The Extent of the Market and the Optimal Degree of Specialization*

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Abstract

In this paper I examine the socially optimal allocation in a random matching economy. The optimal allocation is supported by punishment to defections, using the public record of agents' transaction. However, public record-keeping is imperfect, as an agent's transaction is updated into the public record with probability $\rho < 1$. I interpret ρ as the extent of the market. When ρ is small, an extension of the market increases the optimal degree of specialization and the quantities of goods exchanged in the market. When ρ is large, however, a further extension of the market has no effect on the socially optimal degree of specialization or the quantities of trade. I also examine the optimal allocation when there is an adjustment cost for reducing specialization. In this case, the optimal process of specialization is gradual, even though there is no adjustment cost for increasing specialization.

Keywords: Record-keeping, specialization, incentive constraints.

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> As it is the power of exchanging that gives occasion to the division of labour, so the extent of this division must always be limited by the extent of that power, or, in other words, by the extent of the market. Adam Smith (1776, p21).

1 Introduction

Adam Smith's argument about the relationship between specialization and the extent of the market is intuitive: The more extensive the market is, the higher the return to specialization, and hence the higher the degree of specialization. However, it is not clear what the extent of the market exactly means. Adam Smith's writing seems to suggest that the extent of the market is the extent to which agents can meet with each other to carry out exchanges.[1] Recent models adopt this interpretation, using search frictions to capture the matching difficulty (e.g., Kiyotaki and Wright 1993, Shi 1997, and Camera et al. 2001). In this paper I emphasize a different type of limit on the extent of the market, i.e., the society's imperfect ability to update agents' transaction records. I characterize the socially optimal choices of production, consumption, and specialization, and then examine how they change with respect to the society's ability to update the transaction record.

Let me justify why I focus on the society's imperfect ability to update the transaction record as a limit on the extent of the market. Broadly defined, the extent of the market is the extent to which a society can induce socially desirable trades. The meeting technology emphasized by previous authors clearly limits the society's ability to induce socially desirable trades, but so does the society's imperfect ability to update the transaction record. Even if agents can meet with each other easily, a society may not be able to implement all socially desirable trades if it cannot update the transaction record sufficiently well. For example, when an agent can produce the trading partner's desirable goods, the socially optimal allocation would require him to produce for the partner regardless of whether the partner can produce his desirable goods in return. To induce this trade, the society must compensate in the future for the producer's current production,

[1] For example, in discussing the civilizations of ancient Egypt, India and China, Smith (1776, pp23-25) attributed the increase in the extent of the market to the improvements in the ability to transport goods through rivers and canals in these regions.

which is not possible if no record of transaction is kept. An improvement of the society's public record-keeping ability extends the market because it enables the society to implement socially more desirable allocations, by punishing potential deviations from the allocations. It is not clear how an extension of the market in this sense affects specialization, thus the focus here. Such a study is also useful for understanding how a society specializes as technologies improve in the activities like contract enforcements and credit arrangements, because these activities are examples of a society's ability to update the transaction record.

The economy described here is essentially the one analyzed by Kocherlakota and Wallace (1998), with the introduction of the specialization choice. There are N (≥ 3) types of goods and agents. A type-n agent produces type-$(n+1)$ goods, which he can consume or sell in the market. In addition to consuming his own product, a type-n agent also wants to consume type-n goods, which he can only get from type-$(n-1)$ agents. Matching is random and bilateral, and there is no double coincidence of wants in any match.[2] Specialization is the degree to which an agent makes his product palatable to others. A good produced with higher specialization generates higher utility to some other agents but lower utility to the producer himself. Thus, in autarky each agent will not specialize at all and there is no exchange.

To induce specialization and exchange, the society keeps public records of agents' transaction histories.[3] Such records are imperfect. As in Kocherlakota and Wallace (1998), the society is only able to update each agent's transaction into the public record with probability $\rho < 1$. The parameter ρ is given by the society's record-keeping technology. An increase in ρ represents an extension of the market for the reason described above. I analyze the socially optimal allocation with individuals' incentive constraints. The social planner uses the public record to induce agents to comply with the optimal allocation. If the record shows that an agent defected from the optimal allocation, the entire economy switches into autarky.

One of the main results is that the optimal degree of specialization increases with the extent

[2]This is a simplification: The analytical results do not change when barter is allowed.

[3]A medium of exchange can also partially alleviate the trading difficulty and, for this reason, money is an imperfect substitute for public record keeping (Kocherlakota, 1998). I do not examine the role of money here.

of the market (ρ) when ρ is small but is independent of ρ when ρ is large. The explanation is as follows. When ρ is large, the society is able to detect defections easily. With the potential punishment of autarky to defections, the social planner can implement the unconstrained social optimum, where individuals' incentive constraints do not bind. In this unconstrained optimum, the degree of specialization is independent of ρ. In contrast, when ρ is small, the unconstrained social optimum is not incentive feasible. In particular, the producer in a meeting of a single coincidence of wants has strong incentive to not produce for the consumer. To prevent this defection, the degree of specialization must be lower than in the unconstrained optimum. An increase in ρ relaxes the producer's incentive constraint and hence allows for higher specialization.

Another main result is about the process of specialization when there is a downward adjustment cost, i.e., an adjustment cost for reducing the degree of specialization. Such a downward adjustment cost makes the extent of the market endogenous in the sense that, even for fixed ρ, an increase in specialization reduces the incentive cost of specialization in the future by increasing the cost of future defection. As a result, the optimal process of specialization is gradual, even though there is no adjustment cost for increasing specialization.

The analytical approach in this paper follows closely that in Kocherlakota and Wallace (1998). There are three main differences between these two papers. First, I focus on the choice of specialization, which is absent in Kocherlakota and Wallace's paper. Second, I introduce an adjustment cost in section 5 and examine the sequence of optimal allocations, while Kocherlakota and Wallace examine only the stationary allocation. Third, I characterize the constrained social optimum more explicitly here than Kocherlakota and Wallace do, using first-order conditions. Kocherlakota and Wallace examine how the incentive feasibility set changes with ρ. Although such an approach is more general than mine, it is also less useful for the purpose of establishing strict monotonicity of the optimal choices with respect to ρ or for computing the social optimum when ρ is moderately large.

This paper is complementary in two dimensions to other search models (cited earlier) that examine specialization. First, as explained before, I explore the society's ability to update the transaction records as the extent of the market, while others assume $\rho = 0$ and interpret the extent of the market as the frequency with which agents meet with each other. As a result, those models investigate the role of money in inducing specialization. I abstract from money in order to focus on the role of the record-keeping technology (see section 6 for more discussion). Second, those models analyze the equilibrium allocation, using bargaining to determine prices. In contrast, I follow Kocherlakota and Wallace (1998) to focus on the socially optimal allocation under incentive constraints, which is the appropriate subject because the central issue here is how the extent of market affects the society's ability to achieve socially desirable outcomes. Particular schemes of price determination may, by themselves, generate inefficient outcomes which complicate the main issue unnecessarily.

2 The Model

2.1 Agents, Goods and Specialization

The economy is similar to the one described by Shi (1995) and Trejos and Wright (1995). There are N types of agents and N types of goods, where $N \geq 3$ and the generic index of type is $n \in \{1, 2, ..., N\}$. The population of agents in each type is large, normalized to 1, and is the same for all types. An agent's type is distinguished by the agent's production and consumption ability. A type n agent produces only type $(n + 1)$ goods (modulus N) and the unit cost of production is normalized to one in terms of utility. A type n agent can consume two, and only two, types of goods – his own output and the output of type $(n - 1)$ agents (type n goods). Consumption of one's own product is called *autarkic consumption* and consumption of others' products *market consumption*.

The degree of specialization is denoted $s \in (0, 1)$. From y units of type n goods produced with a degree of specialization s, the utility is $(1 - s)U(y)$ if the goods are consumed by the producer himself and $B(s)U(y)$ if consumed by a type n agent, where $B(s)$ is an increasing function. Thus, specialization is costly to the producer himself but beneficial to some other agents.[4] Because

[4]I model the cost of specialization through the disutility of consuming one's own output, but keep the cost of production independent of specialization. The analytical results are the same if the cost of specialization is modelled, instead, as an increase in the cost of production. In both modelling approaches, net utility from consuming one's own good decreases with specialization.

specialization yields direct benefits to other agents, not to oneself, an agent chooses to specialize only if the market rewards specialization through the exchange. For the moment, let us assume that there is no adjustment cost for changing the degree of specialization, except for the cost embedded in $(1-s)U(y)$ (I will examine the adjustment cost in section 5).

Let me justify why I refer to s as the degree of specialization. Because each agent produces only one type of goods in this model, it is tempting to say that agents are already specialized in production, regardless of the choice on s. However, this interpretation regards each type of goods to be the finest description of goods and services in the market. If, instead, a type consists of a wide range of similar goods or services, then a larger s indeed corresponds to further specialization within the type. For example, if lectures on economics are one type of the goods/services in the model, then an increase in s corresponds to specialization in lectures on the sub-fields of economics. Similarly, if food services are one type of the goods/services, then an agent can further specialize in the production of, say, French food, Chinese food, or wine.[5]

Assumption 1 *(i) The function U is continuous and twice differentiable, with $U' > 0$, $U'' < 0$, $U(0) = 0$, $U(\infty) = \infty$, $U'(0) = \infty$, $U'(\infty) = 0$;*

(ii) $B' > 0$, $B'' < 0$, $B(0) = 0$ and $1 < B(1) < \infty$;

(iii) The function

$$f(s) \equiv \frac{B'(s)}{N} U\left(U'^{-1}\left(\frac{1}{B(s)}\right)\right) \Big/ U\left(U'^{-1}\left(\frac{1}{1-s}\right)\right)$$

is a decreasing function of s, and there exists $\bar{s} \in (0,1)$ such that $f(0) > 1 > f(\bar{s})$.

Parts (i) and (ii) of the above assumption are self-explanatory. In particular, part (i) requires that the marginal benefit of specialization to the consumers be positive and diminishing. Part

[5]The classification of goods types here resembles the occupational affinity used by Adam Smith (1776, pp21-22) in the following quote: Workmen in scattered countryside "are almost every where obliged to apply themselves to all the different bracnches of industry that have so much affinity to one another as to be employed about the same sort of materials. A country smith in every sort of work that is made of iron. The former is not only a carpenter, but a joiner, a cabinet maker, and even a carver in wood, as well as a wheelwright, a ploughwright, a cart and waggon maker." An increase in specialization within each affinity corresponds to an increase in s in this paper. Elsewhere (Shi, 1997), I examined specialization as a decrease in the number of goods types produced by each agent, but ignored the possibility of public record-keeping of agents' transaction histories.

(iii) of the assumption ensures that the unconstrained optimum exists and is unique, which I will discuss in subsection 2.4.

2.2 Matching, Exchange, and Record-Keeping

Time is discrete and agents discount future with a discount factor $\beta \in (0,1)$. In each period, an agent randomly meets another agent. There are two types of meetings, those with no-coincidence of wants and those with a single coincidence of wants. In a meeting with no coincidence of wants, neither of the two agents in the match can produce the good the partner wants to consume. In a meeting with a single coincidence of wants, one (and only one) of the two agents can supply the goods for the partner's market consumption. There is no match in which the two agents have double coincidence of wants. The absence of barter simplifies the analysis.

The social planner describes a degree of specialization, s, and the quantities of consumption and production. In a match with no coincidence of wants, the social planner asks each agent to produce z_p units of goods, all consumed by the agent himself.[6] In a match with a single coincidence of wants, let us call the agent who can produce for the partner the *producer*, and the recipient of the goods the *consumer*. Each agent may also consume a part of his own output as autarkic consumption. Consider, for example, a meeting between a type $n-1$ agent and a type n agent. Then, the type $n-1$ agent is the producer and the type n agent the consumer. In this meeting, the planner's allocation is as follows:

- For the producer: producing y_p units of good n and consuming c_p $(\leq y_p)$ units;
- For the consumer: consuming c_m units of good n, where $c_m = y_p - c_p$; in addition, producing and consuming y_c units of good $n+1$.

The subscript p indicates "producer" and c "consumer". The quantities (y_c, c_p, z_p) are the amounts of autarkic consumption, while c_m is market consumption.

To induce the producer to supply a positive amount of output to the partner, the planner must be able to punish potential deviators, which is made possible by public record-keeping of

[6] In such a meeting with no coincidence of wants, it is not optimal to produce more than one consumes.

transaction histories. Public record-keeping is modelled in the same way as in Kocherlakota and Wallace (1998). At the beginning of each period and before matches take place, each agent's transaction in the previous period is updated into the public record with probability $\rho \in [0, 1]$. If the public record shows that an agent deviated from the planner's allocation, then every agent reverts to autarky forever. Note that two matched agents play the punishment strategy even if an observed defection was committed by someone else outside the match. As in Kocherlakota and Wallace (1998), the justification for using this "global" punishment is that the focus here is to find the best allocations that the society can achieve under the incentive constraints.[7] As explained in the introduction, an increase in ρ represents an extension of the market.

The timing of events is as follows. At the beginning of each period, previous transactions are updated into the public record with probability ρ. If a defection is observed, the economy reverts to autarky. If no defection is observed, the planner chooses the degree of specialization and the quantities of trade described above. Then, matching takes place. Each agent can decide whether to comply with the planner's allocation or to defect. After trading, the period ends and time passes to the next period.

2.3 Value Functions and Incentive Constraints

Focus on stationary allocations. Let V be the value function for someone who has not defected *and* has not seen a defection, and W for someone who defected in the past but has not been discovered and has not seen anyone else defect in the past. Both V and W are measured at the beginning of a period, before the public record is updated this period to include the transactions in the previous period.

To calculate V, consider an agent who has not defected and has not seen anyone else defect. In the current period, the agent can be in one of three matching situations: being the consumer (in a match with a single-coincidence of wants), the producer, or being in a match with no coincidence of wants. If the agent is the consumer, the planner's allocation requires the agent to consume

[7] The global, permanent punishment may seem extreme and one may want to consider more realistic types of punishments such as "local" punishments, which apply only to the defecting agents, and temporary punishments. These alternative types of punishment may fail to support the best allocations that the society can achieve.

$(y_p - c_p)$ units of the partner's output, and to produce and consume y_c units of his own goods. The surplus in utility is $B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c$. If the agent is the producer, he is required to produce y_p units, consume c_p units of his own output, and give the rest to the consumer. The surplus in utility is $(1-s)U(c_p) - y_p$. If the agent is in a match with no coincidence of wants, he is required to produce and consume z_p units of his own goods, yielding net utility $(1-s)U(z_p) - z_p$. The Bellman equation for V in a stationary environment is

$$\begin{aligned}(1-\beta)V = & \tfrac{1}{N}\left[B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c\right] \\ & + \tfrac{1}{N}\left[(1-s)U(c_p) - y_p\right] + \left(1 - \tfrac{2}{N}\right)\left[(1-s)U(z_p) - z_p\right].\end{aligned} \tag{1}$$

The right-hand side is the expected surplus of utility in the three situations, which occur with probability $1/N$, $1/N$, and $(1-2/N)$, respectively.

To calculate W, I first calculate the payoff in autarky. An agent plays autarky if he is publicly observed for defection or if he has seen another agent defect. The autarkic strategy maximizes net utility in each period, $(1-s)u(y) - y$. Define

$$Y_a(s) \equiv U'^{-1}\left(\frac{1}{1-s}\right) = \arg\max_y \left[(1-s)U(y) - y\right]. \tag{2}$$

The solutions are $s = 0$ and $z_p = c_p = y_c = y_p = y_a$, yielding utility A per period, where

$$y_a = Y_a(0) = U'^{-1}(1), \tag{3}$$

$$A = U(y_a) - y_a. \tag{4}$$

That is, an agent in autarky does not specialize, always produces y_a amount of goods for himself, and gives nothing to the matching partner.

For an agent who defected in the last period, the defection is discovered with probability ρ this period before matches take place, in which case the present value is $A/(1-\beta)$. If the defection is not discovered this period, the agent can choose between defection again in this period or not, and the expected utility is

$$\begin{aligned}EUD = & \beta W + \tfrac{1}{N}\max\left\{B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c, A\right\} \\ & + \tfrac{1}{N}\max\left\{(1-s)U(c_p) - y_p, A\right\} + \left(1 - \tfrac{2}{N}\right)\max\left\{(1-s)U(z_p) - z_p, A\right\}.\end{aligned}$$

The term βW is the continuation payoff from the next period onward. The other three terms are the expected surpluses in the three meetings described above. Note that, if the agent chooses to defect again this period, the best deviation is autarky which yields A per period. The Bellman equation for W is $W = \rho A/(1-\beta) + (1-\rho)EUD$. Re-arranging terms, I have:

$$\begin{array}{l}(1-\beta)W = A + \frac{(1-\beta)(1-\rho)}{1-\beta(1-\rho)} \left\{ \frac{1}{N} \max\left\{B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c - A, 0\right\} \right. \\ \left. + \frac{1}{N} \max\left\{(1-s)U(c_p) - y_p - A, 0\right\} + (1 - \frac{2}{N}) \max\left\{(1-s)U(z_p) - z_p - A, 0\right\} \right\}. \end{array} \tag{5}$$

The planner's allocation consists of (s, c_p, y_p, y_c, z_p). Evidently, it must satisfy:

$$0 \leq s \leq 1,\ z_p \geq 0,\ y_c \geq 0,\ 0 \leq c_p \leq y_p. \tag{6}$$

Except the constraint on s, these constraints do not bind under the assumption $U'(0) = \infty$. The planner also faces the incentive constraints that require defections to be not profitable. Because an agent can be in one of the three matching situations described before, there is one incentive constraint for each situation:

$$B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c - A + \beta(V - W) \geq 0, \tag{7}$$

$$(1-s)U(c_p) - y_p - A + \beta(V - W) \geq 0, \tag{8}$$

$$(1-s)U(z_p) - z_p - A + \beta(V - W) \geq 0. \tag{9}$$

Because the explanations for these conditions are similar, I explain only (7), which is the incentive condition for the consumer in a meeting with a single coincidence of wants. If the consumer follows the allocation, net utility in the current period is $B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c$ and the continuation payoff from next period onward is βV. If the agent defects, net utility is A in this period and the continuation payoff is βW. The constraint (7) requires the value from following the allocation to be at least as high as from defection. It is possible in some matching situations that an allocation gives a lower current payoff than that from defection, but then the allocation must provide a sufficiently higher continuation value in order to satisfy the incentive constraints.

Definition 1 *An* incentive feasible allocation *consists of a vector (s, z_p, c_p, y_c, y_p) and value functions (V, W) such that the constraints (6) – (9) hold, with (V, W) satisfying (1) and (5). An* optimal allocation *is an incentive feasible allocation which maximizes the representative agent's value function* V.

The set of incentive feasible allocations can be large. For example, autarky is incentive feasible, because all the incentive constraints hold if $s = 0$ and $z_p = c_p = y_c = y_p = y_a$, where y_a is defined in (3). However, since the main subject in this paper is how the extent of the market affects the optimal degree of specialization, it makes sense to focus on optimal allocations, as I will do in the remainder of this paper.

2.4 Unconstrained Optimal Allocation

To analyze the optimal allocation, it is useful to examine first the optimal allocation without the incentive constraints (7) – (9). Such an optimal allocation is called the *unconstrained optimal allocation*. As mentioned earlier, the assumption $U'(0) = \infty$ ensures that the constraints in (6) on z_p, c_p, y_c and y_p do not bind. Then the first-order conditions for these four variables yield:

$$z_p^u = y_c^u = c_p^u = Y_a(s^u), \quad y_p^u = Y_a(s^u) + U'^{-1}\left(\frac{1}{B(s^u)}\right). \tag{10}$$

The superscript u indicates unconstrained optimality and the function $Y_a(.)$ is defined in (2). If the unconstrained optimal choice of specialization is interior, i.e., $s \in (0,1)$, then

$$f(s) = \frac{B'(s)}{NU\left(Y_a(s)\right)} U\left(U'^{-1}\left(\frac{1}{B(s)}\right)\right) = 1. \tag{11}$$

The above relationship holds with "$<$" if $s = 0$ and "$>$" if $s = 1$.

Under Assumption 1, there is a unique $s^u \in (0,1)$ that solves (11). Note that parts (i) and (ii) of Assumption 1 are not sufficient to ensure existence and uniqueness of the solution. Nor do they ensure that the solution is interior, if it exists. Part (iii) of Assumption 1 helps us avoid such complications that are not central to the main issue. In particular, because $f(0) > 1 > f(\bar{s})$ for some $\bar{s} \in (0,1)$ and $f(s)$ is a decreasing function, the unique solution s^u lies in $(0, \bar{s})$. Also, $f(s) < 1$ if and only if $s > s^u$.

The assumption that $f(s)$ is decreasing for all $s \in (0,1)$ simplifies the exposition but is not necessary for the existence and uniqueness of the interior solution s^u. The following example illustrates how it can be relaxed.

Example 2 *Let* $U(c) = u_0 c^{\sigma}$ *and* $B(s) = B_0 s^b$, *where* $\sigma, b \in (0,1)$. *In addition, choose* $b < 1-\sigma$. *Then,* $f(s)$ *is a decreasing function if and only if* $s < 1 - \sigma/(1-b)$, *not for all* $s \in (0,1)$. *In this case, there can be two interior solutions to* $f(s) = 1$, *denoted* s^u *and* s^d, *with* $0 < s^u < s^d < 1$. *But only* s^u *can be a solution for the unconstrained optimum. In contrast,* s^d *delivers a lower* V *than the choice* $s = 1$. *Since the derivative of* V *with respect to* s *is proportional to* $[f(s) - 1]$, V *is an increasing function of* s *for* $s \in [s^d, 1]$. *Thus, the interior solution* s^u *is still the unique maximum if* V *is higher at* s^u *than at* $s = 1$.

3 Characterization of the Optimal Allocation

To characterize the (constrained) optimal allocation, I establish some features of the value functions. Then I examine the cases where ρ is close to 1 and 0, respectively.

To begin, let us simplify the Bellman equation for W. Using (6) and the fact that A is the maximum of $[U(c) - c]$, I have $(1-s)U(z_p) - z_p \leq A$ and $(1-s)U(c_p) - y_p \leq A$. That is, if an agent defected in the last period but has not been discovered, then he will defect again in the current period in a meeting with no coincidence of wants, or in a meeting with a single coincidence of wants where he is the producer. Thus, (5) becomes

$$(1-\beta)W = A + \frac{(1-\beta)(1-\rho)}{N[1-\beta(1-\rho)]} \max\left\{B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c - A, 0\right\}. \qquad (12)$$

Lemma 3 $V \geq W$ *for all incentive feasible allocations and, if* $s > 0$, *then* $V > W$.

The above lemma states two intuitive facts.[8] First, to prevent defection, the allocation must generate a lifetime utility at least as high as that from defection. Second, since autarky has zero degree of specialization, an allocation that requires a positive degree of specialization must compensate the agent for the cost of specialization, which requires the lifetime utility of following the allocation to be strictly higher than from defection.

Next, I analyze the case $\rho = 0$ and the case where ρ is close to one.

[8] The proof of this lemma is as follows. All incentive feasible allocations must satisfy the incentive constraints (6) – (9). Then,

$$\beta(V - W) \geq A - [(1-s)U(z_p) - z_p] \geq A - [U(z_p) - z_p] \geq 0.$$

The first inequality follows from the incentive constraint (9), the second from $s \geq 0$, and the third from the fact that A is the maximum value of $[U(z) - z]$. If $s > 0$, then the second inequality is strict, in which case $V > W$.

Proposition 4 *If $\rho = 0$, the optimal allocation is autarky. If ρ is sufficiently close to one, then there exists $\beta_0(\rho)$ such that the unconstrained optimum is incentive feasible when $\beta \geq \beta_0(\rho)$. The critical level β_0 satisfies $\beta_0'(\rho) < 0$ and $1 > \beta_0(\rho) > \beta^u$, where $\beta^u \in (0,1)$ is defined as follows:*

$$\beta^u \equiv \frac{N\left[A - (1-s^u)U(z_p^u) + y_p^u\right]}{B(s^u)U(y_p^u - z_p^u) + (N-1)(y_p^u - z_p^u)}. \tag{13}$$

Therefore, when ρ is sufficiently close to 1 and $\beta \geq \beta_0(\rho)$, the optimal degree of specialization is independent of the extent of the market, ρ.

Proof of Proposition 4. When $\rho = 0$, (12) implies

$$\begin{aligned}(1-\beta)W &\geq A + \tfrac{1}{N}\left[B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c - A\right] \\ &= \left(1 - \tfrac{1}{N}\right)A + \tfrac{1}{N}\left[B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c\right].\end{aligned}$$

Because $(1-s)U(c_p) - y_p \leq A$ and $(1-s)U(z_p) - z_p \leq A$ for any allocation that satisfies (6), the right-hand side of the above inequality is greater than or equal to that of (1). Thus, $W \geq V$. But $V \geq W$ by Lemma 3. So, $V = W$ when $\rho = 0$. With $V = W$, the incentive constraint in a match with no coincidence of wants, (9), becomes $(1-s)U(z_p) - z_p \geq A$. This is satisfied if and only if $s = 0$ and $z_p = y_a$, because A is the maximum of $[U(z) - z]$ and the maximum is achieved at $z = y_a$. With $s = 0$ and $V = W$, the producer's incentive constraint, (8), becomes $U(c_p) - y_p \geq A$, which can be satisfied if and only if $c_p = y_p = y_a$. Similarly, the consumer's incentive constraint, (7), can be satisfied if and only if $y_c = y_a$. Therefore, when $\rho = 0$, the autarkic allocation is the only optimum (and, in fact, the only incentive feasible allocation).

To examine the case where ρ is sufficiently close to one, consider first the case $\rho = 1$. I show that the unconstrained optimum satisfies the incentive constraints if $\beta \geq \beta^u$. Evidently, the unconstrained optimum satisfies (6). Because the unconstrained optimum has $s^u > 0$ and $y_p^u > z_p^u = c_p^u = y_c^u$, (8) implies (7) and (9), and so it suffices to show that the unconstrained optimum satisfies (8). With $\rho = 1$, (12) shows $W = A/(1-\beta)$. Substituting this for W and (1) for V, (8) becomes $\beta \geq \beta^u$, where β^u is defined in (13). Note that β^u is independent of β, because the unconstrained optimum is independent of β. I need to show $\beta^u \in (0,1)$. It is evident that $\beta^u > 0$. The requirement $\beta^u < 1$ can be expressed as

$$N\left[(1-s^u)U\left(Y_a(s^u)\right) - Y_a(s^u) - A\right] + B(s^u)U\left(U'^{-1}\left(\frac{1}{B(s^u)}\right)\right) - U'^{-1}\left(\frac{1}{B(s^u)}\right) > 0,$$

where I substituted (10) for z_p^u, c_p^u, y_c^u and y_p^u. Temporarily denote the left-hand side of the above inequality by $LHS(s^u)$. Note that $LHS(0) = 0$ and that $LHS'(s)$ has the same sign as that of $[f(s) - 1]$. Because $f(s) > 1$ for all $s \in (0, s^u)$ under Assumption 1, $LHS(s)$ is an increasing function of s for all $s \in (0, s^u)$. Thus, $LHS(s^u) > LHS(0) = 0$, which implies $\beta^u < 1$.

Now consider the case where ρ is less than but sufficiently close to one. Again, for the unconstrained optimum to be incentive feasible, it suffices to show that the unconstrained optimum satisfies (8). Let the quantities of trade and the specialization choice be equal to those in the unconstrained optimum. Because the unconstrained optimum is independent of β and ρ, these two parameters enter (8) entirely through the term $\beta(V - W)$. Compute $\beta(V - W)$ from (1) and (12), noting that the non-zero term in the maximum of (12) is strictly positive under the unconstrained optimum. For any $\rho > 0$, $\beta(V - W) = \infty$ if $\beta = 1$, in which case (8) is trivially satisfied. So, suppose $\beta < 1$. For ρ sufficiently close to 1, $\beta(V - W)$ is a continuous and strictly increasing function of β (if $\beta < 1$). Thus, for any ρ sufficiently close to 1, there exists $\beta_0(\rho)$ such that the unconstrained optimum satisfies (8) if and only if $\beta \geq \beta_0(\rho)$. Because $\beta(V - W)$ is a continuous and strictly increasing function of ρ (if $\beta < 1$), I have $\beta_0'(\rho) < 0$. Clearly, for ρ sufficiently close to but less than one, $1 > \beta_0(\rho) > \beta_0(1) = \beta^u$. QED

If the society has no ability to keep public records of transactions, then it is not possible to punish an agent who defects. Since there is no gain from complying with any allocation other than autarky, the only incentive feasible allocation in this case is the autarkic allocation. On the other hand, if the public record is updated sufficiently well and agents are sufficiently patient, the unconstrained optimum can be implemented. The public record need not be perfect for the unconstrained optimum to be implemented. This is because the unconstrained optimum generates a discrete increase in value over the autarkic allocation, and so the potential punishment on defection can be large enough to induce the agents to comply with the allocation, provided that agents are sufficiently patient.

An increase in the extent of the market through ρ enlarges the parameter region in which the unconstrained optimum is incentive feasible. In particular, the lower bound on the discount factor for the unconstrained optimum to be incentive feasible, $\beta_0(\rho)$, falls when ρ increases. However, when the discount factor already exceeds this lower bound, the optimal degree of specialization is independent of ρ. That is, when the extent of the market is already very large, a further increase in the extent of the market has *no effect* on the optimal degree of specialization. This result provides a counter-example to the popular statement that a more extensive market induces higher specialization.

Now I analyze the case $0 < \rho < 1$, starting with the following lemma (see Appendix A for a proof):

Lemma 5 *If $\rho > 0$ and $\beta \geq \beta^u$, then the optimal allocation satisfies $V > W > A/(1-\beta)$, $s > 0$ and $c_p < y_p$. In addition, the optimum satisfies*

$$B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c - A > 0, \tag{14}$$

$$(1-\beta)W = A + \frac{(1-\rho)(1-\beta)}{N[1-\beta(1-\rho)]}\left[B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c - A\right]. \tag{15}$$

This lemma states the following properties. First, if the society has some ability to keep a public record of transactions ($\rho > 0$) and agents are sufficiently patient, then it is possible to use potential punishments to support allocations better than autarky, as indicated by the results $V > W > A/(1-\beta)$ in the above lemma. The optimal allocation in this case requires all agents to have positive specialization and requires the producer to produce a positive amount for the consumer. Second, the consumer's incentive constraint (7) does not bind, as (14) shows. This is because a meeting in which an agent is the consumer is the only situation in which an agent can possibly improve net utility in the current period by adhering to the optimal allocation; if such a gain were zero or negative, the value of following the optimal allocation would not be higher than in autarky, contradicting $V > W$.[9] Third, because the current gain is positive to the consumer, I can eliminate the maximum operator in (12) to obtain (15).

[9]In the other two types of meetings, following the optimal allocation reduces the agent's one-period utility because the allocation requires the agent either to specialize more or produce more than in autarky, but such a loss in current utility is compensated by the gain from the higher continuation value, $\beta(V - W)$.

Let λ be the shadow cost of the constraint (8) in the optimal allocation. The following proposition characterizes the optimal allocation:

Proposition 6 *Assume $\beta \geq \beta^u$ and $\rho \in (0,1)$. The following results hold:*

(i) The only constraint that can possibly bind is (8), and this constraint does bind (i.e., $\lambda > 0$) if ρ is sufficiently close to 0;

(ii) $y_c = z_p = c_p = Y_a(s)$;

(iii) (s, y_p, λ) jointly solve the following equations:

$$\begin{aligned} &B(s)U\left(y_p - Y_a(s)\right) - [\beta + (1-\beta)N]\tfrac{[1-\beta(1-\rho)]}{\beta\rho}\left(y_p - Y_a(s)\right) \\ &\geq \left[1 + (N-\beta)\tfrac{[1-\beta(1-\rho)]}{\beta\rho}\right][A - (1-s)U(Y_a(s)) + Y_a(s)], \quad \text{“} = \text{”} \text{ if } \lambda > 0; \end{aligned} \tag{16}$$

$$B(s)U'(y_p - Y_a(s)) = \frac{1 + \lambda(\beta + N(1-\beta))}{1 + \lambda\beta\rho/[1-\beta(1-\rho)]}; \tag{17}$$

$$\frac{B'(s)U(y_p - Y_a(s))}{NU(Y_a(s))} = \frac{1 + \lambda\left[1 - \frac{\beta(1-\beta)(1-\rho)}{N(1-\beta(1-\rho))}\right]}{1 + \lambda\beta\rho/[1-\beta(1-\rho)]}. \tag{18}$$

Proof of Proposition 6. Let me first rewrite the objective function and the constraints of the optimal allocation. Denote

$$\begin{aligned} \Delta_1 &= B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c - A, \\ \Delta_2 &= A - (1-s)U(c_p) + y_p, \\ \Delta_3 &= A - (1-s)U(z_p) + z_p. \end{aligned}$$

Then, $(1-\beta)V$ can be rewritten as

$$(1-\beta)V = \frac{1}{N}\Delta_1 - \frac{1}{N}\Delta_2 - \left(1 - \frac{2}{N}\right)\Delta_3. \tag{19}$$

Substituting this for V and (15) for W, I can rewrite the two constraints, (8) and (9), as follows:

$$\frac{\beta\rho\Delta_1}{N[1-\beta(1-\rho)]} \geq \left(1 - \beta + \frac{\beta}{N}\right)\Delta_2 + \beta\left(1 - \frac{2}{N}\right)\Delta_3, \tag{20}$$

$$\frac{\beta\rho\Delta_1}{N[1-\beta(1-\rho)]} \geq \frac{\beta}{N}\Delta_2 + \left(1 - \frac{2\beta}{N}\right)\Delta_3. \tag{21}$$

Recall that the other incentive constraint (7) does not bind. The optimal allocation maximizes $(1-\beta)V$ in (19), subject to these two constraints.

Next, I find the quantities of autarkic consumption. Notice that y_c appears in Δ_1, in the form $(1-s)U(y_c) - y_c$, but it does not appear in Δ_2 and Δ_3. For any choice $s \geq 0$, if $y_c \neq Y_a(s)$, then an alternative allocation with $y_c = Y_a(s)$ increases the value of $[(1-s)U(y_c) - y_c]$, and hence increases Δ_1. Because this change in y_c does not affect Δ_2 and Δ_3, it increases V through Δ_1. In addition, the increase in Δ_1 relaxes the constraints (20) and (21); because at least one of these constraints binds, the alternative allocation further increases V. This cannot be possible if the original allocation is possible. Thus, $y_c = Y_a(s)$. Similarly, $z_p = Y_a(s)$.

With $z_p = y_c = Y_a(s)$, the following holds:

$$(1-s)U(c_p) - y_p < (1-s)U(y_p) - y_p \le (1-s)U(z_p) - z_p.$$

The first inequality follows from $c_p < y_p$ and the second from the fact that $z_p = Y_a(s)$ maximizes $(1-s)U(z) - z$. The above result implies $\Delta_2 > \Delta_3$, and so the right-hand side of (20) is strictly greater than that of (21). Thus, only (20), hence only (8), can possibly bind.

Let λ be the shadow cost of (20). With $U'(0) = \infty$, the first-order conditions of c_p and y_p hold with equality. Dividing these two first-order conditions, I have $(1-s)U'(c_p) = 1$, and so $c_p = Y_a(s)$. Substitute $c_p = z_p = y_c = Y_a(s)$. The conditions (17) and (18) in the proposition are the first-order conditions of (y_p, s). The condition (16) comes from re-arranging (20).

To show that $\lambda > 0$ when ρ is sufficiently close to 0, suppose that $\lambda = 0$. Then, (17) and (18) imply $s = s^u$ and $y_p = y_p^u$. That is, the solution is the unconstrained optimum. However, the unconstrained optimum fails to satisfy (20) when ρ is sufficiently close to 0 (note that $\Delta_2 > 0$ and $\Delta_3 > 0$). A contradiction.[10] QED

The incentive constraint of the producer in a single-coincidence meeting, (8) or the rewritten form (16), is more restrictive on optimal allocations than the other two incentive constraints. As a result, it is the only one that can possibly bind. This is intuitive. Because the producer in a single-coincidence meeting is required to produce for someone else without being repaid immediately,

[10]The case $\lambda = 0$ (when ρ is sufficiently close to 0) violates the upper hemi-continuity of the solutions for (s, y_p) as a correspondence of ρ, because it implies a constant solution (s^u, y_p^u) that does not approach the unique autarky solution $(0, y_a)$ when $\rho \to 0$. Thus, an alternative proof of $\lambda > 0$ is to use the Theorem of the Maximum to establish the upper hemi-continuity of the set of solutions for (s, y_p) as a correspondence of ρ. This alternative proof seems more difficult, because it requires the incentive constraint to determine (s, y_p) as a continuous correspondence of ρ.

he has stronger incentive to defect than does an agent in a no-coincidence meeting, who produces entirely for himself, or a consumer, who consumes others' good without immediately repaying. When the extent of the market is very limited, the producer's incentive constraint indeed binds, for it does so in autarky which is the optimal allocation when $\rho = 0$.

The optimal allocation entails higher market consumption and lower autarkic consumption than in autarky. Market consumption, $c_m = y_p - c_p$, is positive in the optimal allocation, while it is 0 in autarky. The optimal amount of autarkic consumption in the three types of matches, y_c, c_p and z_p, are all equal to $Y_a(s)$ in the optimal allocation. Because $Y_a(s) < Y_a(0) = y_a$, the optimal allocation has lower autarkic consumption than in autarky. This difference between the optimal allocation and the autarkic allocation arises because the higher degree of specialization in the optimal allocation reduces the desirability of autarkic consumption and increases the desirability of market consumption.

How large a quantity of goods the planner can induce the producer to give to the consumer depends on the degree of specialization (s) and the cost of satisfying the producer's incentive constraint (λ). Both s and λ are endogenously determined by the extent of the market, ρ. To understand how the model works, however, it is instructive to express c_m as a function of (s, λ, ρ). This function can be obtained from (17) as follows:

$$c_m = C_m(s, \lambda, \rho) \equiv U'^{-1}\left(\frac{1}{B(s)} \cdot \frac{1 + \lambda(\beta + N(1-\beta))}{1 + \lambda\beta\rho/[1-\beta(1-\rho)]}\right). \tag{22}$$

It is straightforward to show that $\partial C_m/\partial s > 0$, $\partial C_m/\partial \lambda < 0$, and $\partial C_m/\partial \rho > 0$. These features are not surprising. First, given (λ, ρ), an increase in the degree of specialization increases market consumption, because higher specialization increases the desirability of market consumption. Second, given (s, ρ), an increase in the cost of satisfying the producer's incentive constraint, λ, reduces the producer's output and hence reduces market consumption. Third, given (s, λ), an increase in ρ enhances the planner's ability to induce the producer to produce a higher quantity of goods by punishing the producer's potential defection.[11]

[11] Note that market consumption is zero if the degree of specialization is zero, i.e., $C_m(0, \lambda, \rho) = 0$ for given (λ, ρ). Also note that, if the producer's incentive constraint does not bind, then the quantity of market consumption is equal to the unconstrained optimal amount, provided $s = s^u$. That is, $C_m(s^u, 0, \rho) = y_p^u - c_p^u$.

In the next section, I analyze the effect of ρ on the degree of specialization and other endogenous variables, including the cost λ.

4 The Optimal Degree of Specialization

For ρ to have a non-trivial effect on s, it is necessary that ρ is small initially, otherwise the unconstrained optimum would be incentive feasible. In this section, I assume that ρ is sufficiently close to 0. A numerical example later will illustrate the case where ρ is significantly above 0. When ρ is sufficiently close to 0, part (i) of Proposition 6 implies $\lambda > 0$, and so the incentive constraint (16) holds with equality.

Even when $\lambda > 0$, it is not clear, a priori, whether the optimal degree of specialization is different from the unconstrained level. Because the unconstrained optimal degree of specialization is independent of ρ, for ρ to have any effect on s I have to show that the optimal allocation does not achieve the unconstrained optimal degree of specialization. To do so, substitute the quantity of market consumption from (22) into the condition for optimal s, (18):

$$\frac{B'(s)U(C_m(s,\lambda,\rho))}{NU(Y_a(s))} = \frac{1+\lambda\left[1-\frac{\beta(1-\beta)(1-\rho)}{N(1-\beta(1-\rho))}\right]}{1+\lambda\beta\rho/[1-\beta(1-\rho)]}. \tag{23}$$

I have the following Lemma (see Appendix B for a proof):

Lemma 7 *When ρ is sufficiently close to 0 and β is sufficiently close to 1, there is a unique solution to (23) for s, denoted $S(\lambda,\rho)$. This solution has the following properties: $S_\lambda < 0$, $S_\rho > 0$ and $S(0,\rho) = s^u$. Thus, the optimal degree of specialization is strictly lower than that in the unconstrained optimum.*

With $s < s^u$, an increase in the extent of the market affects the optimal degree of specialization directly and indirectly. The direct effect arises because an increase in ρ enhances the planner's ability to punish potential defection, which reduces the payoff of defection and allows the planner to implement a higher degree of specialization. This effect is captured by the feature $S_\rho(\lambda,\rho) > 0$. The indirect effect arises because an increase in ρ affects λ, the cost of inducing the producer to produce for the consumer, which affects the degree of specialization. This indirect effect may be either positive or negative. On the one hand, an increase in ρ increases the planner's ability to enforce the optimal allocation, which reduces the payoff to defection and makes the producer's incentive constraint less binding. On the other hand, an increase in ρ tends to increase specialization and market consumption (for given λ), which increases the cost for the producer to follow the allocation and makes the producer's incentive constraint more binding. It might be possible that the latter effect dominates for some parameter values, in which case an increase in ρ pushes up λ and exerts a negative indirect effect on optimal specialization.

The indirect effect of ρ on s through λ, whatever its sign may be, is dominated by the positive direct effect, as shown in the following proposition (a proof appears in Appendix B).

Proposition 8 *For any given β that is sufficiently close to 1, there exists $\rho_1 > 0$ such that, if $0 \leq \rho \leq \rho_1$, then the optimal allocation exists and is unique. Furthermore, the optimal degree of specialization increases in ρ for $\rho \in [0, \rho_1]$.*

Therefore, an increase in the society's ability to update the transaction record indeed increases the optimal degree of specialization when such ability is initially low. This provides a qualified support for the statement that an increase in the extent of the market increases specialization. As the degree of specialization increases, each agent produces more for other agents to consume and consumes less of his own output. If ρ becomes large enough, the society reached the unconstrained optimum, and then further increases in ρ have no effect on the optimal allocation.

The following example illustrates the dependence of the optimal allocation on ρ in the entire domain $[0, 1] \ni \rho$. It shows that the properties of the optimal allocation for sufficiently small ρ also extend to intermediate values of ρ.

Example 9 *Consider Example 2, with $\sigma = 0.1$, $u_0 = 1$, $b = 0.5$, $B_0 = 10$, $N = 10$ and $\beta = 0.9$. In this example, $s^u = 0.43$ and $\beta^u = 0.272$. The unconstrained optimum is incentive feasible if $\rho > 0.255$. For $0 \leq \rho < 0.255$, λ monotonically decreases in ρ, reaching 0 at $\rho = 0.255$, and the optimal degree of specialization increases monotonically from 0 to s^u.*

5 Downward Adjustment Cost and Gradual Specialization

In this section I analyze the optimal allocation when there is an adjustment cost for agents to reduce specialization. This analysis is important for two reasons. First, it endogenizes the extent of market in the sense that, even for fixed ρ, the choice of specialization can change the incentive cost of inducing agents to comply with the optimal allocation in the future. Second, it generates a gradual process of specialization, as in reality, even though there is no adjustment cost for agents to increase specialization.

Let s_{t-1} be the degree of specialization at the beginning of period t and s_t at the end of period t. The adjustment cost for changing from s_{t-1} to s_t is assumed to be $\Phi(s_{t-1}, s_t) = \phi(s_{t-1}, s_t) > 0$ if $s_t < s_{t-1}$, and $\Phi(s_{t-1}, s_t) = 0$ if $s_t \geq s_{t-1}$. The downward adjustment cost is realistic, as one can undoubtedly find ample examples. I abstract from an upward adjustment cost, because it is obvious that an upward cost can slow down the specialization process.[12] To facilitate algebra, I assume that, for all $s_t < s_{t-1}$, $\phi(.,.)$ is twice continuously differentiable and

$$\begin{gathered}\phi_2 < 0,\ \phi_{22} > 0;\ \phi_1 > 0,\ \phi_{12} < 0;\\ \phi(s,s) = \phi_1(s,s) = \phi_2(s,s) = \phi_{22}(s,s) = 0.\end{gathered}$$

These assumptions are reasonable. For example, $\phi_2 < 0$ and $\phi_1 > 0$ both state that, the further s_t is below s_{t-1}, the larger the adjustment cost.

Suppose that the economy has zero degree of specialization at the beginning. The planner chooses $(s_t, y_{pt}, c_{pt}, y_{ct}, z_{pt})_{t\geq 0}$ to maximize the representative agent's intertemporal utility. Suppress the time index t. Add a subscript $+i$ to the variables in $t+i$ and a subscript $-i$ to the variables in $t-i$, where $i = 0, 1, 2....$ In each period, the state variable is s_{-1} for an agent who has followed the planner's allocation and has not seen anyone defect. Let $V(s_{-1})$ be the value function for such an agent. Similar to (1), V satisfies:

$$\begin{aligned}V(s_{-1}) = \max_{(s,y_p,c_p,y_c,z_p)} \Big\{ & -\Phi(s_{-1}, s) + \tfrac{1}{N}\left[B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c\right] \\ & + \tfrac{1}{N}\left[(1-s)U(c_p) - y_p\right] + \left(1 - \tfrac{2}{N}\right)\left[(1-s)U(z_p) - z_p\right] + \beta V(s) \Big\}.\end{aligned} \tag{24}$$

Clearly, (s, y_p, c_p, y_c, z_p) must still satisfy (6).

[12] This does not mean that there is no cost at all to increasing the degree of specialization. A producer's current utility is reduced by specilization, as illustrated by the utility function $(1-s)U(c) - y$. However, this cost is not an adjustment cost and does not cause the specialization process to be gradual.

To specify the incentive constraints, let us first compute the value function for an agent who defected in the past and whose defection has been detected. Let $L(k)$ denote his value function, where $k \in [0,1]$ is his degree of specialization entering the period. (Because the agent defected in the past, k need not be the same as s_{-1}.) Then,

$$L(k) = \max_{(k',y)} \left[(1-k')\, U(y) - y - \Phi(k,k') + \beta L(k') \right]. \tag{25}$$

Next, consider an agent whose past defection has not been discovered and who has not seen anyone else defect. Let $k \in [0,1]$ be his degree of specialization entering a period, and $W(k, s_{-1})$ his value function before the public record is updated. In the current period, the agent's past defection is discovered with probability ρ, in which case the agent's value function is $L(k)$. If the defection is not discovered in the current period, the agent can choose whether to defect again. If he defects again, the value function is

$$D(k,s) = \max_{(k',y)} \left[(1-k')U(y) - y - \Phi(k,k') + \beta W(k',s) \right]. \tag{26}$$

As in previous sections, such an agent indeed defects again when he is the producer in a match with a single coincidence of wants or when he is in a match with no coincidence of wants.[13] Thus, the value function for such an agent before the public record is updated this period is:

$$\begin{aligned} W(k,s_{-1}) = {} & \rho L(k) + (1-\rho) D(k,s) \\ & + (1-\rho)\tfrac{1}{N} \max \left\{ \begin{pmatrix} B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c \\ -\Phi(k,s) + \beta W(s,s) - D(k,s) \end{pmatrix},\ 0 \right\}. \end{aligned} \tag{27}$$

This value function is a function of (k, s_{-1}), instead of (k, s), because the optimal allocations (s, y_p, c_p, y_c) are functions of s_{-1}. Note that, if the agent wants to follow the optimal allocation this period, he must change specialization from k to s, in which case the continuation value from the next period onward is $W(s,s)$.

Now consider an agent who has not defected before and who has not seen anyone else defect. If this agent defects in the current period, the payoff is $D(s_{-1}, s)$. To prevent such defection, the

[13]To see this, consider a match with no coincidence of wants. If the agent follows the optimal allocation, the value is $(1-s)U(z_p) - z_p - \Phi(k,s) + \beta W(s,s)$, where k is the agent's specialization entering the period. The maximum of this expression, over (s, z_p), is $D(k,s)$. So, the value from following the optimal allocation cannot be larger than that from defecting again. The same result holds when the agent is the producer in a match with a single coincidence of wants.

allocation must satisfy the following incentive constraints for the consumer, the producer and an agent in a match with no coincidence of wants, respectively:

$$B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c - \Phi(s_{-1}, s) + \beta V(s) \geq D(s_{-1}, s), \tag{28}$$

$$(1-s)U(c_p) - y_p - \Phi(s_{-1}, s) + \beta V(s) \geq D(s_{-1}, s), \tag{29}$$

$$(1-s)U(z_p) - z_p - \Phi(s_{-1}, s) + \beta V(s) \geq D(s_{-1}, s). \tag{30}$$

The optimal allocation solves (24), subject to (28)–(30) and (6). Denote the solution for s as $g(s_{-1})$. Similar to parts (i) and (ii) of Proposition 6, the following lemma holds:[14]

Lemma 10 *Suppose $s \geq s_{-1}$ for all t. The optimal allocation has $c_p = y_c = z_p = Y_a(s)$, and the only incentive constraint that can possibly bind is the producer's incentive constraint, (29).*

Optimal specialization takes more than one step in this economy. Each step of specialization increases the cost to future defection, thus relaxing the incentive constraints in the future and making further specialization possible. To establish this result, I start with the following lemma.

Lemma 11 *Denote the solution for k' in (25) as $K^L(k)$. Then, $L'(k) \leq 0$ and $K^L(k) \leq k$ for all $k \geq 0$, with strict inequality if $k > 0$.*

Proof. Denote the right-hand side of (25) as a mapping $TL(k)$. Clearly, T exhibits monotonicity and discounting. Also, $L(k)$ is bounded above by $L(0) = A/(1-\beta)$, where A is defined in (4), and below by 0. Thus, T satisfies Blackwell's sufficient conditions for contraction mapping, and so there is a unique fixed point for L (Stokey et al., 1989, pp. 54-55). To prove the lemma, I show that, if $L'(k') \leq 0$ for $k' \geq 0$, then $TL'(k) \leq 0$ and $K^L(k) \leq k$ for all $k \geq 0$. The solution for y in (25) is $y = Y_a(k')$, where $Y_a(.)$ is defined in (2). The solution for k' satisfies $[-U(Y_a(k')) - \Phi_2(k, k') + \beta L'(k')] \leq 0$ and, if the inequality is strict, then $k' = 0$. If $k = 0$, the inequality is strict, implying $K^L(0) = 0$ and $TL'(0) = \Phi_1(0, 0) = 0$. Let $k > 0$. Because

[14] The proof is simpler here than in Proposition 6. The first-order conditions for (c_p, y_p, z_p, y_c) immediately yield $c_p = y_c = z_p = Y_a(s)$. Substituting these quantities, one can show that the left-hand side of (29) is less than those of (28) and (30).

$k' \geq k$ would imply that the optimality condition for k' holds with strict inequality, which would further imply the contradiction $k' = 0 < k$, it must be the case that $K^L(k) < k$. In this case, the envelope condition gives $TL'(k) = -\phi_1(k, K^L(k)) < 0$. Now that $L'(k') \leq 0$ implies $TL'(k) < 0$, the contraction mapping argument implies that the fixed point of T for L satisfies $L'(k) \leq 0$ for all $k \geq 0$. Repeating the above argument on the fixed point yields $L'(k) < 0$ and $K^L(k) < k$ for all $k > 0$. QED

The above lemma shows that an agent in autarky gradually reduces the degree of specialization toward 0. The higher the degree of specialization from which he defects into autarky, the lower value he can obtain from autarky. This is a good indication of, but not sufficient for, the desired result that specialization allows for further specialization. To establish the latter, I follow the following steps. First, conjecture that the specialization sequence is an increasing sequence. Under this conjecture, I show that $D(s_{-1}, s)$ is a decreasing function of s_{-1}, and strictly so if $s_{-1} > 0$. This implies that the value function $V(s_{-1})$ is an increasing function. Second, using the properties of D and V, I confirm the conjecture that $s > s_{-1}$ for all $0 \leq s_{-1} < s^u$.

Lemma 12 *Suppose $s \geq s_{-1}$ for all t. Then, the following results hold: (i) $W_1(k, s_{-1}) \leq 0$ and $D_1(k, s_{-1}) \leq 0$ for all $0 \leq k \leq s_{-1}$, with strict inequality if $k > 0$. (ii) $W(k, s_{-1}) = W_A(k) + W_B(s_{-1})$ and $D(k, s) = D_A(k) + D_B(s)$, for some functions W_A, W_B, D_A and D_B. (iii) $V'(s_{-1}) > 0$ for all $s_{-1} > 0$ if one or more of the incentive constraints, (28)–(30), binds.*

Proof. For (i), conjecture that $W_1(k, s_{-1}) \leq 0$ for all $0 \leq k \leq s_{-1}$. Because $s \geq s_{-1}$ by supposition, $W_1(k, s) \leq 0$ as well. Let $K^D(k, s)$ be the solution for k' in (26). Similar to the proof of Lemma 11, one can show that $K^D(k, s) \leq k$ and $D_1(k, s) \leq 0$, with strict inequality if $0 < k \leq s$. Now examine (27). Suppose, first, that the maximum in (27) is 0. Because $L'(k) < 0$ and $D_1(k, s) < 0$ for all $0 < k \leq s$, and $s_{-1} \leq s$, (27) implies $W_1(k, s_{-1}) < 0$ for all $0 < k \leq s_{-1}$. If the maximum is not zero, then again

$$W_1(k, s_{-1}) = \rho L'(k) + (1 - \rho) \left(1 - \frac{1}{N}\right) D_1(k, s) < 0,$$

where I have used the fact that $\Phi_1(k, s) = 0$ when $k \leq s_{-1} \leq s$. Thus, indeed $W_1(k, s_{-1}) \leq 0$ and so $D_1(k, s_{-1}) \leq 0$ for all $0 \leq k \leq s_{-1}$, both with strict inequality if $k > 0$.

Part (ii) of the lemma can be verified directly from (26) and (27), under the supposition $s \geq s_{-1}$. For (iii), note that the objective function in (24) is weakly increasing in s_{-1} for all $s \geq s_{-1}$. Also, under the supposition $s \geq s_{-1}$, $D_1(s_{-1}, s) < 0$ for $s_{-1} > 0$ and $\Phi_1(s_{-1}, s) = 0$. Thus, an increase in s_{-1} strictly relaxes each of the incentive constraints (28)–(30). If any of these constraints binds, then $V'(s_{-1}) > 0$. QED

Now let us consider the first step of specialization, supposing that the economy starts with zero specialization, i.e., with $s_{-1} = 0$. At this initial stage, the optimal choice of specialization must be positive for $\rho > 0$. Otherwise, the economy would remain in complete autarky forever, but with the payoff in complete autarky, zero degree of specialization would not be optimal. Thus, $g(0) > 0$. Also, $g(0) < s^u$ if ρ is sufficiently small. This is because, at $s_{-1} = 0$, a defection from the optimal allocation requires no change of specialization from s_{-1} and hence involves no adjustment cost. If the unconstrained optimum were incentive feasible in the current economy, it would be feasible as well in the previous economy without adjustment costs. Because the unconstrained optimum is not incentive feasible in the previous economy when ρ is sufficiently small, it is not incentive feasible either in the current economy at $s_{-1} = 0$.

Finally, the following proposition establishes monotonicity and convergence of the specialization sequence (see Appendix C for a proof):

Proposition 13 *Assume that $\beta \geq \beta^u$, ρ is sufficiently small, and $\phi_1(s^u, 0)$ is bounded above. Then, $s_{-1} < g(s_{-1}) \leq s^u$ for all $0 < s_{-1} < s^u$, and $0 < g(0) < s^u$. Thus, the sequence $\{s_{+i}\}_{i \geq 0}$ generated by $s = g(s_{-1})$ converges monotonically to s^u.*

Therefore, with a downward adjustment cost, the optimal allocation calls for gradual increases in the degree of specialization. With the general interpretation of the extent of the market as the society's ability to achieve desirable outcomes, this result shows that, even for given ρ, the extent of the market increases gradually along the path of specialization. The society is able to get closer and closer to the unconstrained optimum, because each step of specialization makes it easier for the society to specialize further, by increasing the cost to future defection.

It is remarkable that the society can eventually achieve the unconstrained social optimum, regardless of the size of the downward adjustment cost (provided that it is positive). The size of the adjustment cost affects only the length of the convergence process. The smaller the downward adjustment cost, the longer the society might have to take to reach the unconstrained social optimum.

6 Conclusion

In this paper I examine the socially optimal choices of specialization, consumption and production in a random matching economy. As in Kocherlakota and Wallace (1998), public memory is imperfect in the sense that an agent's transaction is updated into the public record with a probability $\rho < 1$. I interpret ρ as the extent of the market. When the extent of the market is very limited, an extension of the market increases the socially optimal degree of specialization and the quantities of goods exchanged in the market. When the market is very extensive already, a further extension of the market has no effect on the socially optimal degree of specialization or the quantities of trade.

I also characterize the optimal specialization process when there is an adjustment cost for reducing specialization. Even without an upward adjustment cost, the optimal degree of specialization increases gradually, and it converges monotonically to the unconstrained optimum. Thus, even for fixed ρ, the extent of the market increases endogenously along the path of specialization, as each step of specialization reduces the incentive cost for further specialization.

Let me conclude with two remarks. The first is about the contrast between the results here and those will obtain with the usual interpretation of the extent of the market as the physical capacity for agents to meet with each other. This usual interpretation can be captured here by an increase in the matching rate or, equivalently, by a reduction in N.[15] It can be shown (in

[15]To see this equivalence, let the matching rate be μ instead of 1. Then, the optimal allocation is characterized as in previous sections, with μ/N replacing $1/N$. An increase in μ, while holding N fixed, has analytically the

the absence of the adjustment cost) that, when ρ is sufficiently close to 0, a reduction in N has effects similar to those of an increase in ρ. That is, specialization and market consumption both increase. When ρ is close to 1, a reduction in N continues to increase the optimal degree of specialization and the quantities of goods traded in the market. This effect of N, which contrasts with the effect of ρ, arises because the unconstrained optimum depends on N. In particular, a reduction in N raises the left-hand side of (11) for any given s, and so the degree of specialization in the unconstrained optimum increases.

The second remark is about the potential role of money in the economy, which I abstracted from so far. In a match with a single coincidence of wants, money can serve as a medium of exchange and help completing the exchange. When ρ is close to 0, the socially optimal allocation requires the use of both the public record-keeping technology and money, see Kocherlakota and Wallace (1998). It is interesting to see how an extension of the market through ρ simultaneously affects the optimal quantity of money and the optimal degree of specialization. For example, if the increase in ρ reduces the optimal quantity of money, then the public record-keeping technology and money are substitutes, even for ρ close to 0.

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same effect as that of a decrease in N.

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Appendix

A Proof of Lemma 5

Let $\rho > 0$ and $\beta \geq \beta^u$. I first show that $W > A/(1-\beta)$. Suppose, to the contrary, that $W = A/(1-\beta)$ (note that W cannot be strictly less than $A/(1-\beta)$). Then, the unconstrained optimum is incentive feasible provided $\beta \geq \beta^u$ (the proof for this is similar to that in the case $\rho = 1$ in Appendix 3). The fact $\beta^u < 1$ implies $B(s)U(y_p - z_p) > (y_p - z_p) + N[A - (1-s)U(z_p) + z_p]$, where the variables take their values in the unconstrained optimum. Then,

$$\begin{aligned} & B(s^u)U(y_p^u - z_p^u) + (1-s^u)U(z_p^u) - z_p^u - A \\ & > y_p^u - z_p^u + (N-1)\left[A - (1-s^u)U(z_p^u) + z_p^u\right]. \end{aligned}$$

Because $y_p^u > z_p^u$ and $A > (1-s^u)U(z_p^u) + z_p^u$, the above expression is strictly positive. With (12), this implies that $W > A/(1-\beta)$, which contradicts the supposition $W = A/(1-\beta)$.

Next, suppose, contrary to $V > W$, we have $V = W$ instead (note that $V \geq W$ by Lemma 3). Then, only autarky satisfies all the feasibility constraints (6) – (9). In this case, (1) yields $V = A/(1-\beta)$. So, $W = A/(1-\beta)$, which contradicts the above result $W > A/(1-\beta)$.

To show $s > 0$ and $y_p > c_p$, suppose either $s = 0$ or $c_p = y_p$. Then either $B(s) = 0$ or $U(y_p - c_p) = 0$. In either case, $B(s)U(y_p - c_p) + (1-s)U(y_c) - y_c = (1-s)U(y_c) - y_c$. Since the latter expression is less than or equal to A, (12) implies $W = A/(1-\beta)$, which contradicts the earlier result $W > A/(1-\beta)$.

The strict inequality (14) must hold; otherwise (12) would imply the counterfactual result $W = A/(1-\beta)$. With (14), (12) becomes (15). QED

B Proofs of Lemma 7 and Proposition 8

To prove Lemma 7, temporarily denote the left-hand side of (23) by $LHS(s, \lambda, \rho)$ and the right-hand side by $RHS(\lambda, \rho)$. It can be verified that $LHS(s, 0, \rho) = f(s)$, which is the left-hand side of (11) that characterizes the solution for s^u. Also, $RHS(0, \rho)$ is equal to the right-hand side of (11). Note that $LHS(s, \infty, \rho) < LHS(s, \lambda, \rho) < LHS(s, 0, \rho)$ for all $\lambda \in (0, \infty)$, because $LHS(s, \lambda, \rho)$ is a decreasing function of λ. When $\beta \to 1$, $C_m(s, \infty, \rho) \to C_m(s, 0, \rho)$, and so $LHS(s, \infty, \rho) \to LHS(s, 0, \rho) = f(s)$. Because $f(s)$ is a decreasing function of s, as assumed in Assumption 1, $LHS(s, \lambda, \rho)$ is a decreasing function of s if β is sufficiently close to 1. Also, $LHS(s, \lambda, \rho)$ is sufficiently close to $f(s)$ in this case, and so the existence and uniqueness of the solution to (11) imply the existence and uniqueness of the solution to (23). One can verify that

$LHS_\lambda(s,\lambda,\rho) < 0$, $RHS_\lambda(\lambda,\rho) > 0$, $LHS_\rho(s,\lambda,\rho) > 0$ and $RHS_\rho(\lambda,\rho) < 0$. Thus, $S_\lambda(\lambda,\rho) < 0$ and $S_\rho(\lambda,\rho) > 0$. When $\lambda = 0$, (23) is equivalent to (11), in which case $s = s^u$. Because $\lambda > 0$, however, the feature $S_\lambda(\lambda,\rho) < 0$ implies $s < s^u$.

Turn now to Proposition 8. Because $S(\lambda,\rho)$ is the unique solution for s when β is sufficiently close to 1, for the existence and uniqueness of the optimal allocation, it suffices to show that the solution for λ exists and is unique under the conditions specified in the proposition. Substituting $C_m(S(\lambda,\rho),\lambda,\rho)$ for market consumption in (16), I have

$$L(\lambda,\rho) = 0,$$

where $L(\lambda,\rho)$ is the left-hand side of (16) minus the right-hand side. Using the properties $S_\lambda < 0$ and $S_\rho > 0$, a lengthy algebra shows that $L_\lambda(\lambda,\rho) > 0$. Thus, the solution for λ exists and is unique if $L(0,\rho) \leq 0$ and $L(\infty,\rho) > 0$.

Consider the requirement $L(0,\rho) \leq 0$ first. When $\lambda = 0$, $S(0,\rho) = s^u$ and $c_m = y_p^u - c_p^u$. So, the condition $L(0,\rho) \leq 0$ can be rewritten as

$$\frac{1-\beta}{\beta\rho} \geq \frac{B(s^u)U(y_p^u - c_p^u) - [\beta + (1-\beta)N](y_p^u - c_p^u) - (N+1-\beta)\Delta^u}{[\beta + (1-\beta)N](y_p^u - c_p^u) + (N-\beta)\Delta^u},$$

where $\Delta^u = A - (1-s^u)U(z_p^u) + z_p^u$. If the right-hand side of the above inequality is 0 or negative, let $\rho_1 = 1$. If the right-hand side of the inequality is positive, the inequality gives an upper bound on ρ. Let ρ_1 be the minimum of this upper bound and 1. By construction, $0 < \rho_1 \leq 1$ (although ρ_1 might be very close to 0 when β is sufficiently close to 1). Then, $L(0,\rho) \leq 0$ if $0 \leq \rho \leq \rho_1$.

Now consider $L(\infty,\rho)$. When $\beta \to 1$, $S(\infty,\rho) \to s^u$, and so

$$L(\infty,\rho) \to B(s^u)U(y_p^u - c_p^u) - (y_p^u - c_p^u) - \Delta^u > 0.$$

The strictly inequality follows from the fact that the unconstrained optimum satisfies the producer's incentive constraint. Thus, for β sufficiently close to 1, $L(\infty,\rho) > 0$. This establishes the existence and uniqueness of the optimal allocation.

To complete the proof of the proposition, I now show that $ds/d\rho > 0$. Totally differentiating $S(\lambda,\rho)$ with respect to ρ and using the fact that $d\lambda/d\rho = -L_\rho(\lambda,\rho)/L_\lambda(\lambda,\rho)$, I get

$$\frac{ds}{d\rho} = [S_\rho(\lambda,\rho)L_\lambda(\lambda,\rho) - S_\lambda(\lambda,\rho)L_\rho(\lambda,\rho)]/\,L_\lambda(\lambda,\rho).$$

Because $L_\lambda(\lambda,\rho) > 0$, as established above, $ds/d\rho > 0$ iff $S_\rho(\lambda,\rho)L_\lambda(\lambda,\rho) > S_\lambda(\lambda,\rho)L_\rho(\lambda,\rho)$. Substituting L_λ and L_ρ, the latter condition becomes

$$\begin{aligned} 0 < & \left[(\beta+(1-\beta)N)\tfrac{1-\beta(1-\rho)}{\beta\rho} - B(s)U'(C_m)\right]\left[S_\lambda(\lambda,\rho)\tfrac{dC_m}{d\rho} - S_\rho(\lambda,\rho)\tfrac{dC_m}{d\lambda}\right] \\ & -S_\lambda(\lambda,\rho)\tfrac{1-\beta}{\beta\rho^2}\left[(\beta+(1-\beta)N)C_m + (N-\beta)\Delta\right], \end{aligned} \tag{31}$$

where $C_m = C_m(S(\lambda,\rho),\lambda,\rho)$ and $\Delta = A - [1-S(\lambda,\rho)]U(Y_a(S(\lambda,\rho))) + Y_a(S(\lambda,\rho)) > 0$. Since the right-hand side of (17) is an increasing function of λ and, when $\lambda \to \infty$, it is equal to $[\beta+(1-\beta)N][1-\beta(1-\rho)]/(\beta\rho)$, it is clear that

$$[\beta+(1-\beta)N]\,\frac{1-\beta(1-\rho)}{\beta\rho} > B(s)U'(C_m).$$

In addition, $S_\lambda < 0$. Then, a sufficient condition for (31) is

$$S_\lambda(\lambda,\rho)dC_m/d\rho - S_\rho(\lambda,\rho)dC_m/d\lambda > 0.$$

In turn, after computing $dC_m/d\rho$ and $dC_m/d\lambda$, this condition is equivalent to

$$\frac{\partial RHS(18)}{\partial\lambda}\cdot\frac{\partial RHS(17)}{\partial\rho} > \frac{\partial RHS(18)}{\partial\rho}\cdot\frac{\partial RHS(17)}{\partial\lambda},$$

which can be verified directly. Thus, $ds/d\rho > 0$. QED

C Proof of Proposition 13

If the optimal allocation ever reaches s^u, it will remain at s^u. That is, s^u is a fixed point of $g(.)$. If $s_{-1} < g(s_{-1}) \le s^u$ for all $0 \le s_{-1} < s^u$, then the sequence $\{s_{+i}\}_{i\ge0}$ is monotonically increasing and bounded. So, the sequence converges to s^u. I have already argued in the text that $0 < g(0) < s^u$. Let $0 < s_{-1} < s^u$. I show that $g(s_{-1}) > s_{-1}$ and $g(s_{-1}) \le s^u$.

First, suppose $g(s_{-1}) \le s^u$. I verify that conjecture $g(s_{-1}) > s_{-1}$. If $g(s_{-1}) = s^u$, clearly $g(s_{-1}) > s_{-1}$. Let $g(s_{-1}) < s^u$. Since $g(0) > 0$, for $g(s_{-1}) > s_{-1}$ it suffices to show $g'(s_{-1}) > 0$ for all $0 < s_{-1} < s^u$. Because $g(s_{-1}) < s^u$, the unconstrained optimum is not incentive feasible at s_{-1}. By Lemma 10, the producer's incentive constraint must bind, yielding

$$y_p = Y(s_{-1},s) \equiv (1-s)U(Y_a(s)) + \beta V(s) - D(s_{-1},s).$$

Lemma 10 also states that $c_p = y_c = z_p = Y_a(s)$. Substituting these quantities, I have

$$V(s_{-1}) = \max_s\left[F(s_{-1},s) + \beta V(s)\right],$$

where

$$F(s_{-1},s) \equiv (1-s)U(Y_a(s)) - Y_a(s) \\ +\tfrac{1}{N}\left[B(s)U(Y(s_{-1},s)-Y_a(s)) - (Y(s_{-1},s)-Y_a(s))\right].$$

Because $s = g(s_{-1}) < s^u$, which is interior, it satisfies the first-order condition:

$$F_2(s_{-1},s) + \beta V'(s) = 0. \tag{32}$$

The second-order condition requires the left-hand side of the above equation to be a decreasing function of s, at least at $s = g(s_{-1})$. Thus, $g'(s_{-1}) > 0$ iff $F_{21}(s_{-1}, g(s_{-1})) > 0$. Using the property $D(k,s) = D_A(k) + D_B(s)$, I can compute $F_{21}(s_{-1}, g(s_{-1})) = -D'_A(s_{-1})G/N$, where

$$G \equiv \left[B'(s)U'(Y-Y_a) + B(s)U''(Y-Y_a)\left(Y_2 - Y'_a\right)\right]_{s=g(s_{-1})}.$$

Here, $Y = Y(s_{-1},s)$, $Y_a = Y_a(s)$, $Y_2 = \partial Y(s_{-1},s)/\partial s$ and $Y'_a = Y'_a(s)$. Since $D_1(k,s) < 0$ for all $0 < k \le s$, then $D'_A(s_{-1}) < 0$ under the conjecture $s_{-1} < g(s_{-1})$. Thus, $F_{21}(s_{-1}, g(s_{-1})) > 0$ if and only if $G > 0$.

Suppose, to the contrary, that $G \le 0$. Computing F_2, (32) yields

$$Y_2 - Y'_a = \frac{NU(Y_a) - B'(s)U(Y-Y_a) - \beta V'(s)}{B(s)U'(Y-Y_a) - 1}.$$

Because the producer's incentive constraint binds, $V'(s_{-1}) > 0$ (see Lemma 12). The envelope condition for s_{-1} then implies $B(s)U'(Y-Y_a) > 1$. Treating $Y - Y_a$ as a separate variable, I get

$$\frac{\partial(Y_2 - Y'_a)}{\partial(Y-Y_a)} = \frac{-G}{B(s)U'(Y-Y_a) - 1} \ge 0.$$

Because $B(s)U'(Y-Y_a) > 1$, $Y - Y_a < U'^{-1}\left(\frac{1}{B(s)}\right)$. Then,

$$Y_2 - Y'_a \le \left.\frac{NU(Y_a) - B'(s)U\left(Y-Y_a\right) - \beta V'(s)}{B(s)U'\left(Y-Y_a\right) - 1}\right|_{Y-Y_a = U'^{-1}\left(\frac{1}{B(s)}\right)}.$$

The right-hand side of the inequality is $-\infty$ if the numerator is negative, because the denominator approaches 0 from above when $Y - Y_a$ increases toward $U'^{-1}\left(\frac{1}{B(s)}\right)$. The numerator is indeed negative, because $V'(s) > 0$ and $B'(s)U\left(U'^{-1}\left(\frac{1}{B(s)}\right)\right)/[NU(Y_a)] = f(s) \ge f(s^u) = 1$ (as $f'(s) < 0$ from Assumption 1 and $s < s^u$). The result $Y_2 - Y'_a \le -\infty$ clearly implies $G > 0$. A contradiction. Therefore, $g(s_{-1}) > s_{-1}$ for all $s_{-1} < s^u$.

Next, suppose $g(s_{-1}) > s_{-1}$ for all $s_{-1} < s^u$. I verify the conjecture $g(s_{-1}) \le s^u$ for all $s_{-1} < s^u$. The result clearly holds if $g(s_{-1}) = s^u$. So, let $g(s_{-1}) \ne s^u$. Then, the producer's incentive constraint must bind, yielding $y_p = Y(s_{-1},s)$. Because $s = g(s_{-1})$ solves (32), whose left-hand side is a decreasing function of s, it suffices to show that the left-hand side of (32) is non-positive at $s = s^u$. In turn, it suffices to show $F_2(s_{-1}, s^u) \le 0$, because $V'(s^u) = 0$ from the definition of s^u. Compute:

$$F_2(s_{-1}, s^u) = -U(Y_a) + \frac{1}{N} B'(s^u) U(Y - Y_a) + \frac{1}{N} \left[B(s^u) U'(Y - Y_a) - 1 \right] \left(Y_2 - Y_a' \right),$$

where $Y = Y(s_{-1}, s^u)$ and $Y_a = Y_a(s^u)$. The definition of Y implies $Y_1(s_{-1}, s^u) = -D_A'(s_{-1}) > 0$. Since $s_{-1} < s^u$, then $Y(s_{-1}, s^u) < Y(s^u, s^u) = y_p^u$. I have:,

$$-U(Y_a) + \frac{1}{N} B'(s^u) U(Y - Y_a) < U(Y_a) \left[f(s^u) - 1 \right] = 0,$$

$$B(s^u) U'(Y(s_{-1}, s^u) - Y_a(s^u)) > B(s^u) U'(Y(s^u, s^u) - Y_a(s^u)) = 0.$$

The first line uses the definitions of $f(.)$ and s^u; the second line uses the envelope condition: $0 = V'(s^u) = F_1(s^u, s^u)$. Thus, $F_2(s_{-1}, s^u) < 0$ if $Y_2(s_{-1}, s^u) \le Y_a'(s^u)$. Using the definition of Y, I get $Y_2(s_{-1}, s^u) = -\left[D_B'(s^u) + U(Y_a(s^u)) \right]$. Thus, $F_2(s_{-1}, s^u) < 0$ if $D_B'(s^u) + U(Y_a(s^u)) \ge 0$. A sufficient condition is $D_B'(s^u) \ge 0$.

Let us verify $D_B'(s^u) \ge 0$ for sufficiently small ρ. From (26), $D_B'(s^u) = \beta W_B'(s^u)$. If the max in (27) is zero, then $W_B'(s^u) = (1 - \rho) D_B'(s^u)$ (recall $g(s^u) = s^u$), in which case $D_B'(s^u) = 0$. If the max in (27) is not zero, then

$$\begin{aligned} W_B'(s^u) = {} & (1 - \rho) D_B'(s^u) \\ & + \tfrac{1-\rho}{N} \left[(N - 2) U(Y_a(s^u)) - D_B'(s^u) + Y_1(s^u, s^u) + \beta W_A'(s^u) \right]. \end{aligned}$$

Here I have used the following facts: $Y(s^u, s^u) - Y_a(s^u) = c_m^u$, $B(s^u) U'(c_m^u) = 1$, $B'(s^u) U(c_m^u) = N U(Y_a(s^u))$, and $Y_2(s^u, s^u) - Y_a'(s^u) = -U(Y_a(s^u)) - D_B'(s^u)$. Combining the above equation with $D_B'(s^u) = \beta W_B'(s^u)$, I get

$$D_B'(s^u) = \frac{\beta(1 - \rho)/N}{1 - \beta\,(1 - \rho) \left(1 - \frac{1}{N} \right)} \left[(N - 2) U(Y_a(s^u)) + Y_1(s^u, s^u) + \beta W_A'(s^u) \right].$$

$D_B'(s^u) \ge 0$ if $Y_1(s^u, s^u) + \beta W_A'(s^u) \ge 0$. Using the definition of Y, I have $Y_1(s^u, s^u) = -D_A'(s^u)$. The envelope condition for (25) gives $L'(s^u) = -\phi_1(s^u, K^L(s^u))$. Since $\phi_{12} < 0$, $L'(s^u) \ge -\phi_1(s^u, 0)$. Using the envelope condition for (27) to compute $W_A'(s^u)$, I have:

$$\begin{aligned} & Y_1(s^u, s^u) + \beta W_A'(s^u) \\ & = -D_A'(s^u) + \beta \left[\rho L'(s^u) + (1 - \rho) \left(1 - \tfrac{1}{N} \right) D_A'(s^u) \right] \\ & \ge -\rho \beta \phi_1(s^u, 0) + \left[1 - \beta(1 - \rho) \left(1 - \tfrac{1}{N} \right) \right] (-D_A'(s^u)) \\ & > -\rho \beta \phi_1(s^u, 0) + \left[1 - \beta \left(1 - \tfrac{1}{N} \right) \right] \left[-D_A'(s^u) \right]. \end{aligned}$$

Since $\phi_1(s^u, 0)$ is bounded above and $D_A'(s^u) < 0$, the above expression is positive for sufficiently small ρ. QED