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Surplus Extraction and Competition*

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Abstract

A competitive economy is studied in which sellers offer alternative direct mechanisms to buyers who have correlated private information about their valuations. In contrast to the monopoly case where sellers charge entry fees and extract all buyers surplus, it is shown that in the *unique* symmetric equilibrium with competition, sellers hold second price auctions with reserve prices set equal to their cost. Most important, it is a best reply for sellers not to charge entry fees of the kind normally used to extract surplus, even though it is feasible for them to do so.

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1. Introduction

In a competitive environment, it seems natural that mechanism designers might consider offering 'softer' mechanisms that are less efficient at extracting buyer surplus if they thought that this would attract more buyers. This trade-off between surplus extraction and participation is central to the argument in Bulow and Klemperer [1] who show that in an independent information environment, sellers are better off using a simple second price auction without reserve price than they are using an 'optimal' negotiation mechanism if the 'optimal' mechanism is sure to scare away one buyer. This result is true despite the fact that the deviation that sellers are forced to make from the optimal mechanism to keep the marginal buyer is 'large' in any conventional sense.

In the independent value case discussed in [1], the cost of attracting buyers is modelled in a very special way. The cost to the seller of attracting one more buyer is that instead of using an optimal reserve price, he must use a reserve price equal to his cost. This becomes more complex in the case where buyer valuations are correlated, since the optimal mechanism then extracts all the buyer surplus (McAfee and Reny [8] and Cremer and Maclean [3]). Then, reverting to a simple second price auction with a zero reserve price is extremely costly to the seller and will not generally be warranted if all it accomplishes is to attract one additional buyer.[1]

This is unfortunate since this trade-off between participation and surplus extraction, at least as it is expressed by [1], suggests that it might pay sellers to use relatively *simple* mechanisms to attract buyers. This in contrast to the complex belief dependent mechanisms predicted by the theory of mechanism design. Indeed, [8] go so far as to suggest that the full surplus extraction results for the correlated value environment are evidence of the irrelevance of mechanism design since the complex random entry fees that they use to extract the surplus have yet to be observed in practise.

The specific trade-off used in [1] may go too far, but the key idea remains. Even when valuations are correlated, it is clear that if the deviation from optimality that is required to attract one more buyer is small enough, sellers will make it. Obviously, this will mean that the stark predictions of the McAfee Reny theorem will have to be modified. The problem that remains is to gain some more

[1] Reasonable restrictions on the class of feasible mechanisms (for example requiring that only the winning bidder in an auction be forced to pay the seller any money) can be used to resurrect the theorem.

precise understanding of what the trade-off between surplus extraction and buyer participation is.

This paper analyzes the trade-off when the contract market is competitive. The result described for this environment is striking. Despite the fact that sellers can offer arbitrarily complex (direct) mechanisms, in the unique symmetric equilibrium, sellers offer simple second price auctions without reserve prices (i.e., with reserve prices equal to their costs). The notable thing about this result is that sellers do not use the complicated entry fees of the kind described by [8] in equilibrium despite the fact that they could do so, and despite the fact that buyers' valuations are correlated.

The results in the paper suggest that competitive contract markets have some interesting and useful properties. In terms of the trade-off between participation and surplus extraction, [1] argue that second price auctions with zero reserve prices are better than optimal mechanisms if reverting to them attracts just a single additional buyer. They explain why their result cannot be true in the correlated value environment. The competitive results given here show the sense in which their intuition extends to the correlated value case. The equilibrium mechanism here is the zero reserve price auction emphasized in [1], and as they suggested, complex surplus extraction techniques are not used when sellers understand the surplus-participation trade-off..

As in the previous literature on competitive contract markets ([7, 11, 10]), the equilibrium mechanisms have a remarkable robustness property. They are invariant to the number of buyers and seller participating, and to buyers' and sellers' beliefs about the distribution of types of the other participants in the market. Prior papers on competition are restricted to situations in which valuations are independent. In this sense, the results presented here extend this robustness property to markets where valuations are correlated. Thus competition among sellers seems to be enough to explain why simple mechanisms are observed across a spectrum of informational assumptions.

Finally, the results presented here provide some vindication for the methods and approach involved in the theory of mechanism design. The elegant results of McAfee and Reny presents a dilemma for mechanism design since optimal mechanisms are complex and highly sensitive to unobservable beliefs. Since no contracts have ever been observed that even remotely resemble 'optimal' contracts, McAfee and Reny themselves conclude that their results argue for the irrelevance of mechanism design.[2] As will be seen below, the methods of mechanism design

[2] It should be emphasized that it is *not* the fact that optimal contracts extract all the surplus

work very well in *competitive* contract markets, and are central to the proof of the main result. Despite this, the equilibrium contract is an auction. Thus even the most ardent believer in 'direct mechanisms' can be satisfied with the apparently ad hoc restriction to auctions that is made in much of the theoretical literature.

A competitive contract market is one where sellers take the payoff that they need to offer buyers to attract them to be fixed. This idea[3] is natural in the setting examined here since prices are specified as part of the contract. It has a number of advantages. First, though sellers need to know the number of potential buyers, they do not need to know precisely who these buyers are. This is characteristic of markets like real estate where sellers have good information about aggregate demand, but little information about the identities or tastes of individual buyers. Secondly, sellers do not need to know exactly who their competitors are, or even to know what prices their competitors have offered (though it is important that they believe that buyers know these prices). This is important since competition may be relevant even in auctions or sales that appear to be isolated. For example a privately held company might try to sell off a controlling interest in the company to some group of potential investors. The sale appears unique, but the potential investors have many alternative investment opportunities that the seller only poorly understands. Many of the alternative opportunities will be actively competing for the investors' money.

The trade-off between surplus extraction and participation is captured in the following way. Each seller observes (or formulates a conjecture about) the market payoff. The seller then believes that he can achieve any participation-surplus combination having the property that every buyer type who the seller expects to participate receives at least the market payoff. The trade-off then behaves much like a standard demand curve which can be interpreted as a description of the quantity that the seller can *expect* to sell at different prices. Tougher mechanisms make it less likely that buyers will participate in the same way that higher prices are expected to reduce demand. The properties of this simple trade-off are precisely what makes it possible to exploit the standard techniques of the theory of mechanism design to get precise predictions about the mechanisms that sellers will use.

The results of the paper follow from one key property - that sellers' best replies to any market payoff must involve an 'efficient' mechanism that awards

that creates the problem, it is the fact that optimal contracts are such complex objects (see below) that creates problems.

[3]This idea is due to Gale [5] and is now widely used in labor economics, for example [9].

the commodity to the buyer with the highest valuation among all the buyers who visit him. Ironically, the McAfee-Reny argument is crucial to this. To see how, observe first that the seller's profits are equal to the expected gain to trade generated by his mechanism, less the expected surplus buyers get by participating in it. In the competitive environment, the expected surplus that buyers get by participating in the seller's mechanism must be the same as what they can get elsewhere on the market.

Now it is well know that an efficient mechanism will maximize the expected gains from trade provided participation is exogenously fixed. So consider a seller who is using a soft (inefficient mechanism). When he deviates to an efficient mechanism he could compensate buyers (provided valuations are correlated) by implementing a McAfee Reny participation fee that generates an expected payment (conditional on the buyer's type) exactly equal to the difference between the expected surplus before and after the deviation assuming constant participation. The deviation along with this new participation fee prevents buyers' surplus from changing, and therefore maintains participation. The expected surplus component of the seller's payoff increases, while the surplus that he offers buyers does not change - so the deviation is profitable.

It is then immediate that there will be an equilibrium in which sellers use second price auctions with zero entry fees. Suppose that all but one seller is offering a second price auction with zero entry fee. Then when they visit the remaining seller, buyers market payoff is the payoff that they get from participating in a second price auction with no entry fee. The remaining seller believes that there is nothing he can about this, so he will simply choose a mechanism to maximize expected gains from trade. Since the second price auction without entry fees will accomplish this latter task and give buyers the market payoff, there is no incentive for the remaining seller to deviate from this strategy.

Perhaps the more remarkable fact is that among symmetric equilibria, this outcome is unique. It is difficult to give a good intuitive explanation for this result, however, the assertion that an equilibrium is symmetric along with the result that sellers must use efficient mechanisms in equilibrium completely determines the distribution of buyer types that each seller faces (as is shown below). The seller can change this distribution by modifying his mechanism if he likes, so to verify uniqueness it is sufficient to show there is a unique market payoff function for which the distribution implied by symmetry and efficiency of mechanisms is actually a best reply to this market payoff.

The contribution here is not that sellers will be unable to extract all the surplus when there is competition (or more generally when participation is endogenous). That much is obvious. The contribution here is that competition generates *equilibrium* mechanisms that are much simpler than *optimal* mechanisms. Ironically, the key to the simplicity of equilibrium mechanisms is precisely the fact that sellers can imagine offering arbitrarily complicated mechanisms, specifically random entry fees of the kind described by McAfee and Reny ([8]). On the equilibrium path mechanisms will be very simple, but off the equilibrium path they need not be. Thus the techniques of mechanism design are critical to understanding the result. On the other hand, since sellers' best reply to the market is to offer simple second price auctions, the paper also justifies the extensive analysis of auctions in environments where they are known to be non-optimal mechanisms.

2. The Model

2.1. Primitives

There is a finite number J of sellers, each of whom owns a single unit of a homogenous and indivisible commodity. Seller j has a cost represented by a real number c_j. It will be assumed that the sellers' costs are all common knowledge though the model could be extended to allow this to be private information.

There are N buyers participating in the market. Buyers all have private information characterized by their valuation. The valuation of buyer i is $x_i \in [0,1]$.

The ex post payoffs that buyers and sellers get depend only on their own information and both are assumed risk neutral. Thus a buyer of type x who trades with a seller with cost c at a price p gets surplus $x - p$. In the same situation the seller gets $p - c$.

2.2. Prior Beliefs

Let F denote the joint distribution function for buyer valuations. Two assumptions are made about F.

Assumption 2.1 There is a random variable y distributed according to G on the interval $[0,1]$ such that $x_1, \ldots x_N$ are identically and independently distributed conditional on y.

This assumption is used primarily to clarify the role that the buyers' choice strategy plays in determining beliefs. It appears that this assumption could be weakened, but the extension is not trivial. Formally, it implies that there is a conditional distribution $\tilde{F}(\cdot|y)$ such that

$$F(x_1, \ldots x_N) = \int \prod_{i=1}^{N} \tilde{F}(x_i|y)\, dG(y)$$

It is assumed henceforth that F has a continuous and strictly positive density denoted by $f(x_1, \ldots, x_N)$.

Assumption 2.2 For each x_1 and any probability measure μ on the Borel sets of $[0,1]$ satisfying $\mu(\{x_1\}) \neq 1$

$$f(x_2, \ldots x_N|x_1) \neq \int f(x_2, \ldots x_N|x)\, d\mu(x)$$

Assumption 2.2 is the necessary and sufficient condition for (approximate) surplus extraction given by [8]. In this paper, it's significance is that it implies that for any continuous function $\beta(x)$ on $[0,1]$, and any $\varepsilon > 0$, there are finitely many functions $y_i(x_2, \ldots, x_N)$ for $i = 1, \ldots I$ such that

$$\sup \left| \beta(x) - \min_i \int \cdots \int y_i(x_2, \ldots x_N)\, dF(x_2, \ldots x_N|x) \right| < \varepsilon$$

uniformly in x. This is the method that McAfee and Reny use to extract buyers' surplus. In their approach, $\beta(x)$ is the payoff that a buyer of type x receives by participating in the seller's mechanism and the z_i are a series of payment schedules from which the buyer is allowed to choose.

It is not hard to construct functions that satisfy both assumptions 1 and 2. For example, suppose that ε_i for $i = 1, \ldots N$ and y are independently and uniformly distributed on $[0,1]$ and let $x_i = \varepsilon_i \cdot y$. Then $F(x_i|y) = \Pr\{\tilde{x} \leq x_i|\tilde{y} = y\} = \Pr\{\tilde{\varepsilon} \cdot \tilde{y} \leq x_i|\tilde{y} = y\} = \Pr\{\tilde{\varepsilon} \leq x_i/y\} =$

$$\begin{cases} x_i/y & \text{if } x_i \leq y \\ 1 & \text{otherwise} \end{cases}$$

and $G(y|x) = \Pr\{\tilde{y} \leq y|\tilde{x} = x\} = \Pr\{\tilde{y} \leq y|\varepsilon \cdot y = x\} = \Pr\{\varepsilon \geq x/y\} =$

$$\begin{cases} 1 - x/y & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Then using the densities defined by these functions, we have

$$f(x,\dots x|x')|_{x'=x} = \int \prod_{i=2}^{N} \tilde{f}(x|y)\, dG(y|x_1)$$

$$= \int_{\max(x,x')}^{1} \left(\frac{1}{y}\right)^{N-1} \left(\frac{x'}{y^2}\right) dy$$

It is straightforward to show that $f(x,\dots x|x') < f(x,\dots x|x)$ whenever $x' \neq x$ (for $x' < x$ the expression is linear in x', while straightforward differentiation verifies that the expression is strictly decreasing as x' increases above x). This implies that if u is not a point mass at x,

$$f(x,\dots x|x) > \int f(x,\dots x|x')\, d\mu(x')$$

Unfortunately, McAfee and Reny's theorem cannot be applied directly to the competitive environment because beliefs in the competitive environment are not absolutely continuous (with respect to Lebesque measure) and because beliefs are endogenous and depend on the mechanism that the seller offers.

2.3. The market process

The market game[4] that determines all trades now proceeds. At the beginning of the game, each seller offers a direct mechanism to buyers. Mechanisms are described in more detail below. They specify how the seller will determine a price and trading partner (if there is one) among the buyers who visit him. The outcome will depend on messages that the buyers send to the sellers. For example, the seller could promise to hold an auction with a fixed reserve price. Alternatively he could simply offer his unit of output at a fixed price

Once the buyers see the mechanisms that are being offered by sellers, they select *one and only one* seller as a potential trading partner. After buyers select sellers, they communicate with the sellers they have chosen as specified by the seller's announced mechanism. Trades and payments occur, then the game ends. Each seller's problem is similar to any static mechanism design problem except for the fact that buyers have many alternative sellers to whom they might turn, and sellers recognize this[5].

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[4]It is important to note that despite the fact that the trading process is described as a game, the solution concept that is employed is not Nash. The solution uses competitive ideas that resemble the price taking assumptions of market theory.

[5]It is not difficult to convert this problem to a dynamic one in which buyers and sellers who

2.4. Mechanisms

It is assumed here, as in the rest of the literature[6], that sellers offer buyers anonymous direct mechanisms $\gamma : T^N \to \{\mathbf{R}\times[0,1]\}^{N+1}$ where $T \equiv [0,1] \cup \{x_\emptyset\}$ and $x_\emptyset$ is a message that tells the seller that the buyer decided not to participate. These mechanisms have the property that each vector of messages determines a sequence of transfers and trading probabilities $\{p_i, q_i\}$, one for each buyer $i = 1, \ldots N$. These mechanisms should be incentive compatible and satisfy a number of obvious restrictions (like the transfer assigned to a buyer who signals that he is not participating must be zero). Since these restrictions will be evident in context, discussion of them will be suppressed and reference will be made to the set , of feasible (though not necessarily incentive compatible) anonymous direct mechanisms.

Armed with this notation, it is possible to define the buyers' selection strategies $\pi^i : [0,1] \times ,^J \to \left\{\pi \in \mathbf{R}^{J+1} : \pi_j \geq 0; \sum_{j=0}^J \pi_j = 1\right\}$. In words, for each valuation x and each array of mechanisms that are offered by the sellers the seller chooses the probability π_j^i with which he will choose to participate in the mechanism offered by seller j. The notation π_0^i is interpreted as the probability with which the buyer chooses to not to participate in any mechanism at all.

In what follows attention will be focussed on symmetric continuation equilibria in which buyers all adopt the same mixed strategy π. This selection captures the buyers' inability to predict each others' participations decisions. This inability to coordinate is an important aspect of all decentralized matching models.

This symmetry assumption ensures that the distribution of types faced by any particular seller will satisfy the exchangeability requirements of Assumption 2.2. Exchangeability is a reasonable defense of anonymity in a monopoly problem. In a competitive environment, anonymity is a strong assumption. It prevents a seller from picking out a particular buyer and promising to treat him well if he participates in the seller's mechanism. The solution concept used in this paper does not require that sellers be able to predict individual buyer strategies. In this sense, it is consistent with an environment in which sellers have little idea about who the buyers are and how to contact them (though sellers certainly need to know something about the number of potential buyers). In such an environment,

fail to trade right away, can try to do so again in the following period. This extension is straightforward provided it is reasonable to assume that buyers and sellers believe that their current actions have no impact on future payoffs (which is reasonable in the large game environment that we study here).

[6] For example McAfee [7], Peters [?] or Peters and Severinov [11].

a restriction to anonymous mechanisms is reasonable.

The symmetry and anonymity assumptions together will ensure that it is sufficient to consider the problem from the point of view of buyer 1.

2.5. Seller's Beliefs

Let $Z_j(\cdot)$ denote the joint distribution of types that seller j expects to face. The process by which buyers choose among the various sellers imposes some important structure on seller's beliefs. This structure is used in the proof of the main theorem below. By assumption, buyer valuations are independently and identically distributed conditional on some outside variable, y. Thus conditional on y and the selection strategy used by all buyers including buyer 1, the probability that buyer 1 either has a valuation below x or chooses not to participate in the mechanism is given by $1 - \int_x^1 \pi_j(s) \tilde{f}(s|y)\, ds$.[7] Thus any equilibrium belief $Z_j(\cdot)$ should have the property that there is a choice strategy π_j such that

$$Z_j(x_1, \ldots x_n) = \int \prod_{i=1}^{N} \left[1 - \int_{x_i}^{1} \pi_j(s)\, d\tilde{F}(s|y)\right] dG(y) \tag{2.1}$$

Definition 2.1. *A belief function Z_j that satisfies (2.1) will be said to be* admissible.

One property of (2.1) that bears mention is that

$$Z_j(0, \ldots 0) = \int \prod_{i=1}^{N} \left[1 - \int_{0}^{1} \pi_j(s)\, d\tilde{F}(s|y)\right] dG(y) > 0$$

whenever $\pi_j(x) < 1$ on a set of non-zero measure. This means that the distribution Z_j will typically have atoms at zero. This simply reflects the fact that the seller should believe that a lot of potential buyers just won't turn up at all with high probability.

One final definition is required for the argument below. Assuming that Z_j is differentiable, the density function is given for every vector $(x_1, \ldots x_N) \in [0,1]^N$ by

$$z_j(x_1, \ldots x_N) = \int \prod_{i=1}^{N} \pi_j(x_i)\, \tilde{f}(x_i|y)\, dG(y)$$

[7] Here we suppress the fact that π_j depends on the array of mechanisms on offer to ease the notation.

From this description, it is apparent that a buyer of type x_i participates in seller j's mechanism with positive probability if $\pi_j(x_i) > 0$ or equivalently if

$$z_j(x_i, x_{i,} \ldots x_i) > 0$$

This suggests the following definition

Definition 2.2. *A buyer of type x_i is* expected to participate in seller j's mechanism *if $(x_i, x_{i,} \ldots x_i)$ is in the support of Z_j.*

Alternatively, symmetry in the choice strategy and the fact that the x_i are i.i.d conditional on y guarantees that all the marginal distributions of Z_j are the same. Hence it is legitimate to say that x_i is expected to participate in seller j's mechanism if x_i is in the support of the marginal distribution for Z_j.

2.6. Payoffs

Let $\gamma_j = (p_j, q_j)$ denote seller j's mechanism. Fix a common selection strategy π for buyers. This implies a distribution of buyer types at seller j given by some fixed Z_j. The expected payoff that buyer 1 gets by choosing to participate in the mechanism γ_j depends on his valuation x_1, and his beliefs about the other buyers, given by $Z_j(x_2, \ldots x_N | x_1)$. Say that a pair (γ_j, Z_j) is *incentive compatible* if γ_j is incentive compatible when each buyer who participates in seller $j's$ mechanism has beliefs $Z_j(\cdot|x)$. The payoff that buyers expect from the incentive compatible pair (γ_j, Z_j) is given by

$$\beta_j(x_1, Z_j, \gamma_j) = \mathbf{E}_{x_2, \ldots x_N | x_1} \{q_{1j}(x_1, x_2, \ldots x_N) x - p_{1j}(x_1, x_2, \ldots x_N)\} \tag{2.2}$$

where the expectation is taken using the conditional distribution associated with Z_j.

Taking Z_j to be the fixed belief by seller j about the distribution of types that he faces, the surplus that seller j enjoys from the incentive compatible mechanism γ_j is equal to

$$\sigma_j(\gamma_j, Z_j) =$$
$$N\mathbf{E}_x \{p_{1j}(x_1, \ldots x_N) - q_{1j}(x_1, \ldots x_N) c_j\}$$
$$= N \{\mathbf{E}_x \{[x_1 - c_j] q_{1j}(x_1, \ldots x_N)\}$$
$$-B_j(x_1, \ldots x_N)\}$$

$$= N \left\{ \mathbf{E}_x \left\{ [x_1 - c_j] \, q_{1j}(x_1, \ldots x_N) - \beta_j(x_1, Z_j, \gamma_j) \right\} \right\} \tag{2.3}$$

where

$$B_j(x_1, \ldots x_N) \equiv q_{1j}(x_1, \ldots x_N) \, x_1 - p(x_1, \ldots x_N)$$

is the expected payoff of a buyer of type x_1 given the valuations of the other buyers.

3. Equilibrium Mechanisms

When a monopoly seller raises price he weighs the gain he gets by reducing the surplus of buyers who continue to purchase from him at the higher price, against the lost profits from buyers who decide that they no longer wish to trade with him at the higher price. When there is competition the gain is mitigated because the existence of alternative suppliers limits the seller's ability to extract the surplus of buyers with higher types. In equilibrium, buyers in large markets can always find some other seller who is willing to offer the good at the original price. Thus if the seller wants to trade with buyers he must offer them the market price, or equivalently he must offer them the same payoff that they can get elsewhere in the market. This leads to the following definition:

Definition 3.1. A competitive equilibrium in mechanisms *is a payoff function* $\beta^*(\cdot)$, *an array of mechanisms* $\{\gamma_1^*, \ldots, \gamma_J^*\}$ *and a symmetric buyer strategy* $\pi^* = (\pi_0^*, \ldots, \pi_J^*)$ *such that*

1. for each j and each $(x_1, \ldots x_N)$

$$Z_j^*(x_1, \ldots x_N) = \int \prod_{i=1}^{N} \left[1 - \int_{x_i}^{1} \pi_j^*(s) \, f(s|y) \, ds \right] dG(y)$$

2. for each $j = 1, \ldots J$, $\left(\gamma_j^*, Z_j^*\right)$ is an incentive compatible pair and

$$\sigma_j\left(\gamma_j^*, Z_j^*\right) \geq \sigma_j(\gamma_j, Z_j)$$

   for any incentive compatible pair (γ_j, Z_j) satisfying $\beta_j(x, Z_j, \gamma_j) \geq \beta^*(x)$ for each type x who is expected under Z_j to participate in seller j's mechanism.

3. for all $x \in [0, 1]$

$$\beta^*(x) = \max_{j=1,J} \left[0, \beta_j\left(x; Z_j^*, \gamma_j^*\right) \right]$$

Note that the definition of Z_j^* along with condition (3) ensures that the buyers' common strategy π^* is a continuation equilibrium strategy for the subgame in which the buyers choose among the sellers. The market payoff β^* is the continuation equilibrium payoff associated with this.

A seller who considers deviating takes the market payoff to be fixed in calculating his best reply according to (2). The market payoff plays a role analogous to a market price which the sellers believe is beyond their control. Hence the title 'competitive equilibrium in mechanisms". As this market payoff adjusts to 'clear the market', sellers offer mechanisms to maximize profits against this fixed market payoff.

4. Characterization

To begin, the main result of McAfee and Reny is adapted to the model presented in this paper. The proof follows the argument in their paper closely, but needs to be adapted for the atoms that the distributions Z_j have when any of their arguments are zero.

Theorem 4.1. *Suppose that the joint distribution F of types satisfies Assumption 2.2. Let γ_j be any mechanism for firm j and let $Z_j(\cdot)$ be any admissible conjecture. Let $\beta_j(\cdot, Z_j, \gamma_j)$ be defined by (2.2) and let $\beta^*(\cdot)$ be any continuous function whose domain and range are both $[0,1]$. Then for any $\varepsilon > 0$ there exist a finite set of continuous fee (subsidy) schedules $\{s_k\}_{k=1,\ldots K}$ depending only on $x_{-1} = \{x_2, \ldots x_N\}$ such that*

$$\left|\min_k \mathbf{E}_{x_{-1}|x} s(x_{-1}) - (\beta_j(x; Z_j, \gamma_j) - \beta^*(x))\right| < \varepsilon$$

for all x in the support of the marginal distribution for Z_j.

Proof. See the Appendix ■

Remark 1. *The idea is that a seller who wishes to ensure participation in his auction can add these fees on to his existing mechanism to affect participation. The fees are random and the buyer who agrees to them does not know what the actual fees will be at the time that he decides to participate. So for example, if the buyer chooses to participate and latter finds that he is the only buyer who has done so, he will pay a fee (or possibly receive a subsidy) equal to $s_k(0, \ldots 0)$ where s_k is the schedule that he chose.*

Remark 2. *The payoff function $\beta^* (\cdot)$ in the statement of the theorem is intended to be the market payoff. However, when the theorem is stated in this more general way it is possible to imagine β^* as the market payoff plus some small positive constant δ. Then entry fees that get the buyer within ε of the market payoff plus δ are sure to give the buyer at least as much as the market payoff. This device is used below to get around the fact that the McAfee Reny theorem only gives approximate surplus extraction.*

5. Auctions

The result that follows is readily derived from Theorem 4.1 and (2.3). It illustrates the manner in which the surplus extraction result can be used in the competitive setting.

Lemma 5.1. *Suppose that the joint distribution of valuations satisfies Assumptions 2.2 and 2.2. Then in any equilibrium*

$$q_{j1}(x_1, \ldots x_N) = \begin{cases} 1 & \textit{if } x_1 > \max_{k \neq 1} x_k \\ 0 & \textit{if } \exists k : x_k > x_1 \\ \in [0,1] & \textit{otherwise} \end{cases} \tag{5.1}$$

for Z_j-almost all $(x_1, \ldots x_N)$.

Proof. In any equilibrium $\beta_j(x; Z_j, \gamma_j) \geq \beta^*(x) \forall x : z_j(x, \ldots x) > 0$. Using assumption 2.2, seller j's payoff can be written

$$N \int \left\{ \int_0^1 \cdots \int_0^1 \{q(x_1, \ldots x_N) x_1 - c_j - \beta_j(x_1; Z_j, \gamma_j)\} d\tilde{Z}_j(x_1|y) \ldots d\tilde{Z}_j(x_N|y) \right\} dG(y) \tag{5.2}$$

Suppose that seller j replaces the mechanism $\{q_j(\cdot), p_j(\cdot)\}$ with the second price auction where q^a is given by (5.1) and

$$p^a(x_1, \ldots x_N) = \begin{cases} 0 & \text{if } q(x_1, \ldots x_N) = 0 \\ \max_{k \neq 1} x_k & \text{otherwise} \end{cases}$$

Let $\beta^a(x)$ be the payoff associated with the auction assuming that the seller believes that the probability distribution $Z_j(\cdot)$ of valuations for buyers will be the same for this auction as it was for his original mechanism (which means that

Z_j is admissable). Let $\alpha(x) = \beta^a(x) - \beta_j(x) - \delta$ where $\delta > 0$ is small. By Theorem 4.1, for any $\varepsilon > 0$, there exists a finite set of entry fees $\{s_k(\cdot)\}_{k=1,K}$ depending only on $(x_2, \ldots x_N)$ such that

$$\left|\min_k \mathbf{E}_{x_{-1}|x} s_k(x_2, \ldots x_N) - \alpha(x)\right| < \varepsilon$$

for all x in the support of the marginal distribution for Z_j. If the seller's conjecture Z_j is true, then the buyer's payoff in the second price auction augmented by the choice of a random entry fee is given by

$$\beta^a(x) - \min_k \mathbf{E}_{x_{-1}|x} s_k(x_2, \ldots x_N) \geq \beta^a(x) - \alpha(x) - \varepsilon$$

$$= \beta_j(x) + \delta - \varepsilon \geq \beta_j(x)$$

The last inequality follows from the fact that ε can be chosen to be arbitrarily small. Since this augmented second price auction yields every buyer type who selects it with positive probability a payoff that is no smaller than the market payoff, the seller's profits under this new mechanism are

$$N \int \left\{ \int_0^1 \left\{ x_1 \tilde{Z}_j^{N-1}(x_1|y) - c_j - \beta_j(x_1) \right\} d\tilde{Z}_j(x_1|y) \right\} dG(y) - \delta \tag{5.3}$$

Since δ can be chosen arbitrarily small, and since q^a strictly increases the integrand in (5.2) for each $(x_1, \ldots x_N)$ such that q differs from q^a, this will exceed the value given by (5.2) if q differs from q^a on a set that has non-zero probability under Z_j. ■

This theorem says roughly that for any vector of valuations that the seller might observe with positive probability, the seller's mechanism must award trade to the buyer with the highest valuation, as in an auction. The reason is simply that the entry fees allow the seller to extract all the additional surplus that is created by this change in his mechanism. Analytically this is advantageous because it makes it possible to use (5.3) to represent seller's profits.

The main characterization theorem, however, is given by the following.

Theorem 5.2. *In any competitive equilibrium in mechanisms* $\left\{\beta^*(\cdot), Z_1^*, \ldots Z_j^*, \gamma_1^*, \ldots \gamma_j^*\right\}$, *seller j's payoff is given by*

$$N \int \left\{ \int_0^1 \left\{ x_1 Z_j^{*N-1}(x_1|y) - c_j - \beta^*(x_1) \right\} dZ_j^*(x_1|y) \right\} dG(y)$$

Furthermore, the pair $\left(\gamma_j^*, Z_j^*\right)$ maximizes seller j's profits given β^* relative to the set of all direct mechanisms if and only if there does not exist a buyer strategy π' such that

$$\hat{Z}(x|y) = 1 - \int_x^1 \pi'(s) f(s|y)\, ds \tag{5.4}$$

and

$$N \int \left\{ \int_0^1 \left\{ x_1 \hat{Z}_j^{N-1}(x_1|y) - c_j - \beta^*(x_1) \right\} d\hat{Z}_j(x_1|y) \right\} dG(y) >$$
$$N \int \left\{ \int_0^1 \left\{ x_1 Z_j^{*N-1}(x_1|y) - c_j - \beta^*(x_1) \right\} dZ_j^*(x_1|y) \right\} dG(y) \tag{5.5}$$

Proof. The Proof is similar to the proof of the previous Lemma and is relegated to the appendix ■

This theorem simplifies the problem of characterizing equilibrium. Suppose that firms compete in 'levels of demand' Z_j instead of in mechanisms. A series of demands $\{Z_1, \ldots Z_J\}$ that maximize

$$N \int \int_0^1 \left\{ [x_1 - c]\, Z_j^{N-1}(x_1|y) - \beta^*(x_1) \right\} dZ_j(x_1|y)\, dG(y) \tag{5.6}$$

for each seller will, according to the 'if' part of Theorem 5.2, coincide with a series of second price auctions with entry fees that maximize sellers profits relative to the entire class of direct mechanisms. According to the 'only if' part of the Theorem, a series of mechanisms and associated beliefs $\left\{\gamma_j^*, Z_j^*\right\}$ that all constitute best replies to the market payoff β^* must all maximize (5.6). Thus the equilibrium mechanism problem can be reduced to the problem of characterizing an equilibrium in which firms compete in 'levels of demand' (Z_j).

A *symmetric competitive equilibrium* is simply a competitive equilibrium where all sellers offer the same mechanism. The characterization result given above is used to prove the following result.

Theorem 5.3. *Suppose that all sellers have a common cost c. Then there exists a unique symmetric competitive equilibrium in mechanisms in which all sellers offer second price auctions with reserve prices equal to c and zero entry fees.*

Proof. Begin with existence. Define the selection rule $\pi_j(x) = 1/J$ for all $x \geq c$, and $\pi_j(x) = 0$ otherwise. This selection rule generates the beliefs

$$Z^*(x|y) = \begin{cases} 1 - \frac{1}{J}\left[1 - F(x|y)\right] & \text{if } x \geq c \\ 1 - \frac{1}{J}\left[1 - F(c|y)\right] & \text{otherwise} \end{cases} \tag{5.7}$$

The distribution function $Z^*(\cdot) = \int Z^*(\cdot|y)\, dG(y)$ is admissible by construction and therefore satisfies condition (1) in the definition of a competitive equilibrium in mechanisms.

Next define the function $\beta^*(x|y)$ such that $\beta^*(x|y) = 0$ if $x \leq c$ and

$$\beta^{*\prime}(x|y) = Z^*(x|y)^{N-1}$$

otherwise. Finally let $\beta^*(x) = \int \beta^*(x|y)\, dG(y|x)$.

With β^* defined this way, we now show that $\int Z^*(x|y)\, dG(y)$ maximizes (5.6). Rewriting (5.6) using the definition of the conditional expectation gives

$$N \int \int_0^1 \left\{ [x-c]\, Z^{N-1}(x|y) - \beta^*(x|y) \right\} dZ(x|y)\, dG(y) \tag{5.8}$$

Consider the problem of maximizing this by choosing some function $Z(x|y)$ constrained only to be non-negative. From the fact that $\beta^*(x|y) = 0$ for $x < c$ and the fact that $Z^{N-1}(x|y) \geq 0$ for all x, it is apparent that a necessary condition for maximization of the expression (5.8) in Z is that $dZ(x|y) = 0$ for all $x < c$. Note that $Z^*(x|y)$ has this property by construction for all y. Since $\beta^*(x|y)$ is differentiable by construction, (5.8) can be integrated by parts to get

$$1 - c - N\beta^*(1) - \int \int_0^1 \left[Z^{*N}(x|y) - NZ^*(x|y)\, \beta^{*\prime}(x|y) \right] dx dG(y) \tag{5.9}$$

Since $Z(x|y)$ is constrained only to be non-negative, the function can be maximized by selecting the function $Z(x|y)$ that minimizes the integral pointwise. The first order necessary condition for this is $Z^{N-1}(x|y) = \beta^{*\prime}(x|y) = Z^{*N-1}(x|y)$. Since the integrand is convex in Z for any fixed x, this necessary condition is also sufficient. It follows that $Z^*(x|y)$ minimizes the integrand in the above expression point-wise in x and y for $x \geq c$ when the buyer's payoff is given by β^*. Thus Z^* must maximize (5.6).

It remains to calculate the entry fee that generates the payoff β^* for buyers when beliefs are Z^*. This can be accomplished by calculating the payoff to participating in the auction then finding the difference between this payoff and the market payoff β^*. Let $\beta(x, Z^*|y)$ be the expected payoff earned in a second price auction with reserve price c by a buyer of type x *conditional on* y. Since the auction awards the good to the buyer with the highest valuation uniformly in y, and since buyer valuations are independently distributed condition on y, the argument in [12] gives

$$\beta(x, Z^*|y) = \int_c^x Z^{*N-1}(s|y)\, ds = \int_c^x \beta^{*\prime}(s|y)\, ds = \beta^*(x|y)$$

where the second equality is by construction and the third is by definition. Thus the payoff from the second price auction is equal to the market payoff and the desired entry fee is uniformly 0.

Since the payoff that the buyers get with every seller is $\beta^*(\cdot)$ in the symmetric equilibrium, condition (3) in the definition of equilibrium is also satisfied. Thus the market payoff β^* along with the array consisting of J second price auctions is a competitive equilibrium in mechanisms.

To show uniqueness, suppose that there is a second equilibrium in which the market payoff is $\beta^{**}(x)$ with associated conditional payoffs $\beta^{**}(x|y)$. By Theorem 5.2 we can assume that sellers offer second price auctions with reserve prices equal to $\hat{c}$ in this equilibrium. Thus sellers' profits are given by

$$N\int\int_{\hat{c}}^{1}\left\{[x-c]\,Z^{N-1}(x|y)-\beta^{**}(x|y)\right\}dZ(x|y)\,dG(y)$$

From this expression and Theorem 5.2, we can take $\hat{c}=c$ without loss of generality.[8] Integrating this by parts gives

$$\int\left\{[x-c]\,Z(x|y)^{N}\Big|_{c}^{1}-\int_{c}^{1}Z(s|y)^{N}\,ds\right\}dG(y)$$

$$-\int N\left\{\beta^{**}(x|y)\,Z(x|y)|_{c}^{1}-\int_{c}^{1}Z(s|y)\,d\beta^{**}(s|y)\right\}dG(y)$$

$$=\int\left\{(1-c)-\int_{c}^{1}Z(s|y)^{N}\,ds\right\}dG(y)$$

$$-\int N\left\{\beta^{**}(1|y)-\beta^{**}(c|y)\,Z(c|y)-\int_{c}^{1}Z(s|y)\,d\beta^{**}(s|y)\right\}dG(y)$$

Since $\beta^{**}(c|y)=0$ for all y, by the definition of the reserve price, this expression will be maximized whenever the expression

$$\int\left\{\int_{c}^{1}Z^{N}(s|y)\,ds-N\int_{c}^{1}Z(s|y)\,d\beta^{**}(s|y)\right\}dG(y)$$

is minimized. This expression can be written

$$\int\left\{\int_{c}^{1}\left[1-\int_{x}^{1}\pi(s)\,f(s|y)\,ds\right]^{N}dx-N\int_{c}^{1}\left[1-\int_{x}^{1}\pi(s)\,f(s|y)\,ds\right]d\beta^{**}(x|y)\right\}dG(y)$$

[8] It is clear from this expression that the seller would not want a reserve price below $\hat{c}$. If he wants a reserve price $\hat{c}>c$ he can take $dZ(x|y)=0$ for $x\in[c,\hat{c}]$.

In any symmetric equilibrium sellers offer the same mechanism and share the same beliefs about the distribution of types who will participate. Thus the conditional beliefs $Z_j(x|y)$ and $Z_k(x|y)$ for any two firms must be the same, or

$$\left[1-\int_x^1 \pi_j(s) f(s|y)\,ds\right]=\left[1-\int_x^1 \pi_k(s) f(s|y)\,ds\right]$$

Since this must be true for all x, we must have $\int_x^{x'}(\pi_j(s)-\pi_k(s)) f(s|y)\,ds$ on every interval $[x,x']$. This implies that $\pi_j(x)=\pi_k(x)$ almost everywhere, or by the summing up restriction, $\pi_j(x)=1/J$ for $x\geq c$. This implies that in any symmetric equilibrium, beliefs must be given by Z^* as defined by (5.7).

Theorem 5.2 implies that if β^{**} is an equilibrium payoff, then there is no admissible belief that yields the seller a higher payoff than that associated with Z^*. Thus the buyer strategy $\pi(x)=\frac{1}{J}$ should satisfy the necessary conditions for unconstrained optimization. That is, for all w,

$$\int\left\{\int_c^w -N\left[1-\int_x^1 \pi(s) f(s|y)\,ds\right]^{N-1} f(w|y)\,dx+N\int_c^w f(w|y)\,d\beta^{**}(x|y)\right\}dG(y)=0$$

or using the fact that $\pi(s)=1/J$ in a symmetric equilibrium,

$$\int\left\{\int_c^w -N\left[1-\frac{1-F(x|y)}{J}\right]^{N-1} dx+N\int_c^w d\beta^{**}(x|y)\right\} f(w|y)\,dG(y)=0$$

Using the fact that $\left[1-\frac{1-F(x|y)}{J}\right]^{N-1}=\beta^{*\prime}(x|y)$ from the first part of the proof gives

$$\int\left\{\int_c^z -N\beta^{*\prime}(x|y)\,dx+N\int_c^z d\beta^{**}(x|y)\right\} f(z|y)\,dG(y)=0$$

Using the fact that $\beta^*(c|y)=0$ for all y gives

$$\int\left\{\beta^{**}(z|y)-\beta^*(z|y)\right\} f(z|y)\,dG(y)=0$$

which gives

$$\beta^{**}(z)=\beta^*(z)$$

for all z. This contradiction gives the result. ■

The existence of an equilibrium without entry fees is perhaps not so surprising - in a large market sellers need to conform. This argument can be readily generalized to the case where sellers have different costs so that the equilibrium is not symmetric. Uniqueness on the other hand, is more difficult to prove. The argument involves two ideas. First, from Lemma 5.1, attention can be restricted to situations in which sellers offer 'efficient' mechanisms. In a symmetric equilibrium these mechanisms are the same so that buyers will choose every seller with the same probability. Since the seller will need to understand this in equilibrium, this determines the seller's beliefs in such an equilibrium. Theorem 5.2 then insures that the beliefs that this generates will have to be optimal given the market payoff β^* and this ultimately ties down the market payoff function β^*. This argument makes use of the symmetry assumption.

It is important to note, when interpreting this result, that the entry fees are not being bid away in the way that profits are in a Bertrand equilibrium. Sellers who offer second price auctions with reserve price equal to their costs still earn positive profits. The trade-off that each seller faces at the margin is conventional. To raise the probability of being chosen by buyers of type x the seller needs to offer exactly the market payoff $\beta^*(x)$. This is exactly offset by the increase in surplus enjoyed by the seller at the margin when the other sellers are offering second price auctions. The seller could attract certain buyer types with probability 1. The drawback is that when the seller is attracting a particular buyer type with very high probability, he must still pay the market payoff to attract another buyer of that type. Since the new buyer will face a lot of competition, it is unlikely the seller will be able to recoup this payoff by trading with the new buyer.

6. Conclusions and Problems

The interpretation offered by [8] for their surplus extraction result is a negative one. Since the complicated entry fees that are required to implement the optimal mechanism are essentially never observed in practice, the theory of optimal mechanism design is irrelevant. This paper shows that the complicated entry fees that the theory predicts disappear in the presence of competition among the sellers. The fees are essentially competed away, leaving second price auctions with reserve prices set equal to sellers's opportunity costs. This prediction is much closer to the auctions that are observed in practice.

The result leaves open a number of issues. The assumption that sellers behave competitively seems defensible on it's own merits. The informational requirements of a full game theoretic treatment seem inappropriate to many competitive environments. A competitive model in which sellers have little specific information about other sellers and about their potential customers, seems well suited to markets like the real estate market, or to financial markets. Nonetheless, the relationship between the competitive solution and the full game theoretic one is of some interest.

It has been shown that the impact that sellers have on market payoffs shrinks as the economy gets large ([10]) when valuations are independent and sellers are required to use direct mechanisms. I do not know if similar results are true for the case where valuations are correlated. Furthermore the existing mathematical results are not sufficient to conclude that competitive converge to exact equilibria, or that competitive equilibria will be approximate equilibria when the number of traders is large even in the case of independent valuations, though it would seem plausible that both conjectures ought to be true under reasonable conditions.

It seems likely that when the number of buyers and sellers is small enough, there will still be room for entry fees in equilibrium. The study of non-competitive environments where there are small numbers of sellers is complicated by two major problems. First, it is known that when sellers are restricted to competition in auctions and valuations are independent, pure strategy equilibria typically do not exist when the number of sellers is small ([7]). This is partly due to the inherent complexity of the two stage interaction in which sellers have to anticipate the impact that their offers have on buyers' continuation equilibrium behavior..

Secondly, the appropriate set of mechanisms for sellers to use in the small numbers problem is *not* the set of 'direct' mechanisms as has been described in this paper ([4]). Sellers can learn about the mechanisms that have been offered by their opponents by asking buyers to report this information to them. Forcing sellers to use simple direct mechanisms in which buyers report their willingness to pay is therefore restrictive. Though it is possible to describe the appropriate set of mechanisms, not enough is currently known about this set to make qualitative predictions about what equilibrium mechanisms look like.

7. Appendix

7.1. Proof of Theorem 4.1

Proof. The belief Z_j is admissible so that there is a buyer strategy π_j such that

$$Z_j\left(x_1,\ldots x_n\right)=\int\prod_{i=1}^{N}\left[1-\int_{x_i}^{1}\pi_j\left(s\right)d\tilde{F}\left(s|y\right)\right]dG\left(y\right)$$

The theorem is vacuously satisfied when $\pi_j(x) = 0$ Lebesque almost everywhere, hence we can assume without loss of generality that $\pi_j(x) > 0$ on a set of strictly positive Lebesque measure.

Secondly, note that (again by the admissibility of Z_j) that each term in the expansion of the conditional distribution

$$Z_j(x_1, \ldots, x_n|y) = \prod_{i=1}^{N} \left[1 - \int_{x_i}^{1} \pi_j(s)\, d\tilde{F}(s|y)\right]$$

can be decomposed into a mass point at 0 and a distribution that is absolutely continuous with respect to Lebesque measure elsewhere. Define

$$\tilde{Z}_j(0|y) = \left[1 - \int_0^1 \pi_j(s)\, d\tilde{F}(s|y)\right]$$

and let $\tilde{z}_j(x|y)$ denote the (Radon-Nikodym) derivative of this conditional distribution . This is almost everywhere equal to $\pi_j(x) f(x|y)$.

Finally, observe that incentive compatibility guarantees that $\beta_j(x; Z_j, \gamma_j)$ is continuous. Since the payoff function β^* is continuous as part of the hypothesis of the theorem, $(\beta_j(x; Z_j, \gamma_j) - \beta^*(x_1)) \equiv \alpha(x_1)$ is continuous and hence in $C[0,1]$.

Following [8], define

$$r(Z_j) = \left\{ y \in C[0,1] : \left(\exists K, \{s_k\}_{k=1,\ldots K}\right) (\forall x \in [0,1])\, y(x) = \min_k \mathbf{E}_{x_{-1}|x} s_k(x_{-1}) \right\}$$

The expectation in this expression is given by

$$\mathbf{E}_{x_{-1}|x} s(x_{-1}) = \int s(x_2, \ldots x_N)\, dZ_j(x_2, \ldots x_N|x)$$

$$= \int \int_0^1 \cdots \int_0^1 s(x_2, \ldots x_N)\, d\tilde{Z}_j(x_2|y) \ldots d\tilde{Z}_j(x_N|y)\, dG(y|x)$$

$$= \int \tilde{Z}_j(0|y)^{N-1} s(0, \ldots 0)\, dG(y|x) +$$

$$\int \sum_{i=1}^{N-1} \binom{N-1}{i} \tilde{Z}_j(0|y)^{N-i-1} \times$$

$$\left\{ \int_0^1 \cdots \int_0^1 \{s(x_2 \ldots x_{i+1}, 0, \ldots 0)\}\, \tilde{z}_j(x_2|y) \ldots \tilde{z}_j(x_{i+1}|y)\, dx_2 \ldots dx_i \right\} dG(y|x)$$

The assertion in the Theorem amounts to showing that α lies in the closure of $r(Z_j)$. McAfee and Reny [8] provide restrictions on Z_j such that the closure of

$r(Z_j) = C[0,1]$. There are two problems to resolve before applying their theorem: Z_j has atoms when any of it's arguments are zero, and Z_j depends indirectly on an endogenous variable. The argument in this proof simply provides a slightly modified version of the part of their argument that depends on these things.

Define

$$U(\varepsilon, \delta, x_0) =$$
$$\{u \in C[0,1] : (u(x) \geq 0 \forall t \in [0,1]) ; (u(x_0) \leq \varepsilon) ; (u(x) \geq 1 \forall x : |x - x_0| > \delta)\}$$

It is straightforward to show that $r(Z_j)$ satisfies the restrictions (2.7) through (2.10) of [8] Theorem 1 (pp 403).[9] This same theorem shows that if $U(\varepsilon, \delta, t)$ has a non-empty intersection with the closure of $r(Z_j)$ for all $\varepsilon > 0$, $\delta > 0$, and $t \in [0,1]$ then the closure of $r(Z_j)$ will be equal to $C[0,1]$.

If theorem 4.1 is false, then this cannot be true, and so there must be some ε_0, δ_0, and x_0 such that the intersection of $U(\varepsilon_0, \delta_0, x_0)$ and the closure of $r(Z_j)$ is empty. Define

$$R(Z_j) = \left\{ y \in C[0,1] : (\exists s)(\forall x \in [0,1])\, y(x) = \mathbf{E}_{x_{-1}|x} s(x_{-1}) \right\}$$

It is apparent from the definition that $R(Z_j)$ is a linear subspace of $C[0,1]$ that is entirely contained in $r(Z_j)$. Thus $R(Z_j)$ contains no interior point in common with $U(\varepsilon_0, \delta_0, x_0)$ which is readily seen to be convex, and to have a non-empty interior ([8, p402,ftnote 7]). Thus by the separating hyperplane theorem[6, Theorem 1, page 133], there exists an additive set function μ whose total variation is finite and a constant c having the property that

$$\int y(t)\, d\mu(t) = c \;\forall y \in R(Z_j) \tag{7.1}$$

and

$$\int y(t)\, d\mu(t) < c \tag{7.2}$$

for all y in the interior of $U(\varepsilon_0, \delta_0, x_0)$. From (7.2) and the definition of $U(\varepsilon_0, \delta_0, x_0)$, the function $\mu(\cdot)$ is non-zero.. Since $R(Z_j)$ is in fact a linear subspace, $c = 0$, otherwise for $\lambda > 1$

$$\int \lambda y(t)\, d\mu(t) \neq c$$

[9]The only difference between the $r(Z)$ here and the one in their paper is the atoms in the distribution used to take expectations. It is easy to see that the properties (2.7) through (2.10) follow from the definition of $r(Z)$ and not from any properties of the distribution used to define it.

Thus by definition

$$\int\int\int_0^1\cdots\int_0^1 s\left(x_2,\ldots x_N\right)dZ_j\left(x_2,\ldots x_N|x\right)d\mu\left(x\right)=0$$

for all $s\left(\cdot\right)\in C\left[0,1\right]^{N-1}$. Using the definition of Z_j gives

$$\int\int\tilde{Z}_j\left(0|y\right)^{N-1}s\left(0,\ldots 0\right)g\left(y|x\right)dyd\mu\left(x\right)+$$

$$\int\int\sum_{i=1}^{N-1}\left(\begin{array}{c}N-1\\ i\end{array}\right)\tilde{Z}_j\left(0|y\right)^{N-i-1}\cdot$$
$$\left\{\int_0^1\cdots\int_0^1\left\{s\left(x_2\ldots x_{i+1},0,\ldots 0\right)\right\}\tilde{z}_j\left(x_2|y\right)\ldots\tilde{z}_j\left(x_{i+1}|y\right)dx_2\ldots dx_i\right\}g\left(y|x\right)dyd\mu\left(x\right)$$

$$=0$$

Now using Fubini's theorem this implies

$$s\left(0,\ldots 0\right)\int\int\tilde{Z}_j\left(0|y\right)^{N-1}g\left(y|x\right)dyd\mu\left(x\right)+\sum_{i=1}^{N}\left(\begin{array}{c}N-1\\ i\end{array}\right)\cdot$$

$$\cdot\int_0^1\cdots\int_0^1\left\{s\left(x_2\ldots x_{i+1},0,\ldots 0\right)\right\}\times$$

$$\int\int\tilde{Z}_j\left(0|y\right)^{N-i-1}\tilde{z}_j\left(x_2|y\right)\ldots\tilde{z}_j\left(x_{i+1}|y\right)g\left(y|x\right)dyd\mu\left(x\right)dx_2\ldots dx_i$$

$$=0$$

This must be true for all $s\left(\cdot\right)\in C\left[0,1\right]^{N-1}$, so that

$$\int\int\tilde{Z}_j\left(0|y\right)^{N-1}g\left(y|x\right)dyd\mu\left(x\right)=0$$

and

$$\int\int\tilde{Z}_j\left(0|y\right)^{N-i-1}\tilde{z}_j\left(x_2|y\right)\ldots\tilde{z}_j\left(x_{i+1}|y\right)g\left(y|x\right)dyd\mu\left(x\right)=0$$

for all $i=1,\ldots N-1$. Let μ^+ and μ^- be the Jordan Hahn decomposition [2, Theorem 5-1G,p53] of μ. Since $\mu^+-\mu^-=\mu$, this implies

$$\int\int\tilde{Z}_j\left(0|y\right)^{N-1}g\left(y|x\right)dyd\mu^+\left(x\right)=\int\int\tilde{Z}_j\left(0|y\right)^{N-1}g\left(y|x\right)dyd\mu^-\left(x\right)$$

and

$$\int\int \tilde{Z}_j(0|y)^{N-i-1}\tilde{z}_j(x_2|y)\ldots\tilde{z}_j(x_{i+1}|y)\,g(y|x)\,dyd\mu^+(x) =$$

$$\int\int \tilde{Z}_j(0|y)^{N-i-1}\tilde{z}_j(x_2|y)\ldots\tilde{z}_j(x_{i+1}|y)\,g(y|x)\,dyd\mu^-(x) \tag{7.3}$$

for $i = 1, \ldots N-1$. Summing these equalities over i gives

$$\int dZ(x_2, \ldots x_N|x)\,d\mu^+(x) \equiv \int\int \sum_{i=1}^{N-1} \tilde{Z}_j(0|y)^{N-i-1} \prod_{k=2}^{i} \tilde{z}_j(x_k|y)\,g(y|x)\,dyd\mu^+(x)$$

$$= \int\int \sum_{i=1}^{N-1} \tilde{Z}_j(0|y)^{N-i-1} \prod_{k=2}^{i} \tilde{z}_j(x_k|y)\,g(y|x)\,dyd\mu^-(x) = \int dZ(x_2, \ldots x_N|x)\,d\mu^-(x) \tag{7.4}$$

Integrating both sides over $(x_2, \ldots x_N)$ yields

$$\int d\mu^+(x) = \int d\mu^-(x) \tag{7.5}$$

Since μ has bounded variation, it is finite [2, Theorem 51B,p 51], hence both μ^+ and μ^- are finite. Thus it is readily verified that if μ satisfies 7.1 and 7.2, then so does $\lambda\mu$ for any $\lambda > 0$. Hence the integrals in (7.5) can both be taken to be equal to 1 without loss of generality, implying that both μ^+ and μ^- can be interpreted as probability measures.

It is next shown that neither μ^+ nor μ^- can have all of their mass concentrated at some x^0. For suppose the contrary that μ^+ is a point mass at x^0. By the Jordan Hahn decomposition there is a set $A \subset [0,1]$ such that $\mu^+(A) = 1$ and $\mu^-(A^c) = 1$. Thus if μ^+ is a point mass at x^0, μ^- cannot be (and conversely). So set $i = N-1$ in condition (7.3) to get

$$\int\int \prod_{k=2}^{N} \tilde{z}_j(x_k)\,g(y|x)\,dyd\mu^+(x) =$$

$$\int \prod_{k=2}^{N} \tilde{z}_j(x_k)\,g\left(y|x^0\right)dy =$$

$$\prod_{k=2}^{N} \pi_j(x_k) \int \prod_{k=2}^{N} \tilde{f}(x_k|y)\,g\left(y|x^0\right)dy \neq$$

$$\prod_{k=2}^{N} \pi_j(x_k) \int\int \prod_{k=2}^{N} \tilde{f}(x_k|y)\,g(y|x)\,dyd\mu^-(x) =$$

$$\int \int \prod_{k=2}^{N} \tilde{z}_j (x_k) \, g(y|x) \, dy d\mu^- (x)$$

where the second equality follows from the fact that Z_j is admissible and the inequality follows from the fact that F satisfies Assumption 2.2 and the assumption that $\pi_j(x)$ is strictly positive on a set of positive Lebesque measure. This inequality violates condition (7.3) so we conclude that μ^+ cannot have unit mass at x_0. The same argument shows that μ^- cannot be a point mass at x_0.

Finally, there exists a set $B \subset A$, (where $\mu^+(A) = 1$ as in the previous paragraph) such that $\mu^+(B) > 0$ and $x^0 \notin B$ (for if this were false, μ^+ would be a point mass at x^0). Now use the set B and follow the argument in [8, p406] to verify the existence of a function $y \in U(\varepsilon_0, \delta_0, x_0)$ such that (7.2) is violated, a contradiction. ■

7.2. Proof of Theorem 5.2

Proof. The characterization of the seller's payoff and the 'if' part of the second assertion follow from Lemma 5.1 and the fact that the seller's conjecture Z_j must be admissible (i.e. must satisfy (2.1) in equilibrium).

To show the 'only if' part, suppose to the contrary that there is a strategy rule π' satisfying (5.4) and (5.5). We will show that by appropriately choosing (or modifying) the entry fee, the seller can ensure that the probability distribution induced by π' is admissible. The argument mimics the one given in Lemma 5.1. Let $\hat{Z}_j$ be the distribution induced by π'. Let $\beta_j^a(x)$ denote the payoff that a buyer of type x gets by participating in the seller's mechanism when the distribution of types is $\hat{Z}_j$. Define $\alpha(x) = \beta_j^a(x) - \beta^*(x) - \delta$ where $\delta > 0$ is small. Since the profits associated with π' are given by

$$N \int \left\{ \int_0^1 \left\{ x_1 \hat{Z}_j^{N-1}(x_1|y) - c_j - \beta^*(x_1) \right\} d\hat{Z}_j(x_1|y) \right\} dG(y) =$$

$$N \int \left\{ \int_0^1 \left\{ x_1 \hat{Z}_j^{N-1}(x_1|y) - c_j - \beta^*(x_1) \right\} \pi'(x_1) f(x_1|y) \, dx_1 \right\} dG(y)$$

the inequality implied by the contrary hypothesis cannot be true unless $\pi'(x) > 0$ on a set of non-zero (Lebesque) measure. Then by Theorem 4.1, for any $\varepsilon > 0$, there exists a finite sequence of entry fees $\{s_k(\cdot)\}_{k=1,K}$ depending only on $(x_2, \ldots x_N)$ such that

$$\left| \min_k \mathbf{E}_{x_{-1}|x} s_k(x_2, \ldots x_N) - \alpha(x) \right| < \varepsilon$$

for all x for which $\pi'(x) > 0$, where the expectation is taken using the distribution $\hat{Z}_j$. If the seller's conjecture $\hat{Z}_j$ is true, then the buyer's payoff in the mechanism augmented by this random entry fee is given by

$$\beta^a(x) - \min_k \mathbf{E}_{x_{-1}|x} s_k(x_2, \ldots x_N) \geq \beta^a(x) - \alpha(x) - \varepsilon$$

$$= \beta^*(x) + \delta - \varepsilon \geq \beta^*(x)$$

The last inequality follows from the fact that ε can be chosen to be arbitrarily small. This verifies that the conjecture $\hat{Z}_j$ is admissible.

As δ can be taken arbitrarily small, the seller's profits are arbitrarily close to

$$N \int \left\{ \int_0^1 \left\{ x_1 \hat{Z}_j^{N-1}(x_1|y) - c_j - \beta^*(x_1) \right\} d\hat{Z}_j(x_1|y) \right\} dG(y) >$$

$$N \int \left\{ \int_0^1 \left\{ x_1 \tilde{Z}_j^{N-1}(x_1|y) - c_j - \beta^*(x_1) \right\} d\tilde{Z}_j(x_1|y) \right\} dG(y) \geq$$

$$N \int \left\{ \int_0^1 \left\{ x_1 \tilde{Z}_j^{N-1}(x_1|y) - c_j - \beta_j(x_1) \right\} d\tilde{Z}_j(x_1|y) \right\} dG(y)$$

The latter inequality follows from the fact that $\beta_j(x) \geq \beta^*(x)$. This contradicts condition (2) in the definition of the competitive equilibrium in mechanisms. ■

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