Liquidity, Interest Rates and Output

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Abstract

This paper integrates limited participation into monetary search theory to analyze the liquidity effects of open market operations. The centralized bonds market features limited participation and shocks to government bond sales, while the decentralized goods market features bilateral matches. Unmatured bonds can be used together with money to purchase goods in a fraction of matches, but in other matches a legal restriction forbids the use of bonds as the means of payments. In this economy, a shock to bond sales has two distinct liquidity effects. One is the immediate liquidity effect on the bond price and the nominal interest rate. The other is a liquidity effect in the goods market starting one period later, i.e., the effect on the amount of unmatured bonds circulating in the goods market. Thus, even independent shocks can affect the household's money allocation between the two markets, affect real output and the term structure of interest rates, and cause nominal interest rates to be serially correlated. I establish the existence of the equilibrium and, with numerical examples, examine equilibrium properties.

Keywords: Liquidity; Interest rates; Money; Search.

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1. Introduction

This paper integrates goods market search into a model with limited participation to analyze the liquidity effect of open market operations. Limited participation refers to the assumption that the participants in the bonds market cannot adjust their money holdings immediately to shocks to government bond sales. In an influential paper, Lucas (1990) has shown that limited participation enables open market operations to generate a liquidity effect, i.e., positive shocks to bond sales depress the bond price and drive up the nominal interest rate while negative shocks reduce the nominal interest rate. A central assumption in his model is that all transactions are constrained by cash. In this paper, I will replace the Walrasian goods market in his model by a decentralized search market in order to support a role for money and to relax the cash-in-advance constraint in the goods market.

One motivation for this analysis is to provide a microfoundation for the role of money in Lucas's analysis. As Wallace (2001) forcefully argued, cash-in-advance constraints are not suitable for monetary theory and policy analysis. Search theory of money originated in Kiyotaki and Wright (1989) provides a microfoundation for money, but so far the models based on it are not tractable for analyzing nominal bonds, not mentioning the liquidity effect arising from stochastic bond sales. The current paper is an attempt to construct such a tractable model. Another motivation is that relaxing the cash-in-advance constraint in the goods market might improve the empirical performance of limited participation models. There is convincing evidence that the correlation between nominal interest rates and money is negative for very narrow monetary aggregates but positive for broad aggregates (see Chari et al. 1995). One interpretation of this evidence is that the cash-in-advance constraint does not apply in some transactions.

The economy in this paper has a bonds market and a goods market, which operate separately in each period. The bonds market is centralized and functions in exactly the same way as in Lucas's model. That is, the government issues nominal bonds at the market price and accepts only money as payments. The amount of new bonds is stochastic, which is the only aggregate uncertainty in the economy. This shock is realized after the households have already allocated the assets between the markets, and hence the familiar liquidity effect arises in the bonds market.

The goods market is decentralized, where agents meet randomly and determine how much to trade bilaterally. Money can be used in all trades for goods, but bonds can only be used in a fraction of trades called unrestricted trades. In the other fraction $g \in (0, 1)$ of matches, which are called restricted trades, a legal restriction forbids the use of bonds as the means of payments for goods. With this legal restriction, bonds are redeemed immediately after they mature, but unmatured bonds can circulate as an imperfect substitute for money. The imperfect substitutability generates an additional discount on long-term bonds, relative to short-term bonds. As this additional discount responds to open market operations, the term structure of interest rates varies. I set the bonds' maturity to be two periods – the shortest length that allows bonds to circulate in the goods market before maturity.

Allowing unmatured bonds to purchase a subset of goods is a crude way to capture the idea that an actual economy has more than cash as the means of payments. However, it may not be useful to insist on the literal interpretation that unmatured bonds act directly as a medium of exchange. In reality, bonds are a large part of money market checking accounts that the households have in financial institutions, on which checks can be written to pay for goods.

When unmatured bonds circulate in the goods market, open market operations generate a delayed liquidity effect in the goods market, in addition to the immediate liquidity effect in the bonds market. In particular, a high shock to bond sales in the previous period increases the quantity of unmatured bonds circulating in the current goods market. The additional bonds provide liquidity to the buyers who are in unrestricted trades. Thus, in addition to changing the price level, shocks to bond sales in the previous period change the current dispersion of real quantities of goods produced and traded in unrestricted matches versus restricted matches. This liquidity effect of unmatured bonds in the goods market induces a number of new features regarding output, interest rates and the term structure.

Before examining these features, I will devote a large part of this paper to establishing the existence of the equilibrium. Although the existence proof follows the general route used by Lucas (1990), the details necessarily differ for the following reasons. First, the goods market here is non-Walrasian, and so prices are determined bilaterally. Second, output is determined endogenously in the equilibrium, rather than being given by endowments. Third, there are two types of trades in the goods market – the restricted trades and unrestricted trades – and so prices are different in the two types of trades.

Because this paper attempts to integrate limited participation into search models of money, it is naturally related to both literatures. The literature on limited participation is large and a reference list can be found in Christiano et al. (1999). The goods market in this literature is centralized, Walrasian, and with cash-in-advance constraints. The literature on search models of money is also sizable, but only a few are tractable enough to incorporate elements such as money growth and nominal bonds. The precursors to the current paper, Shi (2002, 2003), incorporate such elements in deterministic environments.

2. A Search Economy with Legal Restrictions

In this section I describe an economy with a legal restriction in the goods market, analyze individuals' decisions, and define the equilibrium.

2.1. Households, Matches, and Markets

The economy has discrete time and many types of households. The number of households in each type is large and normalized to one. The households in each type are specialized in producing a specific good, which they do not consume, and exchange for consumption goods in the market. Goods are perishable between periods. The utility of consumption is $u(.)$ for consumption goods and 0 for other goods. The cost (disutility) of production is $\psi(.)$. The utility function satisfies $u' > 0$ and $u'' \leq 0$. The cost function satisfies $\psi(0) = 0$, $\psi' > 0$ and $\psi'' > 0$. Moreover, $u'(0) = \infty > \psi'(0)$ and $u'(\infty) < \psi'(\infty)$. To simplify the algebra, I will use the form $\psi(q) = \psi_0 q^{\Psi}$, where $\Psi > 1$ and $\psi_0 > 0$.

Each household consists of a large number of members normalized to one. A fraction σ of these members are sellers and the remaining are buyers, where $\sigma \in (0, 1)$. A seller produces and sells goods, and a buyer purchases goods for consumption. The members share consumption and regard the household's utility function as the common objective. As a result, individual matching risks are smoothed out within each household, and the distribution of asset holdings across households is degenerate. This degeneracy maintains tractability as it enables me to focus on the equilibrium that is symmetric across households.[1]

There are two assets in the economy. One is money and the other is nominal bonds issued by the government. These assets can be stored without cost. Both are intrinsically worthless; i.e., they do not yield direct utility or facilitate production. Nominal bonds are default-free and their maturity is two periods. A bond in its second period is called an *unmatured bond*. Each bond can be redeemed for one unit of money at, and only at, maturity.[2]

Let me describe the goods market first. In this market agents meet their trading partners bilaterally and randomly. Of interest are *trade matches*, in which the buyer likes the seller's goods. These matches are the only meetings in which a trade can take place. A buyer encounters a trade match at rate $\alpha\sigma$, and a seller at rate $\alpha(1-\sigma)$, where $\alpha < 1$. The total number of trades matches that all buyers (or sellers) of a household have in a period is $\alpha\sigma(1-\sigma)$. There is no chance for a double coincidence of wants to support barter, nor public record-keeping of transactions to support credit trades. As a result, every trade entails a medium of exchange, which may be money or unmatured bonds.

A legal restriction forbids the use of bonds as a means of payments for goods in a fraction $g \in (0, 1)$ of matches. In these matches, money is the only means of payments. In other matches the buyers can combine money and bonds to purchase goods. An example of how the legal

[1] The assumption of large households, used by Shi (1997, 1999), is a modelling device extended from Lucas (1990). Lagos and Wright (2001) use a different set of assumptions to achieve essentially the same purpose of smoothing individual matching risks.

[2] One can allow bonds past the maturity to be redeemed. As shown in Shi (2002), however, no agent in the described environment would choose to hold bonds beyond maturity. Hence, matured bonds do not circulate in the goods market.

restriction is enforced, which I do not explicitly model here, is that a fraction g of all agents in the economy are government agents who accept only money as payments. A trade is called a *restricted trade* if the legal restriction is imposed and an *unrestricted trade* otherwise.

I model the legal restriction as a matching shock in the following way: *All* members of a household will be located in restricted matches with probability g and all in unrestricted matches with probability $1 - g$. These shocks are independent across households and over time. Thus, a specific household experiences either restricted or unrestricted trades in a period, but not both. Among all households, a fraction g are in restricted matches and a fraction $(1-g)$ in unrestricted matches. This formulation simplifies the analysis.[3]

In contrast to the goods market, the bonds market has no transaction cost and trades take zero measure of agents. In this market, the government conducts open market operations by selling new two-period bonds at the competitive price. As in Lucas (1990), the government only accepts money as payments for the purchase. However, agents can bring unmatured bonds into the bonds market, sell them to other households for money, and then use the receipt to purchase new bonds, although the net amount of such transactions is zero in a symmetric equilibrium.

The amount of newly issued bonds is stochastic, which is the only aggregate uncertainty in the economy. Let M be the average amount of money holdings per household. The amount of newly issued bonds is zM, where z is a random variable following a Markov process. The realizations of z lie in a compact set Z, with a lower bound $z_L > 0$ and an upper bound $z_H < \infty$. The transition probabilities of z are described by a function $\Phi(dz, z_{-1})$, where the subscript -1 indicates the previous period. Assume that Φ has the Feller property (i.e., $f : Z \times Z \to R$ is continuous implies $\int f(z, z_{-1})\Phi(dz, z_{-1})$ is continuous).

To focus on "pure" liquidity effects, I maintain Lucas's assumption that the money stock grows at a constant rate γ; i.e., $M = \gamma M_{-1}$. This is made possible by lump-sum monetary transfers that eliminate the effect of the shock z on money growth.

2.2. Timing of Events

Let me clarify three pieces of notation. First, like Lucas (1990), I normalize nominal quantities by the aggregate money holding per household, M. Second, I suppress the generic time subscript t, denoting $t \pm j$ as $\pm j$ for $j \geq 1$. Third, I pick an arbitrary household as the representative household and use lower-case letters to denote the decisions of this household. The corresponding capital-case letters denote other households' decisions or aggregate variables.

Figure 1 depicts the timing of events in each period. At the beginning of the period the household redeems bonds that were issued two periods ago and receives a lump-sum monetary transfer, L. After these events, the household's holding of money (divided by M) is measured as

[3] The formulation is not critical. In a deterministic version of the current model (Shi 2003), I explored a different formulation where a household experiences both restricted and unrestricted trades in each period.

m, and of unmatured bonds as b.

Then, the household chooses a fraction of money, a, and a fraction of unmatured bonds, l, that will be taken to the goods market. This part of the assets the household divides evenly among the buyers; so, each buyer carries $am/(1-\sigma)$ units of money and $lb/(1-\sigma)$ units of unmatured bonds. The household takes the remaining assets to the bonds market. At the time of choosing the portfolio divisions (a, l), the household also chooses the quantities of goods and money in a trade. These quantities are contingent on whether the household members will be located in restricted or unrestricted trades. The quantities are (q^g, x^g) for a restricted trade and (q^n, x^n) for an unrestricted trade.

t redeem		portfolio (a, l)	markets open, shocks realized	markets closed	$t+1$
$\vert\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\vert\rightarrow$
money transfer L	(m, b) measured	decisions on goods trade (q, x)	trades: d, b^u and (q, x)	pooling, consume	

Figure 1 Timing of events in a period

Next, the two markets open simultaneously and separately. In the goods market, the matching shock implied by the legal restriction is realized, and agents trade according to the quantities (q^g, x^g) and (q^n, x^n) prescribed by the household. In the bonds market, the shock (z) to government bond sales occurs. Let d denote the amount of new bonds that the household purchases and b^u the amount of unmatured bonds that the household carries out of the bonds market when the market closes. The price of two-period bonds is S and of unmatured bonds S^u.

Then, the markets close and agents go home. The household pools the receipts from the trades and allocates consumption evenly among all members. After consumption, time proceeds to the next period.

With the above timing, one-period bonds do not have a chance to circulate in the goods market before maturity. Once matured, bonds are redeemed immediately, because it is not optimal to hold bonds beyond maturity, even though they can be used as a means of payments in a fraction of trades (see Shi 2002). Thus, only unmatured long-term bonds can circulate in the goods market.

The temporary separation between the bonds market and the goods market implies that there is an opportunity cost for bringing assets into the bonds market. Also, because the household must choose the portfolio divisions (a, l) before the current state z is realized, these decisions can depend on the past shock z_{-1} but not on the current shock z. In contrast, the household chooses the amounts of bonds to purchase after observing the current state z. Thus, d and b^u can depend on z, as well as on z_{-1}. Similarly, bonds prices (S, S^u) depend on both z and z_{-1}.

2.3. Quantities of Trade in the Goods Market

The household chooses the quantities of money and goods in each trade match. To describe these choices, let $v(m, b, z_{-1})$ be the household's value function, given the realization of the previous period's shock z_{-1} and the asset holdings (m, b) at the beginning of the current period after redeeming bonds and receiving monetary transfers. The discount factor is $\beta \in (0, 1)$. Let $\omega^m(z_{-1})$ be the expected shadow value of next period's money discounted to the current period, where the expectation is calculated before observing the current shock z. Similarly, let $\omega^b(z_{-1})$ be the expected shadow value of unmatured bonds. Then,

$$\omega^i(z_{-1}) = \frac{\beta}{\gamma} \int v_i(m_{+1}, b_{+1}, z) \Phi(dz, z_{-1}), \; i = m, b, \tag{2.1}$$

where $v_m = \partial v(m, b)/\partial m$ and $v_b = \partial v(m, b)/\partial b$. Notice that the discounting involves the money growth rate γ, because the variables m and b are normalized by the aggregate money stock which grows over time at rate γ. The expected values, ω^m and ω^b, are computed before the current shock z is realized, in order to make them relevant for the portfolio decisions in the current period. Other households' expected value of future money is Ω^m and of future unmatured bonds Ω^b.

In a trade match, the buyer makes a take-it-or-leave-it offer. The offer specifies the quantity of goods that the buyer asks the seller to supply, q, and the quantity of assets that the buyer gives, x. These quantities are (q^g, x^g) in a restricted trade and (q^n, x^n) in an unrestricted trade. In an unrestricted trade, it is not necessary to specify the division of the amount x^n into money and unmatured bonds, because the two assets are equivalent to each other to the seller who receives the assets as payments: Upon exiting from the trade, the seller will not have the opportunity to use the assets to purchase goods in the current period and, at the beginning of next period, the received bonds mature and can be redeemed for money at par.[4]

Because of the assumption that the buyer makes a take-it-or-leave-it offer, the quantities (q^i, x^i) yields zero surplus for the seller. Thus,

$$x^i(z_{-1}) = \frac{\psi(q^i(z_{-1}))}{\Omega^m}, \; i = n, g. \tag{2.2}$$

Also, the buyer is constrained by the sum of money and unmatured bonds in an unrestricted trade, and by the amount of money in a restricted trade. These *asset constraints* are:

$$x^n(z_{-1}) \leq \frac{a(z_{-1})m + l(z_{-1})b}{1 - \sigma}, \tag{2.3}$$

$$x^g(z_{-1}) \leq \frac{a(z_{-1})m}{1 - \sigma}. \tag{2.4}$$

When an asset constraint binds, I say that the asset yields *liquidity service* in the goods market. Similarly, money may generate liquidity in the bonds market.

[4] For the same reason, a trade in the goods market between a money holder and a bond holder is inconsequential, and so it is omitted here.

2.4. A Household's Decision Problem

In a typical period, the household's choices are the portfolio division, (a, l), the quantities of trade, (q^n, x^n, q^g, x^g), the amount of new bonds to purchase, d, the amount of unmatured bonds exiting the bonds market with, b^u, consumption, (c^n, c^g), future money holdings, m_{+1}, and future holdings of unmatured bonds, b_{+1}. Recall that the decisions (a, l, q, x, c) are functions of the previous period's state z_{-1}, but (d, b^u) can depend on the current state z as well. In principle, future money holdings depend on whether the household members are all located in restricted or unrestricted trades in the current period. I distinguish these two situations with m^g_{+1} and m^n_{+1}.

Taking other households' decisions and aggregate variables as given, the representative household solves the following problem:

$$\begin{aligned}
&(PH) \quad v(m, b, z_{-1}) = \\
&\max_{(a,l,q,x,c)(z_{-1})} \left\{ g \left[u(c^g(z_{-1})) - \alpha\sigma(1-\sigma)\psi(Q^g) + \beta \int \max_{(d,b^u)(z,z_{-1})} v(m^g_{+1}, b_{+1}, z)\Phi(dz, z_{-1}) \right] \right. \\
&\quad \left. +(1-g) \left[u(c^n(z_{-1})) - \alpha\sigma(1-\sigma)\psi(Q^n) + \beta \int \max_{(d,b^u)(z,z_{-1})} v(m^n_{+1}, b_{+1}, z)\Phi(dz, z_{-1}) \right] \right\}.
\end{aligned}$$

The constraints are as follows:

(i) the constraints in the goods market, (2.2) – (2.4), and

$$c^i(z_{-1}) = \alpha\sigma(1-\sigma)q^i(z_{-1}), \quad i = n, g; \tag{2.5}$$

(ii) the constraints in the bonds market: $b^u(z) \geq 0$ and

$$S(z, z_{-1})d(z, z_{-1}) \leq [1 - a(z_{-1})]\, m + S^u(z, z_{-1}) \{[1 - l(z_{-1})]\, b - b^u(z, z_{-1})\}; \tag{2.6}$$

(iii) the laws of motion of asset holdings:

$$b_{+1} = \frac{1}{\gamma} d(z, z_{-1}), \tag{2.7}$$

$$\begin{aligned}
m^i_{+1} = \; & \tfrac{1}{\gamma} \{m - S(z, z_{-1})d(z, z_{-1}) + S^u(z, z_{-1}) [(1 - l(z_{-1}))\, b - b^u(z, z_{-1})] \\
& + \alpha\sigma(1-\sigma) [X^i - x^i(z_{-1})] + [l(z_{-1})b + b^u(z, z_{-1})] + L_{+1}\}, \; i = g, n.
\end{aligned} \tag{2.8}$$

(iv) and other constraints: $0 \leq a(z_{-1}) \leq 1$ and $0 \leq l(z_{-1}) \leq 1$.

The objective function in the above problem contains two groups of terms, one for the case where the household's members are located in restricted matches and the other for the case in unrestricted matches.[5] The outer maximization determines the choices (a, l, q, x, c), which are made before the realization of the shock z. The inner maximization determines the choices (d, b^u), which maximize the future value function for each realization of z.

[5]The implicit assumption here is that the goods in a restricted trade yield the same marginal utility as the goods in an unrestricted trade. For a relaxation of this assumption, see Shi (2003).

The constraints in (i) are explained before, while the constraints in (iv) are self-explanatory.

There are two constraints in the bonds market, as in (ii). First, because the government does not buy back unmatured bonds, the household cannot hold a negative amount of unmatured bonds. Second, the household must finance the purchase of new bonds by the assets it brings into the bonds market, as (2.6) requires. The last term in (2.6) is the receipt of money that the household obtains by selling some of the unmatured bonds it brought to the bonds market.

In (iii), the law of motion of unmatured bonds states that the amount of unmatured bonds at the beginning of the next period is equal to the amount of new bonds purchased in the current period. The factor $1/\gamma$ appears on the right-hand side of (2.7) because b_{+1} is normalized by future money stock M_{+1} while d by M.

To explain the law of motion of money, (2.8), recall that the household's money holding is measured at the time immediately after receiving monetary transfers and redeeming matured bonds (see Figure 1). Between two adjacent points of time of this measurement, money holdings can change as a result of the following transactions: purchasing newly issued bonds, selling unmatured bonds in the bonds market, selling and buying goods, redeeming matured bonds and receiving the monetary transfer next period. The terms following m on the right-hand side of (2.8) list the net changes in money holdings from these five types of transactions. Again, the factor $1/\gamma$ appears on the right-hand side because of the normalization of variables.

To characterize optimal decisions, let $\rho(z, z_{-1})$ be the Lagrangian multiplier of the constraint in the bonds market, (2.6). Let $\lambda^n(z_{-1})$ be the multiplier of the asset constraint in an unrestricted trade, (2.3), and $\lambda^g(z_{-1})$ the multiplier of the asset constraint in a restricted trade, (2.4). To simplify the equations, multiply λ^n by $\alpha\sigma(1-\sigma)(1-g)$ and λ^g by $\alpha\sigma(1-\sigma)g$. Incorporating these constraints and use (2.2) to eliminate x, I have the following modified objective function:

$$v(m, b, z_{-1}) = \max_{(a,l,q,x,c)(z_{-1})} \left[J1 + \int \max_{(d,b^u)(z,z_{-1})} (J2)\ \Phi(dz, z_{-1}) \right],$$

where

$$\begin{aligned} J1 = \;& g\left\{ u(c^g(z_{-1})) + \alpha\sigma(1-\sigma)\left[-\psi(Q^g) + \lambda^g(z_{-1})\left(\tfrac{a(z_{-1})m}{1-\sigma} - \tfrac{\psi(q^g(z_{-1}))}{\Omega^m} \right) \right] \right\} \\ & +(1-g)\left\{ u(c^n(z_{-1})) + \alpha\sigma(1-\sigma)\left[-\psi(Q^n) + \lambda^n(z_{-1})\left(\tfrac{a(z_{-1})m + l(z_{-1})b}{1-\sigma} - \tfrac{\psi(q^n(z_{-1}))}{\Omega^m} \right) \right] \right\}, \end{aligned}$$

$$\begin{aligned} J2 = \;& \beta g v(m^g_{+1}, b_{+1}, z) + \beta(1-g) v(m^n_{+1}, b_{+1}, z) \\ & + \rho(z, z_{-1}) \left\{ [1 - a(z_{-1})]\, m + S^u(z, z_{-1}) \left[(1 - l(z_{-1}))b - b^u(z) \right] - S(z, z_{-1}) d(z, z_{-1}) \right\}. \end{aligned}$$

Notice that $X^i = x^i$ in all symmetric equilibria, and so $m^g_{+1} = m^n_{+1}$. The superscripts (g, n) on m will then be suppressed. In the following conditions for optimal choices, I also suppress the dependence of (a, l, q, x, c, λ) on z_{-1} and the dependence of (d, b^u, S, S^u, ρ) on (z, z_{-1}).

(i) Quantities q^n and q^g:

$$u'(c^i) = \left(\omega^m + \lambda^i \right) \frac{\psi'(q^i)}{\Omega^m}, \quad i = n, g. \tag{2.9}$$

(ii) Portfolio divisions (a, l) and bonds market decisions (b^u, d):

$$\text{for } a\text{:} \quad \alpha\sigma\left[(1-g)\lambda^n + g\lambda^g\right] = \int \rho\Phi\left(dz, z_{-1}\right); \tag{2.10}$$

$$\text{for } l\text{:} \quad \alpha\sigma(1-g)\lambda^n + \omega^m = \int \left(\rho + \frac{\beta}{\gamma}v_{m_{+1}}\right) S^u\Phi(dz, z_{-1}); \tag{2.11}$$

$$\text{for } b^u\text{:} \quad \frac{\beta}{\gamma}v_{m_{+1}} = \left(\rho + \frac{\beta}{\gamma}v_{m_{+1}}\right) S^u; \tag{2.12}$$

$$\text{for } d\text{:} \quad \frac{\beta}{\gamma}v_{b_{+1}} = \left(\rho + \frac{\beta}{\gamma}v_{m_{+1}}\right) S. \tag{2.13}$$

In each of these conditions, the variable attains the lowest value in the specified domain if the equality is replaced by "$<$", and the highest value if "$>$", where $a, l \in [0, 1]$ and $d, b^u \in [0, \infty)$.
(iii) The envelope condition for m and b:

$$v_m = \omega^m + a\alpha\sigma\left[(1-g)\lambda^n + g\lambda^g\right] + (1-a)\int \rho\Phi\left(dz, z_{-1}\right); \tag{2.14}$$

$$v_b = l\left[\omega^m + \alpha\sigma(1-g)\lambda^n\right] + (1-l)\int \left(\rho + \frac{\beta}{\gamma}v_{m_{+1}}\right) S^u\Phi(dz, z_{-1}). \tag{2.15}$$

The condition (2.9) requires that the net gain to a buyer from asking for an additional amount of goods be zero. By getting an additional unit of good, the household's utility increases by $u'(c)$. The cost is to pay an additional amount $\psi'(q)/\Omega^m$ of assets in order to induce the seller to trade (see (2.2)). By giving an additional unit of asset, the buyer foregoes the discounted future value of the asset, ω^m, and causes the asset constraint in the trade to be more binding. Thus, $(\omega^m + \lambda)$ is the shadow cost of each additional unit of asset to the buyer's household and the right-hand side of (2.9) is the cost of getting an additional unit of good from the seller.

In (ii), (2.10) says that for the household to allocate money to both the goods market and the bonds market, money must generate the same expected liquidity service in the two markets. The liquidity services, derived from relaxing the money constraints in the markets, are λ^n and λ^g in the goods market and ρ in the bonds market.

The condition (2.11) is a similar requirement on the allocation of unmatured bonds between the two markets. If the household takes a unit of unmatured bond to the goods market, the bond can generate liquidity services $\alpha\sigma(1-g)\lambda^n$ by relieving the asset constraints and will have a future value $\frac{\beta}{\gamma}v_{m_{+1}}$ upon redemption. If the household instead takes the unit of unmatured bond to the bonds market, the bond can be sold for S^u units of money, which will generate liquidity service ρ in the bonds market and will have a future value $\frac{\beta}{\gamma}v_{m_{+1}}$. Because the household must choose

l before seeing the realization of z, it compares the expected values of allocating a marginal unit of unmatured bonds to the two markets, and this comparison leads to (2.11).

The condition (2.12) specifies the optimal demand for unmatured bonds in the bonds market. The value of keeping a unit of unmatured bond for future redemption is the discounted future value of one unit of money, $\frac{\beta}{\gamma}v_{m+1}$. The value of selling a unit of unmatured bond for money is $\left(\rho + \frac{\beta}{\gamma}v_{m+1}\right) S^u$, as explained above. For the choice b^u to be interior, these two values must be equal to each other. The condition (2.13) is a similar requirement for the quantity of new bonds purchased, except that the price and future value of a new bond are different from those of an unmatured bond. Notice that (2.12) and (2.13) apply to every realization of z.

Finally, the envelope conditions require the current value of each asset to be equal to the sum of the expected future value of the asset and the expected liquidity service generated by the asset in the current markets. Take the condition for money for example. The current value of money is v_{m+1}. The right-hand side of (2.14) consists of the expected future value of money, ω^m, the liquidity service generated by money in the current bonds market, ρ, and the liquidity service generated by money in the current goods market, λ. The liquidity services in the two markets are weighted by the division of money into the two markets.

2.5. Equilibrium Definition and Interest Rates

A (symmetric) monetary equilibrium consists of a value function v: $R_+ \times R_+ \times Z \to R$, portfolio division functions a, l: $Z \to [0,1]$, functions of trade quantities in matches q^n, x^n, q^g, x^g: $Z \to R_+$, consumption function c: $Z \to R_+$, bonds purchase functions d, b^u: $Z \times Z \to R_+$, bonds price functions S, S^u: $Z \times Z \to R_{++}$ such that

(i) Given other households' choices and (m, b), the household's choices solve (PH);

(ii) The choices are the same across households and, in particular, $m = 1$;

(iii) The bonds market clears, i.e., $d\,(z, z_{-1}) = z$ and $b^u(z, z_{-1}) = [1 - l(z_{-1})]\, b$ for all $(z, z_{-1}) \in Z \times Z$;

(iv) $0 < \omega^m(z), \omega^b(z) < \infty$ for all $z \in Z$.

Part (i) of the definition requires that the household's choices be optimal, given other households' choices, and Part (ii) requires symmetry across households. In part (iii), the supply of new bonds (normalized by M) is z, and the supply of unmatured bonds is $(1 - l)\, b$. In part (iv), the restriction that the value of each asset be positive is necessary for a meaningful examination of the coexistence of money and bonds. The restriction that these values be bounded away from infinity is necessary for the first-order conditions to characterize optimal decisions.

Moreover, I restrict attention to equilibria in which money serves as a medium of exchange in the goods market in all states of the economy. This restriction requires $a(z_{-1}) > 0$ for all

$z_{-1} \in Z$. The restriction also requires that, for all z_{-1}, at least one of $\lambda^n(z_{-1})$ and $\lambda^g(z_{-1})$ be positive; otherwise money would be only a store of value.

By invoking equilibrium conditions, I can simplify some of the optimality conditions. First, because $d(z) = z \in (0, \infty)$ in equilibrium, the optimal condition for d must hold as equality, as in (2.13). Second, because $m = 1$ and $b = d_{-1}/\gamma = z_{-1}/\gamma$ in equilibrium, I can shorten the notation for the shadow values of the assets as follows:

$$\mu^i(z_{-1}) \equiv v_i\left(1, \frac{1}{\gamma}z_{-1}, z_{-1}\right), \; i = m, b. \tag{2.16}$$

The expected value of μ, defined in (2.1), can be expressed as $\omega^i = O(\mu^i)$, where

$$O(\mu^i)(z_{-1}) = \frac{\beta}{\gamma}\int \mu^i(z)\Phi\left(dz, z_{-1}\right), \; i = m, b. \tag{2.17}$$

Third, for all $S > 0$, the bonds market clearing conditions imply $a < 1$. Under the restriction $a > 0$, then $0 < a < 1$, and the equality in (2.10) holds. The condition (2.14) can be simplified as

$$\mu^m(z_{-1}) = \omega^m(z_{-1}) + \alpha\sigma\left[(1-g)\lambda^n(z_{-1}) + g\lambda^g(z_{-1})\right]. \tag{2.18}$$

Define the two-period (net) nominal interest rate as $r = \frac{1}{S} - 1$. If money yields liquidity in the bonds market (i.e., if $\rho > 0$), then (2.6) binds and $S = (1-a)/z$. In this case, (2.13) implies $S < \mu^b(z)/\mu^m(z)$. If $\rho = 0$, then (2.6) does not bind. In this case, $S \leq (1-a)/z$, and (2.13) implies $S = \mu^b(z)/\mu^m(z)$. Combining the two cases, I express the two-period bond price as

$$S\left(z, z_{-1}\right) = \min\left\{\frac{1 - a(z_{-1})}{z}, \frac{\mu^b(z)}{\mu^m(z)}\right\}. \tag{2.19}$$

The two-period nominal interest rate is

$$r(z, z_{-1}) = \max\left\{\frac{z}{1 - a(z_{-1})} - 1, \frac{\mu^m(z)}{\mu^b(z)} - 1\right\}. \tag{2.20}$$

From (2.13), I can compute the shadow price of the asset constraint in the bonds market as

$$\rho\left(z, z_{-1}\right) = \frac{\beta}{\gamma}\max\left\{\frac{z}{1 - a(z_{-1})}\mu^b(z) - \mu^m(z), 0\right\}. \tag{2.21}$$

This shadow price may be zero for particular realizations of z, but the expected value of $\rho(z, z_{-1})$ over z must be positive for all $z_{-1} \in Z$, in order to satisfy the earlier restriction that at least one of $\lambda^g(z_{-1})$ and $\lambda^n(z_{-1})$ be positive (see (2.10)).

The price of unmatured bonds in the bonds market, S^u, depends on whether the household takes all unmatured bonds to the goods market. If $l = 1$, the supply of and the demand for unmatured bonds in the bonds market are both zero, in which case S^u is indeterminate. If $l < 1$, the supply of unmatured bonds in the bonds market is positive. In this case, the equality in (2.12) holds and so

$$S^u(z, z_{-1}) = \frac{\mu^m(z)}{\rho(z, z_{-1})\frac{\gamma}{\beta} + \mu^m(z)}. \tag{2.22}$$

Unmatured bonds are discounted if $\rho(z, z_{-1}) > 0$.

Although the price of unmatured bonds may be indeterminate, the price of newly issued one-period bonds is determinate. If one-period bonds were issued, the price would obey (2.22) (regardless of whether $l < 1$). Denote this price by $S^I(z, z_{-1})$. Then (2.13) and (2.22) yield:

$$\frac{S(z, z_{-1})}{S^I(z, z_{-1})} = \frac{\mu^b(z)}{\mu^m(z)}. \tag{2.23}$$

The ratio μ^m/μ^b is the expected future discount on unmatured bonds. As shown later, $\mu^b(z) < \mu^m(z)$ in the equilibrium, because unmatured bonds are not perfect substitutes for money in the goods market. Thus, there is a deeper discount on two-period bonds than on one-period bonds.

3. The Characterization of the Equilibrium

The equilibrium can be one of two cases, $0 \le l < 1$ and $l = 1$. When $l = 1$, the household takes all unmatured bonds to the goods market and such bonds generate liquidity in a fraction g of trades. In the case $0 \le l < 1$, unmatured bonds do not generate liquidity (at the margin) in the goods market, i.e, $\lambda^n = 0$, although some unmatured bonds may still be used to buy goods.[6]

Money generates liquidity service in both cases. When $\lambda^n = 0$, money does not generate liquidity service in an unrestricted trade, but it does in a restricted trade. This is because, when $\lambda^n = 0$, λ^g must be positive in order to satisfy the earlier restriction that at least one of λ^g and λ^n be positive. When $\lambda^n > 0$, money generates liquidity services in both unrestricted and restricted trades. In particular, $\lambda^g > 0$ because the buyer in a restricted trade has a smaller amount of asset to use than in an unrestricted trade; if a buyer faces a binding asset constraint in an unrestricted trade, then he must be even more severely constrained in a restricted trade.

The condition $\lambda^n > 0$ is equivalent to $u'(c^n) > \psi'(q^n)$ (see (2.9) for $i = n$). Define Q_0 as the solution to the following equation:

$$u'\left(\alpha\sigma(1-\sigma)Q_0\right) = \psi'(Q_0). \tag{3.1}$$

Then, $\lambda^n > 0$ iff $q^n < Q_0$. Similarly, $\lambda^g > 0$ iff $q^g < Q_0$.

I will transform the equilibrium as a fixed point of the assets' value functions (μ^m, μ^b, ω^m) and the money allocation function a. First, I express the above conditions for binding asset

[6]The proof for $\lambda^n = 0$ in this case is as follows. When $0 \le l < 1$, the optimality condition for l holds as "$\le$"; That is, (2.11) holds as "$\le$". Because $b^u = (1-l)b \in (0, \infty)$ when $0 \le l < 1$, the equality in (2.12) holds, which leads to (2.22). Substituting (2.22) into the inequality form of (2.11) yields $\lambda^n \le 0$. Thus, $\lambda^n = 0$.

constraints as conditions on the shadow value of money. Intuitively, an asset constraint binds in a trade if the value of assets used in that trade is low. In an unrestricted trade, the total value of assets is $\frac{am+lb}{1-\sigma}\omega^m$, and the asset constraint binds iff this value is less than $\psi(Q_0)$. This binding asset constraint also implies $l = 1$, as discussed above. Since $m = 1$ and $b = z_{-1}/\gamma$ in the equilibrium, then the asset constraint binds in an unrestricted trade iff $0 < \omega^m < w_1$, where

$$w_1(a, z_{-1}) = \frac{1-\sigma}{a + z_{-1}/\gamma}\psi(Q_0). \tag{3.2}$$

Similarly, the asset constraint binds in a restricted trade iff $0 < \omega^m < w_2$, where

$$w_2(a) = \frac{1-\sigma}{a}\psi(Q_0). \tag{3.3}$$

Clearly, $w_2 > w_1$.

Second, I express the quantity of goods in a trade as a function of the expected value of money. When the asset constraint does not bind in a trade (restricted or unrestricted), the quantity of goods traded in a match is Q_0, which is independent of the value of money. When the asset constraint binds in an unrestricted trade, the buyer spends all his assets to buy goods. So, the quantity of goods in the trade is $q^n = Q_1$ where

$$Q_1(\omega^m; a, z_{-1}) = \psi^{-1}\left(\frac{a + z_{-1}/\gamma}{1-\sigma}\omega^m\right). \tag{3.4}$$

Similarly, when the asset constraint binds in a restricted trade, the quantity of goods in the trade is $q^g = Q_2$ where

$$Q_2(\omega^m; a) = \psi^{-1}\left(\frac{a\omega^m}{1-\sigma}\right). \tag{3.5}$$

Clearly, $Q_2 < Q_1$.

Third, I express the liquidity service that an asset generates in the goods market as a function of the shadow value of money. When the asset constraint binds in a trade, the assets used in that trade generate liquidity service. The total amount of liquidity services that unmatured bonds generate over all trades is $\alpha\sigma(1-g)\lambda^n$. When $\lambda^n > 0$, I can substitute λ^n from (2.9) and compute the bonds' liquidity service as $\omega^m(z_{-1})F^n$, where F^n is defined as:

$$F^n(\omega^m; a, z_{-1}) = \alpha\sigma(1-g)\left[\frac{u'\left(\alpha\sigma(1-\sigma)Q_1(\omega^m; a, z_{-1})\right)}{\psi'(Q_1(\omega^m; a, z_{-1}))} - 1\right]. \tag{3.6}$$

When $\lambda^n > 0$, money also generates the above liquidity services in unrestricted trades. In addition, money generates liquidity services in restricted trades. The total amount of such services is $\alpha\sigma g\lambda^g = \omega^m(z_{-1})F^g$, where F^g is defined as:

$$F^g(\omega^m; a) = \alpha\sigma g\left[\frac{u'(\alpha\sigma(1-\sigma)Q_2(\omega^m; a))}{\psi'(Q_2(\omega^m; a))} - 1\right]. \tag{3.7}$$

The total amount of liquidity that money generates in the goods market is $\alpha\sigma\left[g\lambda^g + (1-g)\lambda^n\right]$. This amount is equal to $\omega^m(z_{-1})F$, where

$$F(\omega^m; a, z_{-1}) = \begin{cases} F^g(\omega^m; a) + F^n(\omega^m; a, z_{-1}), & \text{if } 0 < \omega^m \leq w_1 \\ F^g(\omega^m; a), & \text{if } w_1 \leq \omega^m \leq w_2 \\ 0, & \text{if } \omega^m \geq w_2. \end{cases} \tag{3.8}$$

Now I can obtain the functional equations for $(\mu^m, \mu^b, \omega^m, a)$. The equation for μ^m comes from rewriting (2.18) as:

$$\mu^m(z_{-1}) = \omega^m(z_{-1})\left[1 + F(\omega^m(z_{-1}); a(z_{-1}), z_{-1})\right]. \tag{3.9}$$

This is a functional equation for μ^m, since $\omega^m = O(\mu^m)$. Similarly, the functional equation for μ^b comes from rewriting (2.15) as follows:[7]

$$\mu^b(z_{-1}) = \begin{cases} \omega^m(z_{-1})\left[1 + F^n(\omega^m(z_{-1}); a(z_{-1}), z_{-1})\right], & \text{if } 0 < \omega^m \leq w_1 \\ \omega^m(z_{-1}), & \text{if } \omega^m \geq w_1. \end{cases} \tag{3.10}$$

Finally, using (2.10) to eliminate (λ^g, λ^n) in (2.18) and using (2.21) to eliminate ρ, I have:

$$a(z_{-1}) = 1 - \frac{\beta/\gamma}{\mu^m(z_{-1})}\int \max\left\{z\mu^b(z),\ \left[1 - a(z_{-1})\right]\mu^m(z)\right\}\Phi(dz, z_{-1}). \tag{3.11}$$

The strategy for determining the equilibrium is as follows. Start with an arbitrary continuous function $a(.)$ bounded in the interior of $[0,1]$ and solve the fixed point for μ^m from (3.9). Substitute the solution into $\omega^m = O(\mu^m)$ to get ω^m and into (3.10) to get μ^b. Then, substitute (μ^m, μ^b) into the right-hand side of (3.11) to obtain a new function, denoted as $\Gamma_a(z_{-1})$. The equilibrium solution for $a(.)$ solves $a(z_{-1}) = \Gamma_a(z_{-1})$. Once the functions $(\mu^m, \mu^b, \omega^m, a)$ are determined, I can recover the traded quantities of goods and consumption (output) through (3.4) and (3.5), the bond price S through (2.19) and the nominal interest rate through (2.20). The fraction of unmatured bonds taken to the goods market is $l = 1$ if $\omega^m < w_1$ and $l \in [0,1)$ if $\omega^m > w_1$.[8]

The equilibrium requires $\omega^m < w_2$; otherwise, $\mu^m = \omega^m = O(\mu^m)$, which would not have a stationary solution for μ^m when $\gamma > \beta$. For all $\omega^m < w_2$, (3.9) and (3.10) imply that $\mu^m(z) > \mu^b(z)$ for all z. Thus, unmatured bonds are not perfect substitutes for money in the goods market and, as (2.23) shows, this imperfect substitutability induces a deeper discount on two-period bonds than on one-period bonds. Of course, the imperfect substitutability relies on the existence of the legal restriction.

[7] The derivation is straightforward when $\omega^m < w_1$ (i.e., when $l = 1$). When $\omega^m > w_1$, $0 \leq l < 1$ and $\lambda^n = 0$, as discussed above. Then $b^u = (1-l)b \in (0, \infty)$, and so the equality in (2.12) holds. This equality and the fact $\lambda^n = 0$ reduce (2.15) to $\mu^b = \omega^m$.

[8] When $\omega^m > w_1$, the equilibrium is consistent with a range of values of l in $[0,1)$. This indeterminacy of l has no effect on real variables, because unmatured bonds do not generate liquidity service at the margin. The indeterminacy does not affect the equilibrium value of a, either, and hence the bond price S and the corresponding interest rate r do not depend on such indeterminacy.

4. A Special Case: Independent Shocks

Let me first study the special case where the shocks to bond sales are independent over time. This special case helps illustrating some key differences between the current model and Lucas's (1990) model. The equilibrium behaves differently depending on whether unmatured bonds generate liquidity service in the goods market.

Consider first the case where unmatured bonds do not generate liquidity service, i.e., where $\omega^m > w_1$. Since only the money constraint binds in this case, the equilibrium behaves like that in Lucas's model. With independent shocks, the shadow values of assets (μ^m, μ^b) and the fraction a are numbers, rather than functions. To solve for these constants, note that $F = F^g$ in this case. Also, $\omega^m = O(\mu^m) = \frac{\beta}{\gamma}\mu^m$. Then, (3.9) becomes $F^g(\omega^m; a) = \frac{\gamma}{\beta} - 1$. Substituting F^g from (3.7), this equation solves for the quantity of goods in a restricted trade, which is a constant. Because the quantity of goods in an unrestricted trade is equal to the constant Q_0 when $\omega^m > w_1$, consumption and output are constant. Moreover, $\mu^b = \omega^m$ by (3.10), and so $\mu^b = \frac{\beta}{\gamma}\mu^m$. With these numbers (μ^b, μ^m), (3.11) yields:

$$\frac{\gamma}{\beta} = \int \max\left\{\frac{z\beta/\gamma}{1-a}, 1\right\} \Phi(dz).$$

This solves for the constant a under suitable conditions.[9]

In this case, the interest rate depends only on the current shock, because (2.20) becomes:

$$r(z) = \max\left\{\frac{z}{1-a} - 1, \frac{\gamma}{\beta} - 1\right\}.$$

A tightening open market operation, modelled as an increase in z, raises the interest rate when $z < (1-a)\frac{\gamma}{\beta}$. This liquidity effect in the bonds market does not translate into any effect on real activities. Nor does it affect the additional discount on two-period bonds relative to one-period bonds, which is $\gamma/\beta - 1$ (see (2.23)).

Continue the examination of the economy with independent shocks but consider the case where unmatured bonds generate liquidity service, i.e., where $0 < \omega^m < w_1$. This case of the equilibrium behaves quite differently from that in Lucas's model. In particular, μ^m and a are no longer constants. Because the asset constraint binds in an unrestricted trade, the quantity of goods in such a trade depends on the amount of unmatured bonds, as well as the money stock. Since the amount of unmatured bonds in a period is equal to the quantity of new bonds issued in the previous period, the quantity of goods in an unrestricted match depends on the realization of the previous period's shock, z_{-1} (see (3.4)). That is, the previous period's shock affects the amount of liquidity in the current goods market. As a result, the current shadow values of the two assets are functions of the previous shock (see (3.9) and (3.10)). Since these asset values

[9] A sufficient condition is that the left-hand side of the equation is greater than the right-hand side when $a = 0$.

affect the allocation of money between the two markets, a is a function of z_{-1}, even though the shocks are independent over time.

The nominal interest rate now depends on both the current shock and the previous period's shock. Thus, nominal interest rates are serially correlated even though the shocks are independent over time. Moreover, open market operations affect the relative value of unmatured bonds to money, and hence affect the term structure of interest rates. Determining the equilibrium in this case is not so much easier than in the case of dependent shocks. So, I will go directly to the equilibrium with dependent shocks.

5. The Equilibrium with Dependent Shocks

I now study the equilibrium in which the shocks are dependent. All proofs are collected in Appendix A. For various proofs I need to restrict attention to $\mu^m(.) \in \mathcal{V}$ and $a(.) \in \mathcal{A}$, where $\mathcal{V}$ denotes the set of continuous functions whose values lie in $[\frac{\gamma}{\beta}\omega_L, \frac{\gamma}{\beta}\omega_H]$ and $\mathcal{A}$ the set of continuous functions whose values lie in $[a_L, a_H]$. Norm both $\mathcal{V}$ and $\mathcal{A}$ by the supnorm. In addition to some restrictions described later in Lemmas 5.1 and 5.2, the bounds satisfy:

$$0 < a_L \leq a_H < 1, \; 0 < \omega_L \leq \omega_H < \infty. \tag{5.1}$$

By (2.17), $\omega_L \leq \omega^m(z) \leq \omega_H$ for all z.

The first task to determine the equilibrium is to solve for μ^m from (3.9) under a fixed function $a(.)$. Denote the right-hand side of (3.9) as $T(\omega^m; a, z_{-1})$ and define

$$TO(\mu^m; a, z_{-1}) = T(O(\mu^m); a, z_{-1}). \tag{5.2}$$

Then, (3.9) requires μ^m to be the fixed point of TO. I will find conditions under which TO is a monotone, contraction mapping from $\mathcal{V}$ to $\mathcal{V}$. Impose the following assumption.

Assumption 1. *Denote the relative risk aversion by $\delta(c) = -cu''(c)/u'(c)$. Assume that (i) $\delta(c) \leq 1$, and (ii) the function $[1-\delta(c)]\, u'(c) \Big/ \psi'\left(\frac{c}{\alpha\sigma(1-\sigma)}\right)$ is decreasing in c.*

Part (i) of the assumption is sufficient for $T(\omega^m; a, z_{-1})$ to be increasing in ω^m. Part (ii) is necessary and sufficient for T to be concave in ω^m in each of the three segments $(0, w_1)$, (w_1, w_2), and (w_2, ∞). Part (ii) is satisfied if, for example, the utility function exhibits constant relative risk aversion. Figure 2 depicts T as a function of ω^m. Notice that $\partial T/\partial \omega^m$ is larger at w_1+ than at w_1-, and larger at w_2+ than at w_2-. Thus, T is not concave in the entire region $(0, \infty)$.

The mapping TO is monotone under part (i) in Assumption 1, because $T(\omega^m; a, z_{-1})$ is increasing in ω^m and because $\omega^m = O(\mu^m)$ is a monotone linear mapping of μ^m. However, TO fails to satisfy the contraction mapping requirement when ω^m is sufficiently small. This is because when $\omega^m \to 0$, $Q_1, Q_2 \to 0$, in which case $\partial T/\partial \omega^m$ is infinite. To ensure that TO is a contraction, I impose a lower bound on ω^m and hence an upper bound on $\partial T/\partial \omega^m$.

Denote $T_\omega = \partial T(\omega^m; a, z_{-1})/\partial\omega^m$. Since T is concave in ω^m in each of the three segments, $T_\omega \leq \max\{T_\omega(w_1+; a, z_{-1}), 1\}$ for all $\omega^m \geq w_1$. Also, because $T_\omega(w_1\text{-}; a, z_{-1}) < T_\omega(w_1+;a, z_{-1})$, there exists $w_3 < w_1$ such that for all $\omega^m \geq w_3$, $T_\omega \leq \max\{T_\omega(w_1+; a, z_{-1}), 1\}$. Under (ii) of Assumption 1, $T_\omega(w_1+; a, z_{-1})$ decreases in a and increases in z_{-1}, after the dependence of w_1 on (a, z_{-1}) is taken into account. Setting $a = a_L$, $z_{-1} = z_H$ and $w_1 = w_1(a_L, z_H)$, I have

$$T_\omega(w_1+; a, z_{-1}) \leq \bar{T}_\omega \equiv 1 - \alpha\sigma g + \frac{\alpha\sigma g}{\Psi} \frac{[1 - \delta(\widehat{c})]\, u'(\widehat{c})}{\psi'\left(\frac{\widehat{c}}{\alpha\sigma(1-\sigma)}\right)}, \text{ all } (a, z_{-1}),$$

where

$$\widehat{c} = \alpha\sigma(1-\sigma)\psi^{-1}\left(\frac{a_L\psi(Q_0)}{a_L + z_H/\gamma}\right).$$

I choose the upper bound on T_ω as

$$K = \kappa \max\{\bar{T}_\omega, 1\}, \text{ where } 1 \leq \kappa < \infty.$$

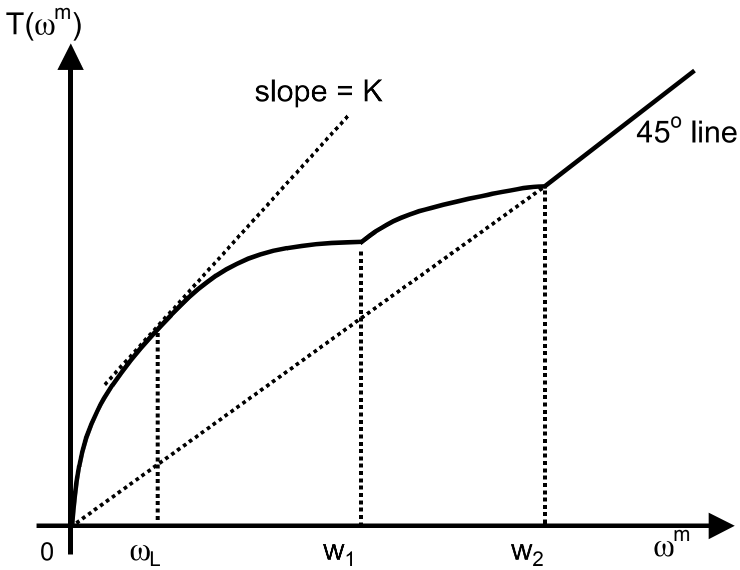

Figure 2

The upper bound K leads to a lower bound on ω^m. Let $\omega_0(a, z_{-1})$ $(< w_1)$ solve $T_\omega(\omega_0; a, z_{-1}) = K$. Because $T_\omega(\omega; a, z_{-1})$ is decreasing in (ω, a, z_{-1}), ω_0 is decreasing in (a, z_{-1}). The lower bound of ω^m is then defined as $\omega_L = \omega_0(a_L, z_L)$. Clearly, ω_L is smaller if a larger κ is chosen (see Figure 2), and $\omega_L > 0$ for all finite κ. Also, for all $\omega^m \geq \omega_L$, $0 < T_\omega \leq K$.

Lemma 5.1. *Let $\varepsilon > 0$ be a small number. Given any function $a(.) \in \mathcal{A}$, the mapping TO defined in (5.2) is a monotone contraction mapping from $\mathcal{V}$ to $\mathcal{V}$ if the following condition holds:*

$$\max\{K + \varepsilon, F(\omega_H, a_L, z_L) + 1\} \leq \gamma/\beta \leq F(\omega_L, a_H, z_H) + 1. \tag{5.3}$$

Under this condition, TO has a unique fixed point $\mu_a^m(.) \in \mathcal{V}$.

The set of parameter values that satisfy (5.3) is nonempty. To see this, note that $F(\omega_L, a_H, z_H) > 0$ by construction. By choosing κ sufficiently close to 1, I can ensure $K + \varepsilon < F(\omega_L, a_H, z_H) + 1$. Then, there are values of γ ($> \beta$) that satisfy $K + \varepsilon \leq \gamma/\beta \leq F(\omega_L, a_H, z_H) + 1$. Finally, because $F(\omega; a, z) = 0$ when ω is large, I can choose a large value for ω_H to ensure $F(\omega_H, a_L, z_L) + 1 \leq \gamma/\beta$. Clearly, these conditions require $\gamma > \beta$ and $\omega_H > \omega_L$.

The notation μ_a^m emphasizes the dependence of the fixed point on the arbitrarily chosen function a. Similarly, the expected future shadow value of money is $\omega_a^m(z_{-1}) = O(\mu_a^m)(z_{-1})$ and the shadow value of unmatured bonds is $\mu_a^b(.)$, obtained by substituting μ_a^m into (3.10). Clearly, $\omega_a^m(.)$ and $\mu_a^b(.)$ are continuous. Also, $\omega_a^m(z) \in [\omega_L, \omega_H]$, all $z \in Z$.

The task now is to determine the equilibrium fraction of money in the goods market, a. Substituting μ_a^m and μ_a^b into (3.11), I have $a(z_{-1}) = \Gamma_a(z_{-1})$, where

$$\Gamma_a(z_{-1}) \equiv 1 - \frac{\beta/\gamma}{\mu_a^m(z_{-1})} \int \max\left\{z\mu_a^b(z),\ [1 - a(z_{-1})]\,\mu_a^m(z)\right\} \Phi(dz, z_{-1}). \tag{5.4}$$

Since I required $a \in \mathcal{A}$, the equilibrium satisfies this requirement only if $\Gamma_a \in \mathcal{A}$. The following lemma gives the sufficient conditions:

Lemma 5.2. *Given any $a \in \mathcal{A}$, $\Gamma_a \in \mathcal{A}$ if a_H is close to one and if*

$$F(\omega_H, a_H, z_H) \geq \max\left\{\frac{z_H}{1 - a_L}, 1\right\}. \tag{5.5}$$

The condition (5.5) is satisfied if z_H and a_L are small. This restriction on z_H is necessary to ensure that the households allocate a positive fraction of money to the goods market. If the size of the open market operation were very large, instead, new bonds would be heavily discounted; given that the money growth rate is fixed, the households would allocate all the money to the bonds market to obtain the discount.

In light of Lemma 5.2, I can interpret Γ as a mapping from $\mathcal{A}$ to $\mathcal{A}$. Then, the equilibrium function a is a fixed point of Γ. Γ is continuous (see the proof of the following theorem in Appendix A). Since $\mathcal{A}$ is compact and convex, Brouwer's fixed point theorem implies that Γ has a fixed point in $\mathcal{A}$. This fixed point is the equilibrium function $a(.)$. Using this function in the computation of $\mu_a^m(.)$ and $\mu_a^b(.)$, I can recover the shadow values of money and unmatured bonds.

The following theorem summarizes the above results.

Theorem 5.3. *Maintain Assumption 1 and (5.1). Let (z_H, a_L) satisfy (5.5) and let a_H be close to 1. Choose (γ, K, ω_H) to satisfy (5.3) and define ω_L by $T_\omega(\omega_L; a_L, z_L) = K$. An equilibrium exists, which satisfies $\mu^m(.) \in \mathcal{V}$, $a(.) \in \mathcal{A}$, and $\omega_L \leq \omega^m(z) \leq \omega_H$ for all $z \in Z$.*

Because the mapping that defines the function a is implicit, it is difficult to check whether the solution is monotone. Likewise, it is difficult to check whether consumption is a monotonic function of the past shock. To study equilibrium properties, I turn to numerical examples.

6. Numerical Examples

Assume the following forms of utility and cost:

$$u(c) = u_0 \frac{c^{1-\delta} - 1}{1-\delta}, \quad \psi(q) = \psi_0 q^{\Psi}.$$

Let the shock z have two realizations, z_1 and z_2, with $z_2 > z_1$. Refer to z_2 as the high shock and z_1 as the low shock. The transition probability from z_i to z_i is θ and to $z_{i'}$ $(i' \neq i)$ is $1-\theta$, $i = 1, 2$. Consider the following parameter values:

preference:	$\delta = 0.5$, $u_0 = 4$, $\psi_0 = 1$, $\Psi = 2$, $\beta = 0.995$;
goods market:	$\alpha = 1$, $\sigma = 0.5$, $g = 0.2$
monetary policy:	$z_1 = 0.02$, $z_2 = 0.08$, $\gamma = 1.005$.

I choose g to be the size of the government relative to the economy, using the interpretation that the legal restriction in the goods market is imposed in trades between private households and the government. The values of (β, z_1, z_2) are the ones chosen by Lucas (1990). With the particular value of β, I can interpret the length of a period as one month and the interest rate r as the bi-monthly interest rate. Also following Lucas, I explore a large range of values of θ: $0.01, 0.1, 0.3, 0.5, 0.7, 0.9$ and 0.99.

Choose $a_L = 0.90$, $a_H = 0.98$, $\kappa = 1$, $\omega_L = 0.543$ and $\omega_H = 3.375$. These parameter values satisfy all conditions required for existence in Theorem 5.3. Moreover, the equilibrium lies in the region $\omega^m \in (0, w_1)$. That is, unmatured bonds generate liquidity in the goods market and the household takes all unmatured bonds to the goods market.[10]

To display the results, let me add a subscript i to variables that depend only on the previous period's shock z_i, where $i = 1, 2$. Add subscripts ji to variables that depend on both the current shock z_j and the previous period's shock z_i, where $i, j = 1, 2$. Table 1 describes equilibrium properties of the fraction of money taken to the bonds market $(1-a)$, interest rates, consumption, the quantities of goods in trades, the shadow values of assets and the term structure of interest rates. The mean, the standard deviation, and serial autocorrelations are calculated using the unique invariant measure $prob(z_i) = 1/2$ for $i = 1, 2$. Since goods have different prices in unrestricted trades and restricted trades, I compute aggregate real output (consumption) as follows:

$$c_i = \alpha\sigma(1-\sigma)\left[\frac{gp^g(z_i)q^g(z_i) + (1-g)p^n(z_i)q^n(z_i)}{gp^g(z_i) + (1-g)p^n(z_i)}\right],$$

where p^g is the price of goods in a restricted trade, normalized by the money stock, and p^n is the normalized price in an unrestricted trade. The term structure of interest rates is represented by the percentage difference between the yield to newly issued two-period bonds, $S^{-1/2}$, and the

[10] This is the case for a large range of parameter values. In fact, the range $(w_1, w_2) \ni \omega^m$, in which unmatured bonds do not generate liquidity service in the goods market, is very narrow.

yield to one-period bonds, $1/S^I$. Letting r^I be the one-period interest rate corresponding to S^I and using (2.23), I can write this percentage difference as:

$$term_{ji} = \left[\left(\frac{\mu^m(z_j)/\mu^b(z_j)}{1 + r^I(z_j, z_i)} \right)^{1/2} - 1 \right] \times 100. \tag{6.1}$$

Three results in Table 1 resemble those found by Lucas (1990). First, interest rates change significantly with the persistence of the shock when the current shock is high. Also, interest rates have a large (unconditional) standard deviation. However, the mean of interest rates does not vary significantly with the persistence of the shock, even if the degree of persistence varies between 0.1 and 0.9. Thus, if one is interested only in the mean of interest rates, one can ignore the persistence and simply examine the case of independent shocks (i.e., $\theta = 0.5$).

Second, the fraction of money allocated to the bonds market is insensitive to the previous period's shock. As in Lucas's model, this insensitivity is puzzling especially when the shocks are negatively dependent. With negatively dependent shocks, a high shock in the previous period implies that the amount of bond sales is likely to be low in the current period and the bond price likely to be high. Since the discount on bonds will be small, there is not much need to allocate more money to the bonds market to take advantage of the discount on new bonds. Thus, when the shocks are negatively correlated, one would expect that the household would reduce $(1 - a)$ significantly upon observing a high shock in the previous period. This does not happen in the numerical examples.

The insensitivity is more puzzling here than in Lucas's model, because the goods market provides an additional reason for the household to adjust the money allocation significantly. A high past shock increases the amount of assets used in an unrestricted trade relative to the assets in a restricted trade. This widens the gap between the quantities of goods obtained in the two types of trades, and hence increases the variation in consumption. To smooth consumption between the two types of trades, the household should increase the fraction of money allocated to the goods market, so as to maintain a stable ratio of assets used in an unrestricted trade relative to a restricted trade. Despite this additional reason, the negative response of $(1 - a)$ to the past shock is not significant. Even when $\theta = 0.1$, an increase of z_{-1} from z_1 to z_2 reduces $(1 - a)$ from 7.83% to 7.65%. This reduction is small in comparison with the variation in the shock.

Third, the insensitivity of the money allocation leads to a strong liquidity effect in the bonds market, as measured by changes in nominal interest rates. Interest rates are significantly higher when the current shock is high than when the current shock is low; that is, r_{2i} is much higher than r_{1i} for $i = 1, 2$. When the money allocation is insensitive, a higher supply of new bonds must be absorbed by a fall in the bond price, resulting in a higher interest rate. Notice that, when the current shock is low, the one-period interest rate is zero (see (2.22)) and the two-period interest rate does not depend on past shocks. This is because there is more money than what is needed in the bonds market when the current shock is low.

Table 1. Simulation results

	θ						
	0.01	0.1	0.3	0.5	0.7	0.9	0.99
$1-a_1$ (%)	7.84	7.827	7.789	7.722	7.578	7.003	3.923
$1-a_2$ (%)	6.17	7.646	7.852	7.890	7.900	7.885	7.863
r_{11} (%)	0.474	0.452	0.444	0.438	0.409	0.379	0.326
r_{12} (%)	0.474	0.452	0.444	0.438	0.409	0.379	0.326
r_{21} (%)	2.045	2.210	2.714	3.601	5.566	14.23	103.9
r_{22} (%)	29.66	4.632	1.884	1.392	1.261	1.454	1.736
$E(r)$ (%)	1.397	1.452	1.455	1.468	1.481	1.556	1.542
$StD(r)$ (%)	2.151	1.124	1.045	1.292	1.759	2.955	7.291
$corr(r, r_{-1})$	-0.126	-0.461	-0.535	-0.341	-0.165	-0.028	0.004
$corr(r, r_{-2})$	0.124	0.368	0.214	0	-0.066	-0.023	0.004
$corr(r, r_{-3})$	-0.121	-0.295	-0.086	0	-0.026	-0.018	0.004
c_1	0.616	0.617	0.617	0.617	0.618	0.619	0.622
c_2	0.627	0.627	0.626	0.626	0.626	0.625	0.622
$E(c)$	0.622	0.622	0.622	0.622	0.622	0.622	0.622
$StD(c)$	0.0058	0.0049	0.0047	0.0044	0.0041	0.0026	0.0002
$corr(r, c)$	-0.294	-0.518	-0.533	-0.427	-0.270	-0.071	0.025
q_1^n	2.469	2.472	2.473	2.474	2.476	2.482	2.492
q_2^n	2.519	2.519	2.519	2.519	2.519	2.518	2.508
q_1^g	2.442	2.446	2.447	2.448	2.449	2.456	2.466
q_2^g	2.473	2.455	2.450	2.446	2.437	2.416	2.406
μ_1^m	3.293	3.298	3.299	3.298	3.294	3.285	3.197
μ_2^m	3.269	3.276	3.272	3.262	3.242	3.190	3.174
μ_1^b	3.277	3.283	3.284	3.247	3.246	3.273	3.187
μ_2^b	3.260	3.263	3.258	3.247	3.225	3.169	3.151
term_{11} (%)	0.236	0.226	0.222	0.219	0.204	0.189	0.163
term_{12} (%)	0.236	0.226	0.222	0.219	0.204	0.189	0.163
term_{21} (%)	-0.726	-0.698	-0.906	-1.304	-2.167	-5.824	-29.47
term_{22} (%)	-11.93	-1.854	-0.504	-0.235	-0.108	-0.069	-0.149

The model generates several results that are absent in Lucas (1990). I will describe these results below for the case of independent shocks, since the contrasts with Lucas's model are the sharpest in this case.

First, open market operations have real effects – A high shock in the previous period increases current real output. The difference between output in the two realizations of the shock is about 0.7% of the mean. This real effect arises because (unmatured) bonds generate liquidity in the goods market. A high shock in the previous period increases the stock of unmatured bonds in the current goods market. This allows a buyer to purchase a larger quantity of goods in an unrestricted trade than if the previous period's shock was low, i.e., $q_2^n > q_1^n$. The presence of a larger quantity of nominal assets in the goods market also pushes up the price level and reduces the quantity of goods purchased in a restricted trade, i.e., $q_2^g < q_1^g$. In the numerical examples, the increase in q^n dominates the decrease in q^g, and so aggregate output rises.[11]

Second, a high past shock reduces the current interest rate when the current shock is high. In contrast, past (independent) shocks in Lucas's model do not affect the current interest rate. To explain this new effect, recall that a high past shock increases the amount of unmatured bonds circulating in the current goods market and increases the price level. The higher price level reduces real values of both money and unmatured bonds. However, the real value of unmatured bonds (μ^b) falls by less than does the real value of money (μ^m), because the increased amount of unmatured bonds increases liquidity in unrestricted trades. (In fact, μ^b barely changes at all with past shocks when shocks are independent.) Thus, the relative value of unmatured bonds to money increases, which induces the households to allocate more money to purchase new bonds. When the current shock is high, the additional money in the bonds market pushes up the bond price and depresses the current interest rate. When the current shock is low, the additional money in the bonds market does not affect the current interest rate, as discussed above.

Third, the above effects of past shocks on current activities induce the following correlations: (i) Interest rates in two adjacent periods are negatively correlated; (ii) Contemporaneous output and interest rates are negatively correlated; (iii) Output is positively correlated with lagged interest rates (not reported in Table 1).[12] These correlations arise because a shock in the previous period increases the interest rate in the previous period, increases current output, and reduces the current interest rate. Note that the negative contemporaneous correlation between output and interest rates is -0.427, which is comparable with the sample value in the US data (see Christiano et al., 1995). However, the positive correlation in (iii) is unrealistic.

Finally, the term structure of interest rates responds to open market operations. The yield curve is negatively sloped when the current shock is high and positively sloped when the current shock is low. In light of (6.1), this negative response of the yield curve to the current shock is not surprising. For example, a high current shock increases the one-period interest rate; at the same time, it reduces the expected future discount on unmatured bonds (μ^m/μ^b) by generating liquidity in next period's goods market. Both effects reduce the slope of the yield curve.

Moreover, the slope of the yield curve can depend on the previous period's shock. When the

[11]Because consumption varies between the states of the shock, the real interest rate varies with the shock. However, the real interest rate varies by less than the nominal interest rate does. Also, real output (consumption) is serially correlated (not reported in Table 1). The coefficient of correlation between current consumption and k-period past consumption is equal to $(2\theta-1)^k$. Thus, positively correlated shocks induce positive autocorrelations in consumption.

[12]The formulas for the correlations between r and c are as follows:

$$\begin{aligned} corr(r,y) &= \tfrac{y_2-y_1}{2}\left[\theta(r_{22}-r_{11})+(1-\theta)(r_{12}-r_{21})\right], \\ corr(r,y_{+1}) &= \tfrac{y_2-y_1}{2}\left[\theta(r_{22}-r_{11})+(1-\theta)(r_{21}-r_{12})\right]. \end{aligned}$$

Moreover, $corr(r,y_{-j}) = (2\theta-1)^j corr(r,y)$ and $corr(r,y_{+j}) = (2\theta-1)^{j-1} corr(r,y_{+1})$, for $j = 1, 2, ...$

current shock is high, a high past shock makes the yield curve less negatively sloped. To explain this result, recall that a high past shock increases the money allocation to the current bonds market. This higher money allocation reduces current interest rates when the current shock is high. However, the expected future discount on unmatured bonds depends only on the current shock, not on past shocks. Thus, by (6.1), the yield curve becomes less negatively sloped.[13] Clearly, the role of unmatured bonds in the goods market is important for this dependence of the yield curve on past shocks, because it is the reason why the households condition their money allocation on past shocks. In contrast, in Lucas's model, bonds play no role in the goods market regardless of the maturity, and so the yield curve is independent of past shocks when the shocks are independent.

Most of the above features with independent shocks continue to exist when shocks are dependent. However, there are a few changes. First, when shocks are highly negatively dependent (i.e., $\theta \leq 0.1$), a high past shock increases (rather than decreases) the current interest rate when the current shock is high. That is, $r_{22} > r_{21}$. This is because, given the high past shock and the negative serial dependence, the households anticipate bond sales to be low in the current period and so they allocate less money to the bonds market. When the current bond sales turn out to be high, the interest rate will be high. Second, when the shocks are highly persistent (i.e., $\theta \geq 0.99$), the correlation between the current and one-period past interest rates becomes positive. This is not surprising because a permanent shock will generate a positive correlation between interest rates in all periods. Similarly, the contemporaneous correlation between output and interest rates becomes positive when the shocks are highly persistent.

Table 2 reports the results of the sensitivity analysis. In this analysis, shocks are independent and I perturb the money growth rate (γ), the scope of the legal restriction (g), the relative risk aversion (δ) and the variation in the shock. These perturbations have small effects on real variables but large effects on the nominal interest rate.

First, an increase in the money growth rate increases the mean of interest rates and reduces the mean of real consumption (output); it also increases the standard deviations of interest rates and real output. By eliminating net money growth from the baseline case, the mean and standard deviation of interest rates fall by about a half, and the standard deviation in output falls by more than a half. Real output and the nominal interest rate are still negatively correlated with each other but the magnitude seems to first increase, and then decrease, with money growth.

Second, an increase in the scope of the legal restriction increases the mean of interest rates but affects the standard deviation of interest rates in a hump-shaped pattern. The mean of real consumption barely changes with the increase in the scope of the legal restriction, the standard deviation of consumption decreases, and the negative correlation between consumption and in-

[13] Of course, when the current shock is low, the one-period interest rate is zero and unaffected by the money allocation , in which case the slope of the yield curve is independent of past shocks.

terest rates weakens. Real consumption responds in this way because the wider coverage of the legal restriction reduces the liquidity effect of unmatured bonds in the current goods market and reduces the variation in the quantity of goods between a restricted trade and an unrestricted trade. Because consumption varies less and interest rates vary more between different states, the two variables become less correlated with each other.

Table 2. Sensitivity results

	baseline	$\gamma = 1$	$\gamma = 1.05$	$g = 0.01$	$g = 0.5$	$\delta = 0.05$	$z_1 = 0.001$ $z_2 = 0.099$
$1 - a_1$ (%)	7.722	7.863	6.983	7.688	7.749	7.720	9.542
$1 - a_2$ (%)	7.890	7.932	7.216	8.000	7.807	7.895	9.794
$E(r)$ (%)	1.468	0.788	7.024	1.038	1.826	1.435	1.375
$StD(r)$ (%)	1.292	0.601	5.839	0.012	1.062	1.320	1.404
$corr(r, r_{-1})$	-0.341	-0.314	-0.154	-0.331	-0.175	-0.343	-0.352
c_1	0.617	0.624	0.581	0.614	0.620	0.502	0.616
c_2	0.626	0.627	0.594	0.629	0.623	0.513	0.627
$E(c)$	0.622	0.626	0.588	0.622	0.622	0.507	0.622
$StD(c)$	0.0044	0.0018	0.0066	0.0079	0.0015	0.0056	0.0052
$corr(r, c)$	-0.427	-0.368	-0.158	-0.581	-0.181	-0.434	-0.474
term_{11} (%)	0.219	0.137	0.665	0.021	0.399	0.196	0.166
term_{12} (%)	0.219	0.137	0.665	0.021	0.399	0.196	0.166
term_{21} (%)	-1.304	-0.581	-5.259	-1.966	-0.778	-1.335	-1.471
term_{22} (%)	-0.235	-0.147	-3.693	0.003	-0.407	-0.225	-0.181

Baseline parameters: $\theta = 0.5$, $\gamma = 1.005$, $g = 0.2$, $\delta = 0.5$, $z_1 = 0.02$, $z_2 = 0.08$, $\theta = 0.5$.

Third, a decrease in the relative risk aversion reduces the mean and increases the variation in interest rates. It also reduces the mean of output, increases the variation in output, and strengthens the negative correlation between output and the nominal interest rate. A remarkable feature is that the allocation of money between the two markets remains very insensitive to the previous period's shock even when the utility function is almost linear (i.e., when $\delta = 0.05$). This is not a feature special to the case of independent shocks; it also occurs when shocks are highly persistent, e.g., when $\theta = 0.99$.

Fourth, an increase in the mean-preserving spread in the shock reduces the mean and increases the variation in interest rates. It also increases the variation in output, without affecting the mean of output much, and strengthens the negative correlation between output and the interest rate.

Finally, the above changes in the parameter values change the magnitude of the slope of the yield curve but have very little effect on the sign of the slope. Not surprisingly, when the scope of the legal restriction becomes very narrow ($g = 0.01$), the yield curve becomes very flat in most cases. When $g \to 1$, the equilibrium approaches the one analyzed by Lucas (1990).

7. Conclusion

In this paper I combine a decentralized goods market and a centralized bonds market to analyze the liquidity effects of open market operations. The bonds market features limited participation, while the goods market features bilateral matches. In a fraction of trades, a legal restriction forbids the use of bonds as the means of payments for goods. In such a restricted trade, the buyer faces a money constraint. In an unrestricted trade, the buyer can use both money and unmatured bonds to buy goods, and so unmatured bonds can provide liquidity. A shock to bond sales in this economy has two distinct liquidity effects. One is the immediate liquidity effect in the bonds market, as emphasized by Lucas (1990), and the other is a liquidity effect in the goods market starting one period later.

The liquidity effect in the bonds market arises because there is limited participation in the bonds market and because the households' money allocation between the markets is insensitive to past shocks, even when shocks are highly persistent. With this insensitive money allocation, the bond price and hence the nominal interest rate absorbs most of the shock to current bond sales. This liquidity effect is short-lived, as in Lucas's model.

The liquidity effect in the goods market is new and it occurs with a delay.[14] For example, a high shock to bond sales in the previous period increases the amount of unmatured bonds circulating in the current goods market, relaxes the asset constraints in unrestricted trades, and hence increases the quantity of goods traded in an unrestricted trade relative to that in a restricted trade. Important for this liquidity effect is the temporary separation between trades in the goods market, implied by random matches. If all exchanges in the goods market were centralized in the Walrasian style, then a high past shock would simply push up the price level without affecting real output. In contrast with the liquidity effect in the bonds market, the liquidity effect in the goods market can be long-lived and, in principle, the duration of this effect increases with the length of maturity of the bonds that are used in open market operations.[15]

The liquidity effect of unmatured bonds in the goods market generates a number of new features. These features are best illustrated in the case of independent shocks. First, a high shock to bond sales in the previous period leads to higher current output by increasing liquidity in the current goods market. This also implies that the real interest rate varies with past shocks. Second, the shock in the previous period affects the current interest rate by changing the allocation of money between the two markets. In particular, a high past shock reduces the relative value of money to unmatured bonds and increases the amount of money allocated to the current bonds

[14] A popular variation of Lucas's model assumes that there is a separate cash-in-advance constraint on firms' wage payment and that open market operations affect firms' available funds before affecting the price level (e.g., Fuerst 1992). This variation allows open market operations to affect real activities, but such real effects are quite different from the liquidity effect in the goods market that I emphasize here.

[15] If the government attaches repurchase agreements to bond sales, then the duration of the liquidity effect of bonds in the goods market will be reduced.

market, which reduces the current nominal interest rate. Third, because of the last two features, the contemporaneous correlation between interest rates and output is negative, and the auto-correlation between interest rates in two adjacent periods is negative. Finally, past and current shocks both affect the slope of the yield curve. In particular, when the current shock is high, the yield curve is negatively sloped and it is more so when the previous period's shock was low.

Another way to understand this set of new features is that the goods market eliminates the "one-factor" character of Lucas's model. Although money and bonds of different maturities are traded in the same centralized market, the liquidity services generated by these assets in the goods market are different, depending on the lengths of maturity and the dependence of shocks. Open market operations affect the relative value between these assets and, especially, between unmatured bonds and money. This effect leads to the re-allocation of money between different markets, which in turn affects real output and generates the serial correlation in interest rates.

In order to focus on the liquidity effects, I have retained several assumptions in Lucas's model. First, open market operations do not affect money growth. Second, there is no element (other than the one-period separation between markets) to delay the transmission of shocks from the bonds market to the goods market. Third, the shock to bond sales is the only shock in the economy. Relaxing these assumptions might improve the predictions of the model. For example, the current model generates the unrealistic result that future output is positively correlated with the current interest rate. This result might be overturned if there are money demand shocks as well as the shocks in open market operations. One example of a money demand shock is stochastic changes in the scope of the legal restriction, g.

Finally, there may be a need to model explicitly how financial institutions turn unmatured bonds into instruments that can circulate in the goods market temporarily. Such an explicit role of intermediation may produce a more realistic mechanism of monetary propagation.

Appendix

A. Proofs of Lemmas 5.1, Lemma 5.2 and Theorem 5.3

Consider Lemma 5.1. First, TO maps from $\mathcal{V}$ to $\mathcal{V}$. For any $\mu^m \in \mathcal{V}$, $O(\mu^m)$ is continuous because Φ has the Feller property. Since $a(.) \in \mathcal{A}$ is continuous, $TO(\mu^m)$ is continuous. Because $\mu^m \geq \frac{\gamma}{\beta}\omega_L$, then $O(\mu^m) \geq \omega_L$. Hence,

$$TO(\mu^m) \geq \omega_L \left[1 + F\left(\omega_L; a, z_{-1}\right)\right] \geq \omega_L \left[1 + F\left(\omega_L; a_H, z_H\right)\right].$$

The first inequality comes from the fact that T is an increasing function of ω^m and the second inequality from the fact that, for given ω^m, $F(\omega^m; a, z_{-1})$ is a decreasing function of (a, z_{-1}). Similarly, $O(\mu^m) \leq \omega_H$ and

$$TO(\mu^m) \leq \omega_H \left[1 + F(\omega_H; a, z_{-1})\right] \leq \omega_H \left[1 + F(\omega_H; a_L, z_L)\right].$$

Thus, $TO(\mu^m) \in \left[\frac{\gamma}{\beta}\omega_L, \frac{\gamma}{\beta}\omega_H\right]$ if

$$F\left(\omega_H, a_L, z_L\right) + 1 \leq \gamma/\beta \leq F\left(\omega_L, a_H, z_H\right) + 1.$$

This is part of the condition (5.3) in the lemma.

Second, TO is a contraction under the supnorm. Take any $\mu', \mu'' \in \mathcal{V}$. Let $\omega' = O(\mu')$ and $\omega'' = O(\mu'')$. Then, $\omega', \omega'' \geq \omega_L$ and

$$\left|\omega' - \omega''\right| = \left|O(\mu') - O(\mu'')\right| \leq \frac{\beta}{\gamma} \left\|\mu' - \mu''\right\|.$$

Because T is concave in each of its segments and because T_ω is bounded above by K for all $\omega^m \geq \omega_L$, I have:

$$\left|T(\omega') - T(\omega'')\right| \leq K \left|O(\mu') - O(\mu'')\right| \leq \frac{\beta}{\gamma} K \left\|\mu' - \mu''\right\|.$$

Thus, $\left\|TO(\mu') - TO(\mu'')\right\| \leq \frac{\beta}{\gamma} K \left\|\mu' - \mu''\right\|$. The mapping TO is a contraction if $\gamma/\beta \geq K + \varepsilon$, where $\varepsilon > 0$. This condition is part of the condition (5.3) in the lemma.

Because TO: $\mathcal{V} \to \mathcal{V}$ is a contraction mapping under (5.3), and $\mathcal{V}$ (with the supnorm) is a complete metric space, TO has a unique fixed point $\mu_a^m \in \mathcal{V}$.

Now turn to Lemma 5.2. Since (μ_a^m, μ_a^b) are continuous, $\mu_a^m \geq \frac{\gamma}{\beta}\omega_L > 0$, and Φ has the Feller property, then $\Gamma_a(.)$ defined by (5.4) is continuous. To show $\Gamma_a \in \mathcal{A}$, it suffices to show $\Gamma_a(z) \in [a_L, a_H]$ for all $z \in Z$. Notice that the right-hand side of (5.4) is increasing in $a(z_{-1})$ for given (μ_a^m, μ_a^b). Since $a(z_{-1}) \in [a_L, a_H]$, the sufficient conditions for $\Gamma_a(z_{-1}) \in [a_L, a_H]$ are:

$$\mu_a^m(z_{-1}) \leq \frac{\beta}{\gamma} \int \max\left\{\frac{z}{1 - a_H}\mu_a^b(z),\ \mu_a^m(z)\right\} \Phi(dz, z_{-1}), \tag{A.1}$$

$$\mu_a^m(z_{-1}) \geq \frac{\beta}{\gamma} \int \max\left\{\frac{z}{1 - a_L}\mu_a^b(z),\ \mu_a^m(z)\right\} \Phi(dz, z_{-1}). \tag{A.2}$$

The first condition is satisfied when a_H is close to 1. For the second condition, note that $\mu_a^b(z) \leq \mu_a^m(z)$ for all z (see (3.9) and (3.10)), and so

$$\max\left\{\frac{z}{1 - a_L}\mu_a^b(z),\ \mu_a^m(z)\right\} \leq \mu_a^m(z) \max\left\{\frac{z}{1 - a_L}, 1\right\} \leq \mu_a^m(z) \max\left\{\frac{z_H}{1 - a_L}, 1\right\}.$$

Then, a sufficient condition for (A.2) is

$$\mu_a^m(z_{-1}) \geq \max\left\{\frac{z_H}{1-a_L}, 1\right\}\frac{\beta}{\gamma}\int \mu_a^m(z)\Phi(dz, z_{-1}) = \omega_a^m(z_{-1})\max\left\{\frac{z_H}{1-a_L}, 1\right\}.$$

Because $\mu_a^m/\omega_a^m = F(\omega_a^m, a, z_{-1})$ and F is a decreasing function of its three arguments, a sufficient condition for the above condition is (5.5).

Now turn to Theorem 5.3. It suffices to show that Γ is continuous in a. Interpret μ_a^m, μ_a^b and ω_a^m as functions of a. I show that $(\mu_a^m, \mu_a^b, \omega_a^m)$ are continuous in a in the supnorm. Once this is done, it is clear from (5.4) that Γ is continuous in a in the supnorm. Because the proofs for $(\mu_a^m, \mu_a^b, \omega_a^m)$ to be continuous in a are similar, I describe only the proof for μ_a^m. For the latter, I need to show that for any $\varepsilon > 0$, there exists $\Delta > 0$ such that $\|\mu_{a_2}^m - \mu_{a_1}^m\| < \varepsilon$ whenever $\|a_2 - a_1\| < \Delta$, where the norm is the supnorm. Let $\varepsilon > 0$ be an arbitrary number. Define

$$B(\omega^m, z_{-1}) = \max_{a,\widehat{a}\in\mathcal{A}}\left|\frac{F(\omega^m, a(z_{-1}), z_{-1}) - F(\omega^m, \widehat{a}(z_{-1}), z_{-1})}{\widehat{a}(z_{-1}) - a(z_{-1})}\right|,$$

where F is defined in (3.8). Since F is decreasing in a, $B > 0$. Also, because the intervals $[a_L, a_H], [\omega_L, \omega_H]$, and $[z_L, z_H]$ are bounded away from zero and bounded above, it can be verified that $B(\omega^m, z_{-1}) < \infty$. For any $a_1, a_2 \in \mathcal{A}$, if $\|a_2 - a_1\| < \Delta$, then

$$\begin{aligned}&|F(\omega^m, a_2(z_{-1}), z_{-1}) - F(\omega^m, a_1(z_{-1}), z_{-1})|\\ &\leq B(\omega^m, z_{-1})\,|a_2(z_{-1}) - a_1(z_{-1})| \leq B(\omega^m, z_{-1})\,\|a_2 - a_1\| < B(\omega^m, z_{-1})\Delta.\end{aligned}$$

Because $T(\omega^m, a, z_{-1}) = \omega^m(1+F)$ and $\omega^m \leq \omega_H$, then

$$\begin{aligned}&|T(\omega^m, a_2(z_{-1}), z_{-1}) - T(\omega^m, a_1(z_{-1}), z_{-1})|\\ &= \omega^m\,|F(\omega^m, a_2(z_{-1}), z_{-1}) - F(\omega^m, a_1(z_{-1}), z_{-1})| < \omega_H B(\omega^m, z_{-1})\Delta.\end{aligned}$$

Since $\mu_a^m = T(\omega_a^m, a(z_{-1}), z_{-1})$ and $\|TO(\mu') - TO(\mu'')\| \leq \frac{\beta}{\gamma}K\,\|\mu' - \mu''\|$, I get:

$$\begin{aligned}&\left|\mu_{a_2}^m(z_{-1}) - \mu_{a_1}^m(z_{-1})\right|\\ =\;&\left|T(\omega_{a_2}^m, a_2(z_{-1}), z_{-1}) - T(\omega_{a_1}^m, a_1(z_{-1}), z_{-1})\right|\\ \leq\;&\left|TO(\mu_{a_2}^m, a_2(z_{-1}), z_{-1}) - TO(\mu_{a_1}^m, a_2(z_{-1}), z_{-1})\right|\\ &\quad+\left|T(\omega_{a_1}^m, a_2(z_{-1}), z_{-1}) - T(\omega_{a_1}^m, a_1(z_{-1}), z_{-1})\right|\\ <\;&\frac{\beta}{\gamma}K\left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| + \omega_H B(\omega_{a_1}^m(z_{-1}), z_{-1})\Delta.\end{aligned}$$

Taking the maximum over z_{-1} on both sides of the inequality yields

$$\left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| < \frac{\Delta}{1-\frac{\beta}{\gamma}K}\omega_H\,\|B\|.$$

Let $\Delta = \varepsilon\left(1-\frac{\beta}{\gamma}K\right)\Big/[\omega_H\,\|B\|]$. Because $\gamma/\beta > K$, $\|B\| < \infty$ and $0 < \omega_H < \infty$, then $\Delta > 0$. For all $a_1, a_2 \in \mathcal{A}$ such that $\|a_2 - a_1\| < \Delta$, $\left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| < \varepsilon$. **QED**

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