Department of Economics
and
Institute for Policy Analysis
University of Toronto
150 St. George St.
Toronto, Canada
M5S 1A1

WORKING PAPER
NUMBER UT-ECIPA-PETERS-04

Competition Among Mechanism Designers in a Common Value Environment

Michael Peters
Department of Economics
University of Toronto
150 St. George St.
Toronto, Canada
M5S 1A1

Preliminary Draft-First version December 5,1995,
this version April 15, 1996

ISSN 0829-4909
Author's email: peters@epas.utoronto.ca
On-line Version: http://www.epas.utoronto.ca:8080/ecipa/wpa.html

Abstract

A competitive economy is studied in which sellers offer alternative direct mechanisms to buyers who have private information about their own private use value for the commodity being traded. In addition the commodity has a common value to all buyers, perhaps represented by the future resale value of the commodity. A competitive equilibrium in mechanisms is described. In every such equilibrium it is shown that sellers must offer mechanisms that are allocationally equivalent to English ascending price auctions. The reservation prices that sellers set are equal to the ex post value of a 'marginal' unsold unit. This value is below the ex post opportunity cost to the seller of trading the commodity.

Journal of Economic Literature Classification Numbers: D82,D83, D44

1. Introduction

The purpose of this paper is to illustrate some of the effects of competition among sellers in a simple common value trading environment. In the market that is considered, sellers are allowed to offer whatever *direct* mechanisms that they wish to buyers. The problem differs from that considered in a standard mechanism design problem in that the number of buyers that a seller expects to attract to his mechanism depends on the mechanism that the seller offers. The common value assumption distinguishes the paper from the rest of the small literature on competition among mechanism designers.

There are many examples of competitive markets in which the mechanism design problem is non-trivial, and in which there is a common value aspect to the good being traded. The real estate market may be the best example, since houses share a common resale value to all buyers. The real estate market is also intensely competitive, so that participation by buyers in a particular seller's selling scheme is endogenous.

In financial markets, brokers may be competing with one another to sell large blocks of shares. These shares will have a resale value that is common to the seller and all the potential buyers who wish to bid on the block. Shares in different firms can be relatively good substitutes for one another, so this market will also be competitive.

In labor markets, firms offer jobs to workers who have varying outside opportunities. The future bargaining power of the worker might depend on the possibility that some competing firm might try to hire the worker away from his existing firm some time in the future. Thus the surplus that the worker and firm can expect depend on the valuations and opportunities of other workers and firms in the market.

No attempt is made in this paper to model this future trading opportunity explicitly. Rather it is simply assumed that the value that buyers and sellers attach to trade depends on the tastes of all the buyers and sellers in the market.

Common value environments are complex analytically. Rather than restricting sellers to the use of specific selling mechanisms, two simplifying assumptions are made in order to extend the discussion to equilibrium in selling mechanisms. First it is assumed that traders' types are unconditionally independently distributed[1].

[1]The more common assumption in the literature on common value auctions is that buyers types are conditionally independent, i.e., independent conditional on the common value of the object.

Though each trader's valuation depends on the types of all of the other traders, these types are completely uncorrelated[2].

Secondly, the preferences of a buyer and seller who trade depend on the types of the other buyers and sellers in the market in an additively separable way. Again the possibility of re-trade is the motivation for this second assumption. A buyer and seller who trade each get a payoff from holding the object for one period that depends on their own private valuation. Then the common future resale value is determined by the tastes of the other buyers and sellers in the market.

The key result is that all competitive equilibrium are allocationally equivalent to equilibria in which all sellers offer to hold English auctions. In these equilibria, each seller commits himself to a reservation price that is equal to the expected monetary value of a 'marginal' unsold unit of output[3]. Oddly, this reserve price has the property that if the seller actually does trade at this reserve price, his ex post opportunity cost of trading will strictly exceed the reserve price[4]. In other words, the seller will regret, ex post, having agreed to the trade[5].

The result is an extension of the results in [3], [7], and [6] to the common value environment. In a competitive environment, the seller tries to control both the level of participation in his auction, and the surplus for buyers who do choose to participate. The surplus that the seller offers buyers must be equal to or larger than the surplus that is available to them in the rest of the market. In the *perfectly* competitive environment examined in this paper, this market payoff is beyond the seller's control[6]. As a consequence, the seller's ex ante profit from each buyer type is the difference between the expected value of the joint surplus generated when the seller trades with a buyer of that type, and the buyer's market payoff. A necessary condition for profit maximization is then that the mechanism ensures that there is a positive probability that the seller trades with every buyer type for whom this difference is positive.

The lowest buyer type with whom the seller wants to trade generates a joint

[2]This model has been analyzed in the monopolistic case by [1].

[3]The marginal unit has the property that there is a buyer who is just indifferent about whether or not he purchases the unit.

[4]Simply because the average buyer who trades at the reserve price will have a higher valuation than the marginal buyer.

[5]The seller will always realize ex post that he could have charged the buyer a higher price. But he would still prefer to trade at the agreed upon price, to not trading at all. Here the seller would prefer not to trade at all.

[6]This is the way that mechanism design in a competitive environment differs from mechanism design in a monopoly environment.

surplus that is exactly equal to the buyer's market payoff. As a consequence, the surplus that the seller expects to earn from a buyer of this type must be zero. Since the seller will only trade with such a buyer if he has no other alternative, this implies that the seller must be just indifferent between selling his unit at his reserve price, and keeping it, when he meets this marginal buyer[7].

Now suppose that the seller's estimate of the value of his unit of output depends positively on the valuation of the buyer with whom he trades, perhaps because he believes that buyer optimism is based on better information about the resale value of the unit. If the seller attracts only a single buyer, then that buyer will trade with the seller at the seller's reserve price, no matter what private information the buyer has. Since the seller knows that this buyers valuation exceeds the marginal buyer's valuation with probability 1, it follows that the seller's ex post use value for the unit of output in this case must strictly exceed the reserve price. If the only reason that the seller wants to hold on to the unit of output is because he can try to sell it again in the future, this means that the reserve price in equilibrium must be strictly lower than the seller's ex post estimate of the resale value of the unit.

The relationship between reserve prices and buyer participation has received some attention in the recent literature. [4] argue that the seller's ex post valuation for the marginal unsold unit of output is a lower bound on the ex ante optimal reserve price for the seller. Indeed, for the environment described in this paper, it is possible to derive the *optimal* mechanism, i.e., the one that maximizes the seller's profits in the case where the number of buyers that he attracts is exogenous[8]. This 'optimal' mechanism is an English auction with a reserve price that strictly exceeds the seller's ex post valuation of the marginal unsold item, just as predicted by [4]. .

The *equilibrium* reserve price in this paper can be *no higher* than the seller's ex post valuation. The reason is that in the extremely competitive environment discussed here, raising the reserve above this level precludes participation by buyer types with whom the seller would ex ante have liked to trade. This result captures participation effects of changes in the reserve price discussed by [8]. In his model, firms try to keep the reserve price secret to prevent low valuation buyers from

[7]Alternatively, the seller might consider raising his reserve price. If he does, he will not expect to change the market payoff that buyers receive. The only impact will be that buyer who generate a strictly positive expected surplus for the seller will no longer be willing to visit his store.

[8]The argument is a simple generalization of the argument in [1].

deciding not to participate in the auction.

In the competitive model discussed here, there is no advantage to firms to trying to keep their reserve prices secret. The reserve price is used by firms to ensure that the correct set of buyers choose to participate in the mechanism in the first place.

In the next section, we present the model and the equilibrium concept. The second section gives conditions under which the optimal mechanism for this environment can be implemented using an ascending price auction. The third section presents the competitive equilibrium concept. The fourth section presents the main results on equilibrium mechanisms. Finally some discussion of the issue of observable commitments is offered in the final section.

2. The Model

There is a finite number J of sellers, each of whom owns a single unit of a homogenous and indivisible commodity. Seller j has a type represented by a real number s_j. The seller's type affects his valuation for the homogenous commodity that sellers wish to trade. It will be assumed that the sellers' types are all common knowledge though the model could be extended to allow this to be private information.

There are $I = kJ$ buyers participating in the market, where k is a fixed finite ratio of buyers to sellers. Buyers all have private information characterized by their *type*. The type of buyer i is $b_i \in [0,1]$. It is assumed that buyer types are independently drawn using an identical distribution F. Let $b = \{b_1, \ldots, b_{kJ}\}$ denote the vector of buyer types for all the buyers in the population.

Buyers and sellers' valuations for the unit of output being traded depend on the private information of all the other buyers and sellers. Since seller types are assumed to be common knowledge, no generality is lost by suppressing the notation that describes this relationship. So let $V(b_i, b_{-i})$ and $U(b, s_j)$ represent the ex post valuations[9] of a buyer i with valuation b_i and a seller j with valuation s_j respectively.

For the purposes of this paper, the dependence between types will be even more specific. Any buyer and seller who trade enjoy a surplus consisting of the sum of a private and common value. The common value part is assumed to depend on the types of all the buyers and sellers who are not part of the exchange. So if

[9]The fact that buyers' and sellers' valuations depend on the types of all of the sellers in the market is simply embedded in the functions V and U.

a buyer and seller with types b_i and s_i respectively engage in a trade, the surplus assigned to the unit of output that the seller has to offer is given by

$$V(b_i, b_{-i}) = v(b_i) + \gamma(b_{-i})$$

for the buyer and

$$U(b, s_j) = u(s_i, b_i) + \gamma(b_{-i})$$

for the seller. The function γ is assumed to depend only on the distribution of valuations among the other buyers. It is assumed to be non-decreasing in each of it's arguments.

The idea is that the value of the durable is equal to the current flow value and the discounted value associated with either consuming or reselling the object in the future. The resale value depends on the types of the buyers (and sellers) who are not involved in the current trade. Notice that the resale value is assumed to be the same for both the buyer and the seller[10].

The valuation functions are assumed to be differentiable everywhere. The function v is assumed to be increasing, the function u is assumed non-decreasing. The difference between these valuations is assumed to be increasing in the buyer's type, i.e., $v'(x) - u'(s_j, x)$ is strictly positive. Throughout the paper $u'(s_j, x)$ means $\frac{\partial u(s_j,x)}{\partial x}$. Finally, the functions are normalized in such a way that $v(s_j) - u(s_j, s_j) = 0$.

These are simple generalizations of the private value assumptions. In the private value environment, the buyer's type is normally assumed to coincide with the buyer's valuation. Furthermore, the seller's valuation is typically independent of the buyer's type[11]. Under these circumstances, $\gamma(b_{-i})$ is constant (possibly, though not necessarily 0), $v'(x) = 1$, and $u'(s_j, x) = 0$.

Buyers and sellers are assumed to be risk neutral, so if a buyer with type b_i trades with a seller having information s_j at a price p, the expected payoff each receives is

$$\mathbf{E}_{b_{-i}} V(b_i, b_{-i}) - p$$

for the buyer and

$$p - \mathbf{E}_{b_{-i}} U(b_i, b_{-i}, s_j)$$

[10]It does not seem unreasonable to assume that a resale value should be the same for both the buyer and seller. In a finite market it will typically not be the same since the potential market that each trader faces differs by exactly one person. In the limit, as the number of buyers and sellers gets large, this assumption amounts to restricting the future market value to depend on the distribution of buyer and sellers types in the population.

[11]The case where this derivative can be negative is examined in [2].

for the seller. The expectation is taken conditional on all of the information that the traders have available at the time that they trade.

The market process[12] that determines all trades now proceeds. At the beginning of the game, each seller offers a direct mechanism to buyers. A mechanism is described in more detail below, but basically it describes how the seller will determine a price and trading partner (if there is one) among the buyers who visit him. The outcome will depend on messages that the buyers send to the sellers. For example, the seller could promise to hold an auction with a fixed reserve price. Alternatively he could simply offer his unit of output at a fixed price

Once the buyers see the mechanisms that are being offered by sellers, they select *one and only one* seller as a potential trading partner. Thus, sellers can try to attract buyers by modifying the mechanisms they offer. In the real estate example, sellers might consider lowering the list price, or reducing the probability that a buyer will face competition from other buyers.

After buyers select sellers, they communicate with the sellers they have chosen as specified by the seller's announced mechanism. Trades and payments occur, then the game ends. The problem is similar to any static mechanism design problem except for the fact that buyers have many alternative suppliers to whom they might turn, and sellers recognize this[13].

In the standard literature on mechanism design, the revelation principle is used to motivate restriction of attention to direct mechanisms. This same motivation can be used in the competitive environment considered here provided two key complications are born in mind.

First, the buyer's type is supposed to parameterize all the information that the buyer possesses at the time that he makes the report. At the time that buyers are required to communicate with sellers in this paper, they already have information about the array of mechanisms that are currently available. The distribution of buyer types should then vary in a systematic way with the array of mechanisms that sellers actually offer. This effect is simply ignored here.

Second, the seller does not know which buyers will ultimately communicate

[12]It would be appropriate to call this process a game form, but as will be seen, the solution concept will not be Nash in the usual sense. Rather competitive ideas will be employed. For this reason it seems preferable to refer to the market process.

[13]It is not difficult to convert this problem to a dynamic one in which buyers and sellers who fail to trade right away, can try to do so again in the following period. This extension is straightforward provided it is reasonable to assume that buyers and sellers believe that their current actions have no impact on future payoffs (which is reasonable in the large game environment that we study here).

with him. To deal with this problem, we will adopt the convention that there is a signal $b_\emptyset$ that buyers can use to report to a seller that they have or are planning to communicate with someone else[14]. It will be assumed that the valuation $b_\emptyset$ is not larger than any valuation in $[0,1]$. The space from which buyer $i's$ valuation is drawn can be simply thought of as $[0,1] \cup b_\emptyset$. Denote this set of valuations as B_i with $B = \prod_{i=1}^{I} B_i$. Note that in this interpretation, the seller designs a mechanism as if he were dealing with all the buyers in the market, despite the fact that he knows that most of them will not actually select him as a potential trading partner.

The valuation that buyers assign when some other buyer has type $b_\emptyset$ depends on the context. For example if it is known that all but one particular type of buyer communicate with the same seller, then the fact that a particular buyer does not communicate reveals his type completely.

A mechanism μ is a mapping from the set B into the set of $I+1$ vectors $\{p_j, q_j\}$. The mechanism μ specifies the probability with which each buyer will trade with the seller, and the transfer that each buyer will make to the seller. So for example, $q_{ij}(b)$ is the probability that buyer i will trade with seller j when the vector of reports by the buyers is given by b. The vectors lie in $\mathbf{R}^{I+1}$ because the seller might decide not to trade with any of the buyers who have currently contacted him.

With this latter convention, the mechanism must require that the sum of the assignment probabilities is one, and that the trading probability and transfer must both be zero for any buyer who 'reports' $b_\emptyset$ to the seller. This means that $b_i = b_\emptyset \Rightarrow p_{ij}(b) = q_{ij}(b) = 0$ and $\sum_{n=0}^{I} q_{nj}(b) = 1 \; \forall b \in T$.

Attention will also be restricted to anonymous mechanisms having the property that if b and b' differ only in that the i^{th} and m^{th} component have been interchanged, then $q_{ij}(b) = q_{mj}(b')$. Let Γ denote the set of anonymous mechanisms.

In this paper it will be assumed that buyers use identical selection strategies when choosing among the mechanisms offered by the various sellers. These selection strategies will be required to have the property that buyers cannot improve their expected payoff by unilaterally deviating from these strategies. This will

[14]The fact that a particular buyer does not choose to participate in a particular seller's auction may be informative to the seller when there are small numbers. Even when there are large numbers, seller may be able to make inferences about the distribution of buyer types in the population from the number of buyers who actually choose to bid. These calculations are demonstrated below.

imply that buyers have to randomize equally over all sellers who offer the same mechanism to them.

Let $\pi_j(x)$ denote the probability with which *each* buyer who has type x selects seller j. Define

$$z_j(x) = 1 - \int_x^1 \pi_j(s)\, dF(s)$$

This formula gives the probability that a buyer either has a valuation below x, or chooses some seller other than seller j. If we assume that $b_\emptyset$ is lower than any valuation in $[0,1]$, and treat π as if it were fixed, then $z_j(x)$ gives the distribution function for a randomly chosen buyer's type. The probability that the buyer has valuation $b_\emptyset$ (i.e. chooses some seller other than j) is just

$$z_j(0) = 1 - \int_0^1 \pi_j(s)\, dF(s)$$

This formula suggests that buyers who choose not to participate can be treated as if they had 0 valuations, provided that buyer types are drawn from a distribution $z_j(\cdot)$ that has a mass point at zero. This interpretation can be used to simplify many of the arguments below. For example, if $Q(x)$ is the probability with which seller j expects to trade with a particular buyer whose type is x, then the probability with which the seller expects to trade with that buyer ex ante is just $\int_0^1 Q(s)\, z_j'(s)\, ds$.

The one qualification arises from the fact that the seller's utility does depend directly on the valuations of buyers who do not participate in his mechanism. Then, for example, if $U(x)$ is the utility that the seller gets from the commodity when some *other* buyer's valuation is x, the seller's ex ante expected utility is

$$\begin{aligned}
\mathbf{E}U(x) = \int_0^1 U(x) f(x)\, dx &= \int_0^1 (1-\pi_j(x))\, U(x) f(x)\, dx + \pi_j(x)\, U(x) f(x)\, dx \\
&= \int_0^1 (1-\pi_j(x))\, U(x) f(x)\, dx + \int_0^1 U(x)\, z_j'(x)\, dx \\
&= z_j(0)\, \bar{U}(0) + \int_0^1 U(x)\, z_j'(x)\, dx
\end{aligned}$$

where $\bar{U}(0)$ is the seller's expected utility conditional on the event that the buyer does not choose his mechanism.

Let $\mu_j = (p_j, q_j)$ denote seller $j's$ mechanism. The expected payoff that a buyer of type x gets from choosing to participate in the mechanism μ_j offered by seller j, conditional on the selection strategy π, is given by

$$\beta_j(x) = \int \cdots \int q_{1j}(x, b_{-1}) V(x, b_{-1}) \, dz_j(b_{-1})$$
$$- \int \cdots \int p_{1j}(x, b_{-1}) \, dz_j(b_{-1})$$

where

$$dz_j(b_{-1}) = dz_j(b_2) \ldots dz_j(b_{kJ})$$

The vector b_{-1} can be used in the integration because of the fact that mechanisms are required to be anonymous.

A mechanism is incentive compatible for selection strategy π if

$$\beta_j(x)$$
$$\geq \int \cdots \int q_{1j}(x', b_{-1}) V(x, b_{-1}) \, dz_j(b_{-1})$$
$$- \int \cdots \int p_{1j}(x', b_{-1}) \, dz_j(b_{-1})$$

for all $x' \in [0, 1]$. Notice that the selection strategy is implicit in the distribution function z_j which is why incentive compatibility is contingent on the selection strategy.

If μ_j is incentive compatible for the selection strategy π_j, then by standard arguments $\beta_j(x)$ is continuous and increasing. Since $\beta_j(x)$ is monotonic, it is differentiable almost everywhere with derivative equal to

$$\int \cdots \int q_{1j}(x, b_{-1}) \frac{\partial V(x, b_{-1})}{\partial x} dz_j(b_{-1})$$

by the envelope theorem, whenever this derivative exists. In particular,

$$\beta_j(x) = \bar{\beta}_j(x) + \int_0^x \beta_j'(y) \, dy$$

where $\bar{\beta}_j(x)$ is almost everywhere constant.

Then using the formula for the derivative, this gives

$$\beta_j(x) = \bar{\beta}_j(x) + \int_0^x \int \ldots \int q_{1j}(y, b_{-1}) \frac{\partial V(y, b_{-1})}{\partial y} dz_j(b_{-1}) \, dy$$

Using the separability assumption, this is equal to

$$= \bar{\beta}_j(x) + \int_0^x \frac{dv(y)}{dy} \int \ldots \int q_{1j}(y, b_{-1})\, dz_j(b_{-1})\, dy$$

$$= \bar{\beta}_j(x) + \int_0^x v'(y)\, Q_j(y)\, dy \tag{2.1}$$

where

$$Q_j(y) = \int \ldots \int q_{1j}(y, b_{-1})\, dz_j(b_{-1})$$

is the probability with which a buyer of type y trades with seller j.

As in the independent private values case, the payoff that the buyer gets from participating in a particular mechanism depends on $\bar{\beta}_j(x)$ and the trading probability $q_{1j}(\cdot)$, without any reference to the transfer rule. The function $\bar{\beta}_j(x)$ can have a countable number of discontinuous jumps in it. In particular, on regions over which $z_j'(x) = 0$, (which is equivalent to $\pi_j(x) = 0$) the payoff at seller j might be strictly lower than the market payoff. If so, it does not, strictly speaking, matter what the mechanism does.

To simplify the argument, I will adopt the convention that if the mechanism is incentive compatible for some x, then it must be incentive compatible for all larger x, even when buyers with these higher types are not expected to participate in the seller's mechanism. This smooths out the payoff associated with the mechanism so that attention can be restricted to $\bar{\beta}_j(x)$ functions that have a single jump at some lower bound r. The lower bound r is just the greatest lower bound on the set of types who choose to participate in the seller's mechanism with strictly positive probability.

The surplus that seller j enjoys from the incentive compatible mechanism μ_j and the selection rule π_j is equal to the expected surplus generated by trade with each buyer, less the surplus that the seller must offer the buyer to get him to come in the first place. This is

$$\sigma_j(s_j) = kJ \int \left\{ \int \ldots \int \{U(b_1, b_{-1}, s_j) + [V(b_1, b_{-1}) - U(b_1, b_{-1}, s_j)]\, q_{1j}(b)\}\, dz_j(b_{-1}) \right.$$

$$\left. - \beta_j(b_1) \right\} dz_j(b_1)$$

Separating out the utility function $U(b, s_j)$ and integrating over the buyer types as discussed above, yields a term that is independent of both the mechanism that the seller offers, and the probabilities with which buyers choose to participate in it. Therefore the seller's objective can be reduced to maximizing the expected gain

$$kJ \int \left\{ \int \dots \int \{ [V(b_1, b_{-1}) - U(b_1, b_{-1}, s_j)] q_{1j}(b) \} dz_j(b_{-1}) - \beta_j(b_1) \right\} dz_j(b_1)$$

2.1. The monopoly seller's optimal mechanism

For purposes of comparison, it is useful to solve the mechanism design problem faced by a 'monopoly' seller. By monopoly seller I mean a seller who cannot change the probability with which buyers choose to participate in his mechanism by making modifications to the mechanism. The *optimal* mechanism for such a seller can be derived under relatively strong assumptions about the probabilities with which buyers choose to participate. Under these conditions, the optimal mechanism is complex. The seller should try to sell to the buyer with the highest private valuation. He might accomplish this by soliciting bids. Once the high valuation seller has been found, the seller must have committed himself to renegotiate the price in a manner that is quite sensitive to the seller's beliefs about the distribution of buyers that he faces.

The optimal mechanism for a monopolist in this environment can be derived[15] in the following way. The seller's payoff can be written (using (2.1) and the separability assumption) as

$$= kJ \int_0^1 \left\{ \int_0^1 \dots \int_0^1 \{ [V(b_1, b_{-1}) - U(b_1, b_{-1}, s_j)] q_{1j}(b) \} dz_j(b_{-1}) \right.$$

$$\left. - \bar{\beta}_j(r) + \int_0^{b_1} \int \dots \int q_{1j}(y, b_{-1}) \frac{\partial V(y, b_{-1})}{\partial y} dz_j(b_{-1}) \, dy \right\} dz_j(b_1)$$

$$= kJ \int_0^1 \left\{ \int_0^1 \dots \int_0^1 \{ [v(b_1) - u(s_j, b_1)] q_{1j}(b) \} dz_j(b_{-1}) \right.$$

$$\left. - \bar{\beta}_j(r) + \int_0^{b_1} v'(y) \int \dots \int q_{1j}(y, b_{-1}) dz_j(b_{-1}) \, dy \right\} dz_j(b_1)$$

$$= kJ \int_0^1 \left\{ [v(b_1) - u(s_j, b_1)] \int_0^1 \dots \int_0^1 \{ q_{1j}(b) \} dz_j(b_{-1}) \right\} dz_j(b_1)$$

$$+ kJ \left\{ -\bar{\beta}_j(r) - \int_0^1 v'(b_1) \int \dots \int q_{1j}(b_1, b_{-1}) dz_j(b_{-1}) \, db_1 \right\}$$

[15] A result similar to this for the case where the number of buyers participating in the seller's auction is fixed, is given in [1].

$$+kJ\int_0^1 \left\{ z_j(b_1)\, v'(b_1) \int \dots \int q_{1j}(b_1, b_{-1})\, dz_j(b_{-1}) \right\} db_1$$

$$= \int \int \dots \int \left\{ \sum_{i=1}^{kJ} [v(b_i) - u(s_j, b_i)]\, q_{ij}(b_i, b_{-i}) \right.$$

$$\left. - q_{ij}(b_i, b_{-i})\, [1 - z_j(b_i)]\, \frac{v'(b_i)}{z_j'(b_i)} \right\} dz_j(b_{-1})\, dz_j(b_1)$$

$$-kJ\bar{\beta}_j(r)$$

The penultimate equation follows by integrating by parts while the final equation exploits symmetry to rewrite the previous equation using summation instead of multiplication.

The seller does not care about the valuations of individual buyers, only about the distribution of valuations. Hence the first term in the sum is the expectation of a weighted average of the difference between the seller's valuation and the buyers 'virtual valuation' given by

$$v(b_i) - [1 - z_j(b_i)]\, \frac{v'(b_i)}{z_j'(b_i)}$$

Then simply maximizing this expression point-wise with respect to q_{ij} shows that the seller cannot improve upon the payoff to a mechanism that assigns the unit of output with probability 1 to the buyer with the highest virtual valuation. This lead to the following

Theorem 2.1. *Suppose that the choice probabilities and distribution function F are such that the function*

$$v(x) - u(s_j, x) - [1 - z_j(x)]\, \frac{v'(x)}{z_j'(x)}$$

is non-decreasing in x and suppose that the function above takes the value 0 for some type $0 < r < 1$. Then one implementation of the seller's optimal mechanism is an English auction in which the seller's reserve price is chosen so that a buyer of type r is just indifferent between trading and not trading at the reserve price when he learns that he is the only bidder in the auction.

Apart from the fact that the number of bidders in the auction is random, the result is really quite standard. When the virtual valuation is increasing faster than the seller's valuation, the buyer with the highest virtual valuation will also have the highest type, and the highest willingness to pay. The argument above shows that the seller wants to commit himself ex ante to a mechanism that guarantees that he will trade with probability 1 whenever he encounters a buyer whose valuation is above r.

A buyer whose valuation is above r who discovers that he is the only bidder in the seller's auction has an ex post valuation for the object being traded equal to

$$\int_0^r \cdots \int_0^r V(x, b_2, \ldots, b_{kJ}) \frac{dz_j(b_2) \ldots dz_j(b_{kJ})}{z_j^{kJ-1}(r)}$$

As this is increasing in x, the seller's reserve price should be set equal to the value of this expression when $x = r$ in order to ensure that a buyer with valuation above r will be willing to trade.

A complete proof of this argument will be deferred to the next section, as the argument for the English auction as the right mechanism is identical in both cases up to the reserve price that is set. Roughly speaking, when additional bidders arrive, buyers will interpret this as good news. The buyer with the highest valuation will remain in the bidding longest. Thus the English auction procedure ensures both that trade is ensured when there are two or more bidders with types above r, and that trade occurs with the buyer with the higher valuation.

Notice that the buyer of type r must get zero surplus ex post in any implementation of the optimal mechanism. Thus $\mathbf{E}V(r, b_{-1}) = R$. Now using the equation that defines the value of r and separability

$$v(r) - u(s_j, r) - [1 - z_j(r)] \frac{v'(r)}{z_j'(r)} = 0 \Rightarrow$$

$$\mathbf{E}V(r, b_{-1}) = \mathbf{E}U(r, b_{-1}, s_j) + [1 - z_j(r)] \frac{v'(r)}{z_j'(r)} \Rightarrow$$

$$R > \mathbf{E}U(r, b_{-1}, s_j)$$

This is not precisely the same as saying that the reserve price that the seller sets is strictly larger than his ex post cost, since ex post the seller's cost would be calculated knowing that the single bidder has a valuation somewhere above r. However, it does imply that the optimal mechanism for the seller will not be ex post efficient. If a buyer with a valuation slightly below r is the only buyer matched with the seller, then there will be no trade between the two, despite the fact that the buyer's valuation conditional on this information, strictly exceeds the seller's.

3. Equilibrium Mechanisms

The English auction is a simple enough mechanism, though in the absence of competition, the premium that the seller demands above his cost, $[1 - z_j(r)] \frac{v'(r)}{z'_j(r)}$, depends both on the seller's beliefs about the distribution of buyers' valuations, F, and on his beliefs about how buyers are being allocated to his mechanism (π_j).

In fact, the separability assumption provides another way of interpreting the optimal mechanism in this environment. The seller simply waits to see how many buyers are allocated to his mechanism. Then offers the optimal mechanism conditional on all the information that is conveyed by this outcome[16]. Thus, once the seller learns that only a single buyer has selected his mechanism, he just makes him the optimal take it or leave it offer.

When the monopoly seller modifies his mechanism to make it possible to trade with some buyer type, this changes the amount of surplus that he is able to extract from other buyer types. When he lowers the reserve price in the English auction, he weighs the gain associated with trading with the new type against the surplus loss from higher types. Competition among the sellers mitigates this second consideration because the existence of alternative suppliers limits the seller's ability to extract the surplus of buyers with higher types. The importance of the second consideration depends on the seller's ability to change the market payoff.

This paper considers the case in which the seller has no control whatsoever on the market payoff. We begin by postulating a function $\beta^*(x)$ that describes the payoff that sellers expect a buyer with type x to be able to earn by selecting a mechanism from the set of available alternatives. Sellers then simultaneously choose mechanisms to maximize their expected payoffs subject to the constraint that their mechanisms yield buyers who they wish to attract at least the market surplus β^*.

Once buyers have seen all the available alternatives, they are assigned among sellers using some common rule. This rule represents the outcome of some underlying search or selection process that is conditional on the information that the buyers have about the seller's mechanisms. The rule is required to satisfy the restriction that no buyer wishes to unilaterally deviate from it, given that all other buyers are being assigned according to the rule. The assignment rule is sim-

[16] It is the separability assumption that ensures that the optimal reserve price is independent of the information that is conveyed by the number of buyers who choose the seller's mechanism. It is important to note that when buyers choose the seller's mechanism in this interpretation, they cannot observe the mechanism that the seller plans to use.

ply the symmetric mixed strategy equilibrium for the continuation game in which the buyers choose among the sellers. This selection rule generates a continuation equilibrium payoff associated with the seller's proposed mechanisms. A market equilibrium arises when this continuation payoff is uniformly equal to the market payoff β^*.

It is important to note that sellers do not understand exactly how this continuation payoff is being generated. If they did, they would realize that as they vary the mechanism that they offer, they will perturb the continuation equilibrium for the selection process among buyers. Thus the market payoff is analogous to the market price in the usual Walrasian model where buyers and sellers do not realize that the demands and supplies that they announce will affect market price.

Exact equilibria for this process are intractable. The problem in which sellers do understand the relationship between their mechanisms and the market payoff has been analyzed in [6]. There it is verified that the effect that individual sellers do have on the market payoff shrinks to zero as the number of buyers and sellers in the market gets large.

Formally a *competitive equilibrium in mechanisms* is a payoff function $\beta^*(b)$, a selection strategy π for buyers, and an array of mechanisms $\{(p_1, q_1), \ldots, (p_J, q_J)\}$ such that

1. for each $j = 1, \ldots J$, the mechanism (p_j, q_j) and the selection strategy $\pi_j(\cdot)$ maximize

$$\sigma_j(s_j) = kJ \int \left\{ \int \ldots \int \{[V(b_1, b_{-1}) - U(b_1, b_{-1}, s_j)] q_{1j}(b)\} dz_j(b_{-1}) \right. \\ \left. - \beta_j(b_1) \right\} dz_j(b_1)$$

   subject to the constraint that for all buyer types x for whom $\pi_j(x) > 0$

$$\bar{\beta}_j(x) + \int_0^x \int \ldots \int q_{1j}(y, b_{-1}) \frac{\partial V(y, b_{-1})}{\partial y} dz_j(b_{-1}) dy \geq \beta^*(x)$$

2. for all $x \in [0,1]$ and each $j = 1, \ldots J$,

$$\bar{\beta}_j(x) + \int_0^x \int \ldots \int q_{1j}(y, b_{-1}) \frac{\partial V(y, b_{-1})}{\partial y} dz_j(b_{-1}) dy \leq \beta^*(x)$$

3. for all $x \in [0,1]$ and all j

$$\pi_j(x) > 0 \Rightarrow \bar{\beta}_j(x) + \int_0^x \int \ldots \int q_{1j}(y, b_{-1}) \frac{\partial V(y, b_{-1})}{\partial y} dz_j(b_{-1}) dy = \beta^*(x)$$

The seller perceives the market opportunities of buyers to be given by the function β^*. Based on these the seller offers some mechanism (p_j, q_j) and expects buyers to respond to his mechanism by choosing him with probability π_j. If there is another mechanism which the seller thinks will induce buyers to select him with some alternative probability π', yields buyers the market payoff, and also gives the seller strictly higher profits than his current mechanism, then the seller should deviate to that mechanism. Thus the first restriction is an obvious condition for equilibrium.

The market payoff in this paper plays a role that is analogous to next period's payoff in a dynamic problem except that it is linked to current actions instead of future ones. Thus the second condition simply requires that the market payoff be given by the upper envelope of the payoffs that are currently available to buyers with all the firms.

The final condition restricts the buyers' selection strategy in such a way that it is a best reply to the way that the other buyers are selecting among sellers conditional on the current array of offers.

Our main result is then the following

Theorem 3.1. *Suppose that the market mechanism satisfies the property that* $\beta^{*\prime}(1) = v'(1) > 0$. *Let* r^* *satisfy*

$$[v(r) - u(s_j, r)]\, Q^*(r) = \beta^*(r)$$

Then the seller cannot improve upon the profits that he earns by offering an English auction with reserve price equal to

$$\int_0^{r^*} \cdots \int_0^{r^*} U(s_j, r^*, x_2 \ldots, x_{kJ}) \frac{dz^*(x_2) \ldots dz^*(x_{kJ})}{z^*(r^*)^{kJ-1}}$$

where

$$z^*(x) = \left(\frac{\beta^{*\prime}(x)}{v'(x)}\right)^{\frac{1}{kJ}}$$

and

$$Q^*(x) = z^*(x)^{kJ-1}$$

.

Proof. The problem that the seller faces is to choose $q_{1j}(\cdot)$ and $\pi(\cdot)$ to maximize

$$kJ \int_0^1 \ldots \int_0^1 \{[V(b_1, b_{-1}) - U(b_1, b_{-1}, s_j)]\, q_{1j}(b) - \beta_j(b_1)\}\, dz_j(b)$$

subject to

$$\beta^*(x) \leq \beta_j(x) = \bar{\beta}_j(x) + \int_0^x v'(y)\, Q_j(y)\, dy \tag{3.1}$$

for every x such that $\pi(x) > 0$, and

$$z_j(x) = 1 - \int_x^1 \pi(x)\, f(x)\, dx \tag{3.2}$$

for all x.

Now using the common value assumption, the seller's objective reduces to

$$kJ \int_0^1 \ldots \int_0^1 \{[v(b_1) - u(s_j, b_1)]\, q_{1j}(b) - \beta_j(b)\}\, dz_j(b)$$

Thus the objective can be rewritten

$$kJ \int_r^1 \ldots \int_r^1 \{[v(b_1) - u(s_j, b_1)]\, q_{1j}(b)\}\, z'(b_1) \ldots z'(b_{kJ})\, db_1 \ldots db_{kJ}$$

$$-kJ \int_r^1 \beta_j(b_1)\, z'(b_1)\, db_1$$

since $q_{ij}(x, b_{-1})$ must be zero uniformly in b_{-1} for any buyer type who chooses to participate in the seller's mechanism with zero probability.

Instead of solving the seller's problem directly, consider the more tightly constrained problem in which the seller is obliged to give every buyer type above r exactly the surplus $\beta^*(x)$. In this problem, the seller maximizes

$$kJ \int_r^1 \{[v(b_1) - u(s_j, b_1)]\, Q_j(b_1)\}\, dz_j(b_1)$$

$$-kJ \int_r^1 \beta^*(b_1)\, dz_j(b_1)$$

with respect to both the assignment of the output to buyers, summarized by Q_j, and to the selection probabilities π_j that are reflected in z_j.

Integrating both terms by parts gives the last expression equal to

$$kJ \left\{ [v(x) - u(s_j, x)] \left[\psi(1) - \int_x^1 Q_j(s)\, dz_j(s)\right] \Big|_r^1 \right.$$

$$\left. - \int_r^1 \left[\psi(1) - \int_x^1 Q_j(s)\, dz_j(s)\right] [v'(x) - u'(s_j, x)]\, dx \right\}$$

$$-kJ\left\{\beta^*(x)\,z_j(x)\,|_r^1-\int_r^1 z_j(s)\,\beta^{*'}(s)\,ds\right\}$$

where $\psi(1)=\int_r^1 Q_j(s)\,dz_j(s)$. Simplifying this gives

$$kJ\left[v(r)-u(s_j,r)\right]\left[\int_r^1 Q_j(s)\,dz_j(s)\right]$$

$$+kJ\left\{\int_r^1\left[v'(x)-u'(s_j,x)\right]\int_x^1 Q_j(s)\,dz_j(s)\,dx\right\}dx$$

$$kJ\left\{-\left[1-z_j(r)\right]\beta^*(r)-\int_r^1\left[1-z_j(s)\right]\beta^{*'}(s)\,ds\right\}$$

Now if $\beta_j(x)$ is almost everywhere equal to $\beta^*(x)$ then the derivatives of these functions must also be equal almost everywhere. That is $v'(x)\,Q_j(x)=\beta^{*\prime}(x)$. Secondly, notice that the probability that the seller gives the unit of output to a buyer in the interval $[x,1]$ can be no larger than the probability that the seller gets such a buyer, so

$$kJ\int_x^1 Q_j(s)\,dz_j(s)\leq 1-z_j(x)^{kJ}$$

Then the seller's problem can be rewritten as maximizing

$$\left[v(r)-u(s_j,r)\right]\left[\int_r^1\frac{\beta^{*\prime}(s)}{v'(s)}dz_j(s)\right]-\left[1-z_j(r)\right]\beta^*(r)$$

$$+\left\{\int_r^1\left[v'(x)-u'(s_j,x)\right]\int_x^1\frac{\beta^{*\prime}(s)}{v'(s)}dz_j(s)\,dx\right\}dx$$

$$-\int_r^1\left[1-z_j(s)\right]\beta^{*'}(s)\,ds$$

subject to the constraint that

$$kJ\int_x^1\frac{\beta^{*\prime}(s)}{v'(s)}dz_j(s)\leq 1-z_j(x)^{kJ}$$

The solution to this problem requires that the constraint holds with equality. To see this rewrite the objective as

$$\left[v(r)-u(s_j,r)\right]\left[\int_r^1\left[\frac{\beta^{*\prime}(s)}{v'(s)}-\beta^*(r)\right]z_j'(s)\,ds\right]$$

$$+\left\{\int_r^1 \left[v'(x)-u'(s_j,x)\right]\int_x^1 \left[\frac{\beta^{*\prime}(s)}{v'(s)}-\beta^{*\prime}(x)\right] z_j'(s)\,ds\right\} dx$$

Consider maximizing this function by choosing the boundary r and the function $z_j'(s)$ subject to the constraint

$$kJ\int_x^1 \frac{\beta^{*\prime}(s)}{v'(s)} z_j'(s)\,ds + \left[1-\int_x^1 z_j'(s)\,ds\right]^{kJ} \leq 1$$

for all $x \geq r$. This can be solved by standard Lagrangian techniques.

The necessary condition for $z'(x)$, $x > r$ is given by

$$\left[v(r)-u(s_j,r)\right]\left[\frac{\beta^{*\prime}(x)}{v'(x)}-\beta^{*}(r)\right]$$

$$\int_r^x \left[v'(s)-u'(s_j,s)\right]\left[\frac{\beta^{*\prime}(x)}{v'(x)}-\beta^{*\prime}(s)\right] ds$$

$$-\int_r^x \lambda(s)\left[\frac{\beta^{*\prime}(x)}{v'(x)}-\left[1-\int_s^1 z_j'(s)\,ds\right]^{kJ-1}\right]=0$$

By our assumptions, $v'(x)\leq 1$ and $\beta^{*\prime}(x)$ is strictly increasing. Hence the term $\frac{\beta^{*\prime}(x)}{v'(x)}-\beta^{*\prime}(s)$ is uniformly positive. Thus $\lambda(x)$ cannot be 0 (a.e.) on any interval containing r. If it were, this necessary condition would be violated for any x in this interval. Thus at any solution to the problem, the constraint must hold with equality near r.

Then suppose there is some interval $[a,b]$ along which $\lambda(x)=0$. The same reasoning shows that the first order condition must be violated for x in the interval since the first term in the condition is strictly increasing in x.

The constraint can be satisfied almost everywhere on $[r,1]$ only if

$$\frac{\beta^{*\prime}(x)}{v'(x)} = z_j^{kJ-1}(x) \tag{3.3}$$

almost everywhere on $[r,1]$.

So finally, the seller's problem can be rewritten as the solution to the maximization problem

$$\left[v(r)-u(s_j,r)\right]\left[\frac{1-z_j(r)^{kJ}}{kJ}\right]-\left[1-z_j(r)\right]\beta^{*}(r)$$

$$+kJ\left\{\int_r^1 \left[v'(x) - u'(s_j, x)\right]\right\}\left[\frac{1 - z_j(x)^{kJ}}{kJ}\right] dx$$

$$-\int_r^1 \left[1 - z_j(x)\right] \beta^{*\prime}(x)\, dx$$

subject to (3.3). Since r is the only free variable left in the maximization, the necessary condition is simply

$$[v(r) - u(s_j, r)]\, z_j^{kJ-1}(r) = \beta^*(r) \tag{3.4}$$

which is the condition given in the theorem.

Finally, to show that this solution will also be a solution to the less constrained problem, suppose that the seller can strictly increase his payoffs by offering buyers a payoff $\hat{\beta}(x)$ that strictly exceeds $\beta^*(x)$ on a measurable set. The seller's payoff with this mechanism is

$$kJ\int_r^1 \left\{[v(b_1) - u(s_j, b_1)]\, \hat{Q}_j(b_1)\right\} d\hat{z}_j(b_1)$$

$$-kJ\int_r^1 \hat{\beta}(b_1)\, d\hat{z}_j(b_1)$$

This must be smaller than

$$kJ\int_r^1 \left\{[v(b_1) - u(s_j, b_1)]\, \hat{Q}_j(b_1)\right\} d\hat{z}_j(b_1)$$

$$-kJ\int_r^1 \beta^*(b_1)\, d\hat{z}_j(b_1)$$

which must then strictly exceed the optimal value of the payoff in the problem solved above. Since $\hat{Q}$ and $\hat{z}$ are both feasible in that problem, this is a contradiction.

We next show the optimality of the reserve price given in the theorem. Using the separability condition, (3.4) can be written as

$$\left\{\mathbf{E}\left\{V(r^*, x_{-1})\right\} - \mathbf{E}\left\{U(s_j, r^*, x_{-1})\right\}\right\} z^*(r^*)^{kJ-1} = \beta^*(r^*)$$

where

$$\mathbf{E}\left\{U(s_j, r^*, x_{-1})\right\} = \int_0^{r^*} \cdots \int_0^{r^*} U(s_j, r^*, x_2 \ldots, x_{kJ})\, \frac{dz^*(x_2)\ldots dz^*(x_{kJ})}{z^*(r^*)^{kJ-1}}$$

is the expected utility of the unit to seller j conditional on a single visitor whose type is r^* selecting his mechanism. The expected utility to the buyer has a similar expansion. Since $\beta^*(r^*)$ is the market payoff, it follows that

$$\{\mathbf{E}\{V(r^*, x_{-1})\} - R_j\} z^*(r^*)^{kJ-1} = \beta^*(r^*)$$

where R_j is seller j's reserve price. The reserve price given in the theorem follows from these equalities.

The final step in the proof is to show that the seller can attain the optimal value derived above by holding an English auction. This straightforward argument is a simple extension of the well known argument in [5] and is provided in the appendix.

■

The key assumption used in showing that the English auction is the competitive seller's best mechanism is the assumption that the market mechanism has the property that $\beta^{*\prime}(1) = v'(1)$. It is this condition that ensures that the distribution z^* that maximizes the seller's payoff subject to the market constraint is a well defined probability distribution. This result is needed to ensure that this probability distribution can be derived from an underlying selection rule π used by the buyers.

Since the market payoff is derived from some market mechanism possessing an associated trading probability, $\beta^{*\prime}(x)$ can be written as $v'(x)Q^*(x)$ where the function $Q^*(x)$ is effectively the market trading probability. The condition required by the theorem is that the market trading probability goes to one with the buyer's type. This, in particular, rules out market mechanisms where the good is allocated randomly without regard to the buyer's type (simple first-come-first-served rationing schemes for example).

This raises a question about whether equilibrium mechanisms with random rationing schemes can be maintained as equilibria simply because sellers believe that other sellers are using such mechanisms. The answer is no, as suggested by the following theorem.

Theorem 3.2. *In any competitive equilibrium in mechanisms, sellers must offer mechanisms that are allocationally equivalent to an English auction.*

Proof. The argument follows from the method of proof in Theorem 3.1.

First observe that in any competitive equilibrium in mechanisms, sellers must offer exactly the market payoff to any buyer whom they expect will choose them with strictly positive probability. Since the mechanisms that sellers offer in a competitive equilibrium in mechanisms must maximize their expected profits subject to the constraint that they offer buyers at least the market payoff, the fact that every seller chooses to offer exactly the market payoff implies that sellers' mechanisms must maximize their profits subject to the tighter constraint that they offer exactly the market payoff to all buyers.

The necessary conditions for the solution to this problem were derived in the proof of Theorem 3.1. The conditions are, first, that the seller's trading probability $Q_j(x)$ be equal almost everywhere $\pi(x)$ is positive to the market trading probability $Q^*(x)$. Second, that the constraint

$$kJ\int_x^1 Q_j(s)\, z_j'(s) \leq 1 - z_j^{kJ}(x)$$

be satisfied with equality for almost all x. This latter constraint requires that $Q_j(x) = z_j^{kJ-1}(x)$ which is equivalent to requiring that the seller always trade with the high bidder.

Finally the solution requires that the reserve price be set as specified in Theorem 3.1. Any mechanism satisfying these properties yields an allocation equivalent to the allocation in the English auction constructed in the proof of Theorem 3.1. ■

The final step in the argument is to verify the existence of a competitive equilibrium in mechanisms. One way to think about this is as a fixed point problem. By Theorem 3.2, the equilibrium must involve mechanisms that are allocationally equivalent to English auctions. These auctions must have the property that the probability with which any buyer trades with the seller goes to one as the buyer's type goes to one. Thus, begin with a non-decreasing market payoff function $\beta^*(x)$ whose derivative at 1 is equal to $v'(1)$. Theorem 3.1 implies that the profit maximizing mechanism against such a market payoff is an English auction. Thus each firm can select a profit maximizing English auction against the market payoff. Once the various mechanisms have been calculated, the selection probabilities for buyers that satisfy condition 3 in the definition of a competitive equilibrium in mechanisms can be derived. These selection probabilities will determine the actual market payoff $\tilde{\beta}(x)$ as in condition 2 of the definition. A competitive equilibrium in mechanisms will be found if a market payoff function $\beta^*(x)$ can be found such that $\tilde{\beta}(x) = \beta^*(x)$ almost everywhere.

Here we have the following result.

Theorem 3.3. *There exists at least one competitive equilibrium in mechanisms.*

Proof. The proof uses standard methods and is given in the appendix. ■

The theorem does not rule out the possibility that there may be more than one equilibrium. However, from Theorem 2, sellers must use mechanisms equivalent to English auctions in each such equilibrium. The reserve prices that they set, and as a consequence, the market payoff, will generally be different in different equilibria.

4. Reserve Prices

From Theorem 3.1 the equilibrium reserve price for the seller is equal to his valuation for the object that he has to sell after he learns that he has a single buyer whose type, or valuation is *equal* to r^*. If the seller tries to raise his reserve price, he simply loses the ex ante surplus that is provided to him by buyers whose valuations are slightly higher than r^* *without gaining anything from buyers with higher valuations*. The higher reserve price is simply a deterrent to participation.

On the other hand, the seller would not want to trade with buyers whose types are below r^* ex ante, because his surplus conditional on attracting them to his mechanism would be negative. In the equilibrium discussed in this paper, sellers with different ex ante types (s_j) will want to set different reserve prices. Given the restrictions on valuations, the relationship between the reserve price and the seller's ex ante type is monotonic.

The reserve price is strictly lower than the marginal buyer's valuation for the seller's output. This means that there are buyers who are precluded from trading with the seller, who nonetheless value the unit of output more highly than the seller does. Yet for these buyers the difference between their valuation and the sellers is smaller than the market payoff that these buyers could have attained elsewhere. In this sense, the reserve price is ex post efficient.

There is some relationship between this result and the result in [1]. They show that a seller would prefer to hold an English auction without a reserve price involving $N+1$ bidders, to holding an English auction with N bidders with the opportunity to make an 'optimal' take it or leave it bid to the final bidder. If the sellers in this paper have no use value for the commodity they own, then they will have the same preferences over these two alternatives. The reason here is that the English auction with no reserve price (which is the equilibrium mechanism in this case) will attract additional buyers who would refuse to participate in the more complicated mechanism.

Finally, note that the reserve price is equal to the seller's valuation for the unit of output when the single visitor has type r^*. Of course, if the single visitor did in fact have a reserve price equal to r^* then he would be just indifferent between buying the unit of output. Thus the reserve price is equal to the value of the marginal unsold (or sold) unit. The very fact that the buyer has selected the seller's mechanism means that his type is strictly larger than r^*. Thus, if the seller's valuation is positively related to the buyer's type, the seller will regret ex post having set such a low reserve price.

5. Conclusions and Problems

The equilibrium mechanism is allocationally equivalent to an English auction with a reserve price equal to the ex post value of a marginal unsold unit of output. Allocation schemes resembling English auctions are relatively common. Private real estate sales typically involve a dynamic procedure of bid and counterbid that resemble English auctions. Distress sales by the Federal Deposit Insurance Corporation and the Resolution Trust Corporation in the United States use explicit English auctions, and involve sales in the billions of US dollars [4]. English auctions of houses and used cars are common in Australia and New Zealand.

The equilibrium auction requires something more since it requires an explicit ex ante commitment to abide by a reserve price, and this is less common. At first glance, this does not seem to be a difficulty since the equilibrium reserve price is equal to the value of a marginal unsold unit of output. It is not unreasonable to assume that buyers know this value. For example, [4] use public information on the average value at which unsold units from previous auctions subsequently trade as an estimate of this value. This would suggest that there is no need for sellers to set the reserve prices explicitly since the reservation value may be common knowledge anyway.

However, the value that the seller assigns to the marginal unsold unit is not always the same as the seller's ex post opportunity cost. Thus the reserve price involves a commitment that binds on the seller ex post. Thus the ex ante publication of these prices seems critical.

Public reserve prices at house and used car sales are more common in Australia and New Zealand than in North America. However the question remains why reserve prices are not set in private house sales.

6. Appendix-The Maximal Payoff is attainable with an English Auction

The assumptions of the theorem guarantee that $z^*(x)$ is increasing and has limit 1. To complete the construction of a probability measure, define $z^*(0) = 1 - \int_{r^*}^1 dz^*(x)$ and let $z^*(x) = z^*(0)$ for $x < r^*$.

Finally, let $\pi^*(x) = 0$ for $x < r^*$ and $\pi^*(x) = z^{*\prime}(x)/f(x)$ otherwise.

The seller's reserve price in the English auction is defined to be

$$p^* = \frac{\int_0^{r^*} \cdot \int_0^{r^*} U(r^*, b_2, \ldots b_{kJ}, s_j)\, dz^*(b_2) \ldots dz^*(b_{kJ})}{Q^*(r^*)}$$

The equilibria for this auction involves only a slight generalization of the argument in [5]. The generalization follows from the fact that in the English auction buyers can see the number of other bidders in an auction, and in the competitive environment, this is a random variable.

Let $b_{mn}(x : p_1, \ldots, p_m)$ be the price at which the buyer plans to drop out of the auction when he observes that n other buyers have chosen to participate in the auction, and that m of them have dropped out of the bidding at prices given by p_1 through p_m. The equilibrium strategy is defined inductively as

$$b_{0n}(x) = \frac{\int_0^{r^*} \cdot \int_0^{r^*} V(x, x, \ldots x, b_{n+1}, \ldots b_{kJ})\, dz^*(b_{n+1}) \ldots dz^*(b_{kJ})}{z^*(r^*)^{kJ-1-n}}$$

and

$$b_{mn}(x : p_1, \ldots, p_m) =$$

$$\frac{\int_0^{r^*} \cdot \int_0^{r^*} V\left(x, b_{0n}^{-1}(p_1), \ldots, b_{m-1,n}^{-1}(p_m), x, \ldots x, b_{n+1}, \ldots b_{kJ}\right) dz^*(b_{n+1}) \ldots dz^*(b_{kJ})}{z^*(r^*)^{kJ-1-n}}$$

The inverse functions exist in the argument because each of the *bidding* functions so constructed is monotonic. The verification that these are equilibrium strategies follows directly from [5, Theorem 10,pp1105]. If two buyers have different valuations and use these strategies, the buyer with the lower valuation will always drop out first. Thus the probability that a buyer wins the auction is simply the probability that either all the other buyers who decided to participate in the auction have lower valuations, or, if they have higher valuations, they choose not to participate.

This mechanism then generates the correct trading probabilities in the sense that $Q_j(x) = z_j^{*kJ-1}(x)$. Thus by construction, the mechanism will generate every buyer type exactly the market payoff, provided that the ex ante payoff to a buyer to type r^* is exactly equal to $\beta^*(r^*)$.

A buyer who finds that he is alone bidding for the unit the seller has to offer will pay the reserve price. The buyer of type r^* will trade with the seller if and only if he is the sole bidder, thus

$$\beta(r^*) = \left[\int_0^{r^*} \cdots \int_0^{r^*} V(r^*, x_2, \ldots, x_{kJ}) \frac{dz^*(x_2)\ldots dz^*(x_{kJ})}{Q^*(r^*)} - p^*\right] Q^*(r^*) =$$

$$\left[\int_0^{r^*} \cdots \int_0^{r^*} \left[V(r^*, x_2, \ldots, x_{kJ}) - U(r^*, b_2, \ldots b_{kJ}, s_j)\right] \frac{dz^*(x_2)\ldots dz^*(x_{kJ})}{Q^*(r^*)}\right] Q^*(r^*)$$

$$= \left[v(r^*) - u(s_j, r^*)\right] Q^*(r^*) = \beta^*(r^*)$$

by the definition of r^*. ■

7. Second Appendix - Existence of Equilibrium

Proof. By Theorem 3.1, (and the fact that the market trading probability must approach 1 as a buyers type goes to one) attention can be restricted to the case where all sellers offer mechanisms equivalent to English Auctions.

A straightforward generalization of an argument in [7] can be used to show that, for the case when all sellers offer auctions, any two sellers who are chosen with positive probability by buyers must be chosen with exactly the same probability.

It is not hard to see why the same condition must hold here. If there is an interval $[a, b]$ along which buyers choose sellers m and n with strictly positive probability for almost all valuations in the interval, then the payoffs that buyers get by choosing either of these sellers must coincide along the interval with the market payoff by condition 3 of the definition of equilibrium. If they do, then the derivatives of these payoffs must also coincide, i.e.,

$$v'(x)\, Q_m(x) = v'(x)\, Q_n(x)$$

almost everywhere. If the sellers are both using English auctions, then this condition reduces to

$$z_m^{kJ-1}(x) = z_n^{kJ-1}(x)$$

almost everywhere on the interval, since the buyer with the highest type always trades with the seller. Differentiating once more gives the condition $\pi_m(x) = \pi_n(x)$ almost everywhere.

Now without loss of generality, suppose that the sellers are ordered such that $s_1 \leq s_2 \leq \ldots s_J$. Let $r_1, \ldots r_J$ denote the corresponding points $\inf \{x : \pi_j(x) > 0\}$. For the sake of clarity, refer to these as cutoff values. So the cutoff value for seller 1 is (roughly) the lowest buyer type who chooses seller 1 with positive probability. Consider the set $\tilde{\mathbf{R}} = \left\{ r \in \mathbf{R}^J : 0 \leq r_1 \leq \ldots r_J \leq 1 \right\}$. $\tilde{\mathbf{R}}$ is a compact convex subset of $\mathbf{R}^J$.

When the cutoff valuations are ordered in this way, buyers whose valuations are below r_1 have valuations below the cutoffs for all sellers. In other words, then do not participate in any of the sellers' mechanisms. Buyers whose valuations are in the interval (r_1, r_2) have types below the cutoff valuations for all sellers except seller 1. In this case, these buyers must choose seller 1 with probability 1. A buyer who has a valuation between r_2 and r_3 chooses seller 1 and 2 with positive probability. From the argument above, this must be the same probability, i.e., 1/2. Proceeding with this argument the entire selection strategy can be determined from the J cutoff valuations.

In particular, if the buyer's type is between r_j and r_{j+1} then each of the sellers indexed 1 through j is selected with the same probability $1/j$, while each of the other sellers is selected with probability 0. This makes it possible to write the function $z_j(x)$ in the following closed form: for $x \in [r_{m-1}, r_m]$

$$z_j(x, r) = 1 - \frac{1}{m-1} \left[F(r_m) - F(x) \right] - \sum_{n=m}^{J} \frac{1}{n} \left[F(r_{n+1}) - F(r_n) \right] \tag{7.1}$$

This relatively complicated arithmetic expression is nonetheless, readily seen to be jointly continuous for $r \in \tilde{\mathbf{R}}$ by the continuity of F.

Now let $\bar{\mathbf{R}}^J = \{0 \leq r \leq 1\}$ (i.e., the set of vectors in $\mathbf{R}^J$ that are componentwise bounded below by zero and above by 1). For any $r \in \bar{\mathbf{R}}^J$ the map $\Gamma(r) : \bar{\mathbf{R}}^J \to \tilde{\mathbf{R}}$ that simply reorders the components from smallest to largest, is continuous.

Now for any vector of cutoffs $r \in \bar{\mathbf{R}}^J$, the trading probability is given by $Q_j(x) = z_j^{kJ-1}(x, \Gamma(r))$ where $z_j(x, \Gamma(r))$ is defined according to (7.1). Since this trading probability is the composition of two continuous functions, it is continuous. In turn, since each firm that is chosen with positive probability yields the same payoff, $\beta_j(x) = \beta_1(x) = \int_{r_1}^{x} v'(s) z_j^{kJ-1}(x, \Gamma(r)) ds$, the market payoffs are all continuous in the vector r.

Now define the mapping $\psi : \bar{\mathbf{R}} \to \bar{\mathbf{R}}$ as

$$\psi_j(r) = \min\left[1, \max\left[s_j, r_j + \left\{ \left[v(r_j) - u(s_j, r_j) \right] Q_j(r_j, \Gamma(r)) - \beta_j(r_j, \Gamma(r)) \right\} \right]\right]$$

This is a continuous mapping from a convex compact subset of $\mathbf{R}^J$ into itself. Thus it has a fixed point.

Each fixed point of this mapping is a competitive equilibrium in mechanisms. At each fixed point $r_j \in [s_j, 1]$. If r_j is in the interior of this interval

$$[v(r_j) - u(s_j, r_j)]\, Q_j(r_j, \Gamma(r)) = \beta_j(r_j, \Gamma(r))$$

If $r_j = s_j$ at a fixed point, then $\beta_j(s_j, \Gamma(r))$ must be zero, which implies $s_j = s_1$. Finally if $r_j = 1$ then so is s_j. Thus using the vector r taken from the fixed point, and generating the market payoff as above, each seller is achieving his maximal profit against that market payoff, and this maximal payoff can be implemented by an English auction as shown in Theorem 1. ■

References

[1] J. Bulow and P. Klemperer, *Auctions vs. Negotiations*, Technical Report Stanford University (1993).

[2] A. Manelli and D. Vincent, Optimal procurement mechanisms, *Econometrica* **63** (1994), 591–620.

[3] P. McAfee, Mechanism design by competing sellers, *Econometrica* **61** (1993), 1281–1312.

[4] R. P. McAfee, D. C. Quan, and D. R. Vincent, How to set minimum acceptable bids, with an application to real estate auctions, Unpublished ms, U of Texas, (1995).

[5] P. Milgrom and R. Weber, A theory of auctions and competitive bidding, *Econometrica* **50** (1982), 1089–1122.

[6] M. Peters, A competitive distribution of auctions, University of Toronto manuscript, (1994).

[7] M. Peters and S. Severinov, Limit equilibrium concepts for problems with competing mechanism designers, unpublished University of Toronto, (1993).

[8] D. Vincent, Bidding off the wall: why reserve prices may be kept secret, *Journal of Economic Theory, forthcoming* (1995), journal.