^b) - C(q^s) \quad (2)$$

is the one-period utility function and  $\beta \in (0, 1)$  is the discount factor. The one-period utility depends on the quantity  $q^b$  consumed if the individual chooses to be a buyer during the period, and on the quantity  $q^s$  produced if the choice is to be a seller. It also depends on an idiosyncratic preference shock  $\varepsilon$  which affects the utility of consumption  $\varepsilon U(q^b)$ , but does not affect the disutility of production  $C(q^s)$ . The preference shock is distributed in the interval  $[1, \bar{\varepsilon}]$  with a cumulative distribution  $F(\varepsilon)$ , independent across time and drawn in such a way that the Law of Large Numbers holds across individuals. Both  $U$  and  $C$  are continuously differentiable and increasing. Also,  $U$  is strictly concave and  $C$  is convex, with  $U(0) = C(0) = 0$ , and  $U'(0) = \infty$ . Finally, there is a maximum quantity  $q^{\max}$  that the individual can produce each day which satisfies  $\bar{\varepsilon} U(q^{\max}) \leq C(q^{\max})$ .

Money is an intrinsically useless, perfectly divisible, and storable asset. Units of money are called dollars. The supply of money grows at a constant factor  $\gamma > \beta$ , so that

$$M_{+1} = \gamma M, \quad (3)$$

where  $M$  is the quantity of money per individual.<sup>6</sup> Each day new money is injected via a lump-sum transfer  $\tau$  common to all individuals. For money to grow at the rate  $\gamma$ , this transfer must satisfy:

$$\tau = (\gamma - 1) M. \quad (4)$$

Each day, goods are traded in a decentralized market where buyers and sellers from different villages meet bilaterally. In this market, buyers and sellers search for trading opportunities, and the terms of trade are determined by a competitive search process (as in Moen (1997) and Shimer (1996)). Before the market opens, each seller simultaneously posts a trading offer, which is a contract detailing the terms at which the seller commits to trade. Once the market opens, buyers direct their search towards the sellers posting the most attractive offer for them (possibly randomizing over offers for which they are indifferent). The set of sellers posting the same offer and the set of buyers directing their search towards them form a submarket. In each submarket, buyers and sellers from different villages are then matched in bilateral pairs.

We specialize the matching process to focus on the precautionary demand for money as follows. We assume that individuals experience only one match with an individual from another village, and that the short-side of the market is always served. That is, the probability that a buyer meets a trading partner in a submarket is

$$\pi^b(\alpha) = \min(1, \alpha), \quad (5)$$

where  $\alpha$  is the ratio of sellers over buyers in that submarket. Similarly, the probability that a seller meets a trading partner is

$$\pi^s(\alpha) = \min(1, \alpha^{-1}). \quad (6)$$

When a buyer and a seller meet in a submarket, they trade according to the posted offer that characterizes the submarket.

Outside their village, individuals are anonymous. This anonymity combined with the absence of a double coincidence of wants (implied by the ex ante choice of trading roles) makes money essential in the goods market. In other words, financial contracts engaging individuals from different villages

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<sup>6</sup>The subscript  $t$  is omitted in most expressions, that is,  $M$  stands for  $M_t$  and  $M_{+1}$  stands for  $M_{t+1}$ .



are not enforceable. However, inside a village financial contracts are enforceable.<sup>7</sup> In particular, each village has a centralized credit market where a one-period risk-free bond is traded. This bond is not accepted as a means of payment in inter-village trades because its issuers are not recognized outside their village of origin. We shall show that this simple credit market exhausts the gains from trade inside a village provided individuals are sufficiently patient. In addition to borrowing and lending, individuals can engage in atemporal gambles of some of their wealth. Even if in equilibrium individuals do not strictly benefit by participating in such gambles, their presence facilitates some of our proofs.

Table 1

MORNING			NOON	AFTERNOON		
Financial markets are open				Goods market is open		
Financial claims are settled.	Choice buyer-seller. Post trading offers.	Choice of bonds, money.	Realization preference shocks.	Buyers pick among trading offers.	Buyers-sellers are matched.	Trade takes place.

A typical day proceeds as follows (see Table 1). In the morning, centralized credit markets are open in each village, and during this time credit contracts from the previous day are settled. The government hands out monetary transfers that increase the money supply. Individuals decide whether to be buyers or sellers, and sellers post their trading offers. Then, all individuals adjust their holdings of bonds and money. At noon, once credit markets have closed, buyers experience a publicly observed idiosyncratic preference shock that determines their willingness to pay for goods. In the afternoon, the goods market is open. Buyers direct their search to the sellers posting the most attractive offer, which organizes traders in submarkets where the short side of the market is always served. When a buyer and a seller meet in a submarket, they trade according to the specified offer. As a result of trade, sellers produce, buyers consume, and money changes hands from buyers to sellers.

Our equilibrium concept combines perfect competition in all centralized credit markets with competitive search in the decentralized goods market. In equilibrium, individuals make optimal

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<sup>7</sup>Typically, full enforceability requires the existence of penalties that go beyond shutting an individual out of the financial system forever. Berentsen, Camera, and Waller (2004) argue that if enforcement is limited and exclusion from the financial system is the only possible penalty, inflation may help relax the enforcement constraints and thereby increase welfare.

choices in the environment where they live. This environment includes a sequence of nominal interest rates and a sequence of conditions in the goods market to be detailed below (essentially the reservation surpluses of other traders). Individuals have rational expectations about the future conditions of this environment. We focus on symmetric and stationary equilibria where aggregate real variables are constant over time. Since real money balances are constant, the gross rate of inflation is equal to the rate of growth of the money supply  $\gamma$ .

To characterize an equilibrium, we adopt the following strategy. First, we describe the buyer-seller choice and the financial decisions of a representative individual given the nominal interest rates and the conditions in the afternoon goods market. Second, we derive the interest rate that clears the morning credit market in each village. Third, we characterize the conditions in the goods market in a competitive search equilibrium. Finally, we show that these conditions satisfy our former conjecture, and we provide a formal definition of an equilibrium.

### 3.1 The Buyer-Seller Choice and the Financial Decisions

Consider an individual facing the following environment. In the morning, the individual can borrow and lend inside the village at the nominal interest rate  $i$  subject to a standard No-Ponzi game condition. The afternoon goods market consists of a continuum of submarkets, one for each buyer type  $\varepsilon$ . In each submarket, the terms of trade are identical in all bilateral meetings. The submarket of type  $\varepsilon$  is characterized by a triple  $(q_\varepsilon, d_\varepsilon, \alpha_\varepsilon)$  where  $q_\varepsilon$  is the quantity traded,  $d_\varepsilon$  is the buyer's total payment in dollars, and  $\alpha_\varepsilon$  is the ratio of buyers over sellers. This description allows for the possibility that some submarkets are inactive with  $q_\varepsilon = d_\varepsilon = 0$ , so the corresponding buyer types do not trade. In the next section we verify this conjecture and characterize the set of active submarkets  $\{(q_\varepsilon, d_\varepsilon, \alpha_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ . In particular, we show that this set is such that sellers are indifferent between participating in any of the active submarkets and they have no incentives to post deviating offers (i.e., create new submarkets). We also show that buyers of type  $\varepsilon$  do not have an incentive to deviate and visit submarkets catering to other types  $\varepsilon' \neq \varepsilon$ .

Prior to all financial choices, each morning the individual chooses the trading role that yields maximal utility. The value function  $V$  of the individual at the beginning of a day then obeys:

$$V\left(\frac{A}{M}\right) = \max\left\{V^b\left(\frac{A}{M}\right), V^s\left(\frac{A}{M}\right)\right\}; \quad (7)$$

where  $A$  is the initial wealth in dollars, and  $V^b$  and  $V^s$  are the value functions conditional on being a buyer or a seller during the day, respectively. The money supply is used to deflate nominal

quantities because, in the stationary environment the individual faces, the gross rate of inflation is equal to the rate of growth of the money supply  $\gamma$ . The ratio  $A/M$  can be interpreted as initial real wealth and is denoted by  $a$ .

While the credit market is open, the individual reallocates wealth. This decision depends on the submarkets that the individual plans to visit during the afternoon.

Conditional on being a buyer the individual chooses the demands for money,  $m^b$  and bonds,  $b^b$  to solve:

$$V^b(a) = \max \int_1^{\bar{\varepsilon}} \left\{ \pi^b(\alpha_\varepsilon) \left[ \varepsilon U(q_\varepsilon) + \beta V(a_{+1}^{b\varepsilon}) \right] + \left[ 1 - \pi^b(\alpha_\varepsilon) \right] \beta V(a_{+1}^{b0}) \right\} dF(\varepsilon) \quad (8)$$

subject to

$$a_{+1}^{b\varepsilon} = \frac{m^b + b^b(1+i) + \tau - d_\varepsilon}{M_{+1}}, \quad (9)$$

$$a_{+1}^{b0} = \frac{m^b + b^b(1+i) + \tau}{M_{+1}} \quad (10)$$

$$a = \frac{m^b + b^b}{M}, \text{ and} \quad (11)$$

$$m^b \geq d_\varepsilon \text{ for all } \varepsilon \in [1, \bar{\varepsilon}]. \quad (12)$$

Contingent on the realization of the shock  $\varepsilon$ , the buyer either meets a seller and buys  $q_\varepsilon$  for  $d_\varepsilon$  dollars, or does not meet a seller and purchases nothing. The probabilities of these two events are  $\pi^b(\alpha_\varepsilon)$  and  $1 - \pi^b(\alpha_\varepsilon)$ , respectively. If the buyer meets a seller, next period's real wealth  $a_{+1}^{b\varepsilon}$  is given by (9). If the buyer does not meet a seller, next period's real wealth  $a_{+1}^{b0}$  is given by (10). The choice of how to allocate wealth between money  $m^b$  and bonds  $b^b$  must satisfy the budget constraint (11). The buyer must also carry enough money to make each possible contingent payment, so  $m^b$  must satisfy (12).

As we shall see, in equilibrium buyers with low realizations of  $\varepsilon$  end up with idle balances in the goods market ( $d_\varepsilon < m^b$ ), while buyers with high realizations of  $\varepsilon$  face binding cash constraints. Even if they wish, buyers of the same village cannot readjust their money holdings by trading with each other *after* they experience their preference shock because credit markets are already closed when the shocks are realized. One motivation for this assumption is that buyers experience the shocks after they depart from their village. This timing is essential for precautionary balances to exist. In Section 3.3, we extend the baseline model by allowing a fraction of individuals in the village to readjust their money after shocks are realized.

Analogously, conditional on being a seller who visits submarket  $(q_\varepsilon, d_\varepsilon, \alpha_\varepsilon)$  in the afternoon, the

individual chooses the demands for money  $m_\varepsilon^s$  and bonds  $b_\varepsilon^s$  to solve:

$$V^s(a) = \max \pi^s(\alpha_\varepsilon) [\beta V(a_{+1}^{s\varepsilon}) - C(q_\varepsilon)] + [1 - \pi^s(\alpha_\varepsilon)] \beta V(a_{+1}^{s0}) \quad (13)$$

subject to

$$a_{+1}^{s\varepsilon} = \frac{m_\varepsilon^s + b_\varepsilon^s(1+i) + \tau + d_\varepsilon}{M_{+1}}, \quad (14)$$

$$a_{+1}^{s0} = \frac{m_\varepsilon^s + b_\varepsilon^s(1+i) + \tau}{M_{+1}}, \quad (15)$$

$$a = \frac{m_\varepsilon^s + b_\varepsilon^s}{M}, \text{ and} \quad (16)$$

$$m_\varepsilon^s \geq 0 \quad (17)$$

Since the seller gets the same utility in all active submarkets,  $V^s$  is not indexed by  $\varepsilon$ . The seller does not need to carry money to make payments, but money holdings cannot be negative as stated in (17).

In addition to all constraints specified above, the individual faces an endogenous lower bound on next period's real wealth because he or she must be able to repay the amounts borrowed with probability one without reliance on unbounded borrowing (No-Ponzi game condition):

$$a_{+1} \geq a_{\min} \text{ with probability one.} \quad (18)$$

We denote by  $a_{+1}$  the stochastic real wealth for next period, which depends on the choice of being a buyer or a seller, the realization of  $\varepsilon$ , and the trading match. The endogenous lower bound  $a_{\min}$  is equal to minus the present discounted value of the maximum guaranteed income the individual can obtain as a seller.

The optimal demands for money follow directly from the fact that money earns no interest but that bonds earn  $i > 0$ . It is clearly suboptimal to carry money balances that are never used (because the marginal utility of wealth is positive, as we show below). Therefore,  $m^b$  is equal to the highest contingent payment:  $m^b = \max \{d_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$  and  $m_\varepsilon^s = 0$ . This implies that  $b^b = aM - \max \{d_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$  and  $b_\varepsilon^s = aM$ .

As is standard, the equilibrium interest rate is pinned down by the following arbitrage condition:

$$i = \frac{\gamma}{\beta} - 1. \quad (19)$$

That is, the equilibrium real interest rate must be equal to the subjective discount rate  $r = \beta^{-1} - 1$ .

Substituting (19) into the optimization program described in equations (7) to (18), one can show that  $V$  is a well-defined function of  $a$  which can be characterized using standard recursive

methods. Proposition 1 shows that, if  $r$  is not too high so agents are sufficiently patient, then  $V$  is concave with a linear segment (see the Appendix for the proof which parallels the arguments in Faig (2004) in an environment with competitive search).

**Proposition 1:** *Let  $r \in (0, 1/2)$ , where  $r = \beta^{-1} - 1$  denotes the subjective discount rate. There is an interval  $[\underline{a}, \bar{a}] \subset [a_{\min}, \infty)$  where the equilibrium value function  $V$  takes the linear form*

$$V(a) = v_0 + a, \tag{20}$$

*where  $v_0$  is a term independent of  $a$ . Outside this interval,  $V$  is strictly concave and continuously differentiable. Finally, the interval  $[\underline{a}, \bar{a}]$  is absorbing, that is  $a \in [\underline{a}, \bar{a}]$  implies  $a_{+1} \in [\underline{a}, \bar{a}]$  with probability one.*

The linear segment of  $V$  is due to the non-convex choice of the trading role individuals makes each day. In the absence of credit, this non-convexity would generate an incentive for individuals to use lotteries that would determine who is going to be a buyer or a seller. As in Rogerson (1988), these lotteries would create a linear segment in the value function. Proposition 1 says that individuals can achieve an equivalent outcome by trading in the village's credit market as long as they are sufficiently patient. Intuitively, an individual is indifferent between being in a steady-state where he/she is a buyer with fifty percent probability each period and being in another steady state where he/she is a buyer half of the time in alternating periods. In the second scenario, individuals who produce first are adequately compensated with the interest earned on bonds for their impatience to consume earlier and produce later. In this paper, we use the credit market as the primary mechanism for allocating individuals among the two trading roles because borrowing and lending seems more realistic than gambling over who is going to produce in a given period. As we have already noted, the argument in the proof of Proposition 1 complements the existence of a credit market with the possibility that individual can engage in atemporal gambles of some of their wealth. Even though inside the linear interval  $[\underline{a}, \bar{a}]$  individuals are indifferent between engaging or not in such gambles, their presence allows us to prove the existence of an equilibrium distribution of wealth under mild conditions.<sup>8</sup>

It seems to us that, empirically, the assumption  $r < 1/2$  is far from being restrictive. This assumption ensures that the interval  $[\underline{a}, \bar{a}]$  is absorbing, which simplifies the model dramatically. As long as all individuals have initial levels of wealth in the interval  $[\underline{a}, \bar{a}]$ , as we assume from

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<sup>8</sup>Essentially, inflation is non-negative or agents are sufficiently patient.

now on, the behavior of each buyer and each seller is independent of his/her wealth. Therefore, the distributions of money holdings are easily characterized, in particular because sellers have no incentive to create submarkets that cater to buyers of different levels of wealth (see Section 3.2).

The invariant distribution of wealth inside the interval  $[\underline{a}, \bar{a}]$  is indeterminate because there are many strategies about when to be buyer or a seller that yield the same utility. For example, one individual may typically alternate the trading role each period, while another may typically alternate the trading role every two periods. As long as their wealths stay inside the interval  $[\underline{a}, \bar{a}]$  with probability one, the two strategies yield the same utility. This indeterminacy does not matter for the characterization of an equilibrium because inside the interval  $[\underline{a}, \bar{a}]$  the marginal utility of wealth is unique.

Using the optimal demands for money, (19), (20), and  $a_{+1} \in [\underline{a}, \bar{a}]$  with probability one, the value functions of the buyer (8) and the seller (13) simplify into:

$$V^b(a) = S^b + \beta \left( v_0 + \frac{\gamma - 1}{\gamma} \right) + a, \text{ and} \quad (21)$$

$$V^s(a) = S^s + \beta \left( v_0 + \frac{\gamma - 1}{\gamma} \right) + a. \quad (22)$$

These value functions differ only in the first term. This term represents the expected trading surpluses of buyers and sellers in the goods market:

$$S^b = \int_1^{\bar{\varepsilon}} \pi^b(\alpha_\varepsilon) [\varepsilon U(q_\varepsilon) - z_\varepsilon] dF(\varepsilon) - im, \text{ and} \quad (23)$$

$$S^s = \pi^s(\alpha_\varepsilon) [z_\varepsilon - C(q_\varepsilon)]. \quad (24)$$

Here

$$z_\varepsilon = \frac{\beta d_\varepsilon}{M_{+1}}. \quad (25)$$

denotes real payments (discounted units of consumption that can be purchased with  $d_\varepsilon$  dollars next period). Similarly,  $m$  denotes real money balances conditional on being a buyer. Since buyers carry only enough money to make the highest contingent payment, we have

$$m = \max \{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}. \quad (26)$$

### 3.2 Competitive Search Equilibrium

In this section we characterize the equilibrium conditions for the goods market and define a symmetric monetary stationary equilibrium. We assume that all individuals have initial wealths in the interval  $[\underline{a}, \bar{a}]$ , so the trading surpluses of buyers and sellers are given by (23) and (24).

In the morning, when individuals can still rebalance their portfolios, sellers post their trading offers in all submarkets where they wish to participate. A trading offer is a pair  $(q_\varepsilon, z_\varepsilon)$  that specify the output offered to and the payment demanded from a buyer of type  $\varepsilon$ .<sup>9</sup> If a seller posts a single trading offer directed to buyers of type  $\varepsilon$ , he commits to supply zero output to all other buyers. All individuals have rational expectations regarding the number of buyers that will be attracted by each offer, and thus about the relative proportion of buyers and sellers that will trade in each submarket. In a competitive search equilibrium the offers posted by the sellers must be such that no seller has incentives to post a deviating offer.

Let  $\Omega_\varepsilon$  be the set of vectors  $(q_\varepsilon, z_\varepsilon, \alpha_\varepsilon)$  characterizing the submarkets where the buyers of type  $\varepsilon$  choose to trade in equilibrium. If buyers of type  $\varepsilon$  choose not to trade at all, then  $\Omega_\varepsilon$  is a singleton with  $q_\varepsilon = z_\varepsilon = 0$  (there is a single inactive submarket for this type). The set of all submarkets that are formed in equilibrium is then  $\Omega = \Pi_{\varepsilon \in [1, \bar{\varepsilon}]} \Omega_\varepsilon$ .

A competitive search equilibrium is a set  $\{\Omega, \bar{S}^b, \bar{S}^s\}$  such that the following four conditions hold:

1. Sellers attain the same expected surplus  $\bar{S}^s$  in all active submarkets.
2. Buyers attain the same *ex ante* expected surplus  $\bar{S}^b$ .
3. The expected surpluses of buyers and sellers are identical provided at least one submarket is active:  $\bar{S}^b = \bar{S}^s$
4. Each  $\omega \in \Omega$  solves the following program:

$$\bar{S}^b = \max_{\{(q_\varepsilon, z_\varepsilon, \alpha_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}} \int_1^{\bar{\varepsilon}} \left\{ \pi^b(\alpha_\varepsilon) [\varepsilon U(q_\varepsilon) - z_\varepsilon] \right\} dF(\varepsilon) - im \quad (27)$$

subject to

$$m = \max \{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}, \quad (28)$$

$$\varepsilon U(q_\varepsilon) - z_\varepsilon \geq 0 \text{ for } \varepsilon \in [1, \bar{\varepsilon}], \text{ and} \quad (29)$$

$$\pi^s(\alpha_\varepsilon) [z_\varepsilon - C(q_\varepsilon)] = \bar{S}^s \text{ for all active } \varepsilon \in [1, \bar{\varepsilon}]. \quad (30)$$

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<sup>9</sup>We could allow for offers that are contingent both on the type  $\varepsilon$  and the wealth  $a$  of the buyer. Since the buyer's expected surplus (23) and money balances (26) are independent of  $a$ , from the seller's view point, all buyers of a given type  $\varepsilon$  are identical even if their wealths are different. Hence, restricting to offers which are only contingent on  $\varepsilon$  is without loss of generality.

The rationale for these conditions is the following: (1) Sellers are free to choose the submarket where they participate, so they must attain the same expected surplus in all active submarkets regardless of the type of buyer they trade with. (2) Conditional on the realization of  $\varepsilon$ , buyers are also free to trade in any submarket in  $\Omega_\varepsilon$ , so they must attain the same conditional expected surplus in any of these submarkets. Since the distribution of  $\varepsilon$  is identical for all buyers, they must all attain the same *ex ante* expected surplus. (3) An equilibrium with at least one active submarket must have both buyers and sellers present in that submarket, so both trading roles must yield identical expected surpluses. (4) Buyers choose among submarkets in order to maximize their expected surplus (27) subject to three constraints. Constraint (28) says that the buyer must be able to pay for the good in each submarket that he/she considers visiting. Remember that sellers are committed to the trading offers they post, so buyers must carry enough money to meet the posted payments in all submarkets they wish to trade. Constraint (29) says that the buyer's utility must be non-negative in each submarket. Constraint (30) says that the buyer chooses among offers that give the seller a fixed surplus  $\bar{S}^s$ . Clearly, sellers never post deviating offers that imply a lower expected surplus than  $\bar{S}^s$  because they can attain  $\bar{S}^s$  in any active submarket.<sup>10</sup> If a seller tries to post an offer that attracts buyers and yields a higher expected surplus than  $\bar{S}^s$ , other sellers would profitably undercut this offer (e.g., by offering the same quantity for a slightly lower price). Finally, a seller cannot create a deviating submarket that attracts several buyer types with cross-subsidies because other sellers would try to attract the type paying the subsidy with a more attractive offer.

The solution to program (27) to (30) must maximize the total expected surplus from a matched subject to the cash constraint and the individual rationality constraints. Therefore, in any active submarket buyers and sellers must trade with probability one:

$$\alpha_\varepsilon = \pi^b(\alpha_\varepsilon) = \pi^s(\alpha_\varepsilon) = 1. \quad (31)$$

Some buyer types may not trade because there is no active submarket serving them, which is the case if the total surplus is lower than  $\bar{S}^s$ , for then no seller posts an offer targeting these types:

$$q_\varepsilon = z_\varepsilon = 0 \text{ if } \varepsilon U(q) - C(q) \leq \bar{S}^s \text{ for all feasible } q. \quad (32)$$

Using (31) and (32), solving for  $z_\varepsilon$  in (30), and restating (28), program (27) to (30) simplifies to:

$$\bar{S}^b = \max_{m, \{q_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}} \int_1^{\bar{\varepsilon}} \max\{\varepsilon U(q_\varepsilon) - \bar{S}^s - C(q_\varepsilon), 0\} dF(\varepsilon) - im \quad (33)$$

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<sup>10</sup> Since each individual is infinitesimal in the market, he/she takes as given the expected surplus of other individuals.



subject to

$$\bar{S}^s + C(q_\varepsilon) \leq m \text{ for } \varepsilon \in [1, \bar{\varepsilon}], \text{ and} \quad (34)$$

$$\varepsilon U(q_\varepsilon) - C(q_\varepsilon) \geq \bar{S}^s \text{ if } q_\varepsilon > 0. \quad (35)$$

When (34) and (35) do not bind, the first-order condition with respect to  $q_\varepsilon$  is

$$\varepsilon U'(q_\varepsilon) = C'(q_\varepsilon). \quad (36)$$

The output  $q_\varepsilon$  that solves (36) is unique and increasing with  $\varepsilon$  given the convexity of  $C$  and concavity of  $U$ . Hence, either the cash constraint (34) is never binding, or it binds in an interval  $[\hat{\varepsilon}, \bar{\varepsilon}]$ . Similarly, either the individual rationality constraint (35) is never binding, or it binds for an interval  $[1, \varepsilon_0]$ . Therefore, the quantities of output that solve program (33) to (35) obey:

$$\begin{cases} q_\varepsilon = 0 & \text{for } \varepsilon \in [1, \varepsilon_0) & \text{if } \varepsilon_0 > 1, \\ \varepsilon U'(q_\varepsilon) = C'(q_\varepsilon) & \text{for } \varepsilon \in [\varepsilon_0, \hat{\varepsilon}], & \text{and} \\ q_\varepsilon = q_{\hat{\varepsilon}} \equiv \hat{q} & \text{for } \varepsilon \in [\hat{\varepsilon}, \bar{\varepsilon}]. \end{cases} \quad (37)$$

Furthermore, the buyers' real money balances are given by

$$m = \bar{S}^s + C(\hat{q}). \quad (38)$$

The break-point for zero output  $\varepsilon_0$  is characterized by combining (35) with equality and (37) to obtain:

$$\varepsilon_0 U(q_{\varepsilon_0}) - C(q_{\varepsilon_0}) = \bar{S}^s \quad (39)$$

The break-point for a binding cash constraint  $\hat{\varepsilon}$  is obtained from the first-order condition of program (33) to (35) with respect to  $m$ :

$$i = \int_1^{\bar{\varepsilon}} \delta_\varepsilon d\varepsilon, \quad (40)$$

where  $\delta_\varepsilon$  is the Lagrange multiplier of (34). The Kuhn-Tucker Theorem implies that  $\delta_\varepsilon = 0$  for  $\varepsilon \in [1, \hat{\varepsilon}]$ , and  $\delta_\varepsilon C'(\hat{q}) = [\varepsilon U'(\hat{q}) - C'(\hat{q})] dF(\varepsilon)$  for  $\varepsilon \in [\hat{\varepsilon}, \bar{\varepsilon}]$ . Using these equalities and (37) to solve the integral in (40), we obtain the break-point  $\hat{\varepsilon}$  as an implicit function of  $i$ :

$$i = \int_{\hat{\varepsilon}}^{\bar{\varepsilon}} \left( \frac{\varepsilon}{\hat{\varepsilon}} - 1 \right) dF(\varepsilon). \quad (41)$$

To complete the characterization of a competitive search equilibrium, it remains is to determine  $\bar{S}^s$ . Since the expected surplus is the same for buyers and sellers,  $\bar{S}^s$  is given by

$$\int_{\varepsilon_0}^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) - \bar{S}^s - C(q_\varepsilon)] dF(\varepsilon) - i [\bar{S}^s + C(\hat{q})] = \bar{S}^s. \quad (42)$$

We are ready for a formal definition of equilibrium. A **symmetric monetary stationary equilibrium** is a vector of real numbers  $(\varepsilon_0, \hat{\varepsilon}, m, \bar{S}^s)$  and a set of real functions  $\{(\alpha_\varepsilon, q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$  that satisfy the system of equations: (30), (31), (37), (38), (39), (41), and (42).<sup>11</sup> This equilibrium is consistent with the environment conjectured in Subsection 3.1. Since the solution to program (27) to (30) is unique, there is at most one active submarket for each type  $\varepsilon$ . Below we argue that buyers of type  $\varepsilon$  have no incentives to deviate and visit a submarket catering different types  $\varepsilon' \neq \varepsilon$ . Also, the credit markets exhaust the gains for trading financial securities in the morning because all individuals have identical marginal rates of substitution in their margins of choice at time of the day. Finally, we show in the Appendix that under the assumptions in Proposition 1 there are distributions of wealth consistent with this definition of equilibrium.

An interesting property of the equilibrium is that it can be implemented if sellers post a trading offer that consists of a simple price schedule:

$$Z(q) = \bar{S}^s + C(q), \quad (43)$$

together with the promise to sell any quantity  $q$  for  $Z(q)$  utils ( $Z(q)\beta^{-1}M_{+1}$  dollars). Price schedule (43) has two tiers. The first tier is a flat amount that covers the seller's expected surplus, and the second tier is a variable amount that covers the production cost of output. One can easily check that a buyer facing (43) chooses a quantity of output consistent with (37) and a quantity of money consistent with (41). Interestingly, the price schedule in (43) is independent of the buyer's type, so all buyers face the same prices. This directly implies that buyers have no incentives to deviate and visit submarkets catering to other types. Therefore, the equilibrium can be implemented even if types cannot be observed by the sellers.

The welfare effects of inflation are captured by equations (37), (38), (39), and (41), together with the equation that determines the equilibrium nominal interest rate (19). At the Friedman rule ( $\gamma \downarrow \beta$ ), the opportunity cost of holding money vanishes since  $i \downarrow 0$ . Buyers then hold enough money to avoid being liquidity constrained in all contingencies:  $\hat{\varepsilon} = \bar{\varepsilon}$ . In this instance,  $q_\varepsilon$  is efficient in any active submarket since the marginal utility of consumption is equal to the marginal disutility of production.<sup>12</sup> Therefore, the equilibrium is efficient at the Friedman rule. For  $\gamma > \beta$ ,

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<sup>11</sup>As usual, the existence of a monetary equilibrium requires that the rate of growth of the money supply is not too large. Moreover, for precautionary balances to be positive, we need that  $\hat{\varepsilon} > 1$ , so using (41)  $i$  must be lower than  $(\bar{\varepsilon} - 1)^2/2$ .

<sup>12</sup>As noted above, a market has no active submarket if the total expected surplus is lower than the opportunity cost of sellers  $\bar{S}^s$ . This is also an efficiency condition.

the opportunity cost of holding money is positive since  $i > 0$ . Buyers then react by reducing their money balances relative to the Friedman rule and are thus liquidity constrained for high realizations of the preference shock:  $\hat{\varepsilon} < \bar{\varepsilon}$ . When this happens, the output traded is below the efficient level of output, and the marginal utility of consumption for liquidity-constrained individuals is above the marginal cost of production, which is the source of the welfare cost of inflation in this model.

The interest elasticities of the demand for money and the velocity of circulation of money are implicitly determined by equations (41) and (38). As  $i$  increases,  $\hat{\varepsilon}$  falls, so buyers reduce their real money balances and increase the fraction of those that they spend. As a result, money circulates faster. There are two effects of  $i$  on  $m$ , to see those remember that buyers carry enough money to pay for  $\bar{S}^s + C(\hat{q})$ . If  $i$  increases,  $\hat{q}$  falls, so  $C(\hat{q})$  and  $m$  fall. Furthermore, the total expected surplus decreases as a result of inflation, so the seller's surplus  $\bar{S}^s$  falls, implying a further reduction in  $m$ .

### 3.3 Extension

As seen in Section 2, the time series of velocity in the United States displays an upward trend which we argued was likely due to advances in information technology. We do not view these advances as having eliminated M1\* as the main media of exchange, but as allowing conversions in and out of M1\* assets more easily and speedily. Thus individuals can now face unexpected expenses without holding large precautionary balances. To incorporate these technological advances in our model, we assume that a fraction of individuals in each village are able to communicate with other fellow villagers early in the afternoon, so that they can rebalance their portfolios after they know their preference shock but prior to meeting a seller.<sup>13</sup> Consequently, there is a fraction  $\theta$  of individuals that experience the preference shock after deciding on the demand for money, while the rest experience the preference shock prior to this decision. For tractability, we assume that all individuals have the same *ex ante* probability of a particular timing.

The analysis of this extended model is analogous to the one in the previous sections. A competitive search equilibrium must still satisfy conditions 1 to 3. Condition 4 is now that each  $\omega \in \Omega$

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<sup>13</sup>See Berentsen, Camera, and Waller (2004) for a model that views banks as institutions that facilitate this type of arrangements.

solves the following program:

$$\begin{aligned} \bar{S}^b = & \max_{\{(\alpha_\varepsilon, q_\varepsilon, z_\varepsilon), (\alpha_\varepsilon^*, q_\varepsilon^*, z_\varepsilon^*, m_\varepsilon^*)\}_{\varepsilon \in [1, \bar{\varepsilon}]}} \theta \left( \int_1^{\bar{\varepsilon}} \left\{ \pi^b(\alpha_\varepsilon) [\varepsilon U(q_\varepsilon) - z_\varepsilon] \right\} dF(\varepsilon) - im \right) \\ & + (1 - \theta) \left( \int_1^{\bar{\varepsilon}} \left\{ \pi^b(\alpha_\varepsilon^*) [\varepsilon U(q_\varepsilon^*) - z_\varepsilon^*] \right\} dF(\varepsilon) - im_\varepsilon^* \right) \end{aligned} \quad (44)$$

subject to

$$m = \max \{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]} \quad \text{and} \quad m_\varepsilon^* = z_\varepsilon \quad \text{for all } \varepsilon \in [1, \bar{\varepsilon}], \quad (45)$$

$$\varepsilon U(q_\varepsilon) - z_\varepsilon \geq 0 \quad \text{and} \quad \varepsilon U(q_\varepsilon^*) - z_\varepsilon^* \geq 0 \quad \text{for } \varepsilon \in [1, \bar{\varepsilon}], \quad \text{and} \quad (46)$$

$$\pi^s(\alpha_\varepsilon) [z_\varepsilon - C(q_\varepsilon)] = \bar{S}^s \quad \text{and} \quad \pi^s(\alpha_\varepsilon^*) [z_\varepsilon^* - C(q_\varepsilon^*)] = \bar{S}^s \quad \text{for all active } \varepsilon \in [1, \bar{\varepsilon}]. \quad (47)$$

For buyers who carry precautionary balances, the conditions for the optimality of  $\{(\alpha_\varepsilon, q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$  are the same as in the previous subsection. For buyers who know their preference shock before deciding on the demand for money, the conditions of optimality of  $\{(\alpha_\varepsilon^*, q_\varepsilon^*, z_\varepsilon^*, m_\varepsilon^*)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$  are the following. A submarket  $\varepsilon$  is active if the trading surplus in this submarket can cover the selling costs; that is, there is  $q > 0$  such that  $\varepsilon U(q) - C(q)(1+i) \geq \bar{S}^s(1+i)$ . For all active submarkets, buyers carry only the money they know they are going to spend, and the real value of this money is equal to the cost of producing the goods to be purchased plus the cost of selling them:  $m_\varepsilon^* = z_\varepsilon^* = \bar{S}^s + C(q_\varepsilon^*)$ . Efficient matching implies that the ratio of buyers over sellers is one:  $\alpha_\varepsilon^* = 1$ . Finally, the marginal utility of consumption is equal to the marginal cost of acquiring goods, which includes the cost of carrying money:  $\varepsilon U'(q_\varepsilon^*) = C'(q_\varepsilon^*)(1+i)$ . For the inactive submarkets,  $\alpha_\varepsilon^* = q_\varepsilon^* = m_\varepsilon^* = z_\varepsilon^* = 0$ .

In this extended model, the equality between the expected trading surpluses of buyers and sellers is given by

$$\begin{aligned} \bar{S}^s = & \theta \int_{\varepsilon_0}^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) - \bar{S}^s - C(q_\varepsilon)] dF(\varepsilon) - i [\bar{S}^s + C(\hat{q})] + \\ & (1 - \theta) \int_1^{\bar{\varepsilon}} \left\{ \varepsilon U(q_\varepsilon^*) - [\bar{S}^s + C(q_\varepsilon^*)] (1+i) \right\} dF(\varepsilon). \end{aligned} \quad (48)$$

For ease of exposition, we have assumed so far that  $\gamma$  and  $\theta$  are constant. However, since there is no capital in the model, the equations that characterize an equilibrium are still valid in an environment where  $\gamma_t$  and  $\theta_t$  change over time as long as the values of these variables are known at the beginning of each period. In such an environment, the real interest rate is equal to the subjective discount rate for all periods, sellers post two-tier price schedules similar to (43), and the demand for money depends on the contemporaneous values of  $\gamma_t$  and  $\theta_t$ . Future values of these two

variables do affect the levels of utility of individuals, but their value functions still have a linear interval with a constant marginal utility as in Proposition 1. The values of  $v_0, \bar{a}$ , and  $\underline{a}$  in this proposition now change over time in a complicated fashion, but these values are not needed to characterize an equilibrium as defined above.<sup>14</sup>

## 4 The Estimation of the Model

This section estimates the extended model advanced in Subsection 3.3 using primarily the United States data described in Section 2. The estimated model is then used to address several important issues in monetary economics.

In the empirical implementation, we adopt specific functional forms for  $U$ ,  $C$ , and  $F$ . The functional forms for the utility of consuming and the disutility of producing are assumed to be respectively isoelastic and linear:

$$U(q_\varepsilon) = \frac{q_\varepsilon^{1-\sigma}}{1-\sigma}, \quad \sigma \in (0, 1), \text{ and} \quad (49)$$

$$C(q_\varepsilon) = q_\varepsilon. \quad (50)$$

These functional forms are the most commonly used in the literature. With these functional forms, the average commercial margin is increasing with the curvature parameter  $\sigma$ . In particular, the average commercial margin is  $\sigma/(2-\sigma)$  as long as buyers purchase goods for all  $\varepsilon$  and  $i = 0$ .<sup>15</sup> Intuitively, if  $U$  has a large curvature parameter  $\sigma$ , individuals seek to consume small quantities often because marginal utility is strongly decreasing in  $q_\varepsilon$ . As a result, individuals require a large remuneration, in the form of a large commercial margin, to sacrifice their time to be sellers.

Previous literature offers little guidance about the distribution of preference shocks. After some experimentation, we discovered that to generate a realistic elasticity of the demand for money, large preference shocks must be rare relative to low preference shocks. A convenient way to capture this is to assume that the distribution of shocks is uniform on the interval  $(1, \bar{\varepsilon}]$  but has mass probability at 1. This distribution has the following convenient interpretation. With probability  $p$ , buyers have a "normal" desire to consume, in which case  $\varepsilon$  is normalized to 1. With probability  $1-p$ , buyers

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<sup>14</sup>The main complication of studying non-stationary equilibria is to prove existence of the absorbing interval of wealth. To this end, the atemporal wealth gambles used in the proof of existence of an (stationary) equilibrium distribution of wealth are a useful device.

<sup>15</sup>The derivation of this formula is lengthy and unrelated to the main issues of the paper, so it is not provided here.

experience a larger than normal desire to consume. When this happens,  $\varepsilon$  is uniformly distributed on the interval  $(1, \bar{\varepsilon}]$ . Algebraically, the distribution function is

$$F(\varepsilon) = \begin{cases} p & \text{at } \varepsilon = 1, \text{ and} \\ \frac{1-p}{\bar{\varepsilon}-1} (\varepsilon - 1) & \text{for } \varepsilon \in (1, \bar{\varepsilon}]. \end{cases} \quad (51)$$

Thus, the density for  $\varepsilon \in (1, \bar{\varepsilon}]$  is constant and equal to

$$\varphi = \frac{1-p}{\bar{\varepsilon}-1}. \quad (52)$$

With the distribution function (51), condition (41) that determines the critical shock  $\hat{\varepsilon}$  at which individuals start being liquidity constrained simplifies into

$$\frac{i}{\varphi} = \frac{(\bar{\varepsilon} - \hat{\varepsilon})^2}{2\hat{\varepsilon}}. \quad (53)$$

Equation (53) determines the key properties of the demand for money for individuals who choose their money balances before knowing their preference shock. At  $i \downarrow 0$ , the liquidity constraint never binds ( $\hat{\varepsilon} = \bar{\varepsilon}$ ), so the demand for money depends on the maximum realization of the preference shock  $\bar{\varepsilon}$ ; large values of  $\bar{\varepsilon}$  imply a large demand for money. At positive interest rates, the cost of carrying money balances induces individuals to accept a positive probability of being liquidity constrained ( $\hat{\varepsilon} < \bar{\varepsilon}$ ). The size of this effect falls with the density of the preference shocks  $\varphi$ . Intuitively, if large preference shocks are rare,  $p$  close to one and  $\varphi$  close to zero, the losses from carrying less money are small because individuals are seldom liquidity constrained. Hence, individuals are willing to cut money balances substantially in response to a rise in  $i$ . As a result, the demand for money is highly elastic.

We estimate the parameters of the model  $(\beta, \sigma, \bar{\varepsilon}, p, \theta)$  using primarily the time series of the velocity of circulation of M1\* and the nominal rate of interest examined in Section 2. The velocity of circulation of money in our model, the theoretical counterpart of the velocity of M1\*, is equal to:

$$\text{Velocity} \equiv \frac{\text{GDP}}{M} = \frac{\theta \int_{\varepsilon_0}^{\bar{\varepsilon}} [\bar{S}^s + C(q_\varepsilon)] \varphi d\varepsilon + (1-\theta) \int_1^{\bar{\varepsilon}} [\bar{S}^s + C(q_\varepsilon^*)] \varphi d\varepsilon}{\theta [\bar{S}^s + C(\hat{q})] + (1-\theta) \int_1^{\bar{\varepsilon}} [\bar{S}^s + C(q_\varepsilon^*)] \varphi d\varepsilon}. \quad (54)$$

If  $\theta = 0$ , velocity is one because all individuals carry exactly the amount of money they know they are going to spend. In contrast, if  $\theta > 0$ , some individuals hold precautionary balances. As a result, velocity is below one and is interest elastic.

Assuming that variations in nominal interest rates are driven by exogenous shifts in monetary policy, the time series examined in Section 2 provide information, not only on the level of velocity

and its time trend, but also on the response of velocity to changes in nominal interest rates. Therefore, we are able to identify  $\bar{\varepsilon}$ ,  $p$ , and the time profile of  $\theta$ . The parameters  $\bar{\varepsilon}$  and  $p$  are assumed constant, so preferences are time invariant. The parameter  $\theta$  is assumed to be the following polynomial of time:  $\theta = \theta_0 + \theta_1 T + \theta_2 T^2 + \theta_3 T^3$ , where  $T$  measures time from  $T = -1$  at the beginning of the sample to  $T = 1$  at the end of the sample, that is,  $T = (\text{Year} - 1948)/56$ , and where  $\theta_0$  is normalized so the maximum value of  $\theta$  is one.

To properly identify the remaining parameters of the model, we complement the data on velocity and nominal interest rates with additional information. We calibrate  $\beta$  so that the real rate of interest is a realistic 3 per cent. Similarly, we calibrate  $\sigma$  to match the average commercial margin<sup>16</sup> reported by the Bureau of the Census ([www.census.gov/svsd/www/artstbl.html](http://www.census.gov/svsd/www/artstbl.html)) (around 28 percent).<sup>17</sup> Finally, we choose the length of the period to be two weeks, which captures the fact that most households receive income and make regular purchases at a fairly high frequency.

We estimated the model using nonlinear least squares and treating velocity as the dependent variable. More precisely, we converted the velocities and interest rates in our data from annual to biweekly,<sup>18</sup> and we searched for the vector of parameter values  $(\sigma, \bar{\varepsilon}, p, \theta_0, \theta_1, \theta_2, \theta_3)$  that minimizes the sum of squared residuals (the difference between actual and predicted velocities) subject to the constraints  $\sigma = 0.435$  and  $\max(\theta) = 1$  ( $\beta$  is not needed to calculate velocity for a given nominal interest rate). The parameter estimates are reported in Table 2.

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<sup>16</sup>The average commercial margin in our model is  $(\bar{z} - \bar{c})/\bar{z}$ , where  $\bar{z}$  is average revenue and  $\bar{c}$  is average cost of sellers. The average mark-up is the inverse of one minus the commercial margin.

<sup>17</sup>The parameter  $\sigma$  is also the inverse of the intertemporal elasticity of substitution of consumption. Faig and Jerez (2005) present a model with multiple purchases each period that distinguishes between the two roles that  $\sigma$  plays here.

<sup>18</sup>The biweekly velocity is the annual velocity divided by 26. One plus the biweekly interest rate is equal to one plus the annual interest rate to the power of  $1/26$ .

Table 2

## ESTIMATION OF THE MODEL

Sample: Annual time series United States 1892-2004

Dependent variable: Velocity (GDP/M1\*)

Independent variables: Commercial paper rate and time

Method: Non-linear least squares

Period length: 2 weeks

$$\sigma = 0.435$$

$$\theta_0 = 0.882$$

Parameter	$\bar{\varepsilon}$	$p$	$\theta_1$	$\theta_2$	$\theta_3$
Estimate	3.938	0.963	-0.326	-0.317	-0.109
Std. dev. estimates <sup>19</sup>	0.077	0.008	0.061	0.047	0.088
Sum of squared residuals = 0.0306					
Centered $R^2 = 0.968$					

As we can see in Table 2, the five parameters  $(\bar{\varepsilon}, p, \theta_1, \theta_2, \theta_3)$  are precisely estimated and have reasonable values. The first four parameters are significantly different from zero at all reasonable confidence intervals. The last parameter is significantly different from zero at the 78 per cent confidence interval. The high  $R^2$  signals a close fit between the model and the data.

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<sup>19</sup>The standard deviations of the estimates were calculated using the formula in Doan (2002) pp. 218-9 with  $k = 5$ . This formula is robust to heteroskedasticity and autocorrelations of the error term up to five years apart.



Figure 2

### Fitted Versus Actual Velocity

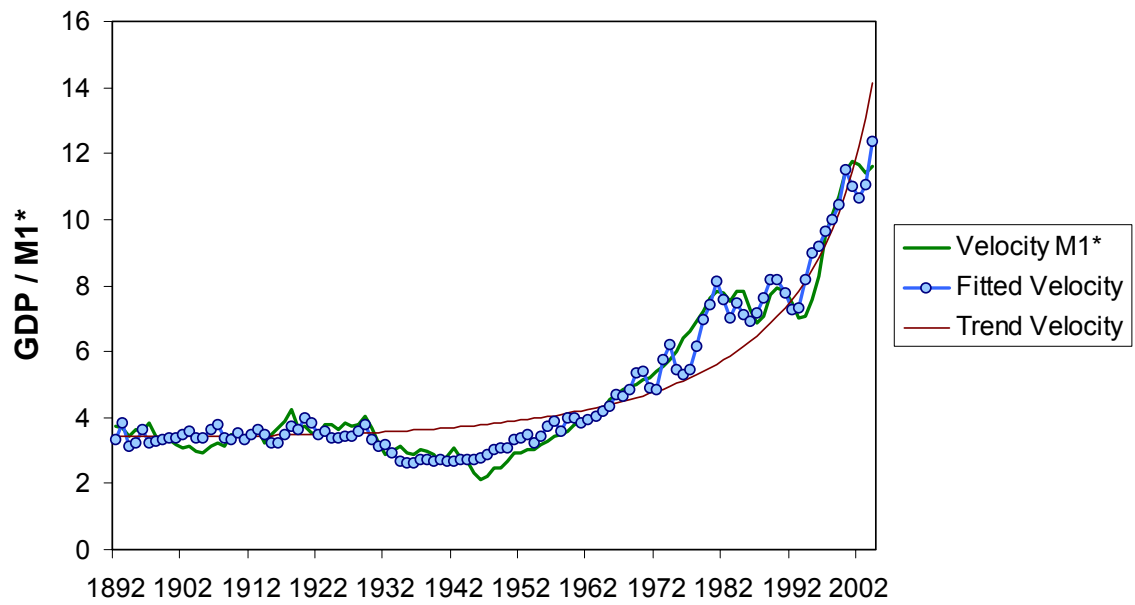
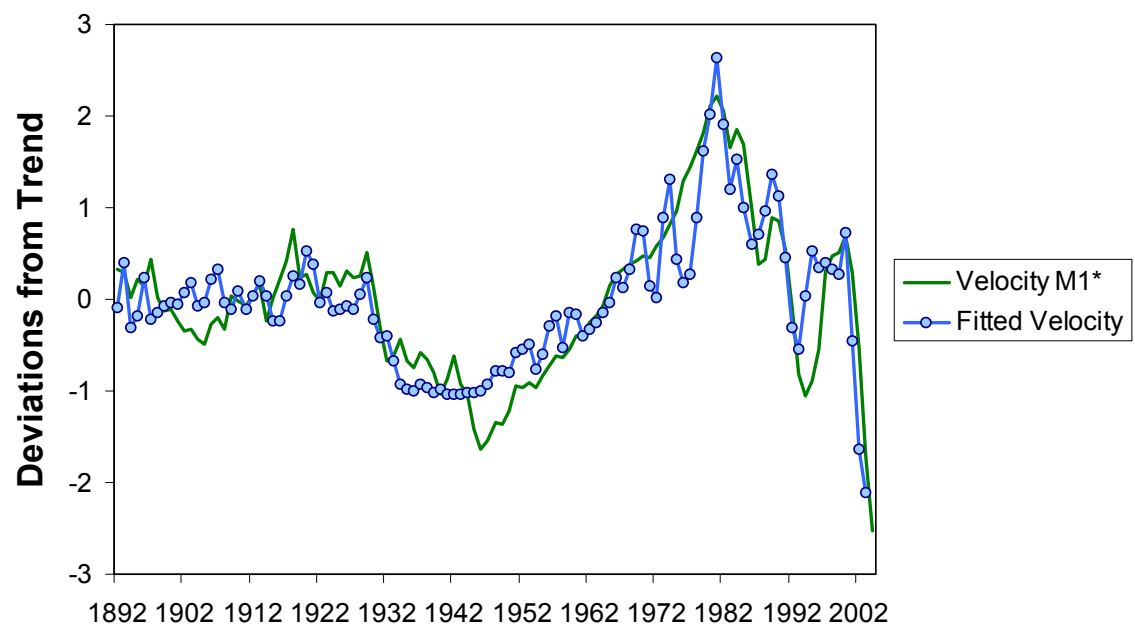


Figure 3

### Fitted Versus Actual Deviations from Trend



To better judge how the model fits the data, we plot in Figure 2 the annual velocity predicted by the model (dark plain line) and the actual velocity (line with circles); the two lines move very closely together. To ascertain how much this close fit is due to the correct estimation of the trend, and how much it is due to the correct predictions of the theoretical model, Figure 3 displays the deviations of the actual and the predicted velocities from trend velocity (trend velocity is defined as the predicted velocity for a constant interest rate at the average level of 4.53 percent). Figure 3 shows that our estimated model predicts accurately the deviations of velocity from trend. For example, the model correctly predicts the fall of velocity during the low-interest rate years of the Great Depression. It also predicts the rise in velocity, well above its trend, during the period of high interest rates that went from the mid-sixties to the mid-eighties. Finally, it predicts with a slight lead, the dramatic ups and downs of the detrended velocity in the last portion of the sample. The largest persistent residuals between the actual and the predicted velocities correspond to those of the Second World War and its aftermath. However, this deviation is not too surprising as this was a period of massive price and financial controls. The time profile of trend velocity, displayed in Figure 2 with a thin line, is interesting in itself. Trend velocity was almost constant in the first half of the sample, whereas it increased at an accelerating rate in the second half. As examined below, this sharp rise in trend velocity has major implications for a variety of issues.

The model predicts that neither the interest elasticity nor the interest semi-elasticity of velocity are constant, but they change with the nominal interest rate. For the median value of  $\theta$  (0.882), Figure 4 graphs the interest semi-elasticity and the interest elasticity of velocity as functions of the nominal interest rate. The elasticity is an increasing and concave function of the rate of interest. The semi-elasticity is a decreasing and convex function of the same rate. Curiously, the graph for this last function is approximately an hyperbola.

Figure 4

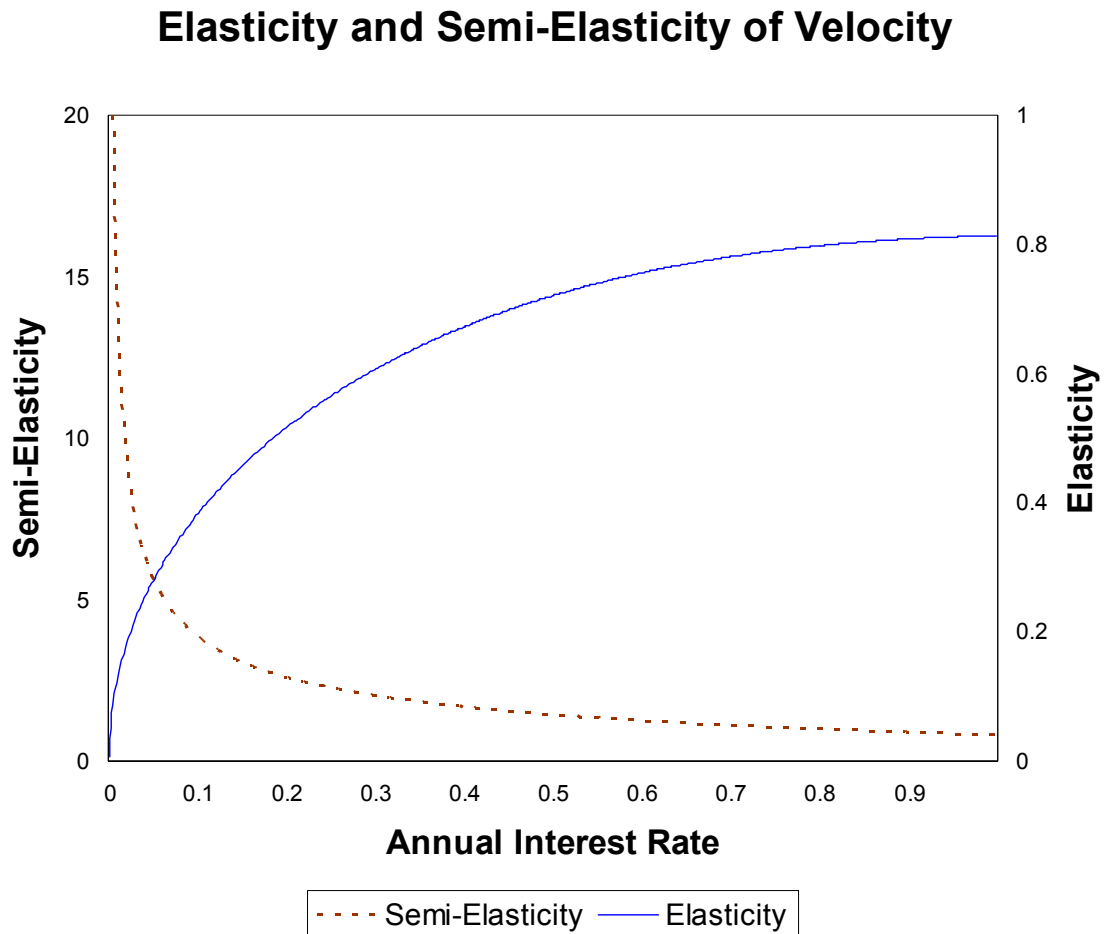


Table 3  
IMPLICATIONS OF THE MODEL

<b>Year</b>	1892	1920	1948	1976	2004
Estimated value of $\theta$ (%)	100	97.9	88.2	62.6	13.1
<b>Equilibrium at average <math>i</math> (4.53 % annual)</b>					
Average commercial margin (%)	28.1	28.1	28.1	28.0	27.9
Precautionary balances over money supply (%)	86.9	86.6	85.3	80.4	45.6
Fraction of buyers liquidity constrained (%)	1.14	1.11	1.01	0.71	0.15
Seigniorage over GDP (%)	0.43	0.42	0.39	0.29	0.11
Interest semi-elasticity of velocity	5.58	5.58	5.54	5.35	3.17
Interest elasticity of velocity	0.25	0.25	0.25	0.24	0.14
<b>Effect of rising <math>i</math> from 0 to 10% annual</b>					
Equivalent variation of consumption (%)	1.00	0.98	0.88	0.63	0.13
Reduction GDP (%)	6.84	6.71	6.12	4.58	1.58
Welfare cost over seigniorage (%)	24.9	24.8	24.5	23.2	13.7

Table 3 calculates the implications of the model for five values of  $\theta$  that correspond to our estimates for the years: 1892, 1920, 1948, 1976, and 2004. The upper part of Table 3 calculates implied equilibrium values if the nominal rate of interest had remained constant at its average level (4.53 percent) throughout the sample. The lower half calculates the implications of raising the nominal rate of interest from 0 to 10 percent. The time profile of  $\theta$  mirrors the time profile of trend velocity: it is almost constant from 1892 to 1948, it falls significantly from 1948 to 1976, and it plummets from 1976 to 2004. This last drop in  $\theta$  has had a major effect on the properties of the demand for money. As the table shows, the drop in  $\theta$  from 1976 to 2004 has led to dramatic reductions in the following: precautionary balances, fraction of buyers that are liquidity constrained, ability to collect seigniorage, and interest semi-elasticity and elasticity of velocity. For example, at

a 4.53 per cent nominal rate of interest, the government could collect around 0.4 percent of GDP from 1892 to 1948, but it could only collect 0.11 percent of GDP in 2004. The semi-elasticity of velocity was between 5 and 6 between 1892 and 1976, but it dropped to 3.16 in 2004.

The sharp drop in the need for precautionary balances that occurred in the last few decades also has major implications for the welfare cost of inflation. For most of our sample years, we find that the welfare cost of a 10 per cent increase in inflation from the Friedman rule is equivalent to a reduction in consumption of about 1 percent. This is consistent with the estimates of Lucas (2000). However, we also find that the welfare cost of inflation has plummeted with  $\theta$ . Intuitively, the shift from a high and elastic demand for money to a low and inelastic demand has compressed the area below the demand for money and hence the welfare cost of inflation. The drop in  $\theta$  has also reduced the effect of inflation on GDP and the ratio of the deadweight-loss of inflation over the seigniorage it generates.<sup>20</sup> In summary, the drop in  $\theta$  has radically altered the answer to most substantive questions in monetary economics. The conventional wisdom about many dimensions of monetary economics needs to be reworked in the new environment created by the revolution of the information and communication technologies, which we believe are behind the drop in  $\theta$  that we estimated in our analysis.

## 5 Conclusion

Precautionary balances, when carefully studied in a model with rigorous microeconomic foundations, are able to explain not only why the velocity of circulation of money has been historically low, but also why it correlates the way it does with the nominal rate of interest. Our empirical implementation of the model discovers that precautionary balances have plummeted in the last three decades. We attribute the origin of this drop to the tremendous improvements in the information and communication technologies that occurred during this time. These improvements have allowed individuals, with the collaboration of banks and other financial institutions, to accommodate unexpected expenditure needs without holding large precautionary balances. This has radically transformed many important issues of monetary economics. The demand for money has not only fallen, but is also less elastic. Moreover, both the seigniorage and the welfare cost of inflation are now a small fraction of what they were 30 years ago.

Our model abstracts from several other features of reality that are likely to be important to

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<sup>20</sup>Seigniorage is defined as  $(\gamma - 1)m$ . Deadweight-loss is defined as the change in  $S^s$  (which is equal to  $S^b$ ).

the issues we study. For example, we abstract from the distinction between currency and checkable deposits, and from the fact that some checkable deposits earn interest. Also, we abstract from the many ways in which individuals can affect their demand for money, such as converting assets at a higher frequency. Finally, we abstract from many complexities in the production of goods such as the presence of physical capital. We list these features here, not only to acknowledge the limitations of our work, but also to stimulate future research.

## Appendix

### Data sources

The interest rate is the short term commercial paper rate. For 1892-1971, it is taken from Friedman and Schwartz (1982), Table 4.8, Column 6. For 1972-2004, it is taken from the DRI series FYCP90 (averaged).

Money is  $M1^* = M1$  - currency outside the country. M1 is the stock at the end of June of each year. For 1892-1928, the source of M1 is the United States Bureau of the Census (1960), Series X267. For 1929-1958, it is the series constructed by the St. Louis FED that extends backwards modern M1 <http://research.stlouisfed.org/aggreg>. For 1959-2004, it is the DRI series FZM1. Currency in circulation abroad is from the FED *Flow of Funds* Table L-204 in the file ltab204d.prn downloaded from <http://www.federalreserve.gov/releases/z1/current/data.htm>.

For 1892-1928, GDP is calculated from the real GDP series in Kendrick (1961) and the implicit price deflator in Friedman and Schwartz (1982), Table 4.8, Column 4. For 1929-2004, it is from BEA NIPA Table 1.1.5 downloaded from [www.bea.doc.gov/bea/dn/nipaweb](http://www.bea.doc.gov/bea/dn/nipaweb) in January 2006.

### Proof of Proposition 1

Consider the problem of an individual in the equilibrium of our basic model where all other individuals have value functions (20) and initial wealths in the interval  $[\underline{a}, \bar{a}]$ , except for a small positive measure of individuals who have wealths identical to that of the individual whose value function we are characterizing. The assumption that there is a positive measure of individuals whose problems are identical eliminates strategic considerations that would be problematic for the notion of competitive search. Since we are describing the optimal decisions of individuals in an equilibrium, we conjecture some of the properties of an equilibrium which are proved in later sections. In particular, we use the absence of uncertainty in trading opportunities, which is the equilibrium outcome with efficient matching.

For all finite  $a \geq a_{\min}$ , the set of feasible time and state contingent policies is non empty. The feasible values of the quantities consumed and produced are bounded. Also, for all feasible policies the present discounted utility is well defined and finite because  $\mathcal{U}$  is a continuous function. Consequently, we can use standard recursive methods to find the value function.

In competitive search, we can recursively characterize the individual optimization problem as follows.<sup>21</sup> The individual chooses to be a buyer or a seller. As a seller, he/she chooses  $(\varepsilon, q_\varepsilon^s, z_\varepsilon^s, m_\varepsilon^s, b_\varepsilon^s)$ ,

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<sup>21</sup>This characterization uses a more general definition of competitive search than in Section 3.2 because it allows

where  $\varepsilon$  is the buyer type the seller targets, and  $(q_\varepsilon^s, z_\varepsilon^s)$  is the posted offer directed to this type. (With certainty of trades, the seller has no incentive to target more than one type of buyer.) As a buyer the individual chooses  $\left(\{q_\varepsilon^b, z_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}, m^b, b^b\right)$  where  $\{q_\varepsilon^b, z_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}$  is a list of offers contingent on the realization of their preference shock, all of which lie in the set of posted offers. A buyer can always choose not to trade for some realization  $\varepsilon$  (or sellers can choose not to post offers targeting certain types  $\varepsilon$ ) in which case  $q_\varepsilon^b = z_\varepsilon^b = 0$ . The above choices are subject to the constraints (9)-(12), (14)-(17), and (18). In the financial markets the individual takes as given the rate of interest. In the goods market, the individual takes as given the reservation expected trade surpluses of other traders and has rational expectations about their actions. Therefore, the offers posted by sellers satisfy the following two conditions:  $\int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon^s) - z_\varepsilon^s] dF(\varepsilon) - i \max \{z_\varepsilon^s\}_{\varepsilon \in [1, \bar{\varepsilon}]} \geq \bar{S}^b$  (buyers are guaranteed the reservation expected trade surplus), and  $z_\varepsilon^s - C(q_\varepsilon^s) \leq \bar{S}^s$  for  $\varepsilon \in [1, \bar{\varepsilon}]$  (the posted offers cannot be profitably undercut). As a buyer, the individual acts as if he/she were choosing  $\{q_\varepsilon^b, z_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}$  that satisfies  $z_\varepsilon^b - C(q_\varepsilon^b) = \bar{S}^s$  for all  $q_\varepsilon^b > 0$ , because competition among sellers implies that all posted offers yield the same trade surplus  $\bar{S}^s$  to the sellers.

Let  $\mathcal{C}(a)$  be the space of bounded and continuous functions  $f : [a_{\min}, \infty) \rightarrow R$ , with the sup norm. Use the Bellman's equations (8) and (13) together with (7) to define the mapping  $T$  of  $\mathcal{C}(a)$  onto itself by substituting  $f$  for  $V$  in the right hand sides of (8) and (13) and denoting as  $Tf(a)$  the left hand side of (7). The choice variables and constraints of these maximization programs are described in the previous paragraph. For a given  $a$ , the set of feasible policies is non-empty, compact-valued, and continuous. The utility function  $\mathcal{U}$  is bounded and continuous on the set of feasible policies and  $\beta \in (0, 1)$ . Therefore, Theorem 4.6 in Stokey and Lucas with Prescott (1989) implies that there is a unique fixed point to the mapping  $T$ , which is the value function  $V$ .

Let  $\mathcal{V}(a)$  be the set of functions  $f : [a_{\min}, \infty) \rightarrow R$  that satisfy (20) where  $v_0$ ,  $\bar{a}$ , and  $\underline{a}$  are given by:

$$\begin{aligned} v_0 &= \frac{1}{r} \left[ \bar{S}^s (1+r) + \frac{\gamma-1}{\gamma} \right], \\ \bar{a} &= \frac{1}{r} \left[ (z_1 + im) (1+r) - \frac{\gamma-1}{\gamma} \right], \text{ and} \\ \underline{a} &= \frac{1}{r} \left[ -z_{\bar{\varepsilon}} (1+r) - \frac{\gamma-1}{\gamma} \right]; \end{aligned} \tag{55}$$

where  $i$ ,  $m$ ,  $\bar{S}^s$ , and  $z_\varepsilon$  satisfy the equilibrium system of equations described in 3.2. Consider the mapping  $T$  defined in the previous paragraph. Let  $a_{+1}^{b\varepsilon}$  be next-period real wealth for an optimal individual to have wealth outside the interval  $[\underline{a}, \bar{a}]$ .



policy conditional on being a buyer and a realization of the preference shock. Similarly, let  $a_{+1}^{s\varepsilon}$  be the next period wealth for an optimal policy conditional on being a seller serving buyers of type  $\varepsilon$ . The recursive budgets (9) to (11) and (14) to (16), together with (4), (19), (25), (26), and  $m^s = 0$ , imply that

$$a_{+1}^{b\varepsilon} = (1+r)(a - z_\varepsilon - im) + \frac{\gamma-1}{\gamma}, \text{ and} \quad (56)$$

$$a_{+1}^{s\varepsilon} = (1+r)(a + z_\varepsilon) + \frac{\gamma-1}{\gamma}. \quad (57)$$

The flow budget (56), together with (55) and  $z_\varepsilon = m$ , implies that  $\bar{a}$  is the maximum initial wealth that guarantees  $a_{+1}^{b\varepsilon} \leq \bar{a}$  and the minimum wealth that guarantees that  $a_{+1}^{b\varepsilon} \geq \underline{a}$  for all  $\varepsilon$  satisfies:

$$\underline{a}^b = m(1+i) - \frac{m}{r} - \frac{\gamma-1}{\gamma r}. \quad (58)$$

Similarly, since the seller can choose the type of buyer to serve, (55) and (57) imply that  $\underline{a}$  is the minimum initial wealth that guarantees  $a_{+1}^{s\varepsilon} \geq \underline{a}$ , and the maximum wealth that guarantees that  $a_{+1}^{s\varepsilon} \leq \bar{a}$  satisfies:

$$\bar{a}^{s\varepsilon} = -z_\varepsilon + \frac{z_1 + im}{r} - \frac{\gamma-1}{\gamma r}. \quad (59)$$

Comparison of (58) and (59), using  $z_1 \geq 0$ , the definition of  $r$ , and (19), implies that  $\bar{a}^{s\varepsilon} \geq \underline{a}^b$  for all  $\varepsilon$  if  $i \geq (2r-1)/(1-r)$ . Using (19) and the definition of  $r$ , this condition is equivalent to  $\gamma \geq \beta r/(1-r)$ , which is satisfied for all  $\gamma > \beta$  as long as  $r \in (0, 1/2)$ . Therefore, the individual with a suitable choice of being buyer or seller can guarantee  $a_{+1} \in [\underline{a}, \bar{a}]$  if  $a \in [\underline{a}, \bar{a}]$ .

The optimal choice of the trading role is characterized as follows. If  $a \in [\underline{a}^b, \bar{a}^{s\varepsilon}]$ , then  $a_{+1} \in [\underline{a}, \bar{a}]$  for all trading roles and realizations of  $\varepsilon$ . Therefore,  $TV(a)$  (which is the maximum of  $V^b(a)$  and  $V^s(a)$  in equations (21) and (22)) is affine and the trade surpluses are those in (23) and (24). As a result, the optimal policies of the individual are the equilibrium ones characterized in Section 3.2, the individual is indifferent between being a buyer or a seller, and as a seller he/she is indifferent to serving any type of buyer. This indifference is broken if one policy would lead to  $a_{+1} \notin [\underline{a}, \bar{a}]$ . In such a case, the strict concavity of  $V$  outside the interval  $[\underline{a}, \bar{a}]$  implies that it is suboptimal to be a seller serving buyers of type  $\varepsilon$  if  $a > \bar{a}^{s\varepsilon}$ , so  $a_{+1}^{s\varepsilon} > \bar{a}$ . Likewise, it is suboptimal to be a buyer if  $a < \underline{a}^b$ , and so  $a_{+1}^b < \underline{a}$ . Consequently, if  $a \in [\underline{a}, \bar{a}]$ , an optimal policy is one that ensures  $a_{+1} \in [\underline{a}, \bar{a}]$ , so  $TV(a)$  is affine in the interval  $[\underline{a}, \bar{a}]$ . Equation (22) implies that the constant term of this affine function is the value of  $v_0$  in (55). If  $a > \bar{a}$ , the optimal policy is to be a buyer. Vice versa, if  $a < \underline{a}$ , an optimal policy is to be a seller serving the set of liquidity constrained buyers. In both cases, the strict concavity of  $U$  and convexity of  $C$  imply the strict concavity of  $TV(a)$  for

$a \notin [\underline{a}, \bar{a}]$ . In summary,  $T$  maps  $\mathcal{V}(a)$  onto itself. Therefore, the value function  $V$  satisfies (20). Finally,  $V$  is continuously differentiable because  $V$  is concave,  $\mathcal{U}$  is continuously differentiable, and the solution is interior.

### Existence of an Equilibrium Distribution of Wealth

With the assumptions of Proposition 1, there is an absorbing interval  $[\underline{a}, \bar{a}]$  where the value functions of all individuals are linear. As long as their respective wealths remain inside this interval, the individuals are willing to accept monetary gambles at the beginning of each period. Let  $\tilde{a}$  denote the wealth after engaging in such gambles. We describe below one invariant distribution of  $\tilde{a}$  which is consistent with an equilibrium as defined in the main text under the following condition:

$$z_1(2+r) + im \geq rm. \quad (60)$$

Consider a distribution of wealth in which measure  $B = [1 + \Pr(\varepsilon < \varepsilon_0)]/2$  of individuals have  $\tilde{a} = a_b \equiv \bar{a}$  and the rest have  $\tilde{a} = a_{s\varepsilon} = -z_1 \frac{1+r}{r} - \frac{\gamma-1}{\gamma r} + \bar{z} - z_\varepsilon$ , with  $\varepsilon$  being distributed according to  $F(\varepsilon)$ , and  $\bar{z}$  being the average  $z_\varepsilon$ . An individual with  $\tilde{a} = a_b$  optimally chooses to be a buyer because  $a_b \in [\underline{a}^b, \bar{a}]$ . An individual with  $\tilde{a} = a_{s\varepsilon}$  optimally chooses to be a seller serving type  $\varepsilon$  buyer because (59) and (60) imply that  $a_{s\varepsilon} \leq \bar{a}^{s\varepsilon}$  (where  $\underline{a}^b$ ,  $\bar{a}$ , and  $\bar{a}^{s\varepsilon}$  are defined in the Proof of Proposition 1). Consequently, the measures of buyers and sellers visiting all active submarkets  $\varepsilon \geq \varepsilon_0$  are identical, and the measure of unmatched buyers is  $\Pr(\varepsilon < \varepsilon_0)$ . Since all money is held by buyers ( $M = Bm^b$ ), (3), (19), (25), (26), and  $m^b = d_{\bar{\varepsilon}}$  imply that  $m(1+i) = B^{-1}$ . Using this equality, together with (19) and  $\beta = (1+r)^{-1}$ , we obtain the aggregate wealth of our proposed distribution is equal to one, which is the real value of the money supply because  $M$  is used as the deflator. Therefore, the credit market clears. At the beginning of period  $t+1$ , the budget (56), together with condition (60), implies that a buyer with shock  $\varepsilon \geq \varepsilon_0$  has initial wealth:

$$a_{+1}^{b\varepsilon} = (1+r)^2 \frac{z_1 + im}{r} - (1+r)z_\varepsilon - im(1+r) - \frac{\gamma-1}{\gamma r} \in [a_{s\varepsilon}, a_b] \subseteq [\underline{a}, \bar{a}]. \quad (61)$$

This individual is able and willing to gamble his/her wealth  $a_{+1}^{b\varepsilon}$  to end up with  $a_b$  with probability  $(a_{+1}^{b\varepsilon} - a_{s\varepsilon}) / (a_b - a_{s\varepsilon})$  and  $a_{s\varepsilon}$  with complementary probability because these gambles are fair and value functions are locally linear. Equation (56) also implies that  $a_{+1}^{b\varepsilon} = a_b$  for buyers with shocks in the interval  $[1, \varepsilon_0]$ , who do not trade. Likewise, (57) and (60) imply that all sellers have the same initial wealth in period  $t+1$ :

$$a_{+1}^{s\varepsilon} = -(1+r)^2 \frac{z_1}{r} + (1+r)\bar{z} - \frac{\gamma-1}{\gamma r} \in [a_{s\varepsilon}, a_b] \subseteq [\underline{a}, \bar{a}]. \quad (62)$$

Consequently, all these individuals are able and willing to pick any  $\varepsilon$  and gamble their initial wealth  $a_{+1}^{s\varepsilon}$  to end up with  $a_b$  with probability  $(a_{+1}^{s\varepsilon} - a_{s\varepsilon}) / (a_b - a_{s\varepsilon})$  and  $a_{s\varepsilon}$  with complementary probability. Let the number of sellers randomly picking  $\varepsilon$  be such that their average wealth is  $a_{+1}^{s\varepsilon} + r(z_\varepsilon - \bar{z})$ . This random assignment exhaust all sellers because  $\bar{z}$  is the average of  $z_\varepsilon$ . For all  $\varepsilon$ , the average probability of ending up with  $a_b$  is  $\{0.5 [a_{+1}^{b\varepsilon} + a_{+1}^{s\varepsilon} + r(z_\varepsilon - \bar{z})] - a_{s\varepsilon}\} / (a_b - a_{s\varepsilon})$ , which (61), (62), and the definition of  $a_{s\varepsilon}$ , imply to be  $1/2$ , so half of the individuals who traded in period  $t$  end up with  $\tilde{a} = a_b$  in period  $t+1$ . Therefore, the measure of individuals with  $\tilde{a} = a_b$  in period  $t+1$  is  $B \Pr(\varepsilon < \varepsilon_0) + B [1 - \Pr(\varepsilon < \varepsilon_0)] / 2 + (1 - B) / 2$ , which is equal to  $B$ . The remaining individuals end up with  $\tilde{a} = a_{s\varepsilon}$  with  $\varepsilon$  distributed as the preference shocks of buyers. Consequently, the proposed distribution of wealth is invariant, and when individuals behave optimally the credit market clears and there is identical number of buyers and sellers in all active submarkets.

The condition (60) is always satisfied if  $\gamma \geq 1$ , which implies  $i \geq r$ . Also, since  $im \geq 0$  and  $m = z_{\bar{\varepsilon}}$ , the condition is satisfied if  $z_1 \geq z_{\bar{\varepsilon}} r / (2 + r)$ , which holds if  $z_1 > 0$  and  $r \rightarrow 0$ . Therefore, non-negative inflation is a sufficient condition for existence; otherwise, one has to assume that individuals are sufficiently patient and/or the range of shocks is sufficiently narrow.

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