Party formation in single-issue politics

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Abstract

We study party formation in a general model of collective decision-making, modelling parties as agglomerations of policy positions championed by decision-makers. We show that in the presence of economies of party size and a one-dimensional policy space, players agglomerate into exactly two parties. This result does not depend on the magnitude of the economies of party size or sensitively on the nature of the individuals' preferences. Our analysis encompasses several models, including decision-making in committees with costly participation and representative democracy in which the legislature is elected by citizens, for a wide range of electoral systems including plurality voting and proportional representation. The result implies that a multiplicity of parties hinges on the presence of more than one significant political issue or of diseconomies of party size.

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1. Introduction

Many societies make collective decisions in legislative assemblies in which the policies chosen are compromises that depend on the policies championed by the legislators. In these assemblies, legislators tend to be grouped into "parties". What determines the number of parties and the positions they take?

A much-studied claim, known as *Duverger's Law*, is that the number of parties is influenced by the electoral system. Duverger (1954, Book II, Ch. I) argues that evidence suggests that "the simple-majority single-ballot system favours the two-party system" (p. 217), whereas "the simple-majority system with second ballot and proportional representation favour multi-partism" (p. 239). Over the last two decades several researchers have studied the first prediction, that plurality voting results in a two party system, in game-theoretic models of collective decision making. In these models, this prediction follows from the strategic interaction of the individuals involved in the political process.

An example is Feddersen's (1992) voting model. In the Nash equilibrium outcome of this game, voters agglomerate into two equal sized groups, interpreted as parties. Voters who care only about the outcome of the political process agglomerate into two groups for purely strategic reasons. The members of each party may have very different political preferences. Each individual is not concerned by the motivations and political preferences of her party colleagues, but rather uses her party as a strategic tool to achieve a desirable outcome. For example, an individual with moderate preferences may be a member of a party at one extreme to counterbalance a party at the opposite extreme. The key assumptions in Feddersen's model are plurality rule, the cost of participation for candidates, and the concavity of voters' preferences.

We model parties as agglomerations of positions championed by legislators. A group of legislators constitutes a party when all members support and vote for the same position ("party discipline"). We study a general model that predicts that two parties will exist for a wide range of political systems, including proportional representation and plurality voting, and for general voter preferences. Our key assumptions are that there is a single political issue (i.e. the policy space is one-dimensional) and economies of party size. An implication of our result is that the existence of multiparty systems hinges on the presence of more than one significant political issue or of diseconomies of party size.

In societies in which an electoral process generates a set of legislators, the operating cost of a party derives in part from an extra-parliamentary organization whose main role is to rally support in elections. Economies of party size may exist because costs have a fixed component. National advertising, for instance, is largely independent of the number of legislators in a party and makes up a substantial fraction of a party's operating cost. Further, large parties get disproportionate public subsidies. The fact that campaign contributions may depend on the strength of a candidate's party also contribute to economies of party size. Candidates belonging to large parties have many favors to sell and may attract more campaign contributions than candidates belonging to small parties with potentially few favors to sell. As Snyder (1990) points out, for instance, in the US Congress all committee and subcommittee chairs are held by members of a majority party; chairpersons generally control the agenda, so they potentially have disproportionately more favors to sell than members of a minority party.

We assume that the policy enacted by a legislature is the outcome of voting. Each legislator champions a position (commits to vote for it), and the policy enacted does not lose to any other under majority rule pairwise voting. We assume further that political issues can be arrayed from left to right on a one-dimensional spectrum, and that championing a position entails voting according to single-peaked preferences centered on the position, so that the policy enacted by the legislature is the median of the positions championed by legislators.[1]

In our model, exactly two parties form. These parties are of equal or almost equal size, and may be accompanied by at most two independents, whose positions are relatively moderate (Section 4). The existence of two parties does not depend on the magnitude of the economies of party size or sensitively on the nature of the individuals' preferences. The parties' positions, however, are affected by the character of costs. The analysis is straightforward, although the proofs are delicate. One application is to committee decision-making, where the main intuition is clear. In this application we fully characterize the equilibria (Section 5). We use the general result also to study a more structured two-stage model in which legislators are elected by citizens according to a mechanism that encompasses both proportional representation and majority rule (Section 6). In this model, parties commit to positions prior to the election; our two-party result is surprising, given the generality of the mechanism for electing legislators.

[1] The median is, more generally, the only subgame perfect equilibrium outcome of any "binary agenda" (a procedure in which the outcome is the result of a sequence of pairwise votes) in which the players use weakly undominated strategies (see, for example, Miller 1995, Section 6.3).

Some history

The evolution of Britain's two-party system motivates our model well.[2] Prior to the *Great Reform Act* of 1832, the English parliament was partitioned into two loosely knit groups, the reformist Whigs and the conservative Tories; little extra-parliamentary party machinery existed. Most bills were local or personal, and party discipline was minimal.

In the unreformed electoral system of Hanoverian England, a member of the élite faced an essentially fixed personal cost to joining parliament. The cost structure changed with the reform acts of 1832, 1867, and 1884, which lead to the evolution of a solidly two-party system—initially Conservative and Liberal, later Conservative and Labour. These reform acts afforded changes that played an important part in the development of modern political parties. The reforms were, broadly, three-fold: the expansion of the franchise;[3] the annual compilation of a voter registry; and the adoption of simple plurality rule within the electoral process.[4]

For our purposes, the most relevant consequence of these reforms is that they introduced significant economies of scale and meant that parliamentarians could reduce their costs by supporting a party. First, the large increase in the size of the electorate and the introduction of plurality rule meant that a candidate had to rally support to get into parliament. Second, the introduction of a voter registry introduced clear administrative costs to rallying support—costs that could be shared within a party. Indeed, by the 1880's "most of the energies of the party agents and the bulk of party election funds were devoted to filling the electoral register with one set of supporters and stripping the same register of the opposing voters through the Registration Courts" (Ball 1987, p. 20; see also p. 26).[5]

This evolution of a political party system was intra-parliamentary in the sense that existing members of parliament grouped themselves into parties. By contrast, the later emergence of the British Labour Party was extra-parliamentary: it resulted mainly from the work of grass-root activists from the working class who were enfranchised by the 1918 Reform Act. Our model contributes to the explanation of both types of party formation.

[2]Our historical account draws on O'Gorman (1989) and Ball (1987). Cox (1987b) provides an alternative analysis.

[3]The 1832 Act increased the electorate by almost 50%, that of 1867 increased it by 38% in counties and 13% in boroughs, that of 1884 increased it by 66%, and that of 1918 (which enfranchised all women over 30) increased it by around 50% (Ball 1987, pp. 18, 24).

[4]In 1866, prior to the reforms of 1867, half the members of the House of Commons were elected by one fifth of the electorate (Ball 1987, p. 21), so that these members had to do little work in terms of rallying support to get into parliament.

[5]The cost of registering supporters and opposing the registration of opponents' supporters was associated with the complex eligibility requirements. Political parties litigated against the inclusion of their opponents' supporters and in defense of the inclusion of their own supporters.

Framework

We study a model of collective decision-making in which each of a finite number of players chooses whether or not to participate in the decision-making process, and if so which policy to champion. We assume that the policy space is completely ordered (e.g. there is a single issue, or many issues ordered lexicographically), and that the outcome is the median of the policies championed by the participants.

We make two basic assumptions about the players' payoff functions, the first of which generalizes the idea that participation is costly.

C (*costly participation*) *If a participant's switching to nonparticipation does not change the outcome, then her payoff increases.*

Our second basic assumption is that there are economies of party size. We study three versions of this assumption, the most straightforward of which is tailored to our model of committee decision-making.

E (*economies of party size*) *If a participant's switching to a larger party does not change the median championed policy, then her payoff increases.*

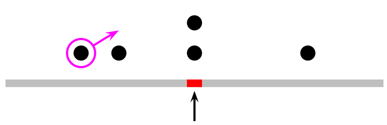

Figure 1. An example in which a player's moving to a larger party does not affect the outcome.

Figure 1 gives an example in which a player's switching to a larger party does not affect the median championed policy. The horizontal line is the policy space; each dot represents a player, its x-coordinate being the policy championed. The vertical arrow points to the median policy championed, which does not change if the player championing the policy on the far left joins forces with the center-left player. Under assumption **E**, this move increases the payoff of the player on the far left.

We study two applications of our basic model. In the first application, a decision is to be made by a committee. Each player may participate and champion a policy, or not participate; championing x entails a cost that declines with the total number of players championing x. This model satisfies assumptions **C** and **E**.

In our second application, candidates first choose positions; then citizens cast votes for the candidates, and the resulting legislature chooses a policy. Taking the candidates to be the players in our basic model, condition **E** may not be satisfied: a change in a candidate's position that does not affect the median *championed* policy may change the outcome because it changes the voters' strategic incentives and thus the voting equilibrium, resulting in a change in the set of candidates elected.

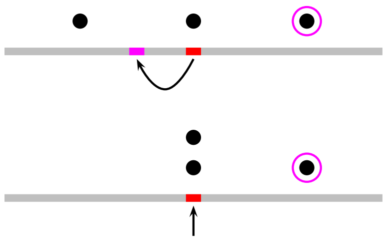

Figure 2. An example in which voters' strategic incentives change when a candidate's position changes.

In the top panel of Figure 2, citizens with favorite policies on the right have an incentive to vote for the right (circled) candidate, because if this candidate, in addition to the other two, is elected, then the policy outcome is closer to their favorite positions than is the outcome if this candidate is not elected. If, however, the left candidate moves to the center to form a two-member party, as in the bottom panel, the outcome is unaffected by the election of the right candidate, so that citizens on the right have no incentive to vote for that candidate.

The following variant of condition **E** accommodates this case.

PE (*participation dependent economies of party size*) *If a participant's switching to a larger party (a) does not change the median championed policy and (b) would not change the median championed policy if the action of any given other player were to be fixed at nonparticipation, then her payoff increases.*

In our two-stage game, condition (b) ensures that a candidate's move does not affect any citizen's incentive to vote. The left party's move to the center in the example in Figure 2 does not satisfy it, because the move would change the outcome were the rightist not to participate.

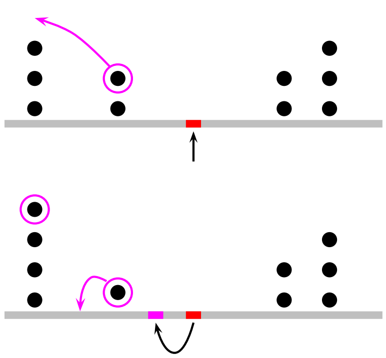

Figure 3. An example in which a candidate's incentives change when another candidate changes positions.

Condition **PE** is appropriate for a two-stage model in which each candidate is committed to the policy she champions. In a model in which a legislator can renege on a commitment to champion a policy, however, we need a version of the condition that ensures that each legislator's incentive to maintain her position remains unchanged when another candidate changes position.

Consider the example in Figure 3, in which we take the median of an even number of candidates' positions to be the average of the middle two positions. A move to the left party by a member (say i) of the center-left party satisfies both (a) and (b) of **PE**.

After i's move, however, the remaining member of the center-left party (say j) can, by changing her position, affect the median position in ways that she could not originally (assuming she remains elected). Specifically, after i's move, j's moving to the left moves the median to the left (assuming j remains elected), whereas prior to i's move, no move of j results in such a change in the median. Thus, depending on j's preferences, i's move may give j an incentive to renege on j's commitment to champion the position of the center-left party. Consequently, i should be concerned that her move to the left will precipitate an undesirable outcome-changing move to the left by her ex-comrade. This consideration leads us to study the following condition, which is weaker than **PE**.

SE (*strategy dependent economies of party size*) *If a participant's switching to a larger party (a) does not change the median championed policy and (b) does not allow any other player, by changing her action, to induce outcomes that she could not induce before the participant's move, then her payoff increases.*

The move from the center-left party to the left party by player i that we considered in Figure 3 does not satisfy (b) because after the move, the other member of the center-left party can, by moving her position to the left, move the median championed policy to the left, a change she cannot induce before i's move.

Results

A key finding is that if participation is costly (**C**) and the game has participation dependent economies of party size (**PE**), then exactly *two* parties form, and these parties are of equal or almost equal size. In addition, up to three "independents" participate. If the game satisfies the stronger condition of economies of party size (**E**), then at most two independents participate. When participation is costly (**C**) and the game has strategy dependent economies of party size (**SE**), then exactly two parties with more than two members form, and these parties are of equal or almost equal size. In addition, up to three small parties or independents participate (see Figure 4).

This two-party result is very general. It does not depend on the magnitude of the economies of party size or sensitively on the nature of the individuals' payoffs beyond conditions **C** and **E**, **PE**, or **SE**. Thus the results in our applications do not depend sensitively on the nature of the individuals' preferences over policies; for example, we do not assume these preferences are single-peaked.

The idea behind the result is straightforward. In deciding whether to participate in a party or to champion a position as an individual, a player potentially faces a tradeoff. Joining a party saves her some cost, but may force her to compromise her position. However, if the policy outcome is the median of the legislators' proposed policies, then joining a party whose

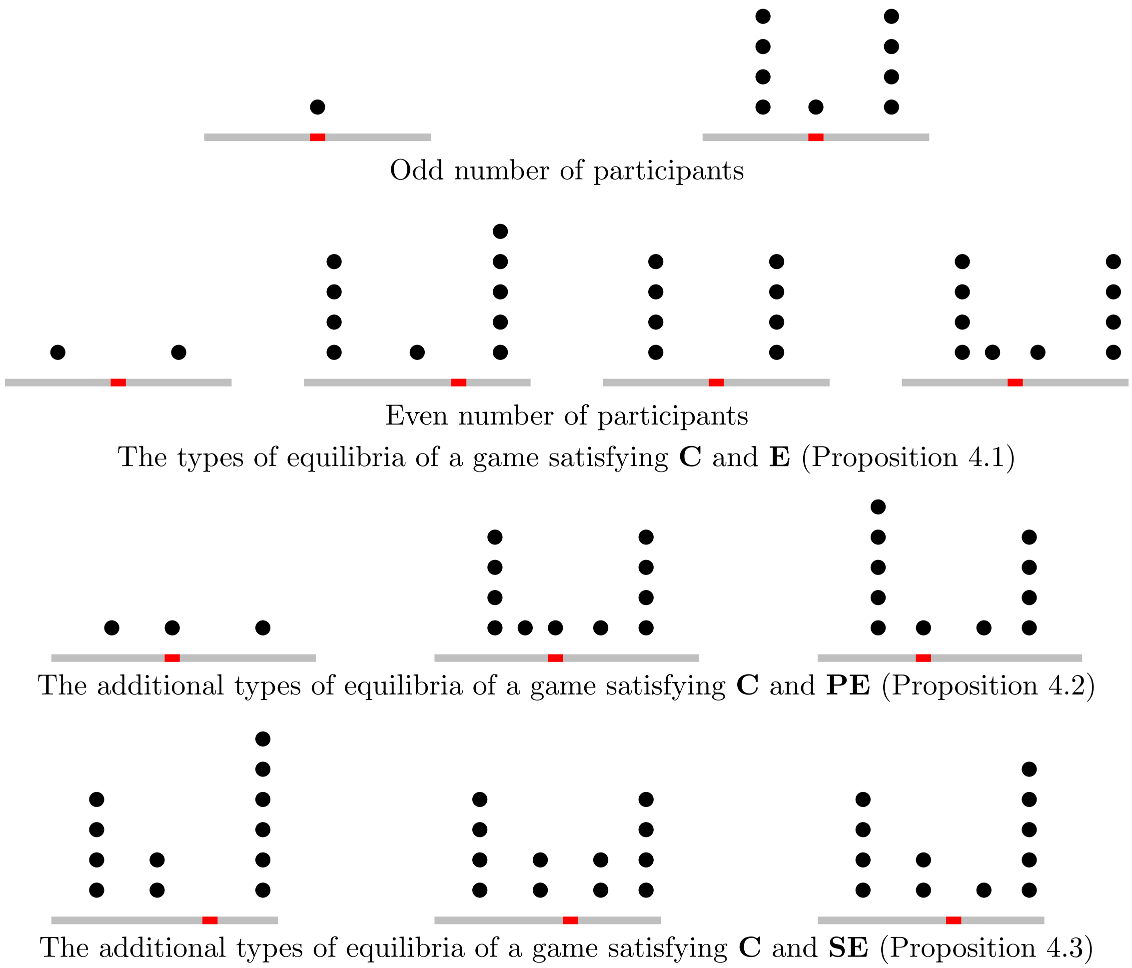

Figure 4. The equilibria of a game satisfying **C** and **E**, **PE**, or **SE**. Agglomerations of four or more players are parties, which may contain any number of players.

position is on the same side of the median as is her favorite position is just as effective in determining the outcome as is championing her own favorite position. Thus in fact the player faces no tradeoff. A left-leaning legislator is better off joining a left-leaning party than acting as an independent, regardless of the size of the cost saving, and a right-leaning legislator is better off joining a right-leaning party.

When we impose more structure on the players' preferences, we can say more about the equilibria. Specifically, assume that the payoff of each player i to any action profile a takes the form $v(M(a) - x_i) - c(k(a_i))$ if she participates and $v(M(a) - x_i)$ if she does not, where v is concave, $v(z) = v(-z)$ for all z, $M(a)$ is the median of the positions championed by participants, $k(a_i)$ is the number of participants championing a_i, and c is positive and decreasing. (The policy x_i is i's favorite position.) Assume further that when the number of participants is even, $M(a)$ is the lottery that assigns probability $\frac{1}{2}$ to each of the two central positions. We show that in every equilibrium in which there are two parties, with positions say x and $y > x$, and no independents, there are positions $\hat{x}$ and $\hat{y} > \hat{x}$ such that the members of the party with position x are exactly the players whose favorite positions are less than $\hat{x}$ and the members of the party with position y are exactly the players whose favorite positions are greater than $\hat{y}$. Interestingly, we may have $x > \hat{x}$ and $y < \hat{y}$: every participant's favorite position may be more extreme than the position of the party to which she belongs. (The positions have this property if the parties are relatively small and the cost of running a small party is not too high.)

In the application of our results to a two-stage game in which players decide whether to become candidates and then citizens cast votes that determine the composition of a legislature (Section 6), we have to confront the possibility that a change in a candidate's position affects the voting equilibrium in the subsequent subgame. We have argued that this possibility means that the induced game between the candidates does not satisfy **E**. For it to satisfy the weaker condition **PE** we need to restrict the way in which the voting equilibrium changes when a candidate's position changes. We assume that citizens do not change their votes without good reason: if a voting equilibrium b remains an equilibrium after a deviation, the outcome in the subgame following the deviation is b, even if other equilibria also exist. Under this assumption we show that the induced game between the candidates satisfies **C** and **PE**, so that in every equilibrium the candidates are grouped into two parties and/or up to three independents.

Relation with literature

Both our model of committee voting and Feddersen's (1992) voting model seek to explain two-party competition. The models differ in their explanatory variables. Feddersen's model emphasizes the role of voting decisions, while our model emphasizes the role of legislators. Feddersen assumes that the outcome is the policy that obtains the most votes, an assumption that captures well a single-district plurality rule election. By contrast, the median rule in our model is an appropriate model of legislative compromise.

Feddersen's first-past-the-post outcome function leads immediately to the conclusion that parties form in equilibrium. (Only a party can win an election.) His two-party result hinges also on this outcome function; it depends in addition on an assumption about the distribution of voters' preferences (see his Proposition 4). Our result does not depend on any assumption about the nature of preferences; it is driven by our assumptions of economies of party size and the existence of a single issue.

Gerber and Ortuño-Ortín (1998) study a model related to Feddersen's, with a continuum of voters. They assume a continuous outcome function that weights parties by their sizes and gives proportionally larger weight to large parties. (Such a function is not consistent with a winner-takes-all electoral rule.) They show that a unique strong Nash equilibrium exists, in which there are two parties.

The model of Osborne, Rosenthal, and Turner (2000) is related to our example of committee decision-making with costly participation. The key respect in which their model differs from our application to committee voting lies in their assumption that each player's participation cost is independent of the other participants' actions. Under this assumption, a player's strategy of proposing her favorite policy weakly dominates her other strategies, so that there is no cost-based incentive for individuals with different preferences to form parties. In the equilibria of their model, moderates do not participate and participation in large populations is low. The equilibria of our model, where the participation cost declines with the number of individuals proposing a given policy, do not necessarily have these characteristics. In particular, while equilibria may exist in which some moderates do not participate, there also exist equilibria in which all individuals participate, even in large populations. Further, in equilibrium two parties form.

Several other models generate parties from principles different from those that drive the application of our model to the electoral process. Morelli (2003), for example, studies a model of party formation as an extensive game in which two potential candidates with extreme preferences propose compromise positions to a moderate potential candidate; subsequently each potential candidate chooses whether to stand, an election is held, and the outcome is the median of the elected politicians' positions. He finds that the number of parties depends on the electoral system and the distribution of preferences. The primary role of parties in his model is that they coordinate citizens' votes in an election.

Baron (1993) studies a model of proportional representation within the Hotelling–Downs framework. Citizens are not strategic, party formation is not costly, and the number of parties is fixed. Party size is determined by the fact that a large party has a diverse, and thus harder to please, membership, whereas such a party is more likely to be part of the government and be able to implement a policy appealing to its members.

Rivière (2001) studies a variant of the citizen-candidate model with sincere voting in which the role of a party is to share the cost of candidacy and inhibit competition between candidates with similar preferences. Jackson and Moselle (2001) study a model of legislative bargaining in which legislators can benefit from forming parties that bind their members to cooperate with each other in the bargaining. In Snyder and Ting's (2002) model, parties are "brands" to imperfectly informed voters, aggregating ideologically similar candidates. Levy (2004) models the idea that political parties increase the ability of candidates to commit to policy positions. She finds that this increased ability affects party formation only when policies are multidimensional.

The first part of Duverger's Law, predicting two parties under a plurality system, is given theoretical support by Cox (1987a) and Palfrey (1989) in a model in which strategic voters elect a single representative. They formalize the idea that votes for candidates with little chance of winning are wasted, resulting in equilibria in which there are two candidates. Feddersen's model, discussed above, also lends the first part of the claim theoretical support, as do the "citizen-candidate" models of Osborne and Slivinski (1996) and Besley and Coate (1997).

2. Model

Each member of a group $I = \{1, 2, \ldots, n\}$ of people chooses whether to champion a *policy*, and if so, which one. A policy is a member of the nonempty set X; the action of non-participation is denoted θ. A person who champions a policy is called a *participant*. Each person i's payoff function is $u_i\colon (X \cup \{\theta\})^n \to \mathbb{R}$.

If two or more people champion the same policy $x \in X$, we say that x is a *party*; if a single person champions x, we say that x is an *independent*. We call a party with more than two members a *large* party and one with exactly two members a *small* party.

The policy outcome is a compromise among the policies championed by participants: a *compromise function* $M : (X \cup \{\theta\})^n \to X$ assigns to each action profile an *outcome*, with $M(\theta, \ldots, \theta)$ (the outcome if no one participates) equal to some given default policy. We assume throughout that the compromise function M is a variant of the median; we discuss it in Section 3.

In summary, we study a strategic game in which the set of players is $I = \{1, \ldots, n\}$, the set of actions of each player is $X \cup \{\theta\}$, and the payoff function of each player i is $u_i : (X \cup \{\theta\})^n \to \mathbb{R}$. The relation between the actions and payoffs depends on the compromise function M, and we denote the game $\Gamma(I, (X \cup \{\theta\})_{i \in I}, (u_i)_{i \in I}, M)$; when there is no possibility for confusion we use the abbreviation $\Gamma(M)$.

We say that the game $\Gamma(M)$ has costly participation if a player prefers to withdraw if her withdrawal does not change the outcome.

Definition 2.1. For an action profile $a \in (X \cup \{\theta\})^n$ the game $\Gamma(M)$ has

C *costly participation at a* if $u_i(\theta, a_{-i}) > u_i(a)$ whenever $a_i \in X$ and $M(a) = M(\theta, a_{-i})$ for any player i.

A game that has costly participation for every action profile has *costly participation*.

Our assumptions about economies of party size are explained in the *Framework* section of the Introduction. Here, and subsequently, we use $\#Z$ to denote the number of members of the set Z.

Definition 2.2. For an action profile $a \in (X \cup \{\theta\})^n$ the game $\Gamma(M)$ has

E *economies of party size at a* if $u_i(x, a_{-i}) > u_i(a)$ whenever $a_i \in X$, $M(a) = M(x, a_{-i})$, and $\#\{j \in I : a_j = a_i\} \leq \#\{j \in I \setminus \{i\} : a_j = x\}$

PE *participation dependent economies of party size at a* if $u_i(x, a_{-i}) > u_i(a)$ whenever $a_i \in X$, $M(a) = M(x, a_{-i})$, $M(\theta, a_{-h}) = M(\theta, (x, a_{-i})_{-h})$ for each player h, and $\#\{j \in I : a_j = a_i\} \leq \#\{j \in I \setminus \{i\} : a_j = x\}$

SE *strategy dependent economies of party size* at a if $u_i(x, a_{-i}) > u_i(a)$ whenever $a_i \in X$, $M(a) = M(x, a_{-i})$, for any participating player h and any $y \in X \cup \{\theta\}$ there exists $z \in X \cup \{\theta\}$ such that $M(z, a_{-h}) = M(y, (x, a_{-i})_{-h})$, and $\#\{j \in I : a_j = a_i\} \leq \#\{j \in I \setminus \{i\} : a_j = x\}$.

We say that a game that satisfies **E** for every action profile has *economies of party size*, one that satisfies **PE** for every action profile has *participation dependent economies of party size*, and one that satisfies **SE** for every action profile has *strategy dependent economies of party size*.

Examples

Example 2.1 (Committee voting). Each member of a committee of n people chooses whether to champion a policy in X or not to participate. Each person i obtains the payoff $v_i(x)$ from the policy x, where $v_i : X \to \mathbb{R}$; we refer to v_i as i's *valuation function*. If person i participates and champions x she bears the cost $c_i(k(x))$, where $k(x)$ is the number of people championing x and $c_i : \mathbb{N} \to \mathbb{R}$ is positive and decreasing.

For a given compromise function M, we may model this situation as the game $\Gamma(I, (X \cup \{\theta\})_{i \in I}, (u_i)_{i \in I}, M)$ in which

$$u_i(a) = \begin{cases} v_i(M(a)) & \text{if } a_i = \theta \\ v_i(M(a)) - c_i(\#\{j \in I : a_j = a_i\}) & \text{if } a_i \in X. \end{cases}$$

This game satisfies conditions **C** and **E** at every action profile: participation is costly and the game has economies of party size.

We may generalize this example by allowing each person's cost of championing a policy to depend on the policies championed by others, not simply on the number of other people championing it. This generalization captures the idea that arguing the case for a policy may be more difficult if it is extreme, or if some other particularly attractive position is being championed. The resulting game also satisfies **C** and **E**.

Example 2.2 (Elections without commitment). Consider a two-stage game in which the players consist of both citizens and candidates for a legislature. In the first stage each citizen votes for a candidate; these decisions are made simultaneously. The set of winners of the election is determined by an arbitrary rule. (For example, the set of winners may consist of the top k vote-getters, with ties broken randomly.) In the second stage, each legislator chooses whether to actively participate, incurring an effort cost, and if so which policy to champion. (A politician cannot commit to a policy prior to the election.) We assume that the effort cost displays economies of party size. As before, the policy outcome is the median of the policies championed by the legislators. Both and citizens and politicians care about this policy outcome. (In particular, the politicians are "ideological".)

More precisely, the players in the subgame following any first-period history are the elected legislators, and the subgame has the form of the game in Example 2.1. Thus each subgame satisfies **C** and **E**. Implicit in the game is the assumption that the electorate knows the preferences of each politician and thus can anticipate the behavior of the elected legislators in each subgame.

Proposition 4.2 below implies that in every subgame the politicians form two parties. (Thus under plurality rule with ties broken randomly, every realization of the legislature contains two parties.)

Example 2.3 (Elections with commitment). Now consider a model that allows a richer interaction between voters and candidates, and permits candidates who are not ideological.

The electoral process has two stages.

- First, each potential legislator chooses whether to become a candidate, and if so the policy she *commits* to champion.

- Second, each citizen chooses whether to vote for a candidate, and if so, which one.

An electoral rule determines the members of the legislature. The policy outcome is the median of the policies championed by the elected candidates.

Each citizen cares about the policy outcome and incurs a fixed cost if she votes. Each potential legislator incurs a cost if she participates as a candidate and may or may not care about the policy outcome. If elected she obtains a "prize" that compensates her for the costs; this prize depends on the number of elected legislators who propose the same policy (i.e. on the size of her party after the election). This prize could reflect public subsidies to parties or a lower cost of running for a member of a larger party.

We study this game in detail in Section 6. Fix a subgame perfect equilibrium and consider the strategic game in which the players are the potential legislators and the payoff of each legislator to the action profile a of policy commitments is her equilibrium payoff in the subgame following a. We show in Proposition 6.1 that this strategic game satisfies **C** (costly participation) and **PE** (participation dependent economies of party size) under a natural refinement of subgame perfection that we call *subgame persistence*.

We consider a modification of the game in which any elected legislator may change her position after the election—i.e. may renege on her policy commitment. We propose an alterative refinement of subgame perfection, which takes into account *incentive compatibility* requirements in subgames, and show that the associated strategic game between the potential legislators satisfies **C** and **SE** (Proposition 6.3).

Example 2.4 (Proportional representation with party lists). The previous example can be specialized to proportional representation based on party lists (the electoral system in Austria, Belgium, Denmark, Finland, the Netherlands, Norway, Sweden, and Switzerland). Candidates proposing the same position are ordered into a party-list and each citizen votes for one of these lists. Each list is awarded a number of seats in proportion to the votes received. Each action profile in the two-stage game of the previous example corresponds to an action profile in this party-list model, with the same payoffs, and vice versa. Thus the results for the previous example apply also to this example.

3. The compromise

We assume that the set X of policies is "completely ordered". That is, the members of X are related by an ordering, denoted $\leq$,[6] which we interpret as embodying the policies' positions on a left–right political spectrum. That is, if $x < y$ then all players agree that x is *more left than* y, and if $x \leq y$ then they agree that x is *at least as left as* y. The members of X

[6]That is, $\leq$ is transitive, complete (either $x \leq y$ or $y \leq x$ for any two policies $x \in X$ and $y \in X$), reflexive ($x \leq x$), and antisymmetric ($x \leq y$ and $y \leq x$, implies $x = y$).

may be numbers, in which case $\leq$ may take its usual meaning, but they may alternatively be points in a higher-dimensional space, as long as they may be ordered.

We assume that the compromise is the median of the policies championed by participants. When the number of participants is odd, the median is the middle proposed policy; when no player participates, it a *default outcome* $d \in X$. When the number of participants is even, we generally take the median to be a policy between the two middle positions; for our committee example (Section 5) we assume instead, for convenience, that the median is a 50–50 lottery between the two middle positions.

Precisely, for any action profile a, order the participants (the players i for whom $a_i \in X$) so that $a_1 \leq \cdots \leq a_k$. If k is odd, the *median* of a is $m(a) = a_{(k+1)/2}$. If k is positive and even, the *left median* of a is $m^l(a) = a_{k/2}$ and the *right median* of a is $m^r(a) = a_{k/2+1}$. We assume that the outcome $M(a)$ is given by

$$M(a) = \begin{cases} d & \text{if } a = (\theta, \ldots, \theta) \\ m(a) & \text{if } \#\{i \in I : a_i \in X\} \text{ is odd} \\ S(m^l(a), m^r(a)) & \text{if } \#\{i \in I : a_i \in X\} \text{ is positive and even}. \end{cases}$$

Unless otherwise stated, we assume that for any $x \in X$ and $y \in X$ we have $S(x, y) \in X$, with $S(x, y) = S(y, x)$, and $x \leq S(x, y) \leq y$ if $x \leq y$. In Section 5 we assume instead that $S(x, y)$ is the lottery that assigns probability $\frac{1}{2}$ to x and probability $\frac{1}{2}$ to y.

The median championed policy is not beaten by any policy in pairwise voting when each participant's preferences are single-peaked (relative to the ordering of X) with a peak at the policy she champions. The understanding underlying our outcome function is that a player's championing a policy x entails her committing to vote according to a single-peaked preference that peaks at x.

4. Properties of equilibrium

We study the properties of a (pure strategy) Nash equilibrium of a game satisfying our conditions.

Proposition 4.1. *At any Nash equilibrium in which participation is positive and* ***C*** *and* ***E*** *are satisfied, one of the following conditions holds.*

a. The number of participants is odd and there is either a single independent or two equal-sized parties between which there is an independent.

b. The number of participants is even, and there are (i) two equal-sized parties and no independents, (ii) two independents, (iii) two equal-sized parties between which there are two independents, or (iv) two parties, one larger by one member than the other, between which there is a single independent.

This result and all subsequent ones are proved in the appendix. The proof rests on two main ideas. First, if there are two parties on one side of the median, then any member of the smaller party (or of either party, if the sizes are the same) can switch to the other party without affecting the outcome. Such a move is profitable by condition **E**. Second, if a single position is proposed and more than one player proposes it, then no player's withdrawal affects the outcome. Therefore, by condition **C** some participating player can profitably withdraw.

When the weaker condition **PE** is satisfied, three further types of equilibria may occur.

Proposition 4.2. *Any Nash equilibrium in which participation is positive and* ***C*** *and* ***PE*** *are satisfied either takes one of the forms given in Proposition 4.1 or has an odd number of participants and (i) three independents, (ii) two equal-sized parties and three independents holding positions between the parties, or (iii) two parties, one larger by one member, and two independents holding positions between the parties.*

Under the even weaker condition **SE** we get only an additional three new types of equilibria.

Proposition 4.3. *Any Nash equilibrium in which participation is positive and* ***C*** *and* ***SE*** *are satisfied either takes one of the forms given in Proposition 4.2 or has an even number of participants and (i) two large parties of equal size and two small parties holding positions between the large parties, (ii) two large parties, one larger than the other by two members, and one small party holding a position between the large parties, or (iii) two large parties, one larger than the other by one member, and a small party and an independent holding positions between the large parties with the independent holding a position closer to the largest party.*

5. A characterization of equilibrium in committee voting

For the game of committee voting in Example 2.1 in which the players' valuation functions are symmetric and strictly concave and their cost functions are all the same, we may fully characterize the equilibria in which there are two parties and no independents, which we refer to as *two-party equilibria*. We do so under the assumptions that no two players have the same favorite position and that when the number of participants is even the outcome is the lottery that assigns probability $\frac{1}{2}$ to each median.

Precisely, we make the following assumption.

A The policy space X is the real line $\mathbb{R}$, each player i has a favorite position x_i, and no two players have the same favorite position. The valuation function of each player i is given by $v_i(x) = v(x - x_i)$, where $v : \mathbb{R} \to \mathbb{R}$ is decreasing and strictly concave on $\mathbb{R}_+$, $v(z) = v(-z)$ for all z, each player's cost function c_i is the same (denoted c), and for all $x \in X$ and $y \in X$ with $x < y$, the outcome $S(x, y)$ (the compromise when the left median is x and the right median is y) is the lottery that assigns probability $\frac{1}{2}$ to x and probability $\frac{1}{2}$ to y.

Under this assumption, a two-party equilibrium may be constructed as follows (refer to Figure 5). Number the players in order of their favorite positions: $x_1 < x_2 < \cdots < x_n$. Choose an integer k with $2 \le k \le \frac{1}{2}n$. We construct an action profile a^* in which there are two parties, each with k members. The parties' positions are symmetric with respect to x_k and x_{n-k+1}; we denote them $x_k + t^*$ and $x_{n-k+1} - t^*$. Players $1, \ldots, k$ belong to the party with position $x_k + t^*$ and players $n - k + 1, \ldots, n$ belong to the party with position $x_{n-k+1} - t^*$.

The amount by which player k's expected payoff exceeds her payoff when she switches to nonparticipation is $\frac{1}{2}[v(t) + v((x_{n-k+1} - t) - x_k)] - c(k) - v((x_{n-k+1} - t) - x_k)$, or $\frac{1}{2}[v(t) - v(x_{n-k+1} - x_k - t)] - c(k)$. Define the function g on $[-\infty, \frac{1}{2}(x_{n-k+1} - x_k))$ by $g(t) = \frac{1}{2}[v(t) - v(x_{n-k+1} - x_k - t)]$. We have $g(\frac{1}{2}(x_{n-k+1} - x_k)) = 0$; the strict concavity of v implies that g is decreasing. Thus for any value of $c(k)$ there exists a value t^* of t such that player k is indifferent between remaining in the party with position $x_k + t^*$ and switching to nonparticipation. Given the symmetry of v and of the parties' positions, for the same value t^* player $n - k + 1$ is indifferent between remaining in the party with position $x_{n-k+1} - t^*$ and switching to nonparticipation. Let $x = x_k + t^*$ and $y = x_{n-k+1} - t^*$. (Note that $x < y$.) The strict concavity of v implies that players $1, \ldots, k - 1$ prefer to remain in party x than to switch to nonparticipation, and players $n - k, \ldots, n$ prefer to remain in party y than to switch to nonparticipation.

For the action profile a^* to be a Nash equilibrium, players $k + 1, \ldots, n - k$, who do not participate, must be no better off participating in either of the parties. By an argument similar to the one in the previous paragraph, this condition is equivalent to the conditions

$$\tfrac{1}{2}[v(x - x_{k+1}) - v(y - x_{k+1})] \le c(k + 1) \tag{1}$$

$$\tfrac{1}{2}[v(x_{n-k} - y) - v(x_{n-k} - x)] \le c(k + 1). \tag{2}$$

If, in addition, no player is better off becoming an independent, a^* is a Nash equilibrium. This condition is satisfied if $c(1)$ is large enough.

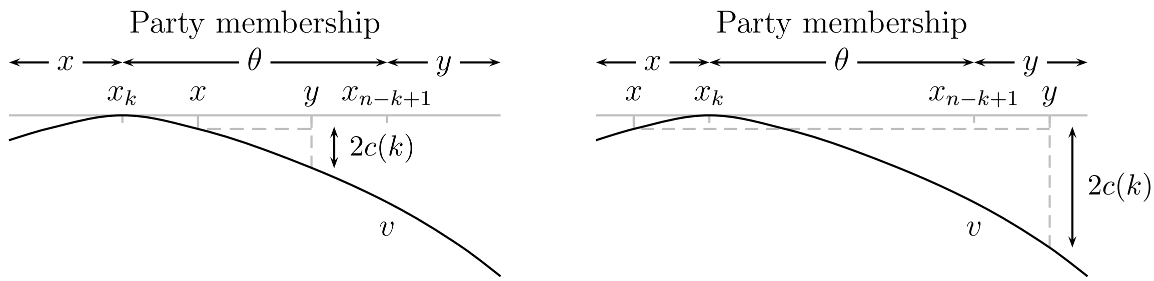

Figure 5. The determination of the party positions x and y, given k, for two cost functions. (We have $x = x_k + t^*$ and $y = x_{n-k+1} - t^*$; $t^* > 0$ in the left panel and $t^* < 0$ in the right panel.)

In this construction, players k and $n-k+1$ are indifferent between remaining in their parties and switching to nonparticipation. This indifference is not necessary for an equilibrium, which requires only that these players not be better off by switching to nonparticipation. Equivalently,

$$\tfrac{1}{2}[v(x - x_k) - v(y - x_k)] \geq c(k) \tag{3}$$

$$\tfrac{1}{2}[v(x_{n-k+1} - y) - v(x_{n-k+1} - y)] \geq c(k). \tag{4}$$

We may argue also that any equilibrium takes the form of a^*. The key point is that the concavity of v implies that if, in a two-party equilibrium with parties x and $y > x$, player i belongs to party x then every player j with $j < i$ also belongs to x.

A precise result follows.

Proposition 5.1. *Assume **A**. The action profile a is a two-party Nash equilibrium if and only if for some integer k with $2 \leq k \leq n/2$ we have*

$$a_i = \begin{cases} x & \text{if } i = 1, \ldots, k \\ \theta & \text{if } i = k+1, \ldots, k-n \\ y & \text{if } i = k-n+1, \ldots, n, \end{cases}$$

x, y, and k satisfy (1), (2), (3), and (4), $c(1) - c(k) \geq \max\{-\tfrac{1}{2}v(x_k - x), -\tfrac{1}{2}v(y - x_{n-k+1})\}$, and $c(1) \geq -\tfrac{1}{2}[v(x_i - x) + v(y - x_i)]$ for $i = k+1$ and $i = n-k$.

Typically, equilibria exist with various values of the common party size k: a range of party sizes is compatible with equilibrium. In equilibria in which k is small, the parties' positions are relatively far apart, as in the right panel of Figure 5. Note that if $c(k)$ is small enough (as in the left panel of the figure), in some equilibria each party's position is more moderate than the favorite positions of all the party's members.

6. Elections with commitment

In Example 2.3, each of a finite number n of *potential legislators* may vie for a place in a legislative assembly. Candidates are elected to the assembly by a population of *citizens*, modelled as a finite positive atomless measure space $(\Omega, \mathcal{F}, \mu)$. (We adopt this formulation to avoid integer problems.) The electoral process has two stages. First, each potential legislator chooses whether to become a candidate, and if so which policy (member of X) to champion. Second, each citizen chooses whether to vote for a candidate, and if so, which one.

In the subgame following the potential legislators' action profile $a \in (X \cup \{\theta\})^n$, a *voting profile* for the citizens is a measurable function

$$b^a : \Omega \to \{i \in I : a_i \in X\} \cup \{\theta\} ,$$

where $b^a(\omega) = \theta$ means that citizen ω does not vote and $b^a(\omega) = i$ means she votes for candidate $i \in I$ (who proposes the policy $a_i \in X$).

Electoral rule An *electoral rule* translates the voting profile into a subset of the candidates (those who are elected). We work here with rules that elect a potential legislator if and only if the number of votes she obtains is at least equal to some "quota" of votes. This quota is given by a *quota function* $Q\colon \mathbb{R}^n_+ \to \mathbb{R}_+$ that is continuous, nondecreasing ($x \geq y$ implies $Q(x) \geq Q(y)$), *anonymous* in the sense that for any one-to-one function $\lambda : I \to I$ we have $Q(\alpha_1, \ldots, \alpha_n) = Q(\alpha_{\lambda(1)}, \ldots, \alpha_{\lambda(n)})$, and vanishes, if at all, only at zero ($Q(x) = 0$ implies $x = 0$). Given a profile a of policies championed by the potential legislators and a voting profile b^a, $(b^a)^{-1}(j)$ is the set of citizens who vote for candidate j and $\mu((b^a)^{-1}(j))$ is the size of this set. Thus legislator i is *elected* if

$$\mu((b^a)^{-1}(i)) \geq Q(\mu((b^a)^{-1}(1)), \ldots, \mu((b^a)^{-1}(n))) .$$

The following three examples of quota functions are continuous, nondecreasing, and anonymous. A potential legislator is elected if and only if she gets at least as many votes as

i. (*first-past-the-post*) every other candidate: $Q(\alpha_1, \ldots, \alpha_n) = \max_{i \in I} \alpha_i$

ii. (*Hare Quota*) the total number of votes divided by $\overline{k}$, the number of seats in the legislature: $Q(\alpha_1, \ldots, \alpha_n) = \left(\sum_{i=1}^n \alpha_i\right) / \overline{k}$

iii. (*fixed quota*) a fixed number (which might be related to the size of the population): $Q(\alpha) = \delta$ for all $\alpha \in \mathbb{R}^n$.

Payoffs The policy championed by legislator i is

$$A_i(a, b^a) = \begin{cases} a_i & \text{if } i \text{ is elected} \\ \theta & \text{otherwise}. \end{cases}$$

Each citizen $\omega \in \Omega$ attaches the value $v(\omega, x)$ to the policy $x \in X$, where $v : \Omega \times X \to \mathbb{R}$. We refer to $v(\omega, \cdot)$ as ω's *valuation function.* We assume that the function $v(\cdot, x)$ is integrable for any x.

A citizen who votes incurs the cost $C_z > 0$. The payoff $u(\omega, (a, b^a))$ of citizen ω depends on the outcome and whether she votes:

$$u(\omega, (a, b^a)) = \begin{cases} v(\omega, M(A(a, b^a))) - C_z & \text{if } b^a(\omega) \neq \theta \\ v(\omega, M(A(a, b^a))) & \text{if } b^a(\omega) = \theta. \end{cases}$$

Each potential legislator i incurs the cost $C_\ell > 0$ if she participates as a candidate and has a *valuation function* $v_i : X \to \mathbb{R}$ over policies. If elected she obtains a "prize" that depends on the number of elected legislators that propose the same policy (i.e. on the size of her party after the election). Specifically, the payoff of potential legislator i is

$$u_i(a, b^a) = \begin{cases} v_i(M(A(a, b^a))) & \text{if } a_i = \theta \\ v_i(M(A(a, b^a))) - C_\ell & \text{if } a_i \neq \theta \ \ \& \ i \text{ not elected} \\ v_i(M(A(a, b^a))) - C_\ell + P_i(a, b^a) & \text{if } a_i \neq \theta \ \ \& \ i \text{ is elected}, \end{cases}$$

where

$$P_i(a, b^a) = p\,(\#\{j \in I : A_j(a, b^a) = a_i\})$$

and $p : \mathbb{N} \to \mathbb{R}_{++}$ is a positive increasing function, called the *prize function.*

Equilibrium Given the continuum of voters, no single voter affects the outcome of the election. To obtain meaningful equilibria, we consider action profiles from which no arbitrarily small group of citizens has an incentive to deviate.

Each action profile $a \in (X \cup \{\theta\})^n$ of the potential legislators leads to a subgame Γ^a in which the citizens vote. Given a voting profile b in the subgame Γ^a and $\epsilon > 0$, we say that a measurable set $S \subset \Omega$ is an ϵ-*club* if $S \subset b^{-1}(j)$ for some $j \in \{i \in I : a_i \in X\} \cup \{\theta\}$ (either all members of S vote for the same candidate, or none votes) and $0 < \mu(S) \leq \epsilon$.

Definition 6.1. The voting profile b in the subgame Γ^a is a *small clubs Nash equilibrium* (or simply an *equilibrium*) of the subgame if there exists $\epsilon > 0$ such that for every *ϵ-club* $S \subset \Omega$ and every $j \in \{i \in I : a_i \in X\} \cup \{\theta\}$

$$\int_S u(\omega, (a, b))d\mu(\omega) \geq \int_S u(\omega, (a, [j, b|_{\Omega \setminus S}]))d\mu(\omega) .$$

Notice that when Ω is finite and μ is the counting measure, the notion of a small clubs Nash equilibrium coincides with the notion of a pure strategy Nash equilibrium.

A strategy profile in the whole game is a pair (a, B) where $a \in (X \cup \{\theta\})^n$ and B is a function that associates each $a^* \in (X \cup \{\theta\})^n$ with a voting profile $B(a^*) : \Omega \to \{i \in I : a_i^* \in X\} \cup \{\theta\}$ in the subgame Γ^{a^*}.[7] For a strategy profile (a, B) we say that there is *positive voter turnout* if $B(a)$ does not vanish almost everywhere.

For each action profile B of the citizens define the strategic game for the legislators

$$G^B(I, (X \cup \{\theta\})_{i \in I}, (w_i)_{I \in I}, M)$$

by letting $w_i(a) = u_i(a, B(a))$ for each $a \in (X \cup \{\theta\})^n$ and each i, where u_i is the payoff of the potential legislator.

Each voting subgame Γ^a may have multiple equilibria, with a variety of outcomes. We restrict attention to equilibria of the whole game in which each legislator expects a deviation in the first stage to have no effect on the citizens' voting behavior *if this behavior remains an equilibrium of the subgame reached after the deviation*. We say that the subgames Γ^{a^*} and Γ^a are *adjacent* if there exists some player i and some $x \in X \cup \{\theta\}$ such that $a^* = (x, a_{-i})$ (i.e. the histories a and a^* differ only in the action of a single candidate).

Definition 6.2. A strategy profile (a, B) is an *equilibrium* if a is a Nash equilibrium of the game G^B and the voting profile $B(a)$ is a small clubs Nash equilibrium of the subgame following the history a. An equilibrium strategy profile (a, B) is a *subgame persistent equilibrium* if for every a^* adjacent to a we have $B(a^*) = B(a)$ whenever $B(a)$ is a small clubs Nash equilibrium of the subgame following a^*.

Results

Proposition 6.1. *If (a, B) is a subgame persistent equilibrium with positive voter turnout, then the game G^B satisfies **C** and **PE** at a.*

[7]We use the notation $B(a^*)$ instead of b^{a^*} to emphasize the fixed action profile B for the second stage of the game.

The next result is an immediate consequence of Propositions 4.2 and 6.1.

Corollary 6.2. *In a subgame persistent equilibrium with positive voter turnout all participating candidates are elected and their positions are configured according to Proposition 4.2.*

To see the logic behind the result, note first that at equilibrium, the continuity of the quota function and the fact that it is nondecreasing imply that all elected candidates get exactly the quota of votes and candidates that are not elected get no votes. Therefore, by subgame persistence any unelected candidate can profitably drop out without affecting the policy outcome, so that all candidates at equilibrium get elected. To show that condition **C** is satisfied at a, we note that if withdrawal does not change the outcome, a small club of citizens voting for that candidate can profitably drop out. To show that condition **PE** is satisfied at a, we argue that if a move by candidate i to a larger party x satisfies the conditions of **PE**, then the equilibrium voting profile of the subgame following the history a is also an equilibrium voting profile of the adjacent subgame following the history (x, a_{-i}). The reason is that if the members of a small club of voters find it profitable to deviate in the adjacent subgame, then they also find it profitable to deviate in the original subgame, contradicting equilibrium. So by subgame persistence candidate i finds it profitable to move to the larger party x.

Incentive compatibility We now study the implications of allowing the candidates to renege on their policy commitments. We look at equilibria that satisfy two conditions. First, no elected legislator can profitably change her policy after she has incurred the cost of participation and obtained the "prize" that depends on the size of her party. Second, each potential legislator expects a deviation in the first stage to have no effect on the citizens' voting behavior if this behavior remains an equilibrium of the subgame reached after the deviation *and* no elected legislator wishes to renege on her policy commitment in this subgame. By weakening the restriction on deviants' beliefs about the resulting voting equilibrium, we potentially allow beliefs for which deviations that were profitable under subgame persistence are no longer profitable, and thus expand the set of equilibria.

Let b be a voting profile for citizens in the subgame following a. We say that (a, b) is *incentive compatible* if for any policy $x \in X$ and for any elected legislator i in that subgame we have

$$v_i(M(A(a,b))) \geq v_i(M(x, A_{-i}(a,b)))\,,$$

where v_i is i's valuation function. That is, a voting profile is incentive compatible for a given strategy profile for potential legislators if no elected legislator can profitably change her policy after the election.

Definition 6.3. An equilibrium strategy profile (a, B) is a *subgame IC-persistent equilibrium* if $(a, B(a))$ is incentive compatible and for every a^* adjacent to a we have $B(a^*) = B(a)$ whenever $B(a)$ is a small clubs Nash equilibrium of the subgame following a^* and $(a^*, B(a))$ is incentive compatible.

A subgame persistent equilibrium (a, B) may not be a subgame IC-persistent equilibrium, because $(a, B(a))$ may not be incentive compatible. However,

$$\begin{array}{l}(a,B) \text{ is subgame persistent and} \\ (a,B(a)) \text{ is incentive compatible}\end{array} \quad \Rightarrow \quad (a,B) \text{ is subgame IC-persistent}\,.$$

Further, a subgame IC-persistent equilibrium (a, B) may not be a subgame persistent equilibrium, because at a^* adjacent to a we may have $B(a^*) \neq B(a)$ even though $B(a)$ is a small clubs Nash equilibrium of the subgame following a^* (i.e. a voting profile may remain an equilibrium for an adjacent subgame but may no longer be incentive compatible.)

The next result follows easily from the proof of Proposition 6.1.

Proposition 6.3. *If (a, B) is a subgame IC-persistent equilibrium with positive voter turnout, then the game G^B satisfies* ***C*** *and* ***SE*** *at a.*

Corollary 6.4. *In a subgame IC-persistent equilibrium with positive voter turnout, all participating candidates get elected and their positions are configured according to Proposition 4.3.*

The Hare Quota Consider briefly the specific election rule given by the Hare Quota, whereby a potential legislator is elected if and only if the number of votes she obtains is at least the total number of votes divided by $\overline{k}$, the number of seats in the legislature.

In this case, if candidates are not ideological then the conclusion of Proposition 6.1 holds for subgame perfect equilibria. This result follows from the fact that for the Hare Quota, in any equilibrium of a voting subgame with positive voter turnout exactly $\overline{k}$ legislators are elected, each receiving exactly the Hare Quota of votes.

Another implication of this property is that the Hare Quota can be considered in the class of the Single Transferable Vote (STV) procedures.[8] At an equilibrium of our model each elected legislator receives exactly the Hare Quota of votes. Therefore, this procedure

[8] In the canonical STV procedure if a candidate receives more than the quota of votes, then the excess votes are transferred to another candidate, usually using the voters' secondary preferences.

provides transferability implicitly—if a candidate has enough votes to get elected, then in equilibrium excess supporters allocate their votes to other candidates or drop out.[9]

Existence of equilibrium A subgame persistent equilibrium of the two-stage game exists under weak conditions. If we assume, for example, that potential legislators are not ideological, citizens' preferences are single-peaked, and the distribution of favorite positions is nonatomic, then we can construct an equilibrium. For a small enough cost of participation, we find a two-party equilibrium in which all citizens vote, citizens whose favorite positions are to the left of the population median vote for the left party, and those whose positions are to the right of the population vote for the right party.

Appendix: Proofs

Proof of Proposition 4.3 ***(SE)****.* Let a be a Nash equilibrium with at least two participants.

Number of participants is even: Ignore nonparticipating players and re-index the participants from $-k$ to -1 and from 1 to k so that $a_i \leq a_j$ if $i \leq j$, so that the left median is a_{-1} and the right median is a_1. In this proof we use $a_{(I \setminus i)}$ to denote the list of actions of the players other than i and a_{-i} to denote the action of player $-i$.

We prove the result by establishing four properties of an equilibrium. (By the symmetry of the situation we need only look at players with positive indexes.)

(i) $a_1 \neq a_{-1}$: If $a_1 = a_{-1}$ and either player 1 or player -1 withdraws then the outcome does not change, so by **C** the player's withdrawal is profitable, which is inconsistent with equilibrium.

(ii) *If* $i \geq 3$ *and* $j \geq 3$*, then* $a_i = a_j$: Suppose by way of contradiction that $a_i \neq a_j$. Without loss of generality, assume that a_j is held by at least as many players as is a_i. If player i moves to a_j, then the policy outcome does not change and she joins a larger party. (I.e. the second and fourth conditions in **SE** are satisfied for $x = a_j$.)

Now take a participating player h. If we fix that player's action at nonparticipation, the new outcome becomes a_1 or a_{-1} irrespective of player i's move to the position of j. Further, if

[9]For a history of the STV procedure see Tideman (1995) and Richardson and Tideman (2000). One of the earliest versions of STV voting was in a local election in Tasmania, Australia. In that election a voter recorded his name on his preferred candidate's list. A candidate secured election by receiving the required number of votes. As in our model, Richardson and Tideman point out that this procedure provides transferability implicitly, in that when a candidate had enough votes to be elected, supporters see this and allocate their votes to other candidates.

we fix player h's action to $y \in X$ then the new outcome is the same before and after i moves to a_j, because the positions of player 2 and all players to the left of her remain unchanged with i's move. That is, $M(y, (a_j, a_{(I\backslash i)})_{(I\backslash h)}) = M(y, a_{(I\backslash h)})$. Thus by **SE**, player i's move to a_j is profitable, which is inconsistent with equilibrium.

(iii) *Players* 2 *is not an independent*: Suppose by way of contradiction that player 2 is an independent. Then by (i) player 1 is also an independent. Now if 2 shifts her position to 1, then she joins a two member party and does not change the outcome. Take some participating player h and fix her position to $y \in X \cup \{\theta\}$. Notice that when $M(y, a_{(I\backslash h)}) \neq M(y, (a_1, a_{(I\backslash 2)})_{(I\backslash h)})$, we have $M(y, (a_1, a_{(I\backslash 2)})_{(I\backslash h)}) = a_1$. Therefore, letting $z = a_1$ we have $M(z, a_{(I\backslash h)}) = M(y, (a_1, a_{(I\backslash 2)})_{(I\backslash h)})$, so that by **SE** player 2's move to a_1 is profitable, which is inconsistent with equilibrium.

(iv) *If $a_2 \neq a_3$ then player* 3 *is a member of a party with three or more members*: Suppose that $a_2 \neq a_3$ and that a_3 is held by two members or an independent. By (iii), players 1 and 2 are members of one party. Now by the same argument as for (ii), the conditions of **SE** are satisfied for $i = 3$ and $x = a_2$, so that a move by player 3 to a_2 is profitable, which is inconsisent with equilibrium.

Number of participants is odd: Re-index the participants from $-k$ to k with $a_i \leq a_j$ if $i \leq j$, so that the median position is a_0.

(i) *Player* 0 *is an independent*: If not, the withdrawal of any player whose position is a_0 does not change the outcome, so that by **C** it is profitable, which is inconsistent with with equilibrium.

(ii) *If $i \geq 2$ and $j \geq 2$, then $a_i = a_j$*: Suppose by way of contradiction that $a_i \neq a_j$. Without loss of generality, assume that a_j is held by at least as many players as is a_i. If player i moves to a_j then the policy outcome does not change and she joins a larger party. (I.e. the second and fourth conditions in **SE** are satisfied for $x = a_j$.) Now take some participating player h and fix her position to $x \in X \cup \{\theta\}$. Then $M(x, (a_i, a_{(I\backslash j)})_{(I\backslash h)}) = M(x, a_{(I\backslash h)})$, so by **SE** the move to a_j is profitable, which is inconsistent with equilibrium.

(iii) *If $i \geq 2$, then i is not an independent*: If i is an independent, then by (i) player 1 is an independent and by the same argument as in (ii) player i can profitably move to a_1. $\square$

Proof of Proposition 4.2 ***(PE)****.* Let a be a Nash equilibrium. **PE** implies **SE**, so we need only exclude as an equilibrium an action profile with an even number of participants in which a two member party holds the right median position and there is a three or more member party to its right.

Once again re-index the participants from $-k$ to k, excluding 0, with $a_i \leq a_j$ if $i \leq j$. Reconsider case (iv) of the proof of Proposition 4.3.

(iv') *Players* 2 *and* 3 *cannot hold different positions*: If $a_2 \neq a_3$ and player 2 moves to a_3 then she joins a larger party and does not change the outcome. Fixing the action of some player h to be nonparticipation, we know that $M(\theta, a_{(I\setminus h)})$ is a_1 if $h < 0$ and a_{-1} if $h > 0$. Therefore $M(\theta, (a_3, a_{(I\setminus 2)})_{(I\setminus h)}) = M(\theta, a_{(I\setminus h)})$, so that **PE** implies that player 2's move to a_3 is profitable. □

Proof of Proposition 4.1 ***(E)****.* Let a be a Nash equilibrium. **E** implies **PE**, which implies **SE**, so we need only exclude as an equilibrium an action profile with an odd number of participants in which an independent holds a position other than the median. Such an action profile is excluded as an equilibrium because the median position is held by an independent in any equilibrium with an odd number of participants, so that any other independent can profitably move to the median, given **E**. □

Proof of Proposition 5.1. We first show that an action profile that satisfies the conditions in the result is a Nash equilibrium. Consider such an action profile. By (3), player k is not better off switching to nonparticipation. Given $|x - x_k| > |y - x_k|$, she is also not better off switching to party y: if she does so then the outcome changes to y from the 50–50 lottery between y and x and her cost of participation falls from $c(k)$ to $c(k+1)$. By the strict concavity of v, these two conclusions apply to players $1, \ldots, k-1$. Further, by the symmetry of v, the symmetry of the parties' positions, and (4), these conclusions apply to players $n-k+1, \ldots, n$. By (1) and (2), players $k+1, \ldots, n-k$ are not better off switching to participation in either party.

It remains to argue that no player is better off becoming an independent. First consider the case in which $x_k \leq x$.

- Suppose $x_i \leq x_k$. Then i is a member of x and her switching to run as an independent either has no effect on the outcome or, if the position she chooses exceeds x, makes the outcome worse for her.

- Suppose $x_k < x_i \leq x$. Then i does not participate. If she switches to running as an independent, the best outcome she can induce is x (which she induces by choosing any position of at most x). She can induce the same outcome by joining party x, which is less costly than running as an independent, so she cannot profitably switch to running as an independent.

- Suppose $x_i > x$. Then i does not participate. The best position for her to choose as an independent is x_i, in which case the outcome changes to x_i. Given the concavity of v, the player who has most to gain from such a deviation is one for whom x_i is closest to x. Now, a player whose favorite position is x does not gain by joining x, and hence definitely does not gain from becoming an independent. Thus i does not gain from becoming an independent.

If $x < x_k$, we argue as follows.

- Suppose $x_i \leq x$. Then i is a member of x and her switching to run as an independent either has no effect on the outcome or, if the position she chooses exceeds x, makes the outcome worse for her.

- Suppose $x < x_i \leq x_k$. Then i is a member of x. If she switches to running as an independent, the best outcome she can induce is a 50–50 lottery between x_i and y (which she induces by choosing the position x_i). The player who has most to gain from such a deviation is k. For this player not to gain from such a deviation we need

$$\tfrac{1}{2}[v(0) + v(y - x_k)] - c(1) \leq \tfrac{1}{2}[v(x_k - x) + v(y - x_k)] - c(k),$$

  or

$$c(1) - c(k) \geq -\tfrac{1}{2}v(x_k - x).$$

- Suppose $x_k < x_i$. Then i does not participate. The best position for her to choose as an independent is x_i, in which case the outcome changes to x_i. Given the concavity of v, the player who has most to gain from such a deviation is player $k + 1$. For this player not to gain from such a deviation we need

$$c(1) \geq -\tfrac{1}{2}[v(x_{k+1} - x) + v(y - x_{k+1})]$$

Again by the symmetry of v and of the parties' positions, the conditions in the result imply that players $n - k + 1, \ldots, n$ are not better off becoming independents. Hence an action profile that satisfies the conditions in the result is a Nash equilibrium.

We now show that any two-party equilibrium satisfies the conditions in the result. Consider such an equilibrium. By Proposition 4.1 the parties have the same number of members. Denote their positions by x and $y > x$ and the number of members in each party by k.

Suppose that player i belongs to party y, where $|y - x_i| > |x - x_i|$. If she switches to party x the outcome changes to x from the 50–50 lottery between x and y and her cost of participation falls from $c(k)$ to $c(k+1)$. Thus she is better off. Hence any player i either does not participate or belongs to the party whose position is closer to x_i.

Denote the party to which player i belongs by x. Then $\frac{1}{2}[v(x-x_i)-v(y-x_i)] \geq c(k)$. We argue that in any equilibrium every player j with $j<i$ also belongs to party x. If such a player j does not belong to party x then she does not participate (by the previous argument), and by joining x she changes the payoff from $\frac{1}{2}[v(x-x_j)+v(y-x_j)]$ to $v(x-x_j)-c(k+1)$. Given the strict concavity of v, the fact that c is decreasing, and the fact that k is not better off leaving party x, this change is positive.

Hence in any equilibrium in which each party has k members, players $1, \ldots, k$ belong to party x, players $k+1, \ldots, n-k$ do not participate, and players $n-k+1, \ldots, n$ belong to party y. Because player k is a member of party x, (3) is satisfied, and symmetrically (4) is satisfied. Because players $k+1$ and $n-k$ do not participate, (1) and (2) are satisfied.

Finally, the arguments in the first part of the proof establish that for no player to have an incentive to become an independent, the conditions in the last part of the result must be satisfied. $\square$

Proof of Proposition 6.1. Given positive voter turnout, the quota q is positive.

We first show that no unelected candidate receives votes. Suppose to the contrary that candidate i is not elected and obtains a positive measure of votes. Let $\delta \in (0,q)$ be the largest measure of votes received by any unelected candidate. Because the quota function is continuous and nondecreasing and Ω is atomless, there exists $\epsilon > 0$ such that the quota remains in $(\delta, q]$ if an ϵ-club of citizens voting for i switches to not voting. Thus these citizens' withdrawal does not affect the set of elected candidates and hence the policy outcome. The members of the ϵ-club decrease their costs, however, contradicting the fact that $B(a)$ is an equilibrium of the subgame following a. Thus no unelected candidate receives votes.

We now show that every candidate is elected. We argue that an unelected candidate who drops out does not affect the outcome, and hence increases her payoff. Suppose that candidate i is not elected. Then by the previous paragraph she receives no votes. We show that the voting equilibrium $B(a)$ of the subgame Γ^a in which i is a candidate is also an equilibrium of the adjacent subgame $\Gamma^{(\theta,a_{-i})}$ in which i is not a candidate. Because $B(a)$ is an equilibrium of Γ^a we know that there exists $\epsilon > 0$ such that no ϵ-club has an incentive to deviate from $B(a)$ in this subgame. Choose $\nu \in (0, \min\{\epsilon, q\})$. For ν small enough, when a ν-club changes its action from $B(a)$ in the subgame Γ^a, candidate i remains unelected, because the quota remains positive. Therefore, if the ν-club changes its action from $B(a)$ in the subgame $\Gamma^{(\theta,a_{-i})}$, the change in the policy outcome is the same as it is in the subgame

Γ^a. Consequently, if some ν-club can profitably change its vote in the subgame $\Gamma^{(\theta,a_{-i})}$, then it can also profitably change its vote in the subgame Γ^a. Thus $B(a)$ is an equilibrium of the subgame $\Gamma^{(\theta,a_{-i})}$; subgame persistence implies that $B(a) = B(\theta, a_{-i})$. Therefore, candidate i's dropping out reduces her costs, and does not change the policy outcome, contradicting the fact that (a, B) is an equilibrium. Hence all candidates are elected.

*The game G^B satisfies **C** at a.* Condition **C** is satisfied if we show that there exists no candidate i for whom $M(\theta, a_{-i}) = M(a)$. Assume the contrary. We know that the quota is positive and that candidate i, who is elected, gets some votes. However, given $M(\theta, a_{-i}) = M(a)$, any small club voting for i can profitably withdraw and reduce its cost without changing the policy outcome: after its withdrawal the quota does not go up and all candidates (with the possible exception of i) remain elected. This contradicts the fact that $B(a)$ is an equilibrium of the subgame Γ^a.

*The game G^B satisfies **PE** at a.* First notice that in an equilibrium all candidates get exactly the quota of votes, because otherwise some ϵ-club voting for a candidate can profitably drop out without changing the election outcome. Thus the policy outcome after the election is $M(a)$.

Now, **PE** is satisfied if there exists no candidate i and policy $x \in X$ with $x \neq a_i$ such that

- $M(a) = M(x, a_{-i})$
- $M(\theta, a_{-j}) = M(\theta, (x, a_{-i})_{-j})$ for each player j
- $\#\{j \in I \setminus \{i\} : a_j = a_i\} < \#\{j \in I \setminus \{i\} : a_j = x\}$.

Assume that the conditions hold for elected candidate i. We show that i can profitably move to x, contradicting the fact that (a, B) is an equilibrium.

For any history a^*, denote the set of small clubs Nash equilibria of Γ^{a^*} by $\beta(a^*)$.

Our strategy is to show that $B(a) \in \beta(x, a_{-i})$, which by subgame persistence implies that $B(x, a_{-i}) = B(a)$. This is because candidate i finds it profitable to switch to the policy x, increase her prize, and not change the electoral or policy outcomes.

So to complete the proof we prove that $B(a) \in \beta(x, a_{-i})$. Suppose that the citizens use the strategy $B(a)$ in the subgame following (x, a_{-i}). We argue that no ϵ-club can profitably deviate from $B(a)$.

If an ϵ-club voting for some legislator j stops voting, then j is no longer elected whereas all other participating legislators are still elected. But we know that $M(\theta, a_{-j}) = M(\theta, (x, a_{-i})_{-j})$, so if this ϵ-club can profitably drop out when the action profile is $B(a)$ in the subgame following (x, a_{-i}), then it can profitably drop out when the action profile is $B(a)$ in the subgame following a, contradicting $B(a) \in \beta(a)$.

If an ϵ-club voting for some legislator j switches to voting for legislator j', then because all candidates get exactly the quota of votes, one of the following cases occurs in the subgames following a and (x, a_{-i}): 1) only candidate j' remains elected, 2) all candidates except j are elected, 3) no candidate remains elected, 4) all candidates remain elected.

In no case is the deviation profitable for the ϵ-club in the subgame following a, because (a, B) is an equilibrium. Therefore case 1 for $j' \neq i$, and cases 2, 3, and 4 are not profitable in the subgame following (x, a_{-i}), because they result in the same change in the policy outcome for both subgames. For case 1 with $j' = i$, there exists another candidate h holding position x. If in the subgame $\Gamma^{(x,a_{-i})}$ the ϵ-club can profitably switch its vote from j to i, then (given the anonymity of the quota function) in the subgame Γ^a it can profitably switch from j to h, which contradicts equilibrium.

A similar argument shows that an ϵ-club that does not vote cannot profitably vote for any candidate j in the $\Gamma^{(x,a_{-i})}$, because such a change does not decrease the quota and either only candidate j remains elected or all candidates remain elected.

The previous paragraphs imply that for some $\epsilon > 0$ the members of no ϵ-club can profitably change their vote from $B(a)$ in the subgame following (x, a_{-i}). $\square$

Proof of Proposition 6.3. We know from the proof of Proposition 6.1 that all elected candidates get the quota of votes and unelected candidates get no votes.

We need to show that all candidates are elected. Suppose by way of contradiction that candidate i is not elected. We know from the proof of Proposition 6.1 that $B(a) \in \beta(\theta, a_{-i})$. Furthermore, (a, B) is incentive compatible, so $((\theta, a_{-i}), B)$ is incentive compatible because i is not elected. Therefore by subgame IC-persistence we have $B(\theta, a_{-i}) = B(a)$ and i finds it profitable to withdraw. Thus all candidates are elected.

The rest of the proof follows with little modification from the proof of Proposition 6.1. $\square$

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