Department of Economics
and
Institute of Policy Analysis
University of Toronto
150 St. George Street
Toronto, Ontario M5S 1A1
Canada

WORKING PAPER
NUMBER UT-ECIPA-YAMASHIG-95-01

Bayesian Approach with Finite Hierarchies of Beliefs: Bounded Rationality in Strategic Form Games

by

Shinji Yamashige

September 9, 1995

ISSN 0829-4909

Author's e-mail: yamashig@epas.utoronto.ca
On-line Version: http://www.epas.utoronto.ca:5680/wpa/wpa.html

Bayesian Approach with Finite Hierarchies of Beliefs: Bounded Rationality in Strategic Form Games

Shinji Yamashige* †
University of Toronto

September 9, 1995

JEL Classification: C72

Keywords: Bayesian Approach, Bounded Rationality, Game

Abstract: We present a Bayesian framework based on the finite hierarchies of beliefs to study the decision problems in strategic form games. The model allows us to analyze the behavior of bounded rational players as well as that of fully rational players. We define the degree of "boundedness" by the complexity of thought process (number of consistent layers of beliefs) and the "fully rational" player is simply a player of complexity-∞ whose thought process continues ad infinitum. We show conditions under which the set of "Bayesian strategies of complexity-k", strategies taken by the Bayesian player of complexity-k, coincides with the set of "kth order iteratively undominated strategies". The conditions are studied for iterative elimination of strictly and weakly dominated strategies. The results are applied to analyze the well known paradoxes in games studied by Rubinstein (1989) and Reny (1992).

*Address: Department of Economics, University of Toronto, 150 St. George Street, Toronto, Ontario, M5S 1A1, Canada. E-mail: yamashig@epas.utoronto.ca

†The earlier versions were presented at University of Toronto and Canadian Economic Theory Conference at McGill University. I am grateful for the comments by the participants. I wish to thank especially for the comments by Professors L. Epstein, E. Kalai, A. Mukherji, and M. Peters, which greatly improved the paper. I am also grateful to Professor J. Harrignton for insightful conversations. I am solely responsible for errors.

1 Introduction

In the decision theoretic studies of games [e.g., Harsanyi (1967-68), Armbruster-Böge (1979), Mertens-Zamir (1985), Tan-Werlang (1987)], players are assumed to consider other players' infinite hierarchies of beliefs and form beliefs over them. I have no doubt that such formal studies have provided a deep mathematical foundation to analyze noncooperative games played by "rational" players. At the same time, however, I cannot but ask myself if we are smart enough to consider such a space of infinite hierarchies of beliefs when we play games. In this paper, we present a formal decision theoretic model based on the finite hierarchies of beliefs to study our behavior in strategic form games. The model allows us to study the behavior of "bounded rational" players as well as that of "fully rational" players.[1] Hence, it can be seen as a Bayesian foundation for games with bounded rational players which have become popular recently in game theory.

Our point of view[2] on the "bounded rationality" of players will be best summarized by the following quotation from Bernheim (1984): "Practically speaking, we might not expect agents to check the consistency of their beliefs for more than a finite number of levels (p. 1017)." We define the degree of "bounded rationality" by the number of consistent thought processes (layers of beliefs) that a player can regress in the decision problems associated with games. A fully rational player can be defined as a player who can regress ad infinitum consistently. As is suggested above, most players do not seem to be fully rational. For example, in playing the chess most players consider their moves for certain number of steps but they do not likely to continue such thought experiment ad infinitum.[3]

The framework is based on the "Bayesian approach" in which best responses of players are studied in terms of the players' beliefs. For example, studies on iteratively undominated strate-

[1]We use the terminology "(Bayesian) rationality" to mean that a player is an expected utility maximizer [cf. Tan-Werlang (1992) and Reny (1992)]. Hence, the bounded rational player is a rational player with limited capability of constructing a consistent hierarchies of beliefs whereas the fully rational player is a rational player with the unlimited capability.

[2]This view is adopted by Harrington (1987) to explain the paradox in the finitely repeated game.

[3]In finite extensive games with perfect information our results predict, roughly speaking, that bounded rational players have some ideas about how to behave in the last several stages of the game (due to the iterative elimination of weakly dominated strategies, which corresponds to the backward induction) but such ideas do not extend to the early stages to consider the behavior in the beginning of the game. See Section 5.2 for such an example.

gies or rationalizable strategies are the examples of this approach where "strategies" (not the "equilibria") are the subjects of interests [cf. Bernheim (1984, 1986), Pearce (1984), Tan-Werlang (1987), Hammond (1993)].[4] In the traditional Bayesian approach, each player is assumed to have an infinite hierarchy of beliefs called a "type" which itself can be seen as a belief over the types of others [e.g., Tan-Werlang (1987, Section 3)]. In this paper, we do not construct type spaces but work with the finite hierarchies of beliefs.[5] We find that such a construction not only works well to study the behavior of both bounded rational players and fully rational players but also frees us from the heavy mathematics.

It seems useful to point out that our approach is different from Kreps, *et al.* (1982) and Reny (1993) in which full rationality of each player is assumed but the common knowledge assumption about the Bayesian rationality is questioned to explain paradoxes in some games.[6] Whether we should question the full rationality of each player or the common knowledge of Bayesian rationality may be answered in the experiments in which players play a game after they are instructed to behave as an expected utility maximizer and publicly informed of the Bayesian rationality of each player. If the public announcement does not stop the emergence of the paradoxical results then we should indeed question the full rationality of the players.[7] Despite this difference, there is also a close relationship. In our framework, it is argued that even if a player is fully rational the player may believe that other players are bounded rational, which is exactly the situation that

[4]Relationship between these two approaches (especially conditions on beliefs under which we can get certain equilibrium strategy as the best response) was extensively studied by Tan-Werlang (1987). See also Remark 3.

[5]This approach is partly motivated by the fact that once we adopt the assumption that players' beliefs are consistent only up to kth order beliefs (k finite), we do not know if we can construct the type spaces.

[6]The argument by Reny (1993) [cf. also Reny (1992)] is much deeper than the one given here. His main insight is that there are many extensive form games in which common knowledge of the Bayesian rationality becomes impossible. It is also useful to point out that our study can be seen as a Bayesian foundation or a bounded rationality interpretation of the approach taken by Harrington (1987) to explain the cooperative behavior in the finitely repeated prisoner's dilemma game.

[7]For example, each player may be instructed to play a strategic form game with a computer which is programmed to behave as a fully rational player and is programmed to believe that the fully rationality of each player is common knowledge. In my experiences, when I ask my friends how they would behave in the e-mail game and the TOL game which we study in Section 5 as examples that exhibit counterintuitive results, their answers are typically not the ones that the logic (inductive argument) implies. Furthermore, they are usually surprised at my explanation about the logical solutions to the problems (e.g., how the computer above would behave in the games) even though they finally show their understanding to the logic with some mixed feeling that they will probably not use the logic if other players are human. We think that these observations indicate that we are not fully rational and that we have some doubt about the full rationality of other players.

the studies above are interested in. Hence, our model provides a Bayesian framework to analyze various problems of games in which some players may not be fully rational.[8]

In the next section, we first discuss the notion of "kth order iteratively undominated strategy" (and some related notions) because one of our main results can be seen as the Bayesian foundation for this notion. In Section 3, we present our framework and introduce the notion of "Bayesian strategies of complexity-k" which can be seen as a decision of the bounded rational players of complexity-k. Then, we identify the conditions under which the set of Bayesian strategies of complexity-k coincides with that of kth order iteratively undominated strategies. It is also shown that the decision problems of the fully rational players can be studied based on the sequence of finite hierarchies of beliefs rather than the type space (a space of infinite hierarchies of beliefs). In Section 4, we extend our framework to consider the iterative elimination of weakly dominated strategies, which sometimes reasonably "refines" the set of optimal strategies of the bounded rational players.[9] We show conditions under which the set of "cautious" Bayesian strategies of complexity-k coincides with that of kth order iteratively undominated "cautious" strategies. To show the usefulness and new aspects of our approach, in Section 5, we apply our results to the games in which paradoxical results are reported. Section 6 concludes the paper.

Throughout the paper, we use the notation $\mathcal{M}_+^1(Z)$ to represent a space of probability measures on a measurable space $(Z, \mathcal{Z})$. For any $\mu \in \mathcal{M}_+^1(Z)$, supp $\mu \subset Z$ represents the support of μ, i.e., the smallest measurable set on which μ is equal to 1. Also, for any collection of elements $\{Z_i\}_{i=1}^n$, we use the notation Z_{-i} to mean the "collection" of all elements other than Z_i.[10] We also use the notation $\mathbf{N} = \{1, 2, \ldots\}$.

[8] For example, Kreps, *et al.* (1982) and Reny (1993) did not ask why some players may believe that other players may be some particular bounded rational player, e.g., Tit-for-Tat player in the finitely repeated game. It is our hope that our framework can be used to study the general reasonable assumption about the bounded rational behavior, which induces particular behavior (e.g., Tit-for-Tat) in different types of games.

[9] This extension is closely related to the notion studied by Pearce (1984) and is useful to analyze the backward induction (subgame perfect equilibrium) type behavior in extensive games.

[10] $Z_{-i} = \prod_{j \neq i} Z_j$ or $Z_{-i} = (Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_n)$ depending on the context.

2 Preliminaries: Iteratively Undominated Strategy

Imagine any game in which players are uncertain about what others will play.[11] Faced with such a game, the traditional Bayesian approach [e.g., Tan-Werlang (1987)] assumes that each player forms a belief over the space of infinite hierarchies of beliefs of other players.[12] But do we construct such spaces and form a belief over such spaces? Let us consider a decision problem of Player-1 in the following strategic form game in which two players take actions simultaneously.

Table 1: Game I

Player-1 \ Player-2	A	B
A	$3, 3$	$1, 4$
B	$1, 1$	$2, 2$
C	$0, 2$	$0, 1$

Suppose that you are Player-1. Assume that the game is public information and that you do not know which action Player-2 will take. The following questions will be the first ones that you would ask yourself. If Player-2 plays A what should I play? If Player-2 plays B what should I play? Which action is likely to be taken by Player-2? The following would be the answers to the first two questions. If Player-2 plays A then A is optimal. If Player-2 plays B then B is optimal. Now the difficult question is the third one. You may have no clue about which action Player-2 will take. But you may conjecture that Player-2 will play B by the following reasoning. Since Player-2 knows the game, Player-2 can also find my best response for each of his own actions and realize that I will not play C (because it is not a best response against any Player-2's action) and thus it is optimal for him to play B (because it is the dominant strategy when Player-1's action is either A or B). Notice that this reasoning is based on the Player-1's belief that [B1] Player-2 is rational and [B2] Player-2 believes that Player-1 is rational. (The meaning of players' being rational will be formalized later.)

[11] Most games will be of this type; otherwise, the game theory will not be interesting to analyze because the problem reduces to a simple decision problem.

[12] Although the problem of the infinite hierarchies of beliefs has been studied in the context of the so-called games with incomplete information [e.g., Harsanyi (1967-68)], it also shows up in games with complete information. For example, Tan-Werlang (1987) only considered the games with complete information.

Notice that here we only considered the finite hierarchies of beliefs to find an optimal strategy which is also predicted by the traditional approach with infinite hierarchies of beliefs and the common knowledge of rationality. Namely, the formulation of the traditional approach may be too complex to describe our thought processes in many games. Moreover, some players may indeed have difficulties to construct the infinite hierarchies of consistent beliefs. Hence, in this paper, we propose a decision theoretic model without the type space (a space of infinite hierarchies of beliefs) and study the decision problems based on the finite hierarchies of beliefs. One of our goals is to provide a decision theoretic foundation for the thought processes of the kind studied above, which we now formalize.

Let us define the notion of "kth order iteratively undominated strategy", which is a set of strategies that remain after $k-1$ iterative elimination of strictly dominated strategies. Let $I = \{1, 2, \ldots, n\}$ be a set of players. Let Ω be a space of states of nature. Let A_i be a space of strategies for $i \in I$. We assume that Ω and A_i's are endowed with some σ-algebras. Let $v_i^\omega : A_i \times A_{-i} \to \mathbf{R}$ be the payoff function of player $i \in I$ when the state of nature is $\omega \in \Omega$. For each $i \in I$, let $A_i^0 = A_i$ and for each $k \in \{1, 2, 3, \ldots\}$ define inductively

$$A_i^k = \{a_i \in A_i \mid \exists \nu_i \in \mathcal{M}_+^1(\Omega \times A_{-i}^{k-1}) : a_i \in \arg\max_{a \in A_i} \int_{\Omega \times A_{-i}} v_i^\omega(a, a_{-i}) \mathrm{d}\nu_i\}$$

To show the nature of the sets defined above, suppose $k = 1$. If $a_i \notin A_i^1$ there is no possibility that a_i can be a best response in a game when others can take actions in A_{-i}. Such an action is called a strictly dominated strategy. Hence, A_i^1 is a set of actions each of which is a best response for some belief on the action profiles A_{-i}. Such an action is called an undominated strategy. Now suppose $k = 2$. If $a_i \notin A_i^2$ then there is no possibility that a_i can be a best response in a game when others can take actions in A_{-i}^1. Hence, A_i^2 is a set of actions each of which is a best response for some belief on the action profiles in A_{-i}^1. By continuing this process of eliminating dominated strategies we obtain the set A_i^m that is a set of undominated strategies of player $i \in I$ after other players' mth elimination of the dominated strategies.

Definition 1 *A strategy $a_i \in A_i^k$ is said to be a kth order iteratively undominated strategy. A strategy $a_i \in A_i^\infty \equiv \bigcap_{k \in \mathrm{N}} A_i^k$ is said to be an iteratively undominated strategy.*

Remark 1. In Game I, the kth order iteratively undominated strategies and the iteratively undominated strategies can be obtained as in Table 2. □

Table 2: Iteratively Undominated Strategies in Game I

	A_1^k	A_2^k
$k = 0$	$\{A, B, C\}$	$\{A, B\}$
$k = 1$	$\{A, B\}$	$\{A, B\}$
$k = 2$	$\{A, B\}$	$\{B\}$
$k = 3$	$\{B\}$	$\{B\}$
$k \geq 4$	$\{B\}$	$\{B\}$

Remark 2 (Rationalizable Strategy). If the definition of A_i^m is replaced by $R_i^0 = A_i$ and

$$R_i^k = \{a_i \in A_i \mid \exists \nu_i \in \mathcal{M}_+^1(\Omega) \times \prod_{j \neq i} \mathcal{M}_+^1(R_j^{k-1}) : a_i \in \arg\max_{a \in A_i} \int_{\Omega \times A_{-i}} v_i^\omega(a, a_{-i}) \mathrm{d}\nu_i\},$$

then the strategy $a_i \in \bigcap_{k \in \mathrm{N}} R_i^k$ is called a rationalizable strategy [Bernheim (1984)]. Notice that, in the definition, a belief is assumed to be the product of the beliefs over other players' actions, which reflects the idea that players make decisions independently. In the two person game, it is obvious that $A_i^k = R_i^k$ for all $k \in \mathbf{N}$. If the definition of each R_i^k is replaced by $\hat{R}_i^0 = A_i$ and

$$\hat{R}_i^k = \{a_i \in A_i \mid \exists a_{-i} \in \hat{R}_{-i}^{k-1} : a_i \in \arg\max_{a \in A_i} v_i^\omega(a, a_{-i})\},$$

then the strategy $a_i \in \bigcap_{k \in \mathrm{N}} \hat{R}_i^k$ is called a point rationalizable strategy since Player-i is assumed to have point beliefs. A strategy $a_i \in \hat{R}_i^k$ is called a k-point rationalizable strategy. □

Remark 3 (Equilibrium Approach vs. Bayesian Approach). The study of the iteratively undominated strategies, for example, can be considered a Bayesian approach because it tries to find the strategies that the rational players will choose under certain assumptions on their beliefs. Even though there is a close relationship between the equilibrium approach and the Bayesian approach [e.g., Aumann (1987) and Tan-Werlang (1987)], there are also many differences. For example, as Bernheim (1986) discussed, there are games in which there is a unique Nash equilibrium but the rationality of players and the common knowledge of rationality do not imply at all that each player chooses the unique Nash equilibrium strategy. The game in Table 3 is another such example. (The pair (C, C) is the unique (mixed-strategy) Nash equilibrium, as Fudenberg-Tirole (1992, p.39) shows, but every strategy is rationalizable for each player.) $\square$

Table 3: A Game with Unique Nash Equilibrium

Player-1 \ Player-2	A	B	C
A	1, -2	-2, 1	0, 0
B	-2, 1	1, -2	0, 0
C	0, 0	0, 0	1, 1

3 Bayesian Approach with Finite Hierarchies of Beliefs

In this section, we introduce the notion of "Bayesian strategy of complexity-k" which can be seen as a decision of the bounded rational players. By giving up the notion of "types", our construction becomes mathematically easier and more flexible than the traditional approach [e.g., Tan-Werlang (1987)]. Especially, it is worth pointing out that we do not need topological assumptions on Ω and A_i's nor the "fundamental mathematical result" [e.g., Tan-Werlang (1987, Theorem 3.1)] which establishes the existence of the homeomorphism from the type space to a space of probability measures over types of others. We show conditions under which the set of Bayesian strategies of complexity-k coincides with that of kth order iteratively undominated strategies.

3.1 Bayesian Rationality

Let $I = \{1, 2, \ldots, n\}$ be a set of players. Let Ω be a space of states of nature. For each $i \in I$, let A_i be an action space. We endow Ω and A_i's with some σ-algebras. Let $v_i^\omega : A_i \times A_{-i} \to \mathbf{R}$ be the payoff function of $i \in I$ when the state of nature is $\omega \in \Omega$. For each $\omega \in \Omega$, a game is represented by a triple $[I, \{A_i\}_{i \in I}, \{v_i^\omega\}_{i \in I}]$. Hence, a game with incomplete information is a quadruple $\Psi = [I, \{A_i\}_{i \in I}, \{v_i^{(\cdot)}\}_{i \in I}, \Omega]$.

Let $B_i^1 = \mathcal{M}_+^1(\Omega \times A_{-i})$ be a space of Player-i's beliefs over states of nature and actions of others. A player is said to be Bayesian rational if he/she acts to maximize the expected utility.

Definition 2 *Player-i is said to be Bayesian rational if he/she has a belief $\tau_i \in \mathcal{M}^1_+(\Omega \times A_{-i})$ and chooses an action $a^* \in A_i$ such that*

$$a^* \in \arg\max_{a \in A_i} \int_{\Omega \times A_{-i}} v_i^{\omega}(a, a_{-i}) \mathrm{d}\tau_i.$$

3.2 Bayesian Player of Complexity-k

The question is how the belief $\tau_i \in \mathcal{M}^1_+(\Omega \times A_{-i})$ is formed. We will illustrate below that the answer depends on how deep the player can analyze a game, i.e., it depends on the player's complexities of the thought process. Suppose that Player-i believes that other players are also Bayesian rational. Then he realizes that Player-j's action is described by $\beta_j^1 : B_j^1 \longrightarrow\!\!\rightarrow A_j$ defined by $\beta_j^1(\mu_j^1) = \arg\max_{a \in A_j} \int_{\Omega \times A_{-j}} v_j^{\omega}(a, a_{-j}) \mathrm{d}\mu_j^1$ for each belief $\mu_j^1 \in B_j^1$. Hence, the Player-i realizes that in order to predict the behavior of other players he must know the belief of other players. Namely, the space B_j^1 becomes the source of uncertainty, which motivates the new space of uncertainty. We endow B_j^1 with a σ-algebra which contains each element $\mu_j^1 \in B_j^1$. Let $B_{-j}^1 = \prod_{i \neq j} B_i^1$ and $B_j^2 = \mathcal{M}^1_+(\Omega \times A_{-j} \times B_{-j}^1)$ for each $j \in I$. Now, suppose Player-i believes that Player-j believes Player-i is Bayesian rational. Then Player-i realizes that Player-j assumes that Player-i's action is determined by the mapping $\beta_i^1 : B_i^1 \longrightarrow\!\!\rightarrow A_i$ and thus Player-i realizes that Player-j's action is determined by the mapping $\beta_j^2 : B_j^2 \longrightarrow\!\!\rightarrow A_j$. Namely, the space B_j^2 becomes the source of uncertainty, which motivates the space $B_i^3 = \mathcal{M}^1_+(\Omega \times A_{-i} \times B_{-i}^2)$.

Needless to say, if Player-i believes that Player-j believes Player-i believes that ... Player-j is Bayesian rational, then the belief over the other players' higher order beliefs becomes relevant, which motivates the following iterative definitions. For each $i \in I$ and $m \in \{2, 3, \ldots\}$, let $B_i^m = \mathcal{M}^1_+(\Omega \times A_{-i} \times B_{-i}^{m-1})$, where we endow B_i^m with a σ-algebra which contains each element in B_i^m, and let $B_{-i}^m = \prod_{j \neq i} B_j^m$. To handle the case of full rationality, it is useful to define a space $T_i^{\infty} \equiv \prod_{m=1}^{\infty} B_i^m$, i.e., a space of infinite hierarchies of beliefs. Let $\mu_i = (\mu_i^1, \mu_i^2, \ldots)$ be a generic notation for an element in T_i^{∞}. Although we work with this space of infinite hierarchies of beliefs to accommodate fully rational players, it is important to keep in mind that we will only consider beliefs over the finite hierarchies of beliefs and never consider beliefs over the infinite hierarchies of beliefs (even for the fully rational players). It is in this sense that we call our framework "a

Bayesian approach with finite hierarchies of beliefs".[13]

We now impose a consistency condition on the hierarchy of beliefs, which requires that the higher order belief and the lower order belief have the same probability on any event that both can evaluate. For example, it is natural to require $\mu_i^2(L \times B_{-i}^1) = \mu_i^1(L)$ for any measurable set $L \subset \Omega \times A_{-i}$. We define a collection of "projection mappings" inductively as follows. For each $i \in I$, define $\phi_i^1 : B_i^2 \to B_i^1$ by $\phi_i^1(\mu_i^2)(L) = \mu_i^2(L \times B_{-i}^1)$ for any measurable set $L \subset \Omega \times A_{-i}$. For each $i \in I$ and for each $m \in \{2, 3, \ldots\}$, define $\phi_i^m : B_i^{m+1} \to B_i^m$ inductively by

$$\phi_i^m(\mu_i^{m+1})(L) = \mu_i^{m+1}(\{(\omega, a_{-i}, \mu_{-i}^m) \in \Omega \times A_{-i} \times B_{-i}^m \mid (\omega, a_{-i}, (\phi_j^{m-1}(\mu_j^m))_{j \neq i}) \in L\})$$

for arbitrary measurable set $L \subset \Omega \times A_{-i} \times B_{-i}^m$. Let $\hat{T}_i^1 = T_i^\infty$ and for any $k \in \{2, 3, \ldots\}$ let

$$\hat{T}_i^k = \{\mu_i \in T_i^\infty \mid \mu_i^{m-1} = \phi_i^{m-1}(\mu_i^m) \; \forall m = 2, \ldots, k\},$$

be a set of beliefs that satisfy the consistency up to kth level for the player. This consistency condition also allows us to focus on the highest order belief of each player.

Now, we consider Bayesian rational players who have limited ability to construct consistent hierarchies of beliefs and make decisions based on the beliefs over the finite hierarchies of beliefs of others. For each $\mu_i^k \in \mathcal{M}_+^1(\Omega \times A_{-i} \times B_{-i}^{k-1})$, let $[\mu_i^k]_{\Omega \times A_{-i}} \in \mathcal{M}_+^1(\Omega \times A_{-i})$ be the marginal[14] of μ_i^k on $\Omega \times A_{-i}$.

Definition 3 *Player-i is said to be a Bayesian player of complexity-k ($k \in \{1, 2, \ldots\}$) or said to be bounded rational if he/she has a belief $\mu_i \in \hat{T}_i^k$ and chooses an action*

$$a_i \in \hat{\beta}_i^k(\mu_i) \equiv \arg\max_{a \in A_i} \int_{\Omega \times A_{-i}} v_i^\omega(a, a_{-i}) \mathrm{d}[\mu_i^k]_{\Omega \times A_{-i}}$$

which we refer to as a Bayesian strategy of complexity-k.

[13] Furthermore, the spaces $\{T_i^\infty\}_{i \in N}$ are not usually regarded as the type spaces because they may be incomplete to describe the uncertainty of each player in the sense that beliefs over T_i^∞ still can be the source uncertainty. Hence, in order to claim that a collection of spaces are the type spaces, we have to establish a homeomorphism from the space of types to the space of beliefs over the types of others to make sure that all the uncertainty are embedded in the spaces [cf. Tan-Werlang (1987, Section 3)]. It is an advantage of our model that we do not need heavy mathematics to establish such a result.

[14] Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measurable spaces. A marginal of $\nu \in \mathcal{M}_+^1(X \times Y)$ on X is a probability measure $[\nu]_X \in \mathcal{M}_+^1(X)$ defined by $[\nu]_X(K) = \nu(K \times Y)$ for each $K \in \mathcal{X}$.

We call a Bayesian player of complexity-k (k finite) a bounded rational player because he/she is rational (in the sense of Definition 2) but his/her ability to construct consistent hierarchies of beliefs is bounded. Notice that for any finite k, a Bayesian player of complexity-k can never be "fully rational" since he/she realizes that the decisions are based on the incomplete consideration of the beliefs of other players. It is also useful to formally define the fully rational players. Notice that there is no need to define the type spaces on which the beliefs are to be defined.

Definition 4 *Player-i is said to be a Bayesian player of complexity-∞ or said to be fully rational if he/she has a belief $\mu_i \in \bigcap_{k=1}^{\infty} \hat{T}_i^k$ and chooses an action $a_i \in \bigcap_{k=1}^{\infty} \hat{\beta}_i^k(\mu_i)$ which we refer to as a Bayesian strategy of complexity-∞.*

The definitions above allow us to have a useful framework to study the behavior of bounded rational players in the sense that their behavior can be summarized by the mapping $\hat{\beta}_i^k$ dependent on their beliefs. Hence, their behavior can be characterized in terms of our assumptions on their beliefs. In the following, we indeed show that under an assumption which restricts the set of hierarchies of beliefs to a smaller set $\hat{K}_i^k \subset \hat{T}_i^k$, the set of all possible Bayesian strategies of complexity-k is exactly equal to the set of kth order iteratively undominated strategies. Needless to say, under different assumptions, we have different characterization of the behavior for bounded rational players. Our approach may be summarized in Figure 1.

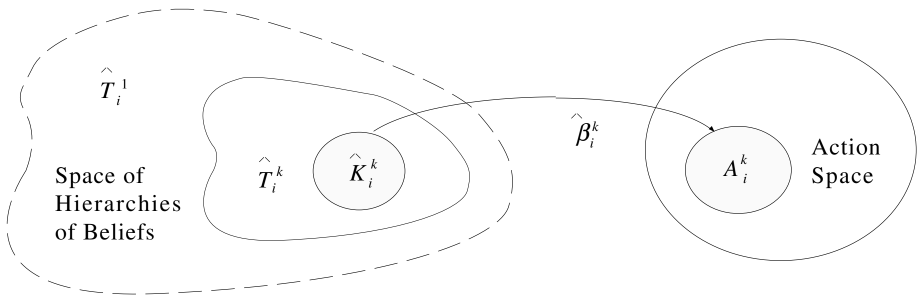


Figure 1: Bayesian Strategies of Complexity-k

3.3 kth Order Common Knowledge of Bayesian Rationality and Consistency

Now, we consider a concept which is weaker than the common knowledge of rationality. The reason for why we need the following notion is simple: If a player is of complexity-k the common knowledge of the Bayesian rationality is not a useful concept because the higher order beliefs about the rationality are of no use for the players whose beliefs are consistent only up to kth layer. In other words, even if the Bayesian rationality can be common knowledge (e.g., by the public announcement), due to the bounded rationality of players, it only has the same effect as the kth order common knowledge defined below. For each $i \in I$, let $\hat{K}_i^1 = \hat{T}_i^1$,

$$\hat{K}_i^2 = \{\mu_i \in \hat{T}_i^2 \mid (a_j, \mu_j^1) \in \text{supp } [\mu_i^2]_{A_j \times B_j^1} \Rightarrow a_j \in \arg\max_{a \in A_j} \int_{\Omega \times A_{-j}} v_j^\omega(a, a_{-j}) \mathrm{d}[\mu_j^1]_{\Omega \times A_{-j}}\}$$

and for each $m \in \{3, 4, \ldots, \}$,

$$\hat{K}_i^m = \{\mu_i \in \hat{T}_i^m \mid \mu_j^{\ell-1} \in \text{supp } [\mu_i^\ell]_{B_j^{\ell-1}} \ (\ell = 2, \ldots, m) \ \Rightarrow (\mu_j^1, \ldots, \mu_j^{m-1}, \ldots) \in \hat{K}_j^{m-1}\}.$$

The set $\hat{K}_i^2$ is a set of beliefs such that Player-i is of at least complexity-2 and knows that everyone is Bayesian rational. $\hat{K}_i^3$ is a set of beliefs such that Player-i is of at least complexity-3 and that everyone knows that everyone is of at least complexity-2 and is Bayesian rational. The higher order set $\hat{K}_i^m$ is inductively defined to formalize the idea that i knows that everyone knows ....

Definition 5 *Bayesian rationality and consistency are said to be kth order common knowledge for $i \in I$ if and only if $\mu_i \in \hat{K}_i^k$. Bayesian rationality and consistency are said to be common knowledge for $i \in I$ if and only if $\mu_i \in \bigcap_{k=1}^\infty \hat{K}_i^k$.*

Notice that by restricting ourselves to the set $\hat{T}_j^m$ in the definition of $\hat{K}_j^m$ for each $m \in \mathbf{N}$ the consistency up to $(k-1)$th layer of beliefs of others is assumed to be known to the player when $\mu_i \in \hat{K}_i^k$. Notice also that since $\mu_i \in \hat{K}_i^k$ if $\mu_i \in \hat{K}_i^{k+\ell}$ for any $\ell \in \mathbf{N}$, the better terminology may be to add "*at least*" before "kth order common knowledge" even though we usually do not add the phrase for simplicity.

3.4 Main Result

Although our framework itself is a general one to analyze the behavior of the bounded rational players under different assumptions on their beliefs, it seems useful to establish a benchmark result which predicts the behavior of the bounded rational players under the kth order common knowledge of Bayesian rationality and consistency. To be more specific, we show that the set of Bayesian strategies $\{\hat{\beta}_i^k(\mu_i) : \mu_i \in \hat{K}_i^k\} \subset A_i$ coincides with the "kth order iteratively undominated strategies" under the kth order common knowledge of the Bayesian rationality and the consistency. The proof for the next proposition is very similar to the one for Theorem 5.2 and Theorem 5.3 of Tan-Werlang (1987) which establish that the set of Bayesian strategies played by the fully rational player under the common knowledge of the Bayesian rationality coincides with the set of iteratively undominated strategies (cf. Corollary 1 below). Our proof verifies that each argument in their proof works well even if we do not have infinite hierarchies of beliefs and the common knowledge of Bayesian rationality.

Theorem 1 *Let $k \in \mathbf{N}$ be arbitrary. Then, $A_i^k = \{\hat{\beta}_i^k(\mu_i) \in A_i : \mu_i \in \hat{K}_i^k\}$, i.e., the set of kth order iteratively undominated strategies coincides with the set of Bayesian strategies of complexity-k if the Bayesian rationality and the consistency are kth order common knowledge for $i \in I$.*

Proof. Let $a_i \in \{\hat{\beta}_i^k(\mu_i) \in A_i : \mu_i \in \hat{K}_i^k\}$. Then, there exists $\mu_i \in \hat{K}_i^k$ such that $a_i \in \hat{\beta}_i^k(\mu_i)$, i.e., a_i is a best response for the belief $[\mu_i^k]_{\Omega \times A_{-i}}$. In order to show $a_i \in A_i^k$, it is enough to show that supp $[\mu_i^k]_{A_{-i}} \subset A_{-i}^{k-1}$. We prove by induction that, for any $j \in I$ and $m \in \mathbf{N}$, $\mu_j \in \hat{K}_j^m \Rightarrow$ supp $[\mu_j^m]_{A_{-j}} \subset A_{-j}^{m-1}$. Let $j \in I$ be arbitrary. Let $m = 1$ and assume $\mu_j \in \hat{K}_j^1$. Since $A_{-j}^0 = A_{-j}$ by definition, it is obvious that supp $[\mu_j^1]_{A_{-j}} \subset A_{-j}^0$. Let $m = 2$ and assume $\mu_j \in \hat{K}_j^2$. By the definition of $\hat{K}_j^2$, if $(a_h, \mu_h^1) \in$ supp $[\mu_j^2]_{A_h \times B_h^1}$ then $a_h \in \arg\max_{a \in A_h} \int_{\Omega \times A_{-h}} v_h^\omega(a, a_{-h}) \mathrm{d}\mu_h^1$. Since $\mu_h^1 \in \mathcal{M}_+^1(\Omega \times A_{-h})$ is a probability under which a_h is optimal, by the definition of A_h^1, j knows $a_h \in A_h^1$. Since this is true for all $h \neq j$, supp $[\mu_j^2]_{A_{-j}} \subset A_{-j}^1$. Now, let $m \in \{3, 4, \ldots\}$ and assume $\mu_j \in \hat{K}_j^{m-1} \Rightarrow$ supp $[\mu_j^{m-1}]_{A_{-j}} \subset A_{-j}^{m-2}$ for all $j \in I$. We want to show $\mu_j \in \hat{K}_j^m \Rightarrow$ supp $[\mu_j^m]_{A_{-h}} \subset A_{-j}^{m-1}$. By the definition of $\hat{K}_j^m$, if $\mu_j \in \hat{K}_j^m$ then $\mu_h^{\ell-1} \in$ supp $[\mu_j^\ell]_{B_h^{\ell-1}}$ ($\ell = 2, \ldots, m$) $\Rightarrow (\mu_h^1, \ldots, \mu_h^{m-1}, \ldots) \in \hat{K}_h^{m-1}$. By the inductive hypothesis, Player-j knows that supp $[\mu_h^{m-1}]_{A_{-h}} \subset A_{-h}^{m-2}$. Hence, if $a_h \in$ supp $[\mu_j^m]_{A_h}$ then $a_h \in A_h^{m-1}$ because $[\mu_h^{m-1}]_{\Omega\times A_{-h}} \in \mathcal{M}_+^1(\Omega \times A_{-h}^{m-2})$ is a probability under which a_h is optimal. Since this is true for all $h \neq j$, supp $[\mu_j^m]_{A_{-j}} \subset A_{-j}^{m-1}$. Finally, since $\mu_i \in \hat{K}_i^k$, the result implies supp $[\mu_i^k]_{A_{-i}} \subset A_{-i}^{k-1}$.

To show the converse, let $a_i \in A_i^k$. We have to show that there exists $\bar{\mu}_i \in \hat{K}_i^k$ such that $a_i \in \hat{\beta}_i^k(\bar{\mu}_i)$. First, let $k = 1$. Then, there exists $\nu_i^1 \in \mathcal{M}_+^1(\Omega \times A_{-i}^{m-1})$ such that $a_i \in \arg\max_{a\in A_i} \int_{\Omega\times A_{-i}^{m-1}} v_i^\omega(a, a_{-i})\mathrm{d}\nu_i^1$. Let $\hat{\mu}_{i1} = (\nu_i^1, \mu_i^2, \mu_i^3, \ldots)$ for arbitrary $(\mu_i^2, \mu_i^3, \ldots) \in \prod_{m=2}^\infty B_i^m$. Then, $\hat{m}u_{i1} \in \hat{K}_i^1$. Now, let $k \in \{2, 3, \ldots\}$. Since $a_i \in A_i^k$, there exist k probabilities $\nu_i^m \in \mathcal{M}_+^1(\Omega \times A_{-i}^{m-1})$ $(m = 1, 2, \cdots, k)$ and $(n-1)(k-1)$ probabilities $\nu_j^m \in \mathcal{M}_+^1(\Omega \times A_{-j}^{m-1})$ $(j \neq i$ and $m = 1, 2, \cdots, k-1)$ such that

$$a_i \in \arg\max_{a\in A_i^{m-1}} \int_{\Omega\times A_{-i}^{m-1}} v_i^\omega(a, a_{-i})\mathrm{d}\nu_i^m \tag{1}$$

and, for all $j \in I$ (including i),

$$\text{supp } [\nu_j^m]_{A_h} \subset \arg\max_{a\in A_h^{m-2}} \int_{\Omega\times A_{-h}^{m-2}} v_j^\omega(a, a_{-h})\mathrm{d}\nu_h^{m-1} \tag{2}$$

for all $h \neq j$ and $m = 1, 2, \cdots, k-1$. Let $\delta_{\{x\}}$ represent the dirac measure on a point x, i.e., the probability measure which has the unit mass on the point x. For each $j \in I$ (including i), $m \in \{1, 2, \cdots, k-1\}$, we can define $\hat{\mu}_{jm}^\ell$ inductively so that $\hat{\mu}_{jm}^1 = \nu_j^m$ and, for each $\ell \in \{2, \cdots, m\}$, $[\hat{\mu}_{jm}^\ell]_{\Omega\times A_{-j}} = \nu_j^m$ and $[\hat{\mu}_{jm}^\ell]_{B_{-j}^{\ell-1}} = \delta_{\{\hat{\mu}_{-j(m-1)}^{\ell-1}\}}$. Let $\hat{\mu}_{jm} = (\hat{\mu}_{jm}^1, \cdots, \hat{\mu}_{jm}^{m-1}, \mu_j^m, \mu_j^{m+1}, \cdots)$ where $(\mu_j^m, \mu_j^{m+1}, \cdots)$ is arbitrary. For Player-i, we also define $\hat{\mu}_{ik}$ by $\hat{\mu}_{ik}^1 = \nu_i^1$ and, for each $\ell \in \{2, \ldots, k\}$, $[\hat{\mu}_{ik}^\ell]_{\Omega\times A_{-i}} = \nu_i^1$ and $[\hat{\mu}_{ik}^\ell]_{B_{-i}^{\ell-1}} = \delta_{\{\hat{\mu}_{-i(k-1)}^{\ell-1}\}}$. Let $\bar{\mu}_i = \hat{\mu}_{ik}$.

We want to show $a_i \in \hat{\beta}_i^k(\bar{\mu}_i)$ and $\bar{\mu}_i \in \hat{K}_i^k$. In order to show $\bar{\mu}_i \in \hat{K}_i^k$, we prove by induction $\hat{\mu}_{jm} \in \hat{K}_j^m$ for all $j \in I$ and $m \in \{1, \cdots, k-1\}$. Let $m = 1$. It is trivial to check $\hat{\mu}_{j1} \in \hat{K}_j^1$. Let $m = 2$. To show $\hat{\mu}_{j2} \in \hat{T}_j^2$, we have only to show that $\hat{\mu}_{j2}^2(L \times B_{-j}^1) = \hat{\mu}_{j2}^1(L)$ for any measurable set $L \subset \Omega \times A_{-j}$, which is guaranteed by the definition $\hat{\mu}_{j2}^1 = [\hat{\mu}_{j2}^2]_{\Omega\times A_{-j}} = \nu_j^2$, i.e., $\hat{\mu}_{j2} \in \hat{T}_j^2$. Since $[\hat{\mu}_{j2}^2]_{A_h} = [\nu_j^2]_{A_h}$, $[\hat{\mu}_{j2}^2]_{B_h^1} = \delta_{\{\hat{\mu}_{h1}^1\}}$ and $\hat{\mu}_{h1}^1 = \nu_h^1$ for all $h \neq j$, to see

$$(a_h, \mu_h^1) \in \text{supp } [\hat{\mu}_{j2}^2]_{A_h\times B_h^1} \Rightarrow a_h \in \arg\max_{a\in A_h} \int_{\Omega\times A_{-h}} v_h^\omega(a, a_{-h})\mathrm{d}\mu_h^1$$

we have to show

$$\text{supp } [\nu_j^2]_{A_h} \subset \arg\max_{a\in A_h} \int_{\Omega\times A_{-h}} v_h^\omega(a, a_{-h})\mathrm{d}\nu_h^1.$$

The condition is indeed guaranteed by the condition (2) for $m = 2$.

Let $m \in \{3, \cdots, k-1\}$ be arbitrary and suppose $\hat{\mu}_{j(m-1)} \in \hat{K}_j^{m-1}$ for all $j \in I$. We show $\hat{\mu}_{jm} \in \hat{K}_j^m$ for all $j \in I$. Since supp $[\hat{\mu}^{\ell}_{j(m-1)}]_{B_h^{\ell-1}} = \hat{\mu}^{\ell-1}_{h(m-2)}$ for all $h \neq j$ and $\ell \in \{2, \cdots, m-1\}$, we know that the condition $\hat{\mu}_{j(m-1)} \in \hat{K}_j^{m-1}$, i.e., the condition

$$\mu_h^{\ell-1} \in \text{supp } [\mu^{\ell}_{j(m-1)}]_{B_h^{\ell-1}} \ (\ell = 2, \cdots, m-1) \Rightarrow (\mu_h^1, \cdots, \mu_h^{m-2}, \cdots) \in \hat{K}_h^{m-2} \ \forall h \neq j$$

implies

$$(\hat{\mu}^1_{h(m-2)}, \cdots, \hat{\mu}^{m-2}_{h(m-2)}, \mu_h^{m-1}, \mu_h^m, \cdots) \in \hat{K}_h^{m-2} \ \forall h \neq j \tag{3}$$

for arbitrary $(\mu_h^{m-1}, \mu_h^m, \cdots)$. By the same reason, in order to show $\hat{\mu}_{jm} \in \hat{K}_j^m$, we have only to show $(\hat{\mu}^1_{h(m-1)}, \cdots, \hat{\mu}^{m-1}_{h(m-1)}, \mu_h^m, \mu_h^{m+1}, \cdots) \in \hat{K}_h^{m-1}$ for all $h \neq j$ and for arbitrary $(\mu_h^m, \mu_h^{m+1}, \cdots)$. Since supp $[\hat{\mu}^{\ell}_{h(m-1)}]_{B_g^{\ell-1}} = \hat{\mu}^{\ell-1}_{g(m-2)}$ for all $g \neq h$ and $\ell \in \{2, \cdots, m-1\}$ by definition, the required condition is equivalent to $(\hat{\mu}^1_{g(m-2)}, \cdots, \hat{\mu}^{m-2}_{g(m-2)}, \mu_g^{m-1}, \mu_g^m, \cdots) \in \hat{K}_g^{m-2}$ for all $g \neq h$ and for arbitrary $(\mu_g^{m-1}, \mu_g^m, \cdots)$, which is guaranteed by the condition (3) for all $j \in I$. Hence,

$$(\hat{\mu}^1_{h(m-1)}, \cdots, \hat{\mu}^{m-1}_{h(m-1)}, \mu_h^m, \mu_h^{m+1}, \cdots) \in \hat{K}_h^{m-1} \ \forall h \in I \tag{4}$$

for arbitrary $(\mu_h^m, \mu_h^{m+1}, \cdots)$. To complete the proof, we have to show $\hat{\mu}_{jm} \in \hat{T}_j^m$. Since $\phi_j^1(\hat{\mu}^2_{jm})(L) = \hat{\mu}^1_{jm}(L)$ for arbitrary measurable set $L \subset \Omega \times A_{-j}$ by construction, we have only to show that, for any $\ell \in \{3, \ldots, m\}$, $\phi_j^{\ell-1}(\hat{\mu}^{\ell}_{jm})(L) = \hat{\mu}^{\ell-1}_{jm}(L)$ for an arbitrary measurable set $L \subset \Omega \times A_{-j} \times B_{-j}^{\ell-2}$. Let $L_{\Omega \times A_{-j}}$ and $L_{B_{-j}^{\ell-2}}$ be the projection of L on $\Omega \times A_{-j}$ and $B_{-j}^{\ell-2}$, respectively. Since supp $[\hat{\mu}^{\ell}_{jm}]_{B_h^{\ell-1}} = \hat{\mu}^{\ell-1}_{h(m-1)}$, supp $[\hat{\mu}^{\ell-1}_{jm}]_{B_h^{\ell-2}} = \hat{\mu}^{\ell-2}_{h(m-1)}$ by definition, and $\phi_h^{\ell-2}(\hat{\mu}^{\ell-1}_{h(m-1)}) = \hat{\mu}^{\ell-2}_{h(m-1)}$ by the result (4) above, if $L_{B_{-j}^{\ell-2}} \bigcap \{\hat{\mu}^{\ell-2}_{-j(m-1)}\} \neq \emptyset$ then

$$\begin{aligned} \phi_j^{\ell-1}(\hat{\mu}^{\ell}_{jm})(L) &\equiv \hat{\mu}^{\ell}_{jm}(\{(\omega, a_{-j}, \mu_{-j}^{\ell-1}) \in \Omega \times A_{-j} \times B_{-j}^{\ell-1} \mid (\omega, a_{-j}, (\phi_h^{\ell-2}(\mu_h^{\ell-1}))_{h \neq j}) \in L\}) \\ &= [\hat{\mu}^{\ell}_{jm}]_{\Omega \times A_{-j}}(L_{\Omega \times A_{-j}}) = [\hat{\mu}^{\ell-1}_{jm}]_{\Omega \times A_{-j}}(L_{\Omega \times A_{-j}}) = \hat{\mu}^{\ell-1}_{jm}(L) \end{aligned}$$

and if $L_{B_{-j}^{\ell-2}} \bigcap \{\hat{\mu}^{\ell-2}_{-j(m-1)}\} = \emptyset$ then $\phi_j^{\ell-1}(\hat{\mu}^{\ell}_{jm})(L) = 0 = \hat{\mu}^{\ell-1}_{jm}(L)$. Hence, $\hat{\mu}_{jm} \in \hat{T}_j^m$ and thus $\hat{\mu}_{jm} \in \hat{K}_j^m$ for all $j \in I$ and $m \in \{1, 2, \ldots, k-1\}$.

Finally, for Player-i, we repeat the same argument for one more step to show $\hat{\mu}_{ik} \in \hat{K}_i^k$, i.e., $\bar{\mu}_i \in \hat{K}_i^k$. Since $[\hat{\mu}^k_{ik}]_{\Omega \times A_{-i}} = \nu_i^1$, the condition (1) for $m = 1$ implies $a_i \in \hat{\beta}_i^k(\bar{\mu}_i)$, as we desired. ∎

The following corollary is an easy consequence of the definition of A_i^∞, the definition of the Bayesian strategy of complexity-∞, and Theorem 1 above. Notice that we have the Bayesian foundation for the iteratively undominated strategies in terms of the sequence of finite hierarchies of beliefs, i.e., without defining the type space (a space of infinite hierarchies of beliefs). It is also useful to point out that, by Tan-Werlang (1987, Theorem 5.2 and 5.3), the notion of Bayesian strategy of complexity-∞ coincides with the strategy concept based on the type spaces and the common knowledge of Bayesian rationality.

Corollary 1 *The set of iteratively undominated strategies coincides with that of all Bayesian strategies of complexity-∞, i.e.,* $A_i^\infty = \bigcap_{k=1}^{\infty}\{\hat{\beta}_i^k(\mu_i) \in A_i : \mu_i \in \bigcap_{k\in\mathrm{N}} \hat{K}_i^k\}$.

Remark 4. It will be useful to list some properties of the Bayesian strategy of complexity-k under the kth order common knowledge of Bayesian rationality and consistency. First, it is useful to remind that $A_i^0 \supset A_i^1 \supset \ldots \supset A_i^\infty$ by construction. Hence, if a_i is a Bayesian strategy of complexity-k then it is also a Bayesian strategy of complexity-$(k-1)$. Especially, if a_i is a Bayesian strategy of complexity-∞ then it is a Bayesian strategy of complexity-k for all $k \in \mathbf{N}$, i.e., it is always possible that a Bayesian player of complexity-k chooses the strategy taken by the fully rational players. As for the nonemptiness of the Bayesian strategy of complexity-k, it is enough to show the nonemptiness of the Bayesian strategy of complexity-∞. To show such a result, in general, we need to have some topological structure, especially compactness of the action spaces and the continuity of the payoff functions to guarantee the nonemptiness of the set of optimal strategies. Such a result is reported in Bernheim (1984, Theorem 3.2). □

3.5 Uncertainty about the Complexities of Other Players

We believe that it is insightful to conclude this section with the discussion on the uncertainty associated with the complexities of other players. It seems that once we admit the possibility of other players' being bounded rational, the complexities of other players become the source of uncertainty and thus we have to reconstruct our framework taking this aspect into consideration. As the following discussion shows, however, our framework already has the structure to deal with the uncertainty about other players' complexities so that such a consideration is not necessary. First, notice that since the definition of the complexity involves the hierarchies of beliefs, uncertainty about the complexities can be only formalized by considering the hierarchies of beliefs. In other words, uncertainty about the complexities is not a part of the underlying uncertainty (Ω) but a part of the extended uncertainty ($\Omega \times A_{-i} \times B^m_{-i}$). Hence, we need not modify our framework but consider how the uncertainty about the complexities can be expressed in our framework.

For example, consider the assumption in the previous subsection. It is easy to check that in order for the Bayesian rationality and the consistency to be kth order common knowledge for a player, the player must believe that other players are of complexity at least $k-1$. Can the player of complexity-k believe that other players are of complexity more than k? The answer is "no" in our framework because the player of complexity-k is characterized by the belief $\mu_i^k \in \mathcal{M}^1_+(\Omega \times A_{-i} \times B^{k-1}_{-i})$ which can check the consistency of other players' hierarchies of beliefs up to $(k-1)$th level. Hence, there is no way that the player of complexity-k concludes to believe that other players are of complexity more than k: What he/she can believe is whether other players are of complexity *at least* $k-1$.

Can the player of complexity-k believe that other players are of complexity less than $k-1$? The answer is "yes" and in this case the Bayesian rationality and consistency are no longer the kth order common knowledge for the player. For example, suppose that Player-i of complexity-10 believes that Player-j is a Bayesian player of complexity-2. Then, $\mu_i \in \hat{K}^2_i$ but $\mu_i \notin \hat{K}^m$ for all $m \geq 3$. Needless to say, in such a situation Theorem 1 does not hold. In fact, it is of no use for Player-i to consider higher order beliefs because the player only need to consider the third level of beliefs and thus the player's decision is given by a Bayesian strategy of complexity-3 despite the fact that Player-i has much higher capability.

One of the most important consequences of this consideration is that even if a player is fully rational or has high complexity, his/her decision will be the same as the one taken the players of the lower complexities if the player believes that other players have lower complexities. Hence, uncertainty about the complexities has important implications even when we study the behavior of fully rational players. This is exactly a situation that Kreps, *et al.* (1982) or Reny (1993) considered as they argue that fully rational players may not behave as, for example, the subgame perfect equilibrium predicts if the Bayesian rationality is not common knowledge. We believe that our framework can be a useful model to analyze various problems of games in which some players may not be fully rational.

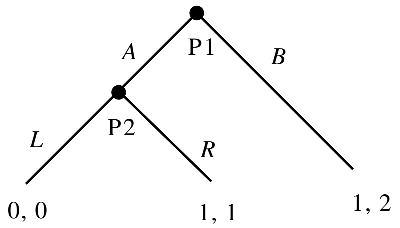
4 Iterative Elimination of Weakly Dominated Strategies

In this section, we extend our framework to study the Bayesian foundation for the iterative elimination of weakly dominated strategies. Such a study will be useful, as we will see later, especially when we study reasonable behaviors in the extensive games, although there are some debates about eliminating the weakly dominated strategies in general [cf. Osborne-Rubinstein (1994)]. To avoid mathematical difficulties, we consider games with a finite number of actions and states of nature. Let Z be a finite set endowed with the power set $\mathcal{Z}$ as the σ-algebra. A probability τ on a measurable space $(Z, \mathcal{Z})$ is said to be completely mixed if supp $\tau = Z$. Let $\tilde{\mathcal{M}}^1_+(Z)$ be the space of completely mixed probabilities on $(Z, \mathcal{Z})$. A sequence $\{\tau^s\}_{s \in \mathrm{N}}$ in $\mathcal{M}^1_+(Z)$ is said to converge to $\tau \in \mathcal{M}^1_+(Z)$ if and only if $\tau^s(\{z\}) \to \tau(\{z\})$ for all $z \in Z$.

4.1 Weakly Dominated Strategies

To motivate our study, consider an extensive game in Figure 2, which is a slight modification of the game in Osborne-Rubinstein (1994, p.109), and its strategic form representation.

Figure 2: Game II



Player-1 \ Player-2	L	R
A	0, 0	1, 2
B	1, 1	1, 1

There are two subgame perfect equilibria; (A, R) and (B, R). In both equilibria, Player-1 gets the payoff 1. Hence, as a first mover, Player-1 can choose either A or B. It seems more likely, however, that the player chooses B because he/she can get 1 for sure whereas by choosing A the payoff may end up with 0 if Player-2 were to choose L. Namely, if Player-1 is a "cautious" player, the player seems to choose A. This type of reasoning (prediction) can be formalized in the elimination of weakly dominated strategies. In this section, we study the Bayesian foundation for such iterative elimination of weakly dominated strategies by the bounded rational players and the fully rational players.

First, it is useful to have the formal definition of the weakly dominated strategy.

Definition 6 *A strategy $a_i^* \in A_i$ is weakly dominated if and only if there is a probability measure $p_i \in \mathcal{M}_+^1(A_i)$ such that $v_i^\omega(a_i^*, a_{-i}) \leq \int_{A_i} v_i^\omega(a_i, a_{-i})\mathrm{d}p_i$ for all $(\omega, a_{-i}) \in \Omega \times A_{-i}$ with the strict inequality for some $(\omega', a'_{-i}) \in \Omega \times A_{-i}$.*

The following lemma, which goes back to Arrow-Barankin-Blackwell (1958, Theorem 1) [cf. Osborne-Rubinstein (1994, 64.2)], is useful.[15]

Lemma 1 *$a_i^* \in A_i$ is not weakly dominated if and only if there exists $\tau_i \in \tilde{\mathcal{M}}_+^1(\Omega \times A_{-i})$ such that $a_i \in \arg\max_{a \in A_i} \int_{\Omega \times A_{-i}} v_i^\omega(a, a_{-i})\mathrm{d}\tau_i$.*

4.2 kth Order Iteratively Undominated Cautious Strategy

Now we consider an iterative elimination of the weakly dominated strategies as we did for the strictly dominated strategies in Section 2. For each $i \in I$, let $W_i^0 = A_i$ and, for each $k \in \mathbf{N}$, define inductively

$$W_i^k = \{a_i \in A_i \mid \exists \nu_i \in \tilde{\mathcal{M}}_+^1(\Omega \times W_{-i}^{k-1}) : a_i \in \arg\max_{a \in W_i^{k-1}} \int_{\Omega \times W_{-i}^{k-1}} v_i^\omega(a, a_{-i})\mathrm{d}\nu_i\}$$

[15] The proof is a minor modification of Arrow *et al.* (1958, Theorem 1). What we need is to consider the space S of vectors whose each coordinate corresponds to a negative of the expected utility with respect to a mixed-strategy of the player for each $(\omega, a_{-i}) \in \Omega \times A_{-i}$ and translate the max operator in our definitions to the min operator in their framework.

Definition 7 *A strategy* $a_i \in W_i^k$ *is said to be a kth order iteratively undominated cautious strategy. A strategy* $a_i \in W_i^\infty \equiv \bigcap_{k \in \mathrm{N}} W_i^k$ *is said to be an iteratively undominated cautious strategy.*

We call a strategy $a_i \in W_i^k$ cautious because it is a strategy which is optimal with respect to the "cautious belief" that puts some weight on each possible action of others.[16] The definition is motivated by the following lemma, which says that W_i^k consists of strategies which are not weakly dominated by any strategy in W_i^{k-1} when possible actions of others are restricted to W_{-i}^{k-1}. Since the lemma is an easy consequence of Lemma 1, we omit the proof.

Lemma 2 *For any* $k \in \mathbf{N}$, $a_i \in W_i^k$ *if and only if* a_i *is not weakly dominated in* W_i^{k-1} *when the actions of others are restricted to* W_{-i}^{k-1}.

Remark 5. We can show that Player-1's unique iteratively undominated cautious strategy is B in the Game II. Table 4 summarizes the elimination process. □

Table 4: Iteratively Undominated Cautious Strategies in Game II

	W_1^k	W_2^k
$k = 0$	$\{A, B\}$	$\{L, R\}$
$k = 1$	$\{B\}$	$\{R\}$
$k \geq 2$	$\{B\}$	$\{R\}$

It has been pointed out that the order of the elimination of weakly dominated strategies is important because changing the order can alter the outcomes [e.g., Osborne-Rubinstein (1994)].[17] This does not happen for the iteratively undominated cautious strategies since players "simultaneously" eliminates weakly dominated strategies given the undominated ones of others. □

[16] This notion of "cautiousness" is also closely related to the "cautious rationalizability" proposed by Pearce (1984) to refine the notion of rationalizable strategy. Unlike Pearce (1984), we started with the original set of actions rather than the set of rationalizable strategies. By Lemma 2, we know that this concept is the same as the one proposed by Moulin (1982) to study the dominant solvable games. The idea that the subgame perfect equilibrium outcome of the extensive form games can be obtained by iteratively eliminating the weakly dominated strategies goes back to Zermelo (1913).

[17] For example, if we eliminate L first then (A, R) and (B, R) are possible outcomes whereas if we eliminate A first then (B, L) and (B, R) are possible outcomes.

4.3 Cautious Bayesian Rationality

Now, we consider the Bayesian foundation for the kth order iteratively undominated cautious strategies. For any $\tau_i \in \mathcal{M}^1_+(\Omega \times A_{-i})$, let $\tilde{\mathcal{M}}(\tau_i) = \{\{\tau^s\}_{s\in \mathrm{N}} \in \prod_{s=1}^{\infty} \tilde{\mathcal{M}}^1_+(\Omega \times A_{-i}) : \tau^s \to \tau_i\}$ be the set of sequences of completely mixed beliefs that converge to the belief τ_i and let

$$W(\tau_i) \equiv \{a_i \in A_i \mid \exists\{\tau_i^s\}_{s\in \mathrm{N}} \in \tilde{\mathcal{M}}(\tau_i) : \ a_i \in \arg\max_{a\in A_i} \int_{\Omega\times A_{-i}} v_i^{\omega}(a, a_{-i})\mathrm{d}\tau_i^s \ \forall s \in \mathbf{N}\}$$

Definition 8 *Player-i is said to be a cautious Bayesian rational player if and only if he/she has a belief* $\tau_i \in \mathcal{M}^1_+(\Omega \times A_{-i})$ *and chooses an action* $a_i^* \in W(\tau_i)$.

This definition is motivated by the following result.[18]

Lemma 3 $a_i^* \in W(\tau_i)$ *if and only if* $a_i^* \in \arg\max_{a\in A_i} \int_{\Omega\times A_{-i}} v_i^{\omega}(a, a_{-i})\mathrm{d}\tau_i$ *and* a_i^* *is not weakly dominated in* A_i.

Proof. Suppose there exists $\{\tau_i^s\}_{s\in \mathrm{N}} \in \tilde{\mathcal{M}}(\tau_i)$ such that $a_i^* \in \arg\max_{a\in A_i} \int_{\Omega\times A_{-i}} v_i^{\omega}(a, a_{-i})\mathrm{d}\tau_i^s$ for all $s \in \mathbf{N}$ and $\tau_i^s \to \tau_i$. Since the mapping $\tau \mapsto \int_{\Omega\times A_{-i}} v_i^{\omega}(a, a_{-i})\mathrm{d}\tau$ is continuous, $\tau_i^s \to \tau_i$ and $a_i^* \in \arg\max_{a\in A_i} \int_{\Omega\times A_{-i}} v_i^{\omega}(a, a_{-i})\mathrm{d}\tau_i^s$ for all s imply $a_i^* \in \arg\max_{a\in A_i} \int_{\Omega\times A_{-i}} v_i^{\omega}(a, a_{-i})\mathrm{d}\tau_i$. Next, suppose, per absurdum, a_i^* is a weakly dominated strategy. Then, there exists $p_i \in \mathcal{M}^1_+(A_i)$ such that $\int_{A_i} v_i^{\omega}(a_i, a_{-i})\mathrm{d}p_i \geq v_i^{\omega}(a_i^*, a_{-i})$ for all $(\omega, a_{-i}) \in \Omega \times A_{-i}$ with the strict inequality for some $(\omega', a'_{-i}) \in \Omega \times A_{-i}$. Hence, for all $s \in \mathbf{N}$,

$$\int_{\Omega\times A_{-i}} v_i^{\omega}(a_i^*, a_{-i})\mathrm{d}\tau_i^s < \int_{\Omega\times A_{-i}} \int_{A_j} v_i^{\omega}(a_i, a_{-i})\mathrm{d}p_i\mathrm{d}\tau_i^s$$

Since, for any $s \in \mathbf{N}$ and $a_i \in A_i$, $\int_{\Omega\times A_{-i}} v_i^{\omega}(a_i^*, a_{-i})\mathrm{d}\tau_i^s \geq \int_{\Omega\times A_{-i}} v_i^{\omega}(a_i, a_{-i})\mathrm{d}\tau_i^s$, we have

$$\int_{\Omega\times A_{-i}} v_i^{\omega}(a_i^*, a_{-i})\mathrm{d}\tau_i^s \geq \int_{A_j} \int_{\Omega\times A_{-i}} v_i^{\omega}(a_i, a_{-i})\mathrm{d}\tau_i^s\mathrm{d}p_i$$

[18] Even though we could have defined

$$W(\tau_i) \equiv \{a_i \in A_i \mid a_i \in \arg\max_{a\in A_i} \int_{\Omega\times A_{-i}} v_i^{\omega}(a, a_{-i})\mathrm{d}\tau_i \text{ and } a_i \text{ is not weakly dominated}\}$$

to define the cautious Bayesian rational player above and skipped the following Lemma 3, we believe that in the framework of the Bayesian approach it is useful to characterize players in terms of their beliefs.

which is a contradiction.

To see the converse, suppose that $a_i^* \in \arg\max_{a \in A_i} \int_{\Omega \times A_{-i}} v_i^\omega(a, a_{-i}) \mathrm{d}\tau_i$ and a_i^* is not weakly dominated in A_i. Then, by Lemma 1, there exists $\nu_i \in \tilde{\mathcal{M}}_+^1(\Omega \times A_{-i})$ such that

$$a_i^* \in \arg\max_{a \in A_j} \int_{\Omega \times A_{-i}} v_i^\omega(a, a_{-i}) \mathrm{d}\nu_i$$

Define $\{\tau_i^s\}$ by $\tau_i^s = \frac{1}{s}\nu_i + (1 - \frac{1}{s})\tau_i$ for each $s \in \mathbf{N}$. Then, $a_i^* \in \arg\max_{a \in A_j} \int_{\Omega \times A_{-i}} v_i^\omega(a, a_{-i}) \mathrm{d}\tau_i^s$ for all $s \in \mathbf{N}$ and $\tau_i^s \to \tau_i$. ∎

4.4 Main Result

In Section 3.3, we showed the equivalence of the set of kth order iteratively undominated strategies and the set of Bayesian strategies of complexity-k under the kth order common knowledge of Bayesian rationality and consistency. Our main result in this section is a similar result for the cautious strategies.

Definition 9 *Player-i is said to be a cautious Bayesian player of complexity-k ($k \in \{1, 2, \dots\}$) or said to be a bounded rational cautious player if and only if he/she has a belief $\mu_i \in \hat{T}_i^k$ and chooses an action $a_i^* \in \tilde{\beta}_i^k(\mu_i) \equiv W(\mu_i^k)$, which we refer to as a cautious Bayesian strategy of complexity-k.*

Definition 10 *Player-i is said to be a cautious Bayesian player of complexity-∞ or said to be a fully rational cautious player if and only if he/she has a belief $\mu_i \in \hat{T}_i^\infty$ and chooses an action $a_i^* \in \bigcap_{k \in \mathrm{N}} \tilde{\beta}_i^k(\mu_i)$, which we refer to as a cautious Bayesian strategy of complexity-∞.*

Next, we define the knowledge of the cautious Bayesian rationality. For each $i \in I$, let $\tilde{K}_i^1 = \hat{T}_i^1$,

$$\tilde{K}_i^2 = \{\mu_i \in \hat{T}_i^2 \mid (a_j, \mu_j^1) \in \text{supp } [\mu_i^2]_{A_j \times B_j^1} \Rightarrow a_j \in W(\mu_j^1)\}$$

and, for each $m \in \{3, 4, \dots\}$, define inductively

$$\tilde{K}_i^m = \{\mu_i \in \hat{T}_i^m \mid \mu_j^{\ell-1} \in \text{supp } [\mu_i^\ell]_{B_j^{\ell-1}} \ (\ell = 2, \dots, m) \Rightarrow (\mu_j^1, \dots, \mu_j^{m-1}, \dots) \in \tilde{K}_j^{m-1}\}$$

Definition 11 *Cautious Bayesian rationality and consistency are said to be kth order common knowledge for $i \in I$ if and only if $\mu_i \in \tilde{K}_i^k$. Cautions Bayesian rationality and consistency are said to be common knowledge for $i \in I$ if and only if $\mu_i \in \bigcap_{m=1}^{\infty} \tilde{K}_i^m$.*

We are ready to present our main result in this section. Since the proof is essentially the same as the one for Theorem 1, we only provide the outline.

Theorem 2 *Let $k \in \mathbf{N}$ be arbitrary. Then, $W_i^k = \{\tilde{\beta}_i^k(\mu_i) \in A_i : \mu_i \in \tilde{K}_i^k\}$, i.e., the set of kth order iteratively undominated cautious strategies coincides with the set of cautious Bayesian strategies of complexity-k if the cautions Bayesian rationality and the consistency are kth order common knowledge for $i \in I$.*

Proof. In order to show $W_i^k \supset \{\tilde{\beta}_i^k(\mu_i) \in A_i : \mu_i \in \tilde{K}_i^k\}$, we also have only to show that supp $[\mu_i^k]_{A_{-i}} \subset W_{-i}^{k-1}$ for any $\mu_i \in \tilde{K}_i^k$. Since the cautious Bayesian rational player chooses a strategy $a_i \in \arg\max_{a \in A_i} \int_{\Omega \times A_{-i}} v_i^{\omega}(a, a_{-i}) \mathrm{d}[\mu_i^k]_{\Omega \times A_{-i}}$ which is not weakly dominated, the result above allows us to conclude $a_i \in W_i^k$. (Notice also that, by the consistency of μ_i, if a_i is a best response for $[\mu_i^k]_{\Omega \times A_{-i}}$ ($k \in \{2, 3, \ldots\}$) then it is also a best response for $[\mu_i^{k-1}]_{\Omega \times A_{-i}} \in \mathcal{M}_+^1(\Omega \times W_{-i}^{k-2})$, i.e., $a_i \in W_i^{k-1}$.) The proof for supp $[\mu_i^m]_{A_{-i}} \subset W_{-i}^{m-1}$ for all $m \in \{1, 2, \ldots, k\}$ is by induction and is the same as in the proof of Theorem 1.

To show the converse, we must construct a hierarchy of beliefs $\bar{\mu}_i \in \tilde{K}_i^k$ such that $a_i \in \tilde{\beta}_i^k(\bar{\mu}_i)$. The construction of $\bar{\mu}_i$ and the proof for $a_i \in \tilde{\beta}_i^k(\bar{\mu}_i)$ are the same as the proof for Theorem 1. The only difference is in showing that $\bar{\mu}_i \in \tilde{K}_i^k$, which can be proved by induction similar to the one in the proof for Theorem 1 except that we have to show that the optimal actions with respect to the constructed belief are consistent with the kth order common knowledge of the cautious Bayesian rationality and the consistency. The result can be easily proved by using Lemma 2 and 3. ∎

The following result for the fully rational cautious players is immediate.

Corollary 2 $W_i^{\infty} = \bigcap_{k \in \mathrm{N}} \{\tilde{\beta}_i^k(\mu_i) \in A_i : \mu_i \in \bigcap_{k \in \mathrm{N}} \tilde{K}_i^k\}$, *i.e., the set of iteratively undominated cautious strategies coincides with the set of cautious Bayesian strategies of complexity-∞ if the cautious Bayesian rationality and the consistency are common knowledge for $i \in I$.*

5 Applications

In this section, we apply our results to the games in which paradoxical (counterintuitive) results are reported. First, it is important to point out that Harrington (1987) has already shown that in the finitely repeated prisoner's dilemma game the cooperation up to certain number of repetitions is a k-point rationalizable strategy (kth order iteratively undominated strategy), i.e., such a cooperative behavior may be taken by bounded rational players. In the following, we will report similar results. We study two games, the e-mail game studied by Rubinstein (1989) and the "take it or leave it (TOL)" game studied by Reny (1992, 1993).

5.1 The E-Mail Game

The following is a simplified version of the e-mail game studied by Rubinstein (1989). Two players, 1 and 2, are involved in a coordination problem. Each has to choose between two actions A (stay) and B (attack). The payoffs are summarized in Table 5. It is assumed that $L > M > 0$ and thus taking an action B is dangerous unless each player is quite sure that the other also plays B.

Table 5: Payoffs in the E-Mail Game

Player 1 \ Player 2	A	B
A	$0, 0$	$0, -L$
B	$-L, 0$	M, M

Two players agreed that when the coordinated action is necessary Player-1 sends a message "Let us take B" and that they choose A (stay) if there is no message. Due to some technical difficulties, however, there is a small probability $\epsilon > 0$ that the message does not arrive at its destination. At the risk of creating discord, the electronic mail network is set up to send a confirmation automatically if any message (including the confirmation) is received. At the end of the communication phase, the screen displays to each player the number of messages that his/her machine has sent. Let T_i be the number of messages i's computer sent (the number on i's screen). Let $\mathbf{T}_i = \{1, 2, \ldots\}$ be a space of private information. In the existence of private information, a strategy must be formulated as a mapping from private information to actions. For each $i \in \{1,2\}$, let $S_i : \mathbf{T}_i \longrightarrow\rightarrow A_i$ be a strategy of Player-i. Since we can think of a strategy as a vector $S_i = (S_i(0), S_i(1), \ldots) \in \prod_{m=1}^{\infty} A_i$, the space of strategies can be identified by $\Sigma_i = \prod_{m=1}^{\infty} A_i$. With this notation, the assumption in the beginning of this paragraph implies that $S_1(0) = S_2(0) = A$ is a public information.

To our surprise, Rubinstein showed that in the unique Bayesian-Nash equilibrium both players play A no matter what the number on each player's screen is, i.e., both players cannot take the action B. He summarizes the result as follows: "What would *you* do if the number on your screen is 17? It is hard to imagine that ... a player will not play B. The sharp contrast between our intuition and the game-theoretic analysis is what makes this example paradoxical (p.389)".

We show that if the players are of finite complexity, then their playing B can be rationalizable.[19] Namely, the following proposition shows that if a player is of complexity-$2m$ for some $m \in \{1, 2, \ldots\}$ then the player plays A for sure when the number of messages is less than or equal to m but the player may play B when the number of messages is greater than m.

Proposition 1 *Let S_i^k be a Bayesian strategy of complexity-k under the kth order common knowledge of Bayesian rationality and consistency for Player-i. Then, for each $m \in \{1, 2, \ldots\}$,*

$$\begin{aligned} S_1^{2m}(T_1) = S_1^{2m+1}(T_1) &= \begin{cases} A \text{ or } B & \text{if } T_1 > m \\ A & \text{if } T_1 \leq m \end{cases} \quad \text{and} \\ S_2^{2m-1}(T_2) = S_2^{2m}(T_2) &= \begin{cases} A \text{ or } B & \text{if } T_2 > m \\ A & \text{if } T_2 \leq m \end{cases} \end{aligned}$$

Proof. By Theorem 1, we have only to find the set of kth order iteratively undominated strategies. We first establish two lemmas. Let $U_i^t : A_i \times \Sigma_j \to \mathbf{R}$ be the expected payoff function such that $U_i^t(a_i, S_j)$ represents the expected payoff from playing a_i when the number of messages i sent is $t \in \mathbf{T}_i$ and the other player j takes a strategy S_j. Figure 3, which illustrates the flow of the messages and strategies, may be helpful to understand the following lemmas.

[19] Monderer-Samet (1989) also studied the e-mail game and showed, using the concept of common p-belief, that we can find an approximate equilibrium in which each player plays B if the number of messages sent is sufficiently large. Since the infinite hierarchies of consistent beliefs are considered in their model, their argument is different from ours. Their argument is close in its spirit to Kreps *et al.* (1982) which showed that rational players may take cooperative behavior in the finitely repeated prisoner's dilemma game if the Bayesian rationality is not common knowledge. Hence, in our context, their argument will correspond to the case in which a fully rational player has some doubt about the full rationality of other players.

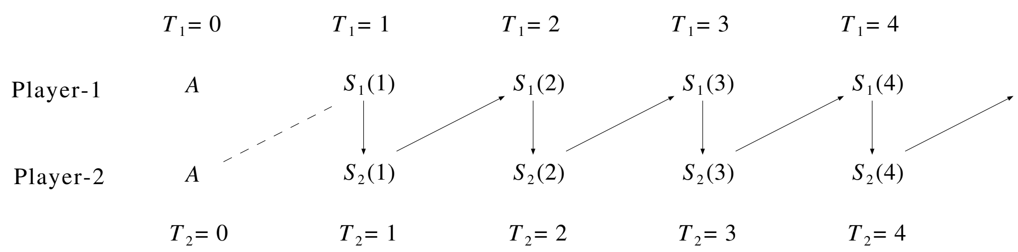


Figure 3: E-mail Game

Lemma 4. *Let $t \in \mathbf{N}$ be arbitrary. If $S_2(t-1) = A$ then $S_1(t) = B$ is a dominated strategy. Also, if $S_1(t) = A$ then $S_2(t) = B$ is a dominated strategy.*

Proof. Let $t \in \mathbf{N}$ be arbitrary. There are two possibilities when $T_1 = t$: Either $T_2 = t-1$ or $T_2 = t$. When $T_2 = t-1$, Player-2 plays A by assumption. When $T_2 = t$, it is not clear if Player-2 plays A or B. Let ρ_2^B be the probability that Player-2 will play B. $T_2 = t-1$ occurs with probability ϵ and $T_2 = t$ occurs with probability $\epsilon(1-\epsilon)$. Let $z = \frac{\epsilon}{\epsilon+\epsilon(1-\epsilon)}$; the conditional probability for $T_2 = t-1$. It is easy to check that $z > \frac{1}{2}$. Given these probabilities, the expected payoff for Player-1 from A and B are $U_1^t(A, S_2) = z0 + (1-z)((1-\rho_2^B)0 + \rho_2^B 0) = 0$ and $U_1^t(B, S_2) = z(-L) + (1-z)((1-\rho_2^B)(-L) + \rho_2^B M)$, respectively. Hence, for any $\rho_2^B \in [0,1]$,

$$\begin{aligned} U_1^t(A, S_2) - U_1^t(B, S_2) &= zL + (1-z)((1-\rho_2^B)L - \rho_2^B M) \geq zL + (1-z)(-M) \\ &= (L+M)z - M > (L+M)\frac{1}{2} - M = \frac{L-M}{2} > 0 \end{aligned}$$

because $z > \frac{1}{2}$ and $L > M$. Namely, A is strictly better than B. Define S_1' by $S_1'(t) = A$ and $S_1'(s) = S_1(s)$ for all $s \neq t$. Since the probability on $T_1 = t$ is positive, the expected payoff from S_1 is strictly smaller than the one under the strategy S_1', i.e., S_1 with $S_1(t) = B$ is a dominated strategy. The proof for the second claim is similar. ∎

Lemma 5. *Let $t \in \mathbf{N}$ be arbitrary. If $S_2(t-1) = S_2(t) = B$, then $S_1(t) = B$ is optimal. Also, if $S_1(t) = S_1(t+1) = B$, then $S_2(t) = B$ is optimal.*

Proof. Let $t \in \mathbf{N}$ be arbitrary. Suppose that S_2 is a Player-2's strategy such that $S_2(t-1) = S_2(t) = B$. Then, the expected payoffs for Player-1 from A and B are $U_1^t(A, S_2) = 0 < M =$

$U_1^t(B, S_2)$, i.e., $S_1(t) = B$ is an optimal strategy. The proof for the second claim is similar. ∎

By Lemma 4, Lemma 5 and the assumption $S_2(0) = A$, we know that $A_1^1 = \{S_1 \in \Sigma_1 \mid S_1(0) = S_1(1) = A\}$ and $A_2^1 = \{S_2 \in \Sigma_2 \mid S_2(0) = A\}$. Given A_1^1, the lemmas show that $A_2^2 = \{S_2 \in \Sigma_2 \mid S_2(0) = S_2(1) = A\}$. Given A_1^2, it is easy to check that $A_1^2 = A_1^1$. Given A_2^2, however, the lemmas imply that $A_1^3 = \{S_1 \in \Sigma_1 \mid S_1(0) = S_1(1) = S_1(2) = A\}$. By continuing this inductive step, it is easy to show that $A_2^{2m} = A_2^{2m+1} = \{S_2 \in \Sigma_2 \mid S_2(t) = A \ \forall t = 0, 1, \cdots, m\}$ and that $A_1^{2m-1} = A_1^{2m} = \{S_1 \in \Sigma_1 \mid S_1(t) = A \ \forall t = 0, 1, \cdots, m\}$, which establishes our proposition. ∎

The following corollary is immediate.

Corollary 3 (Rubinstein (1989)) *$S_i(T_i) = A$ for all $T_i \in \mathbf{T}_i$ if Player-i is a fully rational player (which is a Bayesian Nash equilibrium).*

Proposition 1 predicts a more intuitive behavior than the one given by Rubinstein (1989) but does not predict our most likely behavior that we will choose B when the number on the screen is sufficiently large. As the diagram in Figure 1 suggests, we may have this prediction if we strengthen our assumption. For example, we may impose an extra assumption that a bounded rational player chooses a strategy the other player asks him/her to do. Alternatively, we may want to question our assumption in Proposition 1 that Bayesian rationality and consistency are kth order common knowledge. In the studies of the bounded rational players, it will be an important task to find reasonable assumptions to predict intuitive results in many games.

5.2 The TOL Game

In this subsection, we apply our result on the iterative elimination of the weakly dominated strategies (Theorem 2). The game is best described by Reny (1993). For simplicity, we assume that a referee comes equipped with $2N$ dollars and places one in front of Player-1 and Player-2. Player-1 can take the dollar hereby ending the game or he can leave it. If he leaves it, the referee places a second dollar in front of the players. Player-2 now has the opportunity to take the two dollars and end the game or not, in which case the process repeats. Players' payoffs are assumed strictly increasing in dollars. Here, we assume for simplicity that the payoff is equal to the amount of money. Finally, should the game continue until the $2N$th stage and Player-2 decides to leave the $2N$ dollars, it is then given to Player-1. (cf. Figure 4.)

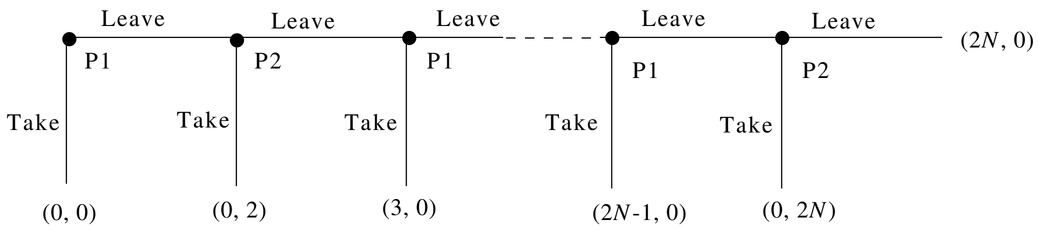


Figure 4: TOL Game

By using the backward induction, it is easy to check that in the subgame perfect equilibrium the first player takes the money as soon as the game starts. This result, however, is counterintuitive if not unreasonable, and thus is considered to be paradoxical. Reny (1993) argued that the players may not take the dollars in the beginning if the Bayesian rationality (in the extensive form game) is not common knowledge. We predict the same behavior (in the strategic form game) by assuming that the player is a cautious Bayesian player of complexity-k even if the cautious Bayesian rationality is public information. It is the bounded rationality which derives the intuitive result not the common knowledge assumption.

Remark 6. Bayesian rationality in the extensive form games is studied by Pearce (1984) and Reny (1992). To illustrate why we need the *cautious* Bayesian rationality in the strategic form of the TOL game, Table 6 shows the game matrix for $N = 2$. Since there are two turns in which each can take or leave the money, there are four strategies. In the matrix, a strategy LT, for example, means that the player leaves the money in the first turn and take the money in the second turn. It is easy to check that the unique subgame perfect equilibrium is (TT, TT).

Since there is no strictly dominated strategy, each strategy is rationalizable. In this extensive game, however, it seems reasonable to eliminate weakly dominated strategies to capture the idea that each player in the extensive games regards every strategy of others as possible because "rational" players will consider other players' decisions at every node. This is the reason for

Table 6: Game Matrix of the TOL game: $N = 2$

Player-1 \ Player-2	TT	TL	LT	LL
TT	1, 0	1, 0	1, 0	1, 0
TL	1, 0	1, 0	1, 0	1, 0
LT	0, 2	0, 2	3, 0	3, 0
LL	0, 2	0, 2	0, 4	4, 0

why we apply Theorem 2 to study this game. Notice also that since the weakly undominated strategies TT and TL give exactly the same payoff to each player, we cannot eliminate one of them as unreasonable. This is the reason for why we will be only interested in the outcome of this game. □

Since the game ends if one of the players takes the dollars, as long as we are interested in the outcome of the game, we can summarize each player's strategy by $S_i \in \{1, 2, \ldots, N\}$. For example, a strategy $S_i = m$ implies that i takes the dollars in the mth turn.[20]

Proposition 2 *Let S_i^k be a cautious Bayesian strategy of complexity-k under the kth order common knowledge of cautious Bayesian rationality and consistency for Player-i. Then,*

$$S_1^k \in \begin{cases} \{1, 2, \ldots, N-(k-1)\} & \text{if } k \leq N \\ \{1\} & \text{if } k > N \end{cases} \quad \text{and } S_2^k \in \begin{cases} \{1, 2, \ldots, N-k\} & \text{if } k \leq N-1 \\ \{1\} & \text{if } k > N-1 \end{cases}$$

Proof. By Theorem 2, we have only to find the set of kth order iteratively undominated cautious strategies. First, $W_2^0 = \{1, 2, \ldots, N\}$. Let $m \in \{1, 2, \ldots, N\}$ and $S_2 = m$. Since the best response for Player-1 is $S_1 = m$ and none of them is weakly dominated, $W_1^1 = \{1, 2, \ldots, N\}$. Let $m \in \{1, 2, \ldots, N\}$ and $S_1 = m$. Since the best response for Player-2 is $S_2 = m - 1$ and $S_2 = N$ is weakly dominated by $S_2 = N - 1$, $W_2^1 = \{1, 2, \ldots, N-1\}$. Similarly, we can establish $W_1^2 = \{1, 2, \ldots, N-1\}$. Let $m \in \{1, 2, \ldots, N-1\}$ and $S_1 = m$. Since the best response is $S_2 = m - 1$ and $S_2 = N - 1$ is weakly dominated by $S_2 = N - 2$, $W_2^2 = \{1, 2, \ldots, N-2\}$.

[20] For example, a strategy $S_1 = 1$ represents all the strategies in which Player-1 chooses "Take" in the first turn and take some actions at the later nodes. Since we are only interested in the outcome in the TOL game and the game ends in the first node whatever the actions in the later nodes, we can view all such possible strategies as a strategy. Similarly, for example, $S_1 = 2$ represents the strategies in which Player-1 chooses "Leave" in the first turn, chooses "Take" in the second turn, and take some actions at the later nodes.

Similarly, we can show $W_2^3 = \{1, 2, \ldots, N-2\}$. By repeating the elimination processes, we have

$$W_1^k = \begin{cases} \{1, 2, \ldots, N-(k-1)\} & \text{if } k \leq N \\ \{1\} & \text{if } k > N \end{cases} \text{ and } W_2^k = \begin{cases} \{1, 2, \ldots, N-k\} & \text{if } k \leq N-1 \\ \{1\} & \text{if } k > N-1 \end{cases}$$

which establishes our claim. ∎

The result shows that if Player-i is of complexity less than N it is possible that the player may not take the money in the beginning. Notice also that he/she knows how to behave in the last stages of the game, i.e., knows that he/she should take the money before the other player takes the money. But this idea does not extend to the earlier stages to consider the behavior in the beginning of the game due to the boundedness in the consistent thinking. It is also easy to check that if players' complexities are sufficiently high (higher than N), then the subgame perfect equilibrium outcome emerges. A simple consequence is the following well known result.

Corollary 4 *If Player-i is fully rational, then* $S_i = 1$ *for any* $N \in \mathbf{N}$ *(which is the subgame perfect equilibrium outcome).*

Remark 7. The main insight of Reny (1992, 1993) about this game is that once the first player did not take the money, the Bayesian rationality cannot be the common knowledge for the fully rational players. This is not the case for the bounded rational players. For example, Player-1's not taking money in the first turn does not contradict to the kth order common knowledge of Bayesian rationality and consistency for Player-2 if $k \leq N-1$. However, there may be a time, as in the case of fully rational players studied by Reny, when a player must revise the belief about the complexities of other players (or the kth order common knowledge of Bayesian rationality and consistency) in the process of the extensive game. Learning and experiments of bounded rational players in the extensive form game seem to be the interesting research agenda. □

6 Concluding Remarks

In this paper, we presented a model of bounded rationality which also allows us to study the behavior of fully rational players. Based on the Bayesian approach, we developed a framework in which players form beliefs over the spaces of finite hierarchies of beliefs rather than infinite hierarchies of beliefs and the degree of bounded rationality can be defined by the number of consistent hierarchies of beliefs. We showed that, under the assumption that Bayesian rationality and consistency are kth order common knowledge, the set of all Bayesian strategies of complexity-k coincides with the set of kth order iteratively undominated strategies. We also extended the framework to study the iterative elimination of the weakly dominated strategies, which is useful to study many extensive games in strategic form. We showed that, under the kth order common knowledge of cautious Bayesian rationality and consistency, the set of cautious Bayesian strategies of complexity-k coincides with the set of kth order iteratively undominated cautious strategies.

As our applications to the paradoxical games show, players seem to be either bounded rational or having some doubts about the full rationality of other players.[21] Needless to say, by relaxing the assumption of full rationality, almost anything can be an optimal strategy. We want to see in the near future if we can extend our framework to have sharper predictions about the behavior of bounded rational players by restricting the set of beliefs to a reasonably smaller set. This question may have some interesting answers in the extension of our approach to the extensive form games because the bounded rational players who can only look forward (or backward) by finite steps would revise their beliefs and strategies in the process of the games.

References

[1] Armbruster, W. and W. Böge, "Bayesian Game Theory," in O. Moeschlin and D. Pallaschke (eds.) *Game Theory and Related Topics*, 1979, North-Holland, Amsterdam.

[2] Arrow, K. J., E. W. Barankin, and D. Blackwell "Admissible Points of Convex Sets," in H. W. Kuhn and A. W. Tucker (eds.) *Contributions to the Theory of Games*, Vol. II (Annals of Mathematics Studies, 28), 1958. Princeton University Press, Princeton.

[21] It is also interesting to see that efficient outcomes can be attained [e.g., in finitely repeated prisoner's dilemma game or e-mail (message) game] in a "bounded rational society" where each individual is bounded rational or the rationality is not common knowledge even though the efficient outcomes cannot be attained in a "rational society" where each individual is rational and the rationality is common knowledge.

[3] Aumann, R. J., "Correlated Equilibrium as an Expression of Bayesian Rationality," *Econometrica* **55**, 1987, 1-18.

[4] Bernheim, D., "Rationalizable Strategic Behavior," *Econometrica* **52**, 1984, 1007-1028.

[5] Bernheim, D., "Axiomatic Characterizations of Rational Choice in Strategic Environments," *Scandinavian Journal of Economics* **88**, 1986, 473-488.

[6] Fudenberg, D. and J. Tirole, *Game Theory,* 1992. MIT Press, Massachusetts.

[7] Hammond, P. "Aspects of Rationalizable Behavior," in K. Binmore, A. Kirman, and T. Piero (ed.) *Frontiers of Game Theory*, 1993. MIT Press, Massachusetts.

[8] Harrington, J. E. Jr., "Finite Rationalizability and Cooperation in the Finitely Repeated Prisoners' Dilemma," *Economic Letters* **23**, 1987, 233-237.

[9] Harsanyi, J. C., "Games with Incomplete Information Played by 'Bayesian' Players, I-III," *Management Science* **14**, 1967-68, 159-182, 320-334, 486-502.

[10] Kreps D., P. Milgrom, J. Roberts, and R. Wilson, "Rational Cooperation in the Finitely Repeated Prisoner's Dilemma," *Journal of Economic Theory* **27**, 1982, 245-252.

[11] Monderer D. and D. Samet, "Approximating Common Knowledge," *Games and Economic Behavior* **1**, 170-190.

[12] Moulin, H. *Game Theory for the Social Sciences, 2nd Edition*, 1986. New York University Press, New York.

[13] Osborne, M. and A. Rubinstein, *A Course in Game Theory,* 1994. MIT Press, Massachusetts.

[14] Pearce, D. "Rationalizable Strategic Behavior and the problem of perfection," *Econometrica* **52**, 1984, 1029-1050.

[15] Reny, P. J., "Rationality in Extensive-Form," *Journal of Economic Perspectives* **6**, 1992, 103-118.

[16] Reny, P. J., "Common Belief and the Theory of Games with Perfect Information," *Journal of Economic Theory* **59**, 1993, 257-274.

[17] Rubinstein, A., "The Electronic Mail Game: Strategic Behavior Under "Almost Common Knowledge,"" *American Economic Review*, 1989, 385-391.

[18] Tan, T. C. and S. Werlang, "The Bayesian Foundations of Solution Concepts of Games," *Journal of Economic Theory* **45**, 1988, 370-391.

[19] Zermelo, E., "Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels," in E. W. Hobson and A. E. H. Love (eds.) *Proceedings of the Fifth International Congress of Mathematicians,* Vol. 2, 1913, Cambridge University Press, Cambridge.