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Subjectively Envy-Free Allocation: Characterization and Existence

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Abstract: We consider a framework in which each individual has his/her own concept of fairness and a social planner chooses an allocation that everyone feels fair. We say that an allocation is subjectively envy-free if no one envies the well-being of others under the allocation. We show that the subjectively envy-free allocation coincides, under different assumptions on individuals' information about others, with the well-known notions of fair allocation due to Rawls (1971) and Foley (1967). The result implies that depending on the circumstances of the society the concept of fairness changes. We also study conditions which guarantee the existence of a subjectively envy-free allocation.

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1 Introduction

The concept of fairness, we believe, is concerned with the comparison of well-being among individuals for we will not need it if there is only one individual on earth. The comparison of well-being, in turn, is associated with the feeling of envy (and compassion as the opposite feeling of envy) to others. In this paper, we propose the following hypothesis about our sense of fairness in the allocation of commodities: An allocation is fair if no one feels envy to the well-being of others under the allocation.[1] We call such an allocation a "subjectively envy-free allocation". The goal of this paper is to characterize it in relation to other notions of fair allocation and study its existence.

Our principle of fairness has a close relationship with the following commodity-based envy-free principle, which can be seen as the one behind the well-known notion of fair allocation ("envy-free allocation") proposed by Foley (1967): An allocation is fair if no one feels envy to the commodities of others under the allocation. We think, however, that this principle does not completely capture our sense of envy and thus our sense of fairness. For example, even if a government provides more help to handicapped persons than it does to someone without handicaps, the individual will not envy handicapped persons and will not feel that such a policy is unfair as long as he/she believes that they are still unhappy. Since the commodity-based envy-free principle does not take individual welfare into account, it fails to justify this type of policy, which we think is an evidence that it fails to capture our sense of fairness.

From the viewpoint of the policy discussed above, our principle is much closer to the welfare-based principle of fairness such as maximin principle by Rawls (1971) which requires the social planner to maximize the welfare of the worst-off individuals. A problem of this welfare-based principle is that it is not always possible to identify the well-being of each individual, which is known as the problem of interpersonal comparability of well-being.[2] Our principle avoids

[1] This definition is based on the observation that envy is usually caused by the happiness of others. For example, Hume (1739) observed as follows: "$\cdots$ envy is excited by some present enjoyment of another, which by comparison diminishes our idea of our own [Hume (1739, Book II, Section VIII 'Of Malice and Envy')]." See also de la Mora (1984) for a long history of this type of observation on envy.

[2] See, for example, Hammond (1991) for a good survey on this problem.

this problem by allowing each individual to use his/her subjective perception of the well-being of others in judging the fairness of an allocation. Because of this subjective nature of our principle, it is also shown that our principle becomes closer to the commodity-based envy-free principle when it is difficult to identify the well-being of individuals in the society.

The discussions above, we hope, show that our principle not only has close relationships with other well known principles of fairness but also has certain advantages over them. By allowing each individual to use his/her belief on the welfare of others, we are implicitly assuming that the social planner's task is to find an allocation that every individual feels fair, which seems to be a natural one in democratic societies.[3] The fact that various notions of fairness have been proposed in the past, however, suggests that the notion of fairness can be totally different from individual to individual and thus it is impossible to develop an interesting theory of fairness in such a framework. Even though we cannot deny such a pessimistic view, we also have a hope that, as some empirical studies [e.g., Yaari and Bar-Hillel (1984)] indicate, the notions of fairness may not be so different among individuals. In fact, a basic point of view in this paper is that the difference may be explained by the difference in individual's perception of the world not by the difference in the principle of fairness.

In the next section, we start with a simple example to further motivate our study in this paper. In Section 3, we formally introduce our new notion of fair allocation based on the subjective comparison of well-being. Because individuals are allowed to judge fairness based on their beliefs, the notion of fairness depends on the information available to each individual. We show that the subjectively envy-free allocation coincides with the maximin allocation due to Rawls (1971) when information is complete and with the envy-free allocation due to Foley (1967) when no information is available. The results provide different interpretations for the two well known principles of fairness. The fact that individuals are allowed to judge fairness based on their beliefs also raises the question about the existence of the subjectively envy-free allocation, which is studied in Section 4. Section 5 concludes the paper.

[3]The spirit of this assumption is close to what Wicksell (1896) called the "principle of voluntary consent and unanimity" for the fairness of government policies.

2 A Preliminary Example

To motivate our study a little further, let us consider a classical problem of dividing a cake fairly between two individuals i and j [Dubins-Spanier (1961)].

Let x_i and x_j be the proportion of a cake given to i and j so that $x_i + x_j = 1$. Suppose that the well-being of each individual is represented by a function defined by $u(i, x_i) = \frac{1}{2}x_i$ and $u(j, x_j) = x_j$. It is useful for our purpose to imagine that i is an elder brother whose satisfaction from a bite of the cake is only half of that of his younger brother j. From their mother's point of view, the division of the cake may be fair if $u(i, x_i) = u(j, x_j)$, i.e., if both of them are equally satisfied. It is easy to check that $(x_i, x_j) = (\frac{2}{3}, \frac{1}{3})$ is the unique solution under which $u(i, x_i) = u(j, x_j)$ is attained. This solution, however, is not envy-free in the sense that $u(j, x_i) = \frac{2}{3} > \frac{1}{3} = u(j, x_j)$, i.e., j (younger brother) envies i (elder brother). Hence, their father may think that it is fair to divide the cake so that both of them do not envy each other. Such an envy-free allocation is uniquely determined at $(x_i, x_j) = (\frac{1}{2}, \frac{1}{2})$ which would be unsatisfactory for the elder brother because $u(i, x_i) = \frac{1}{4} < \frac{1}{2} = u(j, x_j)$. Now the parents are faced with a dilemma.

Even though each argument above has its own appeal, it seems to me that the fair allocation in this problem is $(x_i, x_j) = (\frac{2}{3}, \frac{1}{3})$ under which the well-being of i and j are equalized. It is useful to recognize that this "welfare-egalitarian allocation" coincides with the Rawls' "maximin allocation".[4] Namely, in this problem we think that the welfare-based principles capture our sense of fairness. A crucial condition here is that the two brothers are living together and thus they know each other well (and that they know that the parents know them well). Hence, by explaining why the elder brother can envy the younger brother if his portion is less than $\frac{2}{3}$, we believe that the younger brother would agree with the solution.

Now, let us suppose that i and j are not brothers but two similar children and do not know each other. Since the two children do not know each other and are similar, it seems fair for me to divide the cake in half in this case, i.e., $(x_i, x_j) = (\frac{1}{2}, \frac{1}{2})$ seems to be the fair allocation. It

[4] We show in the next section that this is not a coincidence but a general result if we restrict ourselves to a set of Pareto optimal allocations.

is useful to recognize that this "commodity-egalitarian allocation" is also a commodity-based "envy-free allocation".[5] Namely, we think that the commodity-based principles capture our sense of fairness in this case in which information about individual welfare is missing.

This example shows that our sense of fairness depends on the circumstances of the society. A natural question is if there exists a general principle of fairness which describes our sense of fairness depending on the circumstances of the society. We show in the next section that the subjectively envy-free principle can be such a general principle of fairness. The essential idea is that since each individual's envy to the well-being of others is subjective the "subjectively envy-free allocation" depends on the information (belief) that each individual has. We hope that our study also provides new insights about the welfare-based principles and the commodity-based principles discussed above, and provides reasons for why both of them have been the well-accepted principles.

3 Subjectively Envy-Free Allocation and Information

Let $N = \{1, \ldots, n\}$ be a set of individuals in a society. Let X be a commodity space. Let (i, x_i) stand for "i with x_i". For each $i \in N$, let $\succeq_i$ be a welfare ordering over $N \times X$ that i believes.[6] $(j, x_j) \succeq_i (k, x_k)$ means i believes that j with x_j is at least as happy as k with x_k. As usual, $(j, x_j) \succ_i (k, x_k)$ means $(j, x_j) \succeq_i (k, x_k)$ but not $(k, x_k) \succeq_i (j, x_j)$, and $(j, x_j) \sim_i (k, x_k)$ means $(j, x_j) \succeq_i (k, x_k)$ and $(k, x_k) \succeq_i (j, x_j)$. Notice that the preference of i, for example, can be observed in the comparisons of $(i, \cdot)$'s; e.g., if i prefers x_i to y_i then $(i, x_i) \succ_i (i, y_i)$.

Now, we consider a problem of distributing a given bundle of commodities $\bar{x} \in X$ among the individuals in a Pareto-efficient way. An allocation is an n-tuple vector $x = (x_1, \ldots, x_n) \in X^n$ such that $\sum_{i \in N} x_i \leq \bar{x}$. Let $\hat{X} \subset X^n$ be a set of Pareto-efficient allocations, i.e., $x \in \hat{X}$ implies

[5] It is easy to check that the commodity-egalitarian allocation is always an envy-free allocation.

[6] This personal welfare ordering is sometimes referred to as "subjective interpersonal comparison" for example, by Kaneko (1984) and Hylland (1991). In some context, it may be useful to think that this ordering describes what i believes other individuals should feel, based on his/her assessment on their needs and tastes. In other words, individuals are not assumed to be estimating the true welfare ordering but they are allowed to have their own perceptions on the welfare of others.

that there is no allocation $y \in X^n$ such that $(i, y_i) \succ_i (i, x_i)$ for some $i \in N$ and $(j, y_j) \succeq_j (j, x_j)$ for all $j \neq i$. Although some of the following definitions and results can be stated using the general space X^n rather than $\hat{X}$, we will concentrate on Pareto-efficient allocations to obtain the sharpest results. It may be an interesting exercise to check how the results below would be changed when general allocations are considered.

We are ready to formalize our hypothesis on the fair allocation. In our framework, a notion of fairness is defined by a characterization of allocations accepted as fair allocations. The following definition is based on the idea that the concept of fairness is concerned with each individual's envies to others and that an allocation will be accepted as a fair allocation if each individual, with an effort to put himself/herself in the positions of others, feels no envy to the well-being of others. For each $i \in N$, let $S_i(\succeq_i) = \{x \in \hat{X} \mid (i, x_i) \succeq_i (j, x_j) \; \forall j \in N\}$ be the set of allocations under which i feels no envy to others. We call an allocation $x \in S_i(\succeq_i)$ a subjectively envy-free allocation for i. Then $S(\succeq_1, \ldots, \succeq_n) = \bigcap_{i \in N} S_i(\succeq_i)$ is the set of allocations under which no one feels envy to others (based on his/her subjective perception of the world). Throughout this section we assume that $S(\succeq_1, \ldots, \succeq_n) \neq \emptyset$. This assumption is carefully studied in the next section and sufficient conditions that justify this assumption are provided.

Definition 1 *An allocation $x \in \hat{X}$ is said to be a subjectively envy-free allocation if and only if $x \in S(\succeq_1, \ldots, \succeq_n)$.*

Our hypothesis discussed in the introduction can be stated as follows.

Hypothesis: An allocation x is perceived as a fair allocation if $x \in S(\succeq_1, \ldots, \succeq_n)$.

To test such a hypothesis, we think it useful to study the relationship between this new notion of fair allocation and other notions of fair allocation.

To discuss such relationships, we introduce a welfare ordering $\succeq$ over $N \times X$ which can be considered as a true welfare ordering. Alternatively, one may think of $\succeq$ as a welfare ordering that a social planner believes. We assume that, for each $i \in N$, $(i, x_i) \succeq_i (i, y_i)$ if and only if $(i, x_i) \succeq (i, y_i)$, i.e., each individual knows his/her own preference.[7]

First, we consider a notion of fairness proposed by Rawls (1971). Let $min_{\succeq}(y) = \{i \in N \mid (j, y_j) \succeq (i, y_i)\ \forall j \in N\}$ be the set of worst-off individuals under an allocation $y \in \hat{X}$. Let $MM_{\succeq} = \{y \in \hat{X} \mid (i, y_i) \succeq (i, z_i)\ \forall z \in \hat{X}\ \forall i \in min_{\succeq}(y)\}$ be the set of allocations under which the welfare of the worst-off individual is maximized.

Definition 2 (Rawls (1971)) *An allocation $x \in \hat{X}$ is said to be a maximin allocation if and only if $x \in MM_{\succeq}$.*

Next, we consider a notion of fairness proposed by Foley (1967). Let $EF_{\succeq} = \{y \in \hat{X} \mid (i, y_i) \succeq (i, y_j)\ \forall i, j \in N\}$ be the set of allocations under which no individual wishes to have the commodity bundles given to others.

Definition 3 (Foley (1967)) *An allocation $x \in \hat{X}$ is said to be an envy-free allocation if and only if $x \in EF_{\succeq}$.*

Now we are ready to show how different assumptions on individual information change the nature of the set of subjectively envy-free allocations. We consider two extreme assumptions about each individual's information. First assumption corresponds to the case in which each individual and the social planner have the same belief about the welfare ordering, which we interpret as the case of complete information.

Assumption 1 (Complete Information) *For each $i \in N$, $\succeq_i = \succeq$.*

When i does not have information about others, he/she may assume that others are just like him/her. We interpret the next assumption as the case in which each has no information about others, i.e., the case in which others are all "strangers" to each individual. Namely, it is assumed that each individual i believes that the well-being of j with x_j is the same as his/her well-being with x_j.

[7] If $\succeq$ can be seen as the welfare ordering that the social planner believes, then the condition requires that the social planner have information about each individual's own preferences.

Assumption 2 (No Information) *$(j, x_j) \sim_i (i, x_j)$ for all $i, j \in N$ and $x_j \in X$.*

Now we have the main results of this section. First, to study the consequence of adopting Assumption 1, it is useful to introduce a set $WE_{\succeq} = \{y \in \hat{X} \mid (i, y_i) \sim (j, y_j) \; \forall i, j \in N\}$, i.e., the set of allocations under which each individual is equally well-off. We call $WE_{\succeq}$ the set of welfare-egalitarian allocations. Our first result establishes that if information is complete, the set of subjectively envy-free allocations coincides with the set of welfare-egalitarian allocations.[8]

Lemma 1 *Under Assumption 1, $S(\succeq_1, \ldots, \succeq_n) = WE_{\succeq}$.*

Proof. Let $x \in WE_{\succeq}$. Then $(i, x_i) \sim (j, x_j)$ for all $i, j \in N$. Then, $(i, x_i) \succeq (j, x_j)$ for all $i, j \in N$, i.e., $x \in S(\succeq_1, \ldots, \succeq_n)$ under Assumption 1. Conversely, let $x \in S(\succeq_1, \ldots, \succeq_n)$. Then, under Assumption 1, $(i, x_i) \succeq (j, x_j)$ for all $i, j \in N$, which implies $(i, x_i) \succeq (j, x_j) \succeq (i, x_i)$ for all $i, j \in N$, i.e., $(i, x_i) \sim (j, x_j)$ for all $i, j \in N$. ∎

The next lemma allows us to have a more interesting nature of the subjectively envy-free allocations under complete information.

Lemma 2 *If $WE_{\succeq} \neq \emptyset$, then $WE_{\succeq} = MM_{\succeq}$.*

Proof. Let $x \in WE_{\succeq}$. Then, $(i, x_i) \sim (j, x_j)$ for all $i, j \in N$. Suppose that x does not maximize the welfare of the worst-off individual. Then, there is another allocation $y \in \hat{X}$ under which $(i, y_i) \succ (i, x_i)$ for all $i \in N$, which contradicts that x is a Pareto-efficient allocation. Hence, x maximizes the welfare of the worst-off individual, i.e., $x \in MM_{\succeq}$. Conversely, let $x \in MM_{\succeq}$. By assumption, we can assume the existence of an allocation $x^* \in WE_{\succeq}$. Suppose, per absurdum, $x \notin WE_{\succeq}$, i.e., suppose $(j, x_j) \succ (i, x_i)$ for some $i, j \in N$. Without loss of generality, we can assume that i is the worst-off individual. Since x maximizes the welfare of the worst-off individual, $(i, x_i) \succeq (i, x_i^*)$ which implies that $(j, x_j) \succ (j, x_j^*)$ and $(k, x_k) \succeq (k, x_k^*)$ for all $k \neq j$. (Notice that $(i, x_i^*) \sim (j, x_j^*)$ for all $i, j \in N$, $(j, x_j) \succ (i, x_i)$ and i is the worst-off

[8] The result is already pointed out by Hammond (1987) who refers to an allocation in $S(\succeq_1, \ldots, \succeq_n)$ under Assumption 1 as "resent-free allocation" in his comparison with "envy-free allocation".

individual.) This contradicts that x^* is Pareto efficient. Therefore, $x \in \hat{X}$ must satisfy the condition $(i, x_i) \sim (j, x_j)$ for all $i, j \in N$, i.e., $x \in WE_{\succeq}$. ∎

The following theorem, which establishes the relationship between the set of maximin allocations and the set of subjectively envy-free allocations, is a simple consequence of Lemma 1 and Lemma 2 above. The theorem implies that if a social planner knows that everyone has complete information (in the sense of Assumption 1) about the well-being of others, the social planner chooses a maximin allocation.

Theorem 1 *Under Assumption 1,* $S(\succeq_1, \ldots, \succeq_n) = MM_{\succeq}$.

Proof. By Lemma 1, under Assumption 1, $S(\succeq_1, \ldots, \succeq_n) \neq \emptyset$ implies $WE_{\succeq} \neq \emptyset$. Hence, by Lemma 1 and Lemma 2, $S(\succeq_1, \ldots, \succeq_n) = MM_{\succeq}$. ∎

The result provides an interesting interpretation of the maximin principle. Namely, from our point of view, we can say that the welfare of the least well-off individual must be maximized because he/she is the person who envies others most. Since the spirit of the subjectively envy-free allocation is that under a fair allocation no one should envy others, by maximizing the welfare of the individual who envies everyone else we expect that the fair allocation will be attained. It is in this sense that we can reinterpret the maximin principle in terms of the envy-free principle. Our results show that under complete information if there is an allocation in which no one envies others the welfare of each individual must be equalized and therefore the two principles exactly coincide.

Next, we discuss the implication for adopting Assumption 2. The next theorem implies that if a social planner knows that each individual has no information (in the sense of Assumption 2) about others then he/she chooses an envy-free allocation.

Theorem 2 *Under Assumption 2,* $S(\succeq_1, \ldots, \succeq_n) = EF_{\succeq}$.

Proof. Let $x \in EF_{\succeq}$, i.e., let $(i, x_i) \succeq (i, x_j)$ for all $i, j \in N$. Since each knows his/her own preference, we have $(i, x_i) \succeq_i (i, x_j)$ for all $i, j \in N$, which implies under Assumption 2 that

$(i, x_i) \succeq_i (j, x_j)$ for all $i, j \in N$, i.e., $x \in S(\succeq_1, \ldots, \succeq_n)$. Conversely, let $x \in S(\succeq_1, \ldots, \succeq_n)$. Then, $(i, x_i) \succeq_i (j, x_j)$ for all $i, j \in N$, which implies under Assumption 2, $(i, x_i) \succeq_i (i, x_j)$ and thus $(i, x_i) \succeq (i, x_j)$ for all $i, j \in N$, i.e., $x \in EF_{\succeq}$. ∎

This result also provides an interesting point of view on adopting the envy-free allocation as a fair allocation. Traditionally, the envy-free principle seems to have been adopted as a desirable equity criterion because it does not require interpersonal comparison (nor cardinality of preferences[9]) [e.g., Varian (1974)]. From our point of view, this property is a consequence of Assumption 2 and not the reason for adopting the envy-free allocation as a fair allocation. It is adopted by an individual who tries to compare his/her well-being with others but assumes, due to the lack of information, that others are just like him/her. Hence, our notion of fair allocation coincides (if it exists), at its first-best, with the maximin allocation due to Rawls (1971) in a society where all individuals have information on others, and the envy-free allocation can be seen only as a second-best solution in a real world where it is practically impossible to have the complete information about others. This is a possible reason for why people accept the envy-free allocation as a fair allocation, which we think sheds new lights on the interpretation of the envy-free allocation.

Needless to say, these two cases are very special cases and in general the social planners must know the belief of each individual about the welfare ordering of others in order to determine the fair allocation. Since each individual tends to become unknown to each other, as the society becomes larger, we may roughly argue based on Theorem 2 that the envy-free principle is an adequate notion of fairness in a large society under which no one should envy the commodity bundles of others. If, however, there is someone who requires special needs in the society (and if everyone agrees) then the Theorem 2 does not apply to such an individual and thus the envy-free principle should not be applied to the individual. It is important to keep in mind that the first principle for the social planner is to choose a subjectively envy-free allocation.

We believe that the close relationship between the subjectively envy-free allocation and the

[9] Notice that our approach also does not require the cardinality of preferences.

well-accepted notions of fairness in the normative theory is an evidence that our Hypothesis is a justifiable one. The following remark, we believe, also illustrates a good evidence which supports the Hypothesis.

Remark (Testing the Hypothesis). The empirical study by Yaari and Bar-Hillel (1984), who asked young students their notions of fair allocations in hypothetical situations, contains various interesting results. Although our hypothesis does not explain all the phenomena observed, the following results seem to support our hypothesis. (1) The concept of fairness indeed varies for most students from situation to situation. (2) In the questions where information about the well-being of individuals is most clearly defined (Q1 and Q2), 82 % of the respondents chose the maximin allocation in each case. (3) In the questions where information about the well-being of individuals is most vaguely defined (Q6 and Q7), 34 % (in Q6) and 52 % (in Q7) chose the envy-free allocation ("equal-sprit distribution" in their terminology). Even though the last result is not as good as the second one, we think the numbers are fair considering the fact that some information about individuals are actually given in each question and are possibly used by some students to form the subjective welfare orderings $\{\succeq_i\}_{i\in N}$ and find fair allocations. We think that these results indicate that many individuals are judging fairness by their subjective welfare orderings based on the information they are given and that they are used to find an allocation under which each individual does not envy each other. □

4 Existence of a Subjectively Envy-Free Allocation

The fact that in the subjectively envy-free allocation individuals are allowed to judge fairness based on their beliefs also raises the question about the existence of such an allocation. In this section, we ask if the social planners can find a subjectively envy-free allocation. The following non-existence example illustrates why finding the fair allocation is difficult in general.

Example (Non-existence). Consider a fair division of a cake between two individuals. Suppose $x_1 + x_2 = 1$ and the beliefs of 1 and 2 are characterized by $(1, x_1) \sim_1 (2, \frac{1}{2}x_1)$

for all $x_1 \in [0,1]$ and $(2,x_2) \sim_2 (1,\frac{1}{2}x_2)$ for all $x_2 \in [0,1]$, i.e., each believes that his/her satisfaction is only half of the satisfaction of the other. (Hence, this is a modification of the example in Section 2.) Assume also that $x'_i > x_i$ implies $(i,x'_i) \succ_i (i,x_i)$ for each $i \in \{1,2\}$. Then, it is easy to check that $S_1(\succeq_1) = \{(x_1,x_2) \in [0,1]^2 \mid x_1 \geq \frac{2}{3}, x_1+x_2 = 1\}$ and $S_2(\succeq_2) = \{(x_1,x_2) \in [0,1]^2 \mid x_2 \geq \frac{2}{3}, x_1+x_2 = 1\}$. Hence, $S(\succeq_1,\succeq_2) = \emptyset$. The problem of this example is that the beliefs of the two individuals are very different and both "overestimates" the welfare of the other. As long as the social planner allows each individual to use the belief about the welfare ordering, this overestimation of the others' welfare seem to cause problems for the existence of a fair allocation. In the following, we indeed impose a condition on each individual's belief which prohibits this type of overestimation to guarantee the existence of a subjectively envy-free allocation. □

To find the sufficient conditions, we first consider a welfare ordering $\succeq$, which may be interpreted as the true welfare ordering or the welfare ordering that a social planner believes, and find conditions for $WE_{\succeq} \neq \emptyset$. As far as we know, this result is new in the literature and may be of independent interest. To show our result, we use the standard assumptions and the proof for the existence of a competitive equilibrium in exchange economies [e.g., Debrue (1959)]. An additional assumption requires that individuals be equally well-off when they have no commodities, which seems to be a natural assumption. Our proof can be decomposed into two steps. First we show that for any "price" p, we can find a distribution of "income" $p \cdot \bar{x}$ under which each individual's welfare becomes equal. Given such a distribution mechanism (a mapping from prices to the income distributions), we then find a "competitive equilibrium" under which the allocation of $\bar{x}$ guarantees the equality of welfare and the Pareto optimality. Based on this existence result, we show a condition for $S(\succeq_1,\ldots,\succeq_n) \neq \emptyset$. Our condition requires that each individual's belief should not overestimate the true well-being of others.

Let $X = \mathbf{R}^m_+$ be the non-negative orthant of $\mathbf{R}^m$. Let $\bar{x} > 0$ and $\bar{X} = \{x \in X \mid x \leq \bar{x}\}$. It is easy to check that $\bar{X}$ is convex and compact. To show our first theorem, we use the welfare function which represents the welfare ordering $\succeq$.

Assumption 3 *(1) $\succeq$ is a reflexive, transitive and complete ordering on $N \times X$. (2) For any $x' \in X$ and $i \in N$, $\{x \in X \mid (i,x) \succeq (i,x')\}$ and $\{x \in X \mid (i,x') \succeq (i,x)\}$ are closed in X.*

Lemma 3 *Under Assumption 3, there exists a function $u : N \times X \to \mathbf{R}$ such that $u(i,\cdot) : X \to \mathbf{R}$ is continuous for each $i \in N$ and $u(i,x) \geq u(j,x')$ if and only if $(i,x) \succeq (j,x')$.*

Proof. See Debrue (1959, 4.6). ∎

We also impose the following three conditions on the preference of each individual. The first two conditions require that utility functions be (1) strictly increasing and (2) quasi-concave. The last one requires that if individuals have nothing then they are equally well-off, which I believe is a reasonable assumption.

Assumption 4 *(1) For each $i \in N$, $x' > x$ implies $u(i,x') > u(i,x)$. (2) For any $i \in N$ and $t \in (0,1)$, if $u(i,x') \geq u(i,x)$ then $u(i, tx + (1-t)x') \geq u(i,x)$. (3) For each $i, j \in N$, $u(i,0) = u(j,0)$.*

Let $\mathbf{R}_+$ be a set of non-negative real numbers. Let $\Delta = \{(p_1, \ldots, p_m) \in \mathbf{R}^m_+ \mid \sum_{k=1}^m p_k = 1\}$. Define a correspondence $B : \Delta \times \mathbf{R}_+ \longrightarrow\!\!\!\rightarrow \bar{X}$ by $B(p,\bar{y}) = \{x \in \bar{X} \mid p \cdot x \leq \bar{y}\}$. For each $i \in N$, define an indirect utility function $v_i : \Delta \times \mathbf{R}_+ \to \mathbf{R}$ by $v_i(p,\bar{y}) = \max_{x \in B(p,\bar{y})} u(i,x)$. It is easy to check that v_i is continuous and strictly increasing in the second argument [e.g., Varian (1992, p.102)].

Lemma 4 *Under Assumption 3 and 4, there exists a function $y : N \times \Delta \times \mathbf{R}_+ \to \mathbf{R}$ such that, for any $p \in \Delta$, $a \in \mathbf{R}_+$ and $i, j \in N$, $v_i(p, y(i,p,a)) = v_j(p, y(j,p,a))$, $a = \sum_{i \in N} y(i,p,a)$ and $y(i,\cdot,\cdot) : \Delta \times \mathbf{R}_+ \to \mathbf{R}$ is continuous.*

Proof. Proof by induction on N. Let $a \in \mathbf{R}_+$ be arbitrary. Let $p \in \Delta$ and $b \in [0,a]$ be arbitrary. Let $D_2(p,b,x) = v_2(p,x) - v_1(p,b-x)$ for each $x \in [0,b]$. Then, by Assumption 4, $D_2(p,b,0) = v_2(p,0) - v_1(p,b) \leq 0$ and $D_2(p,b,b) = v_2(p,b) - v_1(p,0) \geq 0$. Since $D_2(p,b,\cdot)$ is continuous, by the intermediate value theorem [Rudin (1974, p.93)] and Assumption 4(1), there exists a unique point $f_2(2,p,b) \in [0,b]$ such that $D_2(p,b,f_2(2,p,b)) = 0$. Let

$F_1(p,b) = b - f_2(2,p,b)$. Then, $v_1(p, F_1(p,b)) = v_2(p, f_2(2,p,b))$ and $F_1(p,b) + f_2(2,p,b) = b$. We want to show that $F_1(\cdot,\cdot)$ and $f_2(2,\cdot,\cdot)$ are continuous. Notice that $f_2(2,p,b) = \arg\max_{x\in[0,b]} -|D_2(p,b,x)|$. Since v_1 and v_2 are continuous, so is $D_2 : \Delta \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$. By Berge's maximum theorem [Debrue (1959, p.19)], $f_2(2,\cdot,\cdot)$ and $F_1(\cdot,\cdot)$ are continuous.

Now, let $\ell \in \{2,\ldots,n-1\}$ be arbitrary. Suppose that for any $p \in \Delta$ and $b \in [0,a]$, there exists $(f_\ell(1,p,b),\ldots,f_\ell(\ell,p,b)) \in \mathbf{R}^\ell$ such that $v_1(p, f_\ell(1,p,b)) = \cdots = v_\ell(p, f_\ell(\ell,p,b))$, $b = \sum_{i=1}^{\ell} f_\ell(i,p,b)$ and $f_\ell(i,\cdot,\cdot)$ is continuous for all $i \in N$. Let $V_\ell(p,b) = v_\ell(p, f_\ell(\ell,p,b))$. Then, by the inductive hypothesis, $V_\ell : \Delta\times\mathbf{R}_+ \to \mathbf{R}$ is continuous and $V_\ell(p,b) = v_1(p, f_\ell(1,p,b)) = \cdots = v_\ell(p, f_\ell(\ell,p,b))$. For any $p \in \Delta$ and $b \in [0,a]$, let $D_{\ell+1}(p,b,x) = v_{\ell+1}(p,x) - V_\ell(p,b-x)$. Since $D_{\ell+1}(p,b,\cdot)$ is continuous, by the intermediate value theorem and Assumption 4(1), there exists a unique point $f_{\ell+1}(\ell+1,p,b) \in [0,b]$ such that $D_{\ell+1}(p,b,f_{\ell+1}(\ell+1,p,b)) = 0$. Let $F_\ell(p,b) = b - f_{\ell+1}(\ell+1,p,b)$. Then, $v_{\ell+1}(p, f_{\ell+1}(\ell+1,p,b)) = V_\ell(p, F_\ell(p,b))$ and $f_{\ell+1}(\ell+1,p,b) + F_\ell(p,b) = b$. Since $f_{\ell+1}(\ell+1,p,b) = \arg\max_{x\in[0,b]} -|D_{\ell+1}(p,b,x)|$ and $D_2 : \Delta\times\mathbf{R}_+\times\mathbf{R}_+ \to \mathbf{R}$ is continuous, by Berge's maximum theorem, $f_{\ell+1}(\ell+1,\cdot,\cdot)$ and $F_\ell(\cdot,\cdot)$ are continuous. Hence, together with the inductive hypothesis, there exists $(f_\ell(1,p,F_\ell(p,b)),\ldots,f_\ell(\ell,p,F_\ell(p,b)), f_{\ell+1}(\ell+1,p,b)) \in \mathbf{R}^{\ell+1}$ such that $v_1(p, f_\ell(1,p,F_\ell(p,b))) = \cdots = v_\ell(p, f_\ell(\ell,p,F_\ell(p,b))) = v_{\ell+1}(p, f_{\ell+1}(\ell+1,p,b))$ and $b = \sum_{i=1}^{\ell} f_\ell(i,p,F_\ell(p,b)) + f_{\ell+1}(\ell+1,p,b)$.

To complete the proof, it is enough to take $\ell = n-1$ and $b = a$. Then, there exists $(f_{n-1}(1,p,F_{n-1}(p,a)),\ldots,f_{n-1}(n-1,p,F_{n-1}(p,a)), f_n(n,p,a)) \in \mathbf{R}^n$ such that

$$v_1(p, f_{n-1}(1,p,F_{n-1}(p,a))) = \cdots = v_{n-1}(p, f_{n-1}(n-1,p,F_{n-1}(p,a))) = v_n(p, f_n(n,p,a))$$

and $a = \sum_{i=1}^{n-1} f_{n-1}(i,p,F_{n-1}(p,a)) + f_n(n,p,a)$. Now, we define $y : N \times \Delta \times \mathbf{R}_+ \to \mathbf{R}$ by $y(n,p,a) = f_n(n,p,a)$ and $y(i,p,a) = f_{n-1}(i,p,F_{n-1}(p,a))$ for each $i \neq n$. It is already shown that, for each $i \in N$, the function $y(i,\cdot,\cdot)$ is continuous. ∎

Lemma 5 *For each $i \in N$, $\hat{B}_i : \Delta \longrightarrow\!\!\!\!\rightarrow \bar{X}$ defined by $\hat{B}_i(p) \equiv B(p, y(i,p,p\cdot\bar{x}))$ is continuous.*

Proof. By the claim in Debreu (1959, p.63), it is enough to show that $y(i,p,p\cdot\bar{x}) > 0$ for any $i \in N$ and $p \in \Delta$. Suppose $y(i,p,p\cdot\bar{x}) = 0$ for some $i \in N$ and $p \in \Delta$. Then, by

Assumption 4(3) and the definition of the function y, we have $y(j, p, p \cdot \bar{x}) = 0$ for all $j \in N$, which contradicts that $\sum_{i \in N} y(i, p, p \cdot \bar{x}) = p\bar{x} > 0$. ∎

Theorem 3 *Under Assumption 3 and 4, there exists a Pareto optimal allocation $x^* \in X^n$ such that $u(i, x_i^*) = u(j, x_j^*)$ for all $i, j \in N$.*

Proof. We use the standard proof for the existence of a competitive equilibrium in exchange economies and thus we only outline our proof. See Debreu (1959) for more details. For each $i \in N$, define $\xi_i : \Delta \longrightarrow\!\!\!\!\!\rightarrow \bar{X}$ by $\xi_i(p) = \arg\max_{x \in \hat{B}_i(p)} u(i, x)$. By Lemma 5, it is easy to check that ξ_i is a nonempty- and convex-valued upper hemicontinuous correspondence. Let $X' = \sum_{i \in N} \bar{X} - \bar{x}$. It is easy to check that X' is nonempty, compact and convex. Define $\phi : \Delta \rightarrow X'$ by $\phi(p) = \sum_{i \in N} \xi_i(p) - \bar{x}$. Define $\psi : X' \rightarrow \Delta$ by $\psi(z) = \arg\max_{q \in \Delta} q \cdot z$. Define $\Psi : \Delta \times X' \longrightarrow\!\!\!\!\!\rightarrow \Delta \times X'$ by $\Psi(p, z) = \phi(p) \times \psi(z)$. Following the proof for the claim of Debreu (1959, 5.6), it is routine to show by the Kakutani's fixed point theorem that there exists $x^* \in X^n$ and $p^* \in \Delta$ such that $x_i^* \in \xi_i(p^*)$ for each $i \in N$ and $\sum_{i \in N} x_i^* \leq \bar{x}$. Since $x_i^* \in \arg\max_{x \in B(p^*, y(i, p^*, p^* \cdot \bar{x}))} u(i, x)$ for each $i \in N$, we have $u(i, x_i^*) = u(j, x_j^*)$ for all $i, j \in N$. Since $(x^*, p^*) \in X^n \times \Delta$ is a "competitive equilibrium", it is easy to check that the allocation x^* is Pareto optimal under Assumption 4 [e.g., Debreu (1959, 6.3)]. ∎

Now we are ready to provide sufficient conditions under which there exists a subjectively envy-free allocation. The next condition requires that each individual should not overestimate the well-being of other individuals.

Assumption 5 *For each $i \in N$ and $j \neq i$, if $(i, x_i) \sim (j, x_j)$ then $(i, x_i) \succeq_i (j, x_j)$.*

Theorem 4 *Given $(\succeq_1, \ldots, \succeq_n)$, if there exists a welfare ordering $\succeq$ which satisfies Assumption 3-5, then $S(\succeq_1, \ldots, \succeq_n) \neq \emptyset$.*

Proof. By Theorem 3, there exists $x^* \in \hat{X}$ such that $(i, x_i^*) \sim (j, x_j^*)$ for all $i, j \in N$. By Assumption 5, $(i, x_i^*) \succeq_i (j, x_j^*)$ for all $i, j \in N$. Hence, for each $i \in N$, $x^* \in S_i(\succeq_i)$, i.e., $x^* \in S(\succeq_1, \ldots, \succeq_n)$. ∎

5 Conclusion

The subjectively envy-free allocation proposed here is closely related with a discussion by Sen (1970, Section 9) concerning Suppes' (1966) model of "grading principle" which corresponds to the Foley's envy-free principle in our context. Sen argued as follows. Even though the grading principle avoids cardinalization and interpersonal comparisons, "there is no requirement in the Suppes model that one must take on the subjective features (in particular, tastes) of the other when one places oneself in his position." The problems caused by this nature of the model, however, can be easily removed by the requirement that "[p]lacing oneself in the position of the other should involve not merely having the latter's objective circumstances but also identifying oneself with the other in terms of his subjective features."

Our point of view in this paper is that it is difficult to correctly place oneself in the position of the other especially concerning the subjective features. This is why we introduced the notion of subjectively envy-free allocation and why information plays a role in our theory of fairness. By introducing the notion of subjectively envy-free allocations, we studied how information can be related to the concepts of fairness and showed that the maximin allocation will be accepted by individuals if they have complete information on others but if they do not know much about others then the envy-free allocation will be the one acceptable to them. We also showed sufficient conditions for the existence of a subjectively envy-free allocation using the existence result for a welfare-egalitarian allocation.

Despite some difficulties associated with the existence of a subjectively envy-free allocation, we believe that our implicit assumption that the social planner's task is to find an allocation that every individual feels fair is a natural one in democratic societies and hope that our approach will be developed further to handle the cases in which there is no subjectively envy-free allocation.

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