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A Competitive Distribution of Auctions

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Abstract

In this paper a competitive distribution of auctions is described for an economy consisting of an infinite number of buyers and sellers, all of whom differ according to their valuation for the single indivisible object being traded. A competitive distribution of auctions is such that no seller can improve his profits by deviating to any alternative direct mechanism. It is shown that the competitive distribution of auctions will have the property that each buyer and seller's best reply is independent of his beliefs about the tastes of other buyers and sellers on the market.

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The purpose of this paper is to provide an equilibrium for an economy in which sellers compete by offering different direct mechanisms to buyers. No attempt is made to give a complete characterization of all possible equilibria, rather we focus on one equilibrium in which the contracts that are offered by sellers in equilibrium, and buyers' behavior under these contract, are both independent of traders' beliefs about one another.

The equilibrium illustrates one of the reasons why contracts that are employed in practice may not be as finely tuned to beliefs as the theory of mechanism design says that they should be. The answer is simply that contracts are not offered in a vacuum. Sellers in the model studied in this paper need to compete with one another in order to attract buyers. Highly tuned mechanisms tend to extract buyer surplus. In such an environment, sellers are able to attract many buyers by offering to use cruder mechanisms that leave some surplus for buyers.

In this paper, sellers are allowed to have different costs for the commodity that is being traded. Nonetheless, the equilibrium (for the limit economy in which there are infinitely many buyers and sellers) is such that every seller offers a second price auction and publicizes a reserve price equal to his cost.

In a standard monopoly auction the optimal reserve price for the seller exceeds his cost by an amount that depends on his beliefs about buyers' valuations [6]. A seller who raises his reserve price has his profits affected in two distinct ways. By raising his reserve price the seller makes it unprofitable for some buyer types to participate in his auction. The seller therefore loses any surplus that he might have earned by trading with these buyers. The buyers who still find it profitable to participate in the seller's auction will end up having to pay a higher price when they win because of this increase in the seller's reserve price. This means that the seller will be able to extract a bit more surplus from buyers who continue to participate. The seller raises the reserve price until the marginal cost associated with raising the reserve price is just equal to the marginal gain associated with the ability to extract more surplus from remaining buyers.

Competition mitigates this argument since there is a limit to the amount of surplus that the seller can extract from remaining buyers before they decide that they should instead choose one of the alternative auctions that are being held on the market. With competition, when the seller raises his reserve price, buyers whose valuations are high enough to allow them to pay this higher reserve price will indeed find their surplus declining, but not by as much as in the monopoly case as they will still have the option of leaving to try to buy at some other auction.

As the probability that buyers choose alternative suppliers rises, an externality is created because buyers choosing the alternative supplier will now face more competition.

In equilibrium the payoff that a buyer gets falls by the same amount with the deviator as it does with every other potential supplier on the market. Thus the monopoly reasoning applies in the competitive case, but to a lesser extent because some existing buyers will seek out other suppliers.

It would seem reasonable that in the limit when there are infinitely many buyers and sellers, this externality would disappear. A deviation might induce the sellers' existing customers to seek out other suppliers, but this would not substantially reduce the payoffs available on the open market. Now when a seller raises his reserve price, he loses buyers who can no longer afford to pay the higher reserve price, but gains nothing from buyers with higher reserve prices as they always have the option of going elsewhere to maintain their surplus. Since the seller extracts all the surplus from the marginal buyer type, he should set a reservation price that guarantees participation by all buyers who have a valuation above his own cost.

As may be apparent, the effect of a change on the reserve price on the surplus of participating buyers only disappears in the limit when there are infinitely many buyers and sellers. The difficulty that this creates is that most of the theory of auctions, and all of the theory of mechanism design uses the fact that there are a finite number of participants explicitly. The only effective way to analyze the properties of the conceptually simple limit model is to take limits of the finite version. This paper suggests a way that this can be done. The implied behavior of the limiting model is quite simple.

The central result of the paper is that there is an equilibrium when sellers compete in direct mechanisms where all sellers offer buyers second price auctions with reservation prices equal to their costs. This result has been proved in McAfee [5]. The argument presented here generalizes his argument in two ways. First McAfee assumes that sellers in the finite version of the model ignore the externalities that their deviations create when choosing among mechanisms. This assumption is critical to the proof that auctions constitute a best reply for sellers. This assumption is relaxed here. The argument that is used here can actually be made by using McAfee's result. Hence I prefer to think of this aspect of the model as a reinterpretation of the result that McAfee has already proved.

Secondly McAfee assumes that sellers are identical. This assumption is actually quite important in interpreting the result. In a simple Bertrand model of price competition when firms have identical (constant) marginal cost curves all firms will set price equal to marginal cost. Firms simply need to know their own marginal cost to calculate their best reply. If firms cost curves differ, on the other hand, the low cost firm wants to set a price just slightly below the marginal cost of his most efficient competitor. The price that he sets will then be exquisitely sensitive to his beliefs about his opponents costs.

It is shown below that the sellers' equilibrium mechanisms remain unchanged over a broad class of distributions of reservation valuations for their opponents, and distributions of beliefs about buyers valuations. The essential assumption is that the support of the distribution of seller valuations is contained in the support of the distribution of buyer valuations. When this is not the case, it is simple to think of situations in which the theorem breaks down. For instance, if sellers share a common valuation that is strictly below the common valuation of buyers then the sellers' equilibrium reserve price (which would just be an equilibrium price in this case) would be at a point in between the two valuations that should depend on where the sellers think this bound is.

At first blush, the result has a Bertrand flavor about it since sellers prices are driven down to their costs. This analogy is misleading since sellers will continue to earn positive profits in equilibrium from buyers who submit bids above their costs. As may be apparent from the discussion above, the movement of reserve prices toward sellers cost is not caused by any discontinuity is seller's profit functions[1]. Deviations will generally cause sellers profits to respond continuously as long as the distribution of seller costs is atomless.

Perhaps a better analogy for the results presented here is the corresponding result for a double auction [13, 12, 11, 10]. As the number of buyers and sellers gets large in a double auction, buyers and sellers will bid their true values and costs respectively. In the limit the market clearing price in the double auction will be strictly higher than the cost for almost all sellers, and strictly lower than the valuation for almost all buyers. Hence both buyers and sellers will continue to receive surplus despite bidding their true valuations.

The setup here involves sellers who actually bid mechanisms instead of prices. However, the matching process is far less centralized than it is in a double auction[2]. The decentralized process analyzed here leaves unrealized gains to trade in the limit. It is the study of decentralized processes like the one discussed here that will eventually explain the emergence of more centralized institutions like the double auction.

The fact that sellers' valuations of the commodity being traded differ in this paper presents a number of new technical and conceptual problems. However, the point of the paper is to show that problems in which sellers compete in mechanisms can be understood using a relatively transparent limit theory. The limit theory is essentially a generalization of the method developed by Severinov and Peters [9] to the case in which sellers can have different costs. In addition, the latter paper restricted attention

[1]In [9] the corresponding result if proved for the special case in which all sellers have the same cost by appealing to a discontinuity is the seller's profit function. This discontinuity does not generalize beyond that special case.

[2]This means that the equilibrium here will not generally have the desirable efficiency properties of double auctions with many buyers and sellers.

to competition among sellers who were only allowed to offer auctions. The result is generalized to competition in direct mechanisms by using one of McAfee's [5] theorems. It is shown that the assumption that McAfee imposed on sellers' expectations defines a pseudo problem (which McAfee solves in his paper) whose solution generates an upper bound on the payoff that a seller can get from any arbitrary deviation when there are a finite number of buyers and sellers even when sellers fully understand how their deviations affect market payoffs. The limit of this upper bound as the number of buyers and sellers goes to infinity is equal to the limit payoff associated with a second price auction in which the seller sets a reserve price equal to his cost. I view this as an alternative way in which McAfee's theorem on the best reply property of auctions can be employed[3].

Of course the model that is employed in this paper is highly stylized. It is not without application. The key assumptions require large numbers of traders, independently distributed information, and decentralization in the sense that there is no coordinating agency in charge of price setting (as there is say in securities markets which otherwise satisfy many of the assumptions in this paper). The real estate market is a prime example of a market that has some of these properties. Products that are sold through classified ads in large cities, like rental housing and used cars, also qualify.

However the fit is not perfect. Though house sales can involve competitive bidding among potential buyers, prices are often negotiated by the seller and a single potential buyer. The model studied in this paper predicts that when there is a single buyer, trade should occur without negotiation at the reserve price that the seller advertized ex ante. This seems characteristic of rental housing markets, but not of resale housing markets or used car markets.

There are clearly common value and dynamic aspects to resale housing and other durable markets that go beyond the scope of this paper. I view the assumptions in this paper (particularly the independent private value, and symmetry assumptions about the buyers' selection process) as characterizing the most extreme form of market decentralization. Hopefully some understanding of the role of competition in this extreme case will provide some insight into the how to think about markets where beliefs and actions are better coordinated.

[3]McAfee establishes a number of upper bounds on seller payoffs in his argument and shows that auctions attain these bounds. The upper bounds are valid conditional on his expectational assumption. The bound that is established here is valid even when this expectational restriction does not hold.

1 The Model

The market consists of J sellers and kJ buyers. The ratio k of buyers to sellers will always be finite and fixed, but the possibility that J will be infinite is central to the analysis of the paper. Formulas containing an infinite value of J should be considered as limits for the values of the formulas as J goes to infinity.

Each seller in this market has a single indivisible unit of output, like a house, to sell to some buyer. Each seller's cost for his unit is private information. The distribution of costs in the population of sellers is given by G which is assumed to have support contained in $[0, 1]$.

Buyers each wish to purchase exactly one unit of the commodity. Buyers' valuations are assumed to be private and independently drawn from the strictly increasing distribution function F whose support is assumed to be $[0, 1]$. The fact that the support of G is contained in the support of F is important in the results that follow. Of course, if there are sellers whose cost is higher than the valuation of any buyer, these sellers can be effectively ignored. The more difficult case occurs when there are sellers who can surely trade with all buyers. These sellers will generally set reserve prices that depend on both the lower bound of the support of F and on the distribution F itself.

Buyers and sellers are assumed to be risk neutral so that if a seller with valuation y trades with a buyer with valuation $x > y$ and a price p, then the corresponding surpluses are $p - y$ for the seller and $x - p$ for the buyer.

At the beginning of the market period, sellers simultaneously offer incentive compatible *direct* mechanisms to buyers. Direct mechanisms are treated here exactly as they are in the static mechanism design problem. Each buyer who selects a seller's mechanism tells the seller his valuation. The seller then chooses a price and trading partner according to the rules of the mechanism. This is somewhat restrictive in that buyers know more than their own valuations by the time they have agreed to participate in the seller's mechanism. For instance, each buyer has seen the mechanisms offered by each of the sellers. Sellers are not allowed to exploit this fact when they design their mechanisms.

After buyers see these announcements, they simultaneously choose potential trading partners. It is assumed that each buyer may choose one and only one seller in an attempt to negotiate a trade. Once buyers have matched up with sellers in this way, each seller, along with the buyers with whom he is matched, negotiate over prices and trades.

2 Competition in Mechanisms

To begin, consider the case where there are a finite number of buyers and sellers. We would like to characterize a direct (anonymous) mechanism for some seller j in the usual way as a pair of functions $\{P^j(\cdot), Q^j(\cdot)\}$ giving the expected price and probability of trade for any buyer who selects the seller as a potential trading partner. This is complicated in a problem with competition since the reduced form payoffs are determined jointly with buyers' selection decisions. The trading probability with seller j will tend to be large if other buyers ignore seller j.

Let $x^i_\emptyset$ denote the outcome in which buyer i does not submit a report to seller j (i.e. chooses not to participate in seller $j's$ mechanism). Let $[0,1] \cup \{x^i_\emptyset\} \equiv \bar{X}$ denote the set of possible types for agent i and let μ^j denote a mapping from the cross product $\bar{X}^{kJ}$ into the set of vectors $\{p^j, q^j\}$ with the interpretation that $q^{ji}(\bar{x})$ is the probability that seller j gives the good to buyer i when the vector of reports received by seller j is given by $\bar{x} \in \bar{X}^{kJ}$. For each $\bar{x}$ we require $\sum_{i=1}^{kJ} q^{ji}(\bar{x}) \leq 1$ and $\bar{x}^i = x^i_\emptyset \Rightarrow q^{ji}(\bar{x}) = 0$. Furthermore, we will restrict attention to anonymous mechanisms having the property that for any two vectors $\bar{x}$ and $\bar{y}$ having the property that $\bar{y}$ is the same as $\bar{x}$ except that the m^{th} and n^{th} components have been interchanged, then $q^{jm}(\bar{x}) = q^{jn}(\bar{y})$. Let $\mu = \{\mu^1, \ldots\}$ denote the array of mechanisms offered by the various sellers.

Let $\pi_j(x, \mu)$ denote the probability with which each buyer selects seller j when the array of mechanisms μ are offered and the buyer's valuation is x. Define

$$z_j(x, \pi) = 1 - \int_x^1 \pi_j(s, \mu) f(s) ds \tag{1}$$

The function $z_j(x, \pi)$ gives the probability that a buyer drawn at random from the set of buyers either has a valuation below x or chooses a seller other than j (i.e., reports a valuation $x^i_\emptyset$).

When a seller offers a mechanism μ^j and anticipates that buyers will use a selection strategy π_j, the various payoffs associated with this mechanism can be calculated using standard arguments once it is understood that the distribution of buyer valuations is the induced distribution z_j instead of the true distribution F. As always, when sellers contemplate deviations from a putative equilibrium, they can restrict attention to incentive compatible mechanisms. This is somewhat complicated in this context because the incentive compatibility of a particular mechanism may depend on the selection strategy π_j that buyers adopt in response to it[4]. For example, a particular mechanism $\mu^j = \{p^j(\cdot), q^j(\cdot)\}$ might be incentive compatible for one selection strategy π_j but not

[4]This creates no problem if the mechanism is a second price auction, for in this case, buyers bid their valuations in equilibrium no matter what distribution of bids they expect from their opponents.

incentive compatible for some other strategy π_j'. The selection strategy in turn will have to satisfy various equilibrium conditions.

In fact, buyers choose both a selection and reporting strategy in the second stage of the game. Whatever reporting strategy that buyers do choose in response to a particular deviation will induce an incentive compatible direct mechanism[5]. The deviating seller will foresee the strategies of the buyers in equilibrium. Therefore he might as well use the induced incentive compatible mechanism (along with the corresponding selection strategy) to calculate his profits. To simplify the argument in what follows attention will simply be restricted to deviations to mechanisms that are incentive compatible for the corresponding selection rule.

Let $Q_j(x,\mu,\pi)$ and $P_j(x,\mu,\pi)$ denote the reduced form probability with which a buyer expects to trade, and the reduced form price that he expects to pay respectively if he selects the mechanism offered by seller j when buyers are using the selection strategy π. Let $v_j(x,\mu,\pi)$ denote the expected payoff to a buyer who selects seller $j's$ mechanism.

If seller $j's$ mechanism is incentive compatible, then the reduced form probability with which a buyer of valuation x trades with seller j is given by[6]

$$\begin{gathered} Q_j(x,\mu,\pi) = \\ \int_0^1 \dots \int_0^1 q^{j1}(x,s_2,\dots,s_{kJ})dz_j(s_2,\pi)\dots dz_j(s_{kJ},\pi) \end{gathered} \tag{2}$$

If this mechanism happens to be an auction for example, then a buyer will win the auction if and only if each of the buyers who has a higher valuation chooses some other seller. This occurs with probability $Q_j(x,\mu,\pi) = z_j^{kJ-1}(x,\pi)$.

From [6][7], if the seller's mechanism is incentive compatible for the selection strategy π, then $Q_j(x,\mu,\pi)$ is a non-decreasing function, and

$$v_j(x,\mu,\pi) =$$

$$\begin{gathered} Q_j(x,\mu,\pi)x - P_j(x,\mu,\pi) = \\ Q_j(y_j,\mu,\pi)y_j - P_j(y_j,\mu,\pi) + \int_{y_j}^x Q_j(s,\mu,\pi)ds \end{gathered}$$

where y_j is the greatest lower bound of the interior of the set $\{x : \pi_j(x,\mu) > 0\}$.

These results are collected formally as

[5]This is defined in the usual way as the composition of the original mechanism and the equilibrium reporting strategy.

[6]Since the mechanism is anonymous, all buyers who have the same valuation are treated the same way. Thus attention can be restricted to the trading probability that applies for buyer number 1.

[7]The proof for piecewise continuous distribution functions of the form presented by z_j is a generalization of the proof in [7] according to [3].

Lemma 1 *For each j, $v_j(x,\mu,\pi)$ is a continuous convex function (of x) which is almost everywhere differentiable with first derivative equal to $Q_j(x,\mu,\pi)$. If seller j is offering an auction to buyers then $Q_j(x,\mu,\pi) = z_j^{kJ-1}(x,\pi)$.*

These are standard implications of incentive compatibility and auctions so the proofs are omitted. The proofs of the remaining theorems in the paper are collected in the Appendix.

When the mechanism that the seller offers is a second price auction, this formula simplifies. Let r_j denote seller j's reserve price. The payoff to a buyer of type y_j who chooses seller j is $y_j - r_j$ times the probability that no buyer with a valuation exceeding y_j chooses seller j. This latter probability is just $z_j^{kJ-1}(y_j,\pi)$. Hence for this seller,

$$v_j(x,\mu,\pi) = (y_j - r_j) z_j^{kJ-1}(y_j,\pi) + \int_{y_j}^{x} z_j^{kJ-1}(s,\pi)ds$$

Again using the assumption that there are a finite number of buyers and sellers, seller j's profits are given by (the derivation can be found in McAfee [5])

$$\begin{gathered}\Phi^j(w,\mu_j,\mu_{-j},\pi) = w + \\ kJ\int_{y_j}^{1}\left[(x-w)\,Q_j(x,\mu_j,\mu_{-j},\pi) - v_j(x,\mu_j,\mu_{-j},\pi)\right] z_j'(x,\pi)dx\end{gathered} \tag{3}$$

where $w \in [0,1]$ is the seller's valuation for the unit of output that he has to sell.

This treatment all looks very standard except for the fact that the sellers do not control the reduced form probabilities Q_j and the prices P_j directly as they do in a simple mechanism design problem. This can be seen by noting that for the auction the reduced form probability is z_j^{kJ} which depends on the selection probabilities π_j that buyers use when choosing among the sellers.

3 Buyers Choices among Mechanisms

Competition among mechanism designers involves a two stage game. The way that the reduced form probabilities evolve as the seller changes μ depend on the properties of the continuation equilibrium that follows the sellers' initial announcements. These continuation equilibria are not well behaved when the number of buyers and sellers is finite.

The buyer's choice strategy is said to be *incentive consistent* if uniformly in μ, and x, $\sum_{j=1}^{J}\pi_j(x,\mu) = 1$ and

$$\pi_j(x,\mu) = 0 \Rightarrow \exists k \neq j : v_k(x,\mu,\pi) \geq v_j(x,\mu,\pi)$$

$$\pi_j(x,\mu) > 0 \Rightarrow v_j(x,\mu,\pi) \geq v_k(x,\mu,\pi)\ \forall k$$

Incentive consistency implies that given that buyers expect other buyers to assign themselves among the sellers according to the rule π, no buyer has an incentive to deviate from this rule in choosing a seller himself.

Incentive consistency imposes a strong symmetry restriction on buyers' choice behavior. It requires that if two sellers offer the same mechanism, then these two sellers must be chosen with the same probability by all buyers[8]. Alternatively, the probability with which a buyer selects a particular seller depends only on the seller's mechanism, and not on the buyer or seller's name. This assumption captures the frictions inherent in a decentralized matching process while retaining the nice restrictions imposed by subgame perfection.

There are other plausible equilibria that are not anonymous. For example, if all sellers offer the same mechanism, then there will be a continuation equilibrium where each buyer chooses a particular seller with probability 1, but different buyers choose different sellers. This matching process presents few trading frictions, but requires considerable coordination to match buyers and sellers properly.

The symmetric equilibrium has one key property that this full coordination equilibrium does not. In the symmetric equilibrium the probability with which the deviating seller is chosen by buyers depends in a sensible way on the mechanism that the deviator offers. In the full coordination equilibrium deviations do not need to affect the choice probabilities at all. The reason for this is that with coordination, each buyer faces a known finite number of opponents in the mechanism that he chooses, no matter how many buyers and sellers there are in the entire economy.

Now suppose that one seller modifies his mechanism so that the payoff that buyers get from it declines slightly. If the buyers who are not immediately affected by this deviation do not change their choice strategies, then there is no need for the deviator's buyers to change their strategies either.

If the deviator's buyers do change, they will simply face one more competitor at the new seller, than they would face if they simply remained with the deviator. The loss in surplus created by the new competitor will more than compensate for the lost surplus caused by the change in the deviator's mechanism. Sellers, knowing this, will all simply offer the optimal auction and ignore the potential competition created by other sellers[9].

[8] This probability can vary with the buyers' types.

[9] This is the well known Diamond [1] paradox from standard search theory. Competing sellers will simply behave as if they were monopolists. In this case, when they do, they will offer reserve prices that depend on their beliefs about the valuations of buyers. The results presented below are not robust to this kind of change in the continuation equilibrium.

3.1 Properties of the Buyers' Selection Problem

So to simplify the development slightly, a couple of properties of incentive consistent matching processes are considered. Let the strategy $\pi(x,\mu)$ be incentive consistent. Consider the set E_j defined as the closure of the interior of the set $\{x : \pi_j(x,\mu) > 0\}$ and let $E_{jk} = E_j \cap E_k$. Define the *cutoff valuation* for seller j to be equal to $y_j = \min\{x : x \in E_j\}$. We first derive various properties of incentive consistent choice strategies.

The following argument is a generalization of the argument presented in Severinov and Peters [9].

Lemma 2 *Suppose that two distinct sellers i and j both offer to hold auctions. Then for any incentive consistent matching process (i) $z_i^{kJ}(x,\pi) = z_j^{kJ}(x,\pi)$ and $\pi_i(x,\mu,\pi) = \pi_j(x,\mu,\pi)$ for almost every $x > \max[y_i, y_j]$; (ii) if the reserve price offered by seller i is no higher than the reserve price offered by seller j, then $E_j \subset E_i$; (iii) for every i, E_i contains the point 1; (iv) if all sellers offer to hold auctions, then E_i and E_j are convex.*

The useful part of this Lemma is the fact that all sellers who hold auctions and are chosen with positive probability by buyers must be chosen with the same probability if the matching process is incentive consistent. Suppose that all but one seller offers to hold an auction, and suppose that for some x there are exactly j auctioneers for whom $x \in E_j$. The deviating seller who offers some alternative mechanism may be chosen with positive probability $\pi(x)$ by buyers. Each of the other sellers must be selected with the same probability according to the Lemma. Since the probabilities must sum to 1, this common selection probability must be $\frac{1-\pi(x)}{j}$.

4 The Limit Game

In the symmetric continuation equilibrium, the deviator's new mechanism changes the probability with which he attracts new buyers. It may also cause buyers to reorganize their choices among the non-deviating sellers as well. This creates a complicated externality that can change the market payoff that buyers can get by ignoring the deviator. To deal with this, this paper considers the limiting version of the model described above.

This paper adopts the distributional approach and imagines that sellers vary their mechanisms against a fixed distribution of mechanisms offered by their opponents. When this distribution is a distribution of auctions, one might accomplish this by constructing finite distributions of reserve prices that approximate the limit distribution of interest, calculating the payoff to a deviation against these approximations, then taking the limit of these payoffs as the approximation becomes better and better.

This procedure does not work well here because the mapping from reserve prices into selection strategies and payoffs is complex. There is an alternative way to proceed using distributions of *cutoff valuations*. The cutoff valuation for seller j is the greatest lower bound on the interior of the set of valuations for which buyers choose seller j with positive probability. The mapping from cutoff valuations to payoffs, and back to the reserve prices that they support, is straightforward.

The procedure adopted below is to create a finite approximation to the limit distribution of cutoff valuations, then calculate the profits associated with deviations from this distribution. The approximation to the limit distribution of cutoff valuations is then made finer and finer and the limit value of the payoff associated with any deviation is calculated.

Two results make this equivalent to calculating the benefits of deviations from a distribution of reserve prices (in the limit). First the relationship between the market payoff that buyers get and the array of reserve prices that support that relationship is continuous (in the topology of weak convergence). Secondly, the market payoff that buyers get when a single seller deviates against a fixed didstribution of reserve prices converges to a function that is independent of the deviating seller's mechanism.

All of this means that, no matter what mechanism is offered by a deviating seller, the induced distribution of reserve prices converges weakly to the same limit distribution. The proceedure then provides a technique for calculating the limit payoff of the deviator against this limit distribution of reserve prices. The main theorem shows that no matter what this limit distribution is, the deviating seller's payoff is bounded above by the payoff associated with holding an auction with reserve price equal to the seller's cost.

Let $\mathcal{M}$ denote the set of all sequences of direct mechanisms μ_J. Recall that μ_J is a direct mechanism offered by the seller when there are kJ buyers in the market. In the limit game, sellers choose sequences of mechanisms from $\mathcal{M}$. Attention will be focussed on the situation in which a single deviator offers a mechanism μ' from $\mathcal{M}$, while other sellers offer a distribution of auctions.

4.1 The Distribution of Cutoff Valuations and its approximation

Let H denote a distribution of cutoff valuations for the limit game. We want to show how to approximate this distribution and how to recover the market payoffs and distribution of auctions associated with it. Note that the distribution H need not be continuous or differentiable in this argument.

This distribution can be approximated for large J in the following way. For any J, let $\bar{y}_J = \{\bar{y}_1, \ldots, \bar{y}_{J-1}\}$ be defined as

$$\bar{y}_1 = \sup\{x : H(x) = 0\}$$

$$\bar{y}_2 = \sup\{x : H(x) \leq 1/J\}$$

and in general,

$$\bar{y}_j = \sup\left\{x : H(x) \leq \frac{j-1}{J}\right\} \tag{4}$$

The numbers $\bar{y}_j$ act like j tiles for the distribution H. For example, if $H(\cdot)$ is continuous, then $\bar{y}_j$ is the cutoff valuation having the property that the proportion $\frac{j-1}{J}$ of all sellers hold auctions that generate cutoff valuations less than or equal to $\bar{y}_j$. In the case where H is not continuous, $\bar{y}_j$ has the property that the proportion $\frac{j-1}{J}$ of all sellers have cutoff valuations less than or equal to $\bar{y}_j$ while the remainder of sellers have cutoff valuations at least as large (but possibly equal to) $\bar{y}_j$.

The market consisting of J sellers and kJ buyers is called a *finite approximation* to the limit market if $J-1$ of the sellers (called non-deviators) hold auctions $\bar{\mu}_J = \{\bar{\mu}_{1,J}, \ldots, \bar{\mu}_{J-1,J}\}$ having the property that the auctions generate cutoff valuations $\bar{y}_1$ through $\bar{y}_{J-1}$ and the other seller, called the deviator, offers to hold some arbitrary direct mechanism μ'_J. Notice that the distribution $\bar{H}_J$ of cutoff valuations in the finite approximation, defined by $x \in [\bar{y}_j, \bar{y}_{j+1}) \Rightarrow \bar{H}_J(x) = j/(J-1)$ converges almost everywhere to $H(x)$ and that $\bar{H}_J(x) \geq \bar{H}_{2J}(x)$.

Now taking this distribution $\bar{H}_J$ to be fixed, it is possible to use Lemma 2 to reconstruct buyer payoffs and calculate an incentive consistent selection strategy. The cutoff valuations for the non-deviators are given by the sequence $\{\bar{y}_1, \ldots, \bar{y}_{J-1}\}$[10]. On the interval $[\bar{y}_{j-1}, \bar{y}_j)$ of buyer valuations, each of the $j-1$ non–deviating sellers whose cutoff is no greater than $\bar{y}_{j-1}$ must be chosen by buyers with the same probability in any incentive consistent matching process according to Lemma 2. If $\pi'_J(x, \mu'_J)$ is the probability with which the deviator is chosen on this interval, then each of the $j-1$ non-deviators is chosen with probability $(1-\pi'_J(x,\mu'_J))/(j-1)$.

Taking this selection strategy to be fixed for the moment, the payoff to a buyer who selects the non–deviating seller offering the lowest reserve price is given by[11]

$$v_1(x, \mu', \pi'_J, J) = \int_{y_1}^{x}\left[1 - \int_{\nu}^{1} \frac{1-\pi'_J(s,\mu'_J)}{n_J(s,H)} f(s)ds\right]^{kJ-1} d\nu \tag{5}$$

where $n_J(x,H) = \max\{j : \bar{y}_j \leq x\}$. Buyers with valuations below y_1 get no payoff (just as they would in a normal monopoly auction). The payoff for buyers who select the

[10]There are only $J-1$ of these cutoff valuations now because one of the J sellers has been designated as a deviator.

[11]In the discussion that follows, it should be kept in mind that all payoff functions are conditional on the fixed distribution of cutoff valuations.

seller offering the lowest reserve price and whose valuations strictly exceed y_1 is just the integral of the buyer's trading probability when the buyer selects the seller with the lowest reserve price.

If the matching process is incentive consistent then the payoff that a buyer gets by selecting the non-deviating seller indexed j is, by Lemma 2

$$v_j(x,\mu',\pi'_J,J) = v_1(y_j,\mu',\pi'_J,J) + \int_{y_j}^{x}\left[1-\int_{\nu}^{1}\frac{1-\pi'_J(s,\mu'_J)}{n_J(s,H)}f(s)ds\right]^{kJ-1}d\nu$$

for any buyer whose valuation exceeds y_j.

The payoff that a buyer gets from the deviating seller is

$$v'(x,\mu',\pi'_J,J) = v'(y',\mu',\pi'_J,J) + \int_{y'}^{x} Q'_J(x,\mu'_J,\pi'_J)d\nu \tag{6}$$

where $y' = \inf\ int\left\{x : \pi'_J(x) > 0\right\}$ and Q'_J is computed from π'_J using (2) and the function associated with μ'_J that defines the trading probabilities.

In order for π'_J to be incentive consistent, it must satisfy three requirements. First, the buyer on the margin must be just indifferent between the choosing the deviator and choosing any one of the non–deviating auctioneers. The non-deviating auctioneers will all give the same expected payoff as the seller who offers the lowest reserve price when the matching process is incentive consistent. Hence π'_J must satisfy

$$v'(y',\mu',\pi'_J,J) = v_1(y',\mu',\pi'_J,J) \tag{7}$$

Secondly, on the region where $1 > \pi'_J(x) > 0$, v' and v_1 must be equal almost everywhere. Thus they must have the same derivative almost everywhere, or

$$Q'_J(x,\mu'_J,\pi'_J) = z_1^{kJ-1}(x,\pi'_J) \tag{8}$$

Finally, if there is a region along which $\pi'_J(x,\mu') = 1$, then

$$v'(y',\mu',\pi'_J,J) \geq v_1(y',\mu',\pi'_J,J) \tag{9}$$

These last three conditions can be used to calculate the incentive consistent matching strategy given the distribution of cutoff valuations $\{\bar{y}_1,\ldots,\bar{y}_{J-1}\}$ and the deviation μ'_J. Let $\bar{\pi}'_J(x,\mu'_J)$ denote an incentive consistent selection strategy for this situation. Let $\bar{v}_1(x,\mu',J) = v_1(x,\mu',\bar{\pi}'_J,J)$ denote the market payoff to buyers that is generated by the distribution of cutoff valuations along with the incentive consistent selection strategy.

So far, these functions have all been constructed without reference to the reserve prices that the sellers offer. These reserve prices can be uniquely recovered from the information given so far. Consider a buyer of type $\bar{y}_j$ who chooses to participate in the auction offered by seller j. Since the highest valuation buyer who chooses to participate will win the auction, the buyer with valuation $\bar{y}_j$ can only win if all other buyers have either have valuations below $\bar{y}_j$ or choose to participate in the mechanisms offered by other sellers. If the buyer wins, he will pay seller $j's$ reserve price. Hence his expected payoff from selecting seller j is given by

$$(\bar{y}_j - r_j)\left[1 - \int_{\bar{y}_j}^{1} \frac{1 - \bar{\pi}'_J(s,\mu')}{n_J(s,H)} f(s) ds\right]^{kJ-1}$$

Incentive consistency of the matching process requires that this be equal to $\bar{v}_1(\bar{y}_j, \mu', J)$. Solving this equation for r_j gives the reserve price that seller j would have to offer to support the distribution of cutoff valuations and the selection strategy.

Note furthermore that the trading probability is the derivative of the market valuation function. Hence we have

$$r_j = \bar{y}_j + \frac{\bar{v}_1(\bar{y}_j, \mu', J)}{\frac{d}{dx}\, \bar{v}_1(x, \mu', J)\big|_{x=\bar{y}_j}} \tag{10}$$

This expression indicates that the reserve prices of sellers are uniquely recovered from the market valuation function $\bar{v}_1$.

Let

$$\phi_J(y) = y + \frac{\bar{v}_1(y, \mu', J)}{\frac{d}{dx}\, \bar{v}_1(x, \mu', J)\big|_{x=y}}$$

Observe that the distribution of reserve prices that is induced by the distribution $\bar{H}_J$ of cutoff valuations and the deviation μ'_J is given by $\bar{H}_J\left(\phi_J^{-1}(r)\right)$.

The first result of this section is the following

Lemma 3 *For every deviation μ',*

$$\lim_{J\to\infty} \bar{\pi}'_J(x, \mu') = 0$$

$$\lim_{J\to\infty}\left[1 - \int_{x}^{1} \frac{1 - \bar{\pi}'_J(s,\mu')}{n_J(s,H)} f(s) ds\right]^{kJ-1} = e^{-k\int_x^1 \frac{dF}{H}}$$

and

$$\lim_{J\to\infty} \bar{v}_1(x, \mu', J) = \int_{y_1}^{x} e^{-k\int_s^1 \frac{dF}{H}} ds$$

for all $x \geq \bar{y}_1$.

The result has two implications. First it says that the trading probability and payoff that a buyer gets from selecting one of the non-deviating sellers are both independent of the mechanism that the deviating seller offers in the limit. Second, since the function $\int_{y_1}^{x} e^{-k\int_s^1 \frac{dF}{H}} ds$ is continuous in x and has a continuous derivative when $x \geq \bar{y}_1$ this implies that the function ϕ_J converges uniformly on $[\bar{y}_1, 1]$ to a continuous function ϕ_∞. Then by, for example [2, result 38, page51], the induced distribution of reserve prices converges weakly to $H \circ \phi_\infty$, and this convergence is independent of the mechanism that the deviating seller offers. Thus, in the limit, a seller who deviates against a fixed distribution H of cutoff valuations is also deviating against a fixed distribution $H \circ \phi_\infty$ of reserve prices.

Before continuing, it should be noted that the deviator could decide to offer an auction. Let $\mu'(y)$ denote the auction that generates the cutoff valuation y for the deviator. Then the payoff that the buyer gets from choosing the deviator when the selection strategy is incentive consistent, is given by

$$v'(x, \mu'(y), \bar{\pi}'_J, J) = \bar{v}_1(x, \mu'(y), J) + \int_y^x \left[1 - \int_\nu^1 \frac{1}{n_J(s,H)+1} f(s) ds\right]^{kJ-1} d\nu$$

since the buyer of type $s > y$ must randomize equally over $n_J(s, H) + 1$ sellers.

Then using (3), the payoff to the deviating seller who has cost w and who uses an auction that induces a cutoff valuation y is given by

$$\bar{\Phi}'_J(w, \mu'(y), H) = w + kJ \int_y^1 \left[(x-w)\left[1 - \int_x^1 \frac{1}{n_J(s,H)+1} f(s) ds\right]^{kJ-1}\right.$$

$$\left. -\bar{v}_1(x, \mu'(y), J)\right] \frac{1}{n_J(x,H)+1} f(x) dx$$

The bar over the function is used to indicate that the payoff is evaluated for a selection strategy that is incentive consistent. The following result follows from Lemma 3

Lemma 4

$$\lim_{J\to\infty} \bar{\Phi}'_J(w, \mu'(y), H) = w +$$

$$k \int_y^1 \left[(x-w) e^{-k\int_x^1 \frac{dF}{H}} - \int_y^x e^{-k\int_s^1 \frac{dF}{H}} ds\right] \frac{1}{H(x)} f(x) dx$$

In the limit, the market payoff that buyers get is independent of the auction that the deviator offers. Furthermore reserve prices and cutoff valuations are related by (10). In the limit the two concepts are then interchangeable. This makes it possible to state the equilibrium conditions for the limit game in the following concise way.

4.2 A Competitive Distribution of Auctions

A *competitive distribution of auctions* is a distribution of cutoff valuations H, and a cutoff rule $y : [0,1] \to [0,1]$ such that for almost all w

1. for every mechanism $\mu' \in \mathcal{M}$

$$w + \lim_{J\to\infty} kJ \int_{y(w)}^{1} \left[(x-w) \left[1 - \int_x^1 \frac{1}{n_J(s,H)+1} f(s) ds \right]^{kJ-1} \right.$$

$$\left. -\bar{v}_1(x, \mu'(y(w)), J) \right] \frac{1}{n_J(x,H)+1} f(x) dx \geq$$

$$\limsup_{J\to\infty} kJ \int_0^1 \left[(x-w) \, Q'_J(x, \mu'_J, \bar{\pi}'_J) - \bar{v}_1(x, \mu'_J, J) \right] \bar{\pi}'_J(x, \mu') f(x) dx$$

and

2. $H(y(w)) = G(w)$.

These conditions are readily interpretable. The left hand side of the inequality in the first condition represents the payoff that a seller would attain in the limit if he offered an auction having a reserve price generating the cutoff valuation $y(w)$. The payoff on the right hand side of the inequality represents the payoff attainable to the seller who offers the arbitrary mechanism μ'. Hence the inequality simply says that in the limit the payoff attainable by deviating to some alternative mechanism cannot dominate the payoff attainable by holding an auction. The lim sup is used in the definition instead of lim because it is possible for there to be more than one incentive consistent selection strategy when there are a finite number of buyers and sellers. The condition then requires that it does not pay to deviate even when the deviator anticipates his most favorable selection strategy.

Secondly, the distribution of cutoff valuations must be consistent with the distribution of seller valuations from which it arises. The terminal condition simply says that the proportion of sellers who have auctions with cutoff valuations below $y(w)$ must be equal to the proportion of sellers who have valuations below w.

5 The Main Result

We are now ready to state the main result of the paper.

Theorem 5 *Suppose that the support of G is contained in the support of F. Then there exists a competitive distribution of auctions in which each seller of type w holds an auction with reserve price w.*

The proof of this Theorem makes up the balance of this paper. The results that have been proved so far make it easy to verify that if the deviating seller wants to hold an auction, then he should set his reserve price equal to his cost.

To proceed, it is assumed temporarily that sellers' best replies to a distribution of auctions offered by the other sellers is to offer an auction themselves. Under this assumption we verify the existence of an equilibrium distribution of cutoff valuations and reserve prices.

Finally, we complete the argument by showing that if a seller's competitors are all holding auctions, the payoff that a seller can attain by offering any direct mechanism is bounded above in the limit by the payoff that he could have obtained by offering an auction. This step completes the argument.

6 Step 1 – Limiting Payoffs for Auctions

Referring back to Lemma 4, notice that the only way that a deviating seller can change his payoff by redesigning his auction is by changing the cutoff valuation that he enjoys[12]. He would like to modify this cutoff valuation in such a way that every buyer valuation x above his cutoff valuation generates an expected surplus $(x-w)\,e^{-k\int_x^1 \frac{dF}{H}}$ that exceeds the market valuation $\int_y^x e^{-k\int_s^1 \frac{dF}{H}}ds$ that he need to offer the buyer to get him to come in the first place. Conversely, for buyer valuations below his cutoff valuation, he wants this inequality to go in the opposite direction.

The seller can accomplish this outcome by setting his reserve price r' equal to his cost. This result can be understood with the help of Figure 1. The convex upward sloping curve $v(x,H) \equiv \int_0^x e^{-k\int_s^1 \frac{dF}{H}}ds$ is the limiting value of the payoff to any buyer who selects the seller who offers the lowest reserve price in the population H of sellers. From the point of view of any individual seller, this payoff is fixed in the limit game. The seller must provide each buyer type whom he expects to attract with at least this much utility. Notice that the derivative of the function $v(x,H)$ is equal to the limit value of the probability with which a buyer of type x will trade with this seller.

Now since the lowest buyer type that the seller attracts must be just indifferent between paying the seller's reserve price if he wins, and picking another auction we have

$$(x-r')\,e^{-k\int_x^1 \frac{dF}{H}} = v(x,H) = (x-w)\,e^{-k\int_x^1 \frac{dF}{H}} \tag{11}$$

Assuming for the moment that the seller's cost is low enough that this equation has a solution, this requires that the seller's reserve price be equal to his cost. This gives the result.

Theorem 6 *If the seller is constrained to hold an auction, his best reply is to set a*

[12]This argument will not be true if the deviating seller offers a mechanism other than an auction. If he does, his payoff will depend explicitly on how the choice probability $\bar{\pi}'_J$ converges.

reserve price equal to his own cost independent of his beliefs about the distribution of buyer types.

This result makes it simple to characterize the cutoff rule that must apply in equilibrium. Once $v(x, H)$ has been determined, simply take the ray that passes through the seller's valuation w that is tangent to the function $v(x, H)$ as illustrated in Figure 1. The point of tangency defines the cutoff valuation that the seller will want to attain. Solving the right hand equality in (11) for x yields the cutoff rule.

It should be apparent from Figure 1 that if the seller's valuation is high enough[13], then it will not be possible to find such a tangent ray. Sellers who have such high valuations fail to trade with probability 1 in equilibrium. There are certainly buyer types whose valuations are high enough to make it worthwhile for these sellers to trade ex post. However, because there is competition for buyers among the sellers, these high valuation sellers must offer all buyer types an ex ante level of surplus $v(x, H)$ that exceeds the gain to trade $w - x$ that the seller can realize by trading with these buyer types. These seller types are better off not trying to trade at all in equilibrium.

7 Step 2 – The Equilibrium Distribution of Cutoff Valuations

The cutoff rule is given by solving the right hand equation in (11) for x. Alternatively if the cutoff point x is given, the cost of the seller who will offer an auction that generates this cutoff is simply the inverse of this solution $x - \frac{v(x,H)}{dv(x,H)/dx}$. The complete equilibrium is now characterized by selecting the distribution H, which determines v, in such a way that the distribution of reservation prices that generates the distribution of cutoff valuations H is just equal to the known distribution G of seller valuations. That is

$$\begin{aligned} H(x) = \Pr\{\omega \leq x\} &= \Pr\left\{w \leq w^* : (x - w^*)\, e^{-k\int_x^1 \frac{dF}{H}} = v(x, H)\right\} \\ &= G\left(x - \tfrac{v(x,H)}{v'(x,H)}\right) \\ &= G\left(x - \frac{\int_0^x e^{-k\int_s^1 \frac{dF}{H}} ds}{e^{-k\int_x^1 \frac{dF}{H}}}\right) \\ &= G\left(x - \int_0^x \frac{e^{-k\int_s^1 \frac{dF}{H}}}{e^{-k\int_x^1 \frac{dF}{H}}} ds\right) \\ &= G\left(x - \int_0^x e^{-k\int_s^x \frac{dF}{H}} ds\right) \end{aligned} \tag{12}$$

[13] In Figure 1 this critical value is given by y^*. It is found by drawing the tangent line to the curve $v^*(x)$ at the point $x = 1$ and extending it down until it cuts the horizontal axis.

Here the appropriate result is the following:

Lemma 7 *There exists a solution to the functional equation (12).*

The equation (12) gives the characterization of the equilibrium distribution of reserve prices. In general this condition can be difficult to work with, however some insight into the condition can be gleaned from special cases.

For example, in the case where sellers all have identical costs equal to zero, then $G(x) = 1 \forall x \geq 0$. Then clearly $H(x) = 1$ for all non-negative x by (12) in the equilibrium distribution of auctions. Then the payoff that buyers get in the equilibrium distribution is given by

$$\int_0^x e^{-k[1-F(s)]} ds = v\,[x, 1]$$

from Lemma 3, which is the formula derived in Severinov and Peters [9] for this special case.

In a simple case like this, some obvious comparative static results arise. For instance if F' is a distribution for buyers valuations that first order stochastically dominates F, then buyers will be made worse off and sellers better off. Of course, from this it should be apparent that though the buyers and sellers equilibrium strategies do not vary as F (and G) change, their payoffs certainly do. So for example, if buyers and sellers had to decide whether or not to enter this market, their optimal decision would be sensitive to beliefs.

An alternative application occurs in the case where the sellers can have only one of two possible costs, say 0 and α. Suppose that λ is the proportion of all sellers who have cost 0. Then in the equilibrium distribution of auctions the proportion λ of all sellers will set reserve price 0 and attract buyers of all types with positive probability. The proportion $(1 - \lambda)$ of all sellers will set reserve price α and attract buyers whose valuation exceeds the solution to the equation

$$y = \alpha + \frac{\int_0^y e^{-k\left[\frac{F(y)-F(s)}{\lambda} + [1-F(y)]\right]}}{e^{-k[1-F(y)]}}$$

by (12). From this it is evident that increasing the proportion of sellers who have low valuations will raise buyers' payoffs, while increasing α will lower them.

8 Step 3 – Deviations to Other Direct Mechanisms

It remains to show that no seller can increase his profit by deviating to some mechanism other than an auction. We do this in the following way. Whatever mechanism the deviator picks, we show that the payoff that any buyer gets in any finite approximation is bounded below by the payoff that the buyer would receive if the deviator were completely removed from the market. This is intuitive since even a bad alternative is better than no alternative at all.

This lower bound on the payoff that buyers receive generates an upper bound on the payoff that the deviator can get. This can be seen by recalling that the seller's payoff is an expectation of the difference between the expected gain from trade, and the payoff that the seller has to make to the buyer to compensate him for what is available elsewhere on the market.

We then nudge this upper bound on the deviator's payoff by calculating the maximum payoff that the deviating seller can obtain if he is required to provide buyers with a payoff that is at least as large as their lower bound. This payoff can be calculated explicitly since the problem formulated in this way is a special case of McAfee's theorem.

We then show that the limiting value of this upper bound payoff is the payoff that we calculated above for the case in which the seller offers an auction with reserve price equal to his cost. This completes our characterization of the equilibrium distribution of auctions.

Now consider the distribution $\bar{H}_J(x)$ of cutoff valuations described above. Recall that there are $J-1$ non-deviating sellers charging reserve prices generating the cutoff valuations $\bar{y}_1$ through $\bar{y}_{J-1}$. The lower bound on buyers payoffs will be found by using a modified situation in which there are only $J-1$ sellers offering contracts to the kJ existing buyers. Since all sellers offer auctions in this modified situation, the payoffs are relatively simple. In this case, because there is no deviating seller, the probability of trading with seller 1 is given by

$$\left[1-\int_x^1 \frac{1}{n_J(s,H)} f(s)ds\right]^{kJ-1}$$

Then define the payoff

$$v_J^*(x,H)=\int_0^x \left[1-\int_y^1 \frac{1}{n_J(s,H)} f(s)ds\right]^{kJ-1} dy$$

Recall that $\bar{v}_1(x,\mu',J)$ is the payoff that a buyer of type x gets by selecting seller 1 when some seller has deviated from the array of auctions by offering a mechanism μ'. We now show that this payoff defines a lower bound on the payoff that buyers receive for whatever mechanism that the deviating seller offers.

Lemma 8 *For any arbitrary mechanism μ offered by the deviating seller, if the selection strategy π_J is incentive consistent then for almost all x in the interval* $(0,1)$,

$$\bar{v}_1(x,\mu,J) \geq v_J^*(x,H)$$

We can now derive an upper bound on the deviator's profits. To do this we will use a result proved in McAfee [5]. In interpreting this result a couple of things should be borne in mind. McAfee approaches the competing auction problem by assuming that sellers believe that changes in their own mechanism will have no effect on the payoffs that buyers can get by choosing other sellers. With this assumption, he is able to prove that the best mechanism that a seller can use in response to almost anything that his competitors are using will be an auction.

Problems arise in the interpretation in that when there are a finite number of buyers and sellers, this expectation on the part of sellers is counter factual[14]. If one seller raises his reserve price in an auction he will force some buyers to choose other sellers. The additional crowding that this creates at the new sellers will reduce the payoff of all buyers with these new sellers.

In the approach adopted in this paper, we imagine that deviating sellers can foresee the effects that their deviations have on the payoffs that buyers can get from other sellers. We use McAfee's theorem to impose an upper bound on the payoff that sellers can expect to get when they deviate (generally this upper bound will not be attainable). We then show that this upper bound converges as the number of buyers and sellers get large to the exact payoff that the seller could have attained by offering to hold an auction. In this way, we establish that auctions constitute an equilibrium mechanism in the limit game.

Note first that

$$\bar{\Phi}'_J(w, \mu', \pi_J) \equiv$$

$$w + kJ\int_0^1 \left[(x-w)\,Q'_J(x,\mu',\bar{\pi}_J) - v'(x,\mu',\bar{\pi}_J,J)\right] dz'(x,\bar{\pi}_J) \le$$

$$w + kJ\int_0^1 \left[(x-w)\,Q'_J(x,\mu',\bar{\pi}_J) - v_1(x,\mu',\bar{\pi}_J)\right] dz'(x,\bar{\pi}_J) \le$$

$$w + kJ\int_0^1 \left[(x-w)\,Q'_J(x,\mu,\bar{\pi}_J) - v^*_J(x,H)\right] dz'(x,\bar{\pi}_J)dx$$

The first inequality follows from Lemma 2 while the second inequality follows from Lemma 8. This expression in turn is bounded above by the solution to the following maximization problem

$$w + \max_{Q',z',\omega,\pi'} kJ\int_\omega^1 \left[(x-w)\,Q'(x) - v^*_J(x,H)\right] \pi'(x)f(x)dx$$

subject to

$$z'(x) = 1 - \int_x^1 \pi'(s)f(s)ds \tag{13}$$

[14]This is the term used by McAfee.

$$0 \leq \pi'(x) \leq 1 \tag{14}$$

$$v'(x, \mu, \pi') = v_J^*(\omega, H) + \int_\omega^x Q'(s, \mu, \pi_J) ds \tag{15}$$

$$\pi'(x) > 0 \Rightarrow v'(x, \mu, \pi') \geq v_J^*(x, H) \tag{16}$$

$$\forall x \in [0,1] \ \int_x^1 Q'(s)\pi'(s) f(s) ds \leq \frac{1}{kJ}\left(1 - z'^{kJ}(x, \pi_J)\right) \tag{17}$$

This is by definition since the corresponding functions from the finite approximation must satisfy the constraints (13) through (17) whenever the selection strategy is incentive consistent.

The solution to this problem is given in McAfee [5] Theorem 1[15]. He shows that the solution is an auction with a reserve price equal to the seller's cost. Furthermore at the optimal solution to the problem

$$v'(x, \mu, \pi') = v_J^*(x, H) \tag{18}$$

$$z'(x) = \frac{dv_J^*(x, H)}{dx} \tag{19}$$

$$Q'(x) = z'^{kJ-1}(x) \tag{20}$$

for almost all $x \in [0,1]$[16].

Notice that we do not claim as McAfee does that the seller can actually attain this payoff by holding a second price auction. In this instance, the payoff corresponds strictly with an upper bound. If the seller were to offer this mechanism, the payoffs that the buyers get at the other sellers would no longer equal v_1^* as required for the result to be true.

Notice furthermore that almost everywhere satisfaction of equation (19) requires that buyers select the deviator with the same probability with which they select the non-deviating sellers in the solution for v_J^*. In this hypothetical problem there are only $J-1$ sellers. So buyers with very high valuations will choose each of these non-deviators with probability $1/(J-1)$. The solution outlined above asks these high valuation buyers to choose the deviator with this same probability. Hence this solution is not even feasible. Hence it is important to note that this solution represents only an upper bound on the deviator's profits.

Now let Φ_J^* denote the value for this upper bound when there are J sellers and kJ buyers. Substituting McAfee's solution yields

$$\Phi_J^* =$$

[15]The problem here is a special case of the problem discussed by McAfee since we have effectively set the discount factor equal to one.

[16]These are just the corresponding expression from McAfee derived by setting the discount factor equal to zero as outlined above.

$$= w + kJ \int_{\omega}^{1} \left\{ (x - w) \left[1 - \int_{x}^{1} \frac{1}{n_J(s, H)} f(s) ds \right]^{kJ-1} \right.$$
$$\left. - v_J^* [x] \right\} \frac{1}{n_J(x, H)} f(x) dx$$

The limit payoff to the deviating seller who offers an auction generating cutoff valuation ω is given by Lemma 4 as

$$\lim_{J \to \infty} \bar{\Phi}'_J \left(w, \mu' (\omega), H \right)$$

The final theorem of the paper is

Theorem 9

$$\lim_{J \to \infty} \Phi_J^* = \lim_{J \to \infty} \bar{\Phi}'_J \left(w, \mu' (\omega), H \right)$$

In words, the maximum payoff that the deviator can attain by deviating to any direct mechanism, is equal to the payoff that the deviator can get by offering an auction with reserve price equal to his cost.

9 Concluding Remarks

The case in which some sellers have costs low enough that they can surely gain from trading with any buyer works somewhat differently from the outcome above. To see how this problem works, it suffices to look at the degenerate case in which all sellers have cost 0 while all buyers have valuation 1. This is not quite a special case of the model discussed above since the distribution F is degenerate and does not have a density. Nonetheless, it is reasonable to assume that every buyer who selects a particular seller submits the same bid, 1, and that the winner is chosen randomly and pays the sellers published reserve price. This model has been analyzed in [8]. The problem does not collapse to a Bertrand competition in this case because of the assumption that all buyers use the same selection strategy. If a seller cuts his reserve price, all buyers will increase the probability with which they select the seller slightly, so that the sellers payoffs will be continuous in the reserve price that he offers. The equilibrium reserve price will have the property that the marginal loss in profits associated with a cut in the reservation price is just equal to the gain associated with the increase in the probability of being chosen by some seller. Not implausibly, the equilibrium reserve price will lie somewhere between 0 and 1 at a point that depends on the ratio of the number of buyers to sellers. Hence the result that sellers will set reserve prices equal to their costs breaks down.

Of course the equilibrium price will also depend on the common value for the buyers valuation, so the reserve price will be sensitive to seller's beliefs about the buyers that they are dealing with.

Buyers selection behavior is poorly coordinated in the equilibrium studied here. This results in many untraded units of output. One would expect sellers to try to sell these units again in the future, perhaps in a new market. This problem can be handled provided it is reasonable to assume that the actions that buyers and sellers take today does not affect the payoffs that are attainable to them in the future. The analysis is almost unaffected except that sellers' cost are equal to the maximum of their production cost, and the opportunity cost that they face by trying to auction their unit of output at some time in the future.

The assumption that buyers and sellers current actions do not affect their future payoffs is not innocuous when there are finitely many buyers and sellers. A seller who offers a very attractive mechanism, for example, will attract many buyers, most of whom will end up trying to trade again in the future. This seller will depress buyers' future payoffs by offering this mechanism. It seems reasonable to expect this new kind of externality to disappear as the number of buyers and sellers gets large, but this does not appear to follow directly from any of the arguments made here.

The poor coordination of buyers' choices also means that sellers do not end up selling to the buyer with the highest valuation in the short run[17]. This suggests the possibility that today's buyers might become tomorrow's sellers. It is not obvious that this phenomena should be important since sellers have the same opportunity to sell in the future that buyers do. The seller's reserve price should be at least as high as this opportunity cost, so that the winning bidder in the auction should always have a use value that exceeds the value of buying and reselling the commodity.

One possible generalization[18] of the results here might be to the case where buyers and sellers have non-unitary demands and supplies. If sellers have cost functions that rise above the maximum buyer valuation quickly enough, and buyers have unitary demands, the optimal mechanism in the monopoly case should be an auction. A plausible conjecture for this case is that the sellers will offer auctions with a series of reserve prices, one for each of the marginal units that they sell. These reserve prices ought to be equal to the corresponding marginal cost of producing the additional unit.

The prediction that sellers will offer auctions is not so implausible. In real estate, for example, if there are multiple bidders on the house, the presumption that the buyer

[17]Of course, since the sellers are using auctions, they sell to the buyer with the highest valuation of all buyers who select their mechanism.

[18]This was suggested by a referee.

with the high valuation will buy the house at a price equal to the second highest bidder's valuation seems compelling. However, in the housing market, sellers do not publish commitments to reserve prices. Hopefully the limit equilibrium concept developed here will provide some help in constructing competitive models to explain these more complicated pricing rules.

10 Appendix – Proofs

10.1 Proof of Lemma 2

Proof. The second conclusion (ii) is proved in Lemma 5 in Severinov and Peters [9]. Part (iv) is proved in Severinov and Peters [9] as Lemma 4.

To prove part (iii), suppose the contrary. Then for some seller i there is an open interval $(1-\epsilon, 1)$ along which $\pi_i(x,\mu) = 0$ almost everywhere, and an open interval $(1-\beta-\epsilon, 1-\epsilon)$ along which $\pi_i(x,\mu) > 0$ almost everywhere. It follows that $z_i^{kJ-1}(x,\pi) = 1$ for almost all $x > 1-\epsilon$ since no one with a valuation above $1-\epsilon$ will ever come to seller i. There must be some seller j' (not equal to i by the contrary hypothesis of this argument) and some small ϵ' such that $\pi_{j'}(x,\mu) > 0$ for almost all $x > 1-\epsilon'$. Since $\pi_i(x,\mu) > 0 \ \forall x \in [1-\beta-\epsilon, 1-\epsilon]$ it must be that $v_i(x,\mu,\pi) \geq v_{j'}(x,\mu,\pi)$ on this interval if the matching process is incentive consistent. Thus $v_{j'}$ (which is continuous by Lemma 1) must be rising at least as quickly as v_i somewhere on the interval $[1-\beta-\epsilon, 1]$. Since v_i and $v_{j'}$ are at least weakly convex, it follows that the derivative of $v_{j'}$ is at least as large at the derivative of v_i on the interval $[1-\epsilon', 1]$. Since the derivative of $v_{j'}$ is equal to the probability of trading with j' by Lemma 1, it follows that seller j' must guarantee every buyer with a valuation in the interval $(1-\epsilon', 1)$ a sure trade, which is impossible since for each x in this interval, there is a strictly positive probability that another buyer in the interval will also select the seller.

Now to prove part (i), note that $v_i(x,\mu,\pi) = v_j(x,\mu,\pi)$ almost every where on the set $E_i \cap E_j$. Almost everywhere equality of the functions $v_i(x,\mu,\pi)$ and $v_j(x,\mu,\pi)$ on any open set requires almost everywhere equality of their first and second derivatives. The definition of the buyers payoffs in auctions (differentiate v twice to get equality of the selection probabilities) then yields (i) on the set $E_i \cap E_j$.

Suppose w.l.o.g. that $E_j \subset E_i$. Suppose further that there is an open set A above y_j such that $\pi_j(x,\mu) = 0$ almost everywhere on A but $\pi_i(x,\mu) > 0$ on some open subset of A. Then on the set A, the second derivative of v_j is

$$\frac{d}{dx}\left[1-\int_x^1 \pi_j(s,\mu)f(s)ds\right]^{kJ-1} = (kJ-1)\left[1-\int_x^1 \pi_j(s,\mu)f(s)ds\right]^{kJ-2} \pi_j(x,\mu)f(x) = 0$$

while the second derivative of v_i on this set is readily seen to be positive on some open interval. It follows that v_i strictly exceeds v_j along the interval A. If A has an upper bound less than 1, then v_i must strictly exceed v_j in every neighborhood of this upper bound by Lemma 1, a contradiction. But the upper bound of A cannot be 1 by property (iii) above.

It follows that if $\pi_j(x,\mu) = 0$ then $\pi_i(x,\mu) = 0$ as well. It follows that the derivatives of $v_i(x,\mu)$ and $v_j(x,\mu)$ are almost everywhere equal above y_j and hence $v_j(x,\mu,\pi)$ and $v_i(x,\mu,\pi)$ are almost everywhere equal. ■

10.2 Proof of Payoff 3

Proof. Note that the payoff that the buyer can get by going to one of the non-deviators is

$$v_1(x,\mu',\pi'_J,J) = \int_{y_1}^{x}\left[1 - \int_{\nu}^{1}\frac{1-\pi'_J(s,\mu'_J)}{n_J(s,H)}f(s)ds\right]^{kJ-1}d\nu$$

This is at least as large as

$$\int_{y_1}^{x}\left[1 - \int_{\nu}^{1}\frac{1}{n_J(s,H)}f(s)ds\right]^{kJ-1}d\nu =$$

$$\int_{y_1}^{x}\left[1 - \frac{F(\bar{y}_{j+1}) - F(\nu)}{j} - \sum_{t=j+1}^{J}\frac{F(\bar{y}_{t+1}) - F(\bar{y}_t)}{t}\right]^{kJ-1}d\nu =$$

$$\int_{y_1}^{x}\left[1 - \frac{F(\bar{y}_{j+1}) - F(\nu)}{\bar{H}(\bar{y}_j)J} - \sum_{t=j+1}^{J}\frac{F(\bar{y}_{t+1}) - F(\bar{y}_t)}{\bar{H}(\bar{y}_t)J}\right]^{kJ-1}d\nu$$

$$= \int_{y_1}^{x}\left[1 - \frac{\int_{\nu}^{1}\frac{d\bar{F}_J}{\bar{H}_J}}{J}\right]^{kJ-1}d\nu$$

by the definition of the integral of a simple (step) function, where $\bar{F}_J(x) = F(y_j)$ for $x \in [y_j, y_{j+1})$.

Now consider the integral $\int_{\nu}^{1}\frac{d\bar{F}_J}{\bar{H}_J}$. This integral will have a limit as J goes to infinity if and only if the integral $\int_{\nu}^{1}\frac{d\bar{F}_{2^n J}}{\bar{H}_{2^n J}}$ has a limit as n goes to infinity that is independent of J. By construction, the sequence of functions $\bar{H}_{2^n J}$ is a decreasing sequence converging almost everywhere to H. The function $\bar{F}_J$ is an increasing sequence that converges almost everywhere to F. Hence the integral converges by the monotone convergence theorem to $\int_{\nu}^{1}\frac{dF}{H}$.

Now for any sequence $a_n \to a$, the expression $\lim_{n\to\infty}[1 - a_n/n]^{kn} \to e^{-ka}$. It follows that the expression

$$\left[1 - \frac{\int_{\nu}^{1}\frac{d\bar{F}_J}{\bar{H}_J}}{J}\right]^{kJ-1} \to e^{-k\int_{\nu}^{1}\frac{dF}{H}}$$

Finally since $\left[1-\frac{\int_\nu^1 \frac{d\bar{F}_J}{H_J}}{J}\right]^{kJ-1}$ is bounded above by 1, the integral

$$\int_{y_1}^{x}\left[1-\frac{\int_\nu^1 \frac{d\bar{F}_J}{H_J}}{J}\right]^{kJ-1} d\nu \rightarrow \int_{y_1}^{x} e^{-k\int_\nu^1 \frac{dF}{H}} d\nu$$

by the bounded convergence theorem[19]. This is strictly positive for all $x > y_1$.

Now consider the deviating seller. The payoff that a buyer gets when he chooses the deviating seller can be written as an expectation over the number of buyers who actually select the deviating seller as follows

$$\sum_{i=0}^{kJ-1}\binom{kJ-1}{i}\left\{1-\int_{y_1}^{1}\pi'_J(s,\mu')f(s)\,ds\right\}^{kJ-1-i}\cdot$$

$$\int_{y_1}^{1}\cdots\int_{y_1}^{1} q'^1_J(x,s_2,\ldots,s_i)\,\pi'_J(s_2,\mu')f(s_2)\,ds_2\ldots\pi'_J(s_i,\mu')f(s_i)\,ds_i \leq$$

$$\sum_{i=0}^{kJ-1}\binom{kJ-1}{i}\left\{1-\int_{y_1}^{1}\pi'_J(s,\mu')f(s)\,ds\right\}^{kJ-1-i}\int_{y_1}^{1}\cdots\int_{y_1}^{1} q'^1_J(x,s_2,\ldots,s_i)f(s_2)\,ds_2\ldots f(s_i)\,ds_i$$

The inequality follows from the fact that $\pi'_J(x,\mu)\leq 1$ while all the other arguments in the integral are positive.

Now the probability function $q'^1_J(x,\cdot)$ must satisfy the feasibility condition

$$i\int_x^1\int_{y_1}^{1}\cdots\int_{y_1}^{1} q'^1_J(\nu,s_2,\ldots,s_i)f(s_2)\,ds_2\ldots f(s_i)\,ds_i df(\nu)\leq 1-F(x)^i$$

That is to say, that when i buyers participate in the deviating seller's mechanism, the probability that the seller trades with a buyer whose valuation lies above x can be no larger than the probability that there is a buyer with a valuation above x.[20] Thus, as i becomes large, the integral on the left hand side of this inequality must converge to zero. This is only possible if

$$\int_{y_1}^{1}\cdots\int_{y_1}^{1} q'^1_J(x,s_2,\ldots,s_i)f(s_2)\,ds_2\ldots f(s_i)\,ds_i$$

is converging almost everywhere to zero[21].

Thus for any ϵ, there is some n^* such that the payoff that a buyer gets from the deviating seller is bounded above by

$$\sum_{i=0}^{n^*}\binom{kJ-1}{i}\left\{1-\int_{y_1}^{1}\pi'_J(s,\mu')f(s)\,ds\right\}^{kJ-1-i}\int_{y_1}^{1}\cdots\int_{y_1}^{1} q'^1_J(x,s_2,\ldots,s_i)f(s_2)\,ds_2\ldots f(s_i)\,ds_i+\epsilon$$

[19] Note here that y_1 is a constant defined to be equal to $\inf \operatorname{supp} H(y)$.

[20] This 'implementability condition' is discussed in [5] and [4].

[21] In words, an anonymous mechanism must have the property that the probability that any buyer trades with the seller must go to zero and the number of competitors that he faces becomes large.

If $\pi'_J(s,\mu')$has a strictly positive limit as J goes to infinity, then $\int_{y_1}^1 \pi'_J(s,\mu') f(s)\, ds$ will have a strictly positive limit and the upper bound written above will converge to ϵ as J goes to infinity. Since ϵ is arbitrary, this bound can be made strictly smaller than the lower bound

$$\int_{y_1}^{x} e^{-k\int_\nu^1 \frac{dF}{H}} d\nu$$

associated with choosing a non-deviating seller. Thus a selection strategy for which $\pi'_J(s,\mu')$ is bounded above zero as J goes to infinity on a set of non-zero measure, must eventually violate incentive consistency. This concludes the proof of the first result.

To prove the rest of the Lemma we can now write out the trading probability for a buyer who selects the non-deviating seller as

$$\left[1 - \int_x^1 \frac{1-\pi'_J(s,\mu')}{n_J(s)} f(s)ds\right]^{kJ-1} =$$

$$\left[1 - \frac{F(\bar{y}^j)-F(x)}{j} - \sum_{t=j+1}^{J-1} \frac{F(\bar{y}^t)-F(\bar{y}^{t-1})}{t} + \int_x^1 \frac{\pi'_J(s,\mu')}{n_J(s)} f(s)\, ds\right]^{kJ-1}$$

$$= \left[1 - \frac{F(\bar{y}^j)-F(x)}{\bar{H}(\bar{y}^j)J} - \sum_{t=j+1}^{J} \frac{F(\bar{y}^t)-F(\bar{y}^{t-1})}{\bar{H}_J(\bar{y}^t)J} + \int_x^1 \frac{\pi'_J(s,\mu')}{n_J(s)} f(s)\, ds\right]^{kJ-1}$$

$$= \left[1 - \frac{\int_x^1 \frac{d\bar{F}_J}{\bar{H}_J}}{J} + \frac{\int_x^1 \frac{\pi'_J(s,\mu')}{\bar{H}_J(s)} f(s)\, ds}{J}\right]^{kJ-1} \tag{21}$$

Since $\pi'_J(s,\mu')$ has almost everywhere limit 0 and $\bar{H}_J$ is positive almost everywhere, the limit of expression (21) is given by

$$e^{-k\int_x^1 \frac{dF_J}{H}}$$

This gives the second result.

Finally, since the payoff is the integral of the trading probability, the third result follows from the bounded convergence theorem. ■

10.3 Proof of Lemma 4

Proof.

$$\lim_{J\to\infty} \bar{\Phi}'_J(w,\mu'(y),H) =$$

$$\lim_{J\to\infty} kJ\int_y^1 \left[(x-w)\left[1-\int_x^1 \frac{1}{n_J(s,H)+1} f(s)ds\right]^{kJ-1}\right.$$

$$\left. -\bar{v}_1(x,\mu'(y),J)\right] \frac{1}{n_J(x,H)+1} f(x)dx =$$

$$\lim_{J\to\infty} k\int_y^1 \left[(x-w)\left[1-\int_x^1 \frac{1}{\bar{H}_J(s,H)J+1}f(s)ds\right]^{kJ-1}\right.$$
$$\left.-\bar{v}_1(x,\mu'(y),J)\right]\frac{1}{\bar{H}_J(x)+1/J}f(x)dx =$$
$$k\int_y^1\left[(x-w)\,e^{-k\int_x^1\frac{dF}{H}}-\int_y^x e^{-k\int_s^1\frac{dF}{H}}ds\right]\frac{1}{H(x)}f(x)\,dx$$

by Lemma 3 and the bounded convergence theorem. ■

10.4 Proof of Lemma 7.

Proof. Let Ξ denote the space of probability distribution functions on $[0,1]$ endowed with the topology of weak convergence. This space is compact. For each pair $\{x,s\}\in[0,1]^2$ with $s\le x$ if $H(t)=0$ almost everywhere on some open subinterval of $[s,x]$ then define $e^{-k\int_s^x\frac{dF}{H}}=0$. Otherwise, if $H(t)>0$ a.e. on $[s,t]$ then $1/H(x)$ is an $F-$ integrable function. With this qualification $e^{-k\int_s^x\frac{dF}{H}}$ is a well defined function mapping Ξ into $\mathbf{R}$.

Let H^n be a sequence of distribution functions from Ξ that converges weakly to H. Note first, that for any $x\in[0,1]$, the sequence $e^{-k\int_s^x\frac{dF}{H^n}}$ converges in $\mathbf{R}$ for almost all $s\le x$. To see this, let $\bar{L}$ denote the subinterval, if any, in $[0,x]$ on which $H(t)=0$. Define L' such that $\bar{L}\cup L'=[0,x]$. The function H is a distribution function so that both $\bar{L}$ and L' are convex intervals. Let $\bar{L}^n$ and L'^n denote the corresponding intervals for the distribution function H^n. Since H^n converges weakly to H, $\bar{L}^n$ converges to $\bar{L}$ and L'^n converges to L'. Thus if $s\in\bar{L}$, then there is some large n for which $s\in\bar{L}^n$ which implies that

$$e^{-k\int_s^x\frac{dF}{H^n}}\to e^{-k\int_s^x\frac{dF}{H}}=0$$

Conversely if $s\in L'$ then for large enough n, $H^n(t)$ and $H(t)$ are both strictly positive for all $t\in[s,x]$. The fact that $H^n(t)$ converges to $H(t)$ at every continuity point of $H(\cdot)$ implies that $1/H^n(t)$ converges to $1/H(t)$ at every continuity point. Since H is increasing, it is continuous almost everywhere. Hence $F-$almost surely, s is a continuity point of H. This implies that for almost every s, $H^n(s)\to H(s)>0$. Thus for almost every s, there is an $\epsilon>0$, $\exists n: H^m(x)\ge H^m(s)>H(s)-\epsilon>0$ for all $m>n$. Now for large enough n, $1/H^n(x)$ is bounded above by the finite constant $1/(H(s)-\epsilon)$ on the interval $[s,x]$. Since $H^n(x)\to H(x)$ almost everywhere on the interval $[s,x]$ we conclude that $e^{-k\int_s^x\frac{dF}{H^n}}\to e^{-k\int_s^x\frac{dF}{H}}$ by the bounded convergence theorem.

As the function $e^{-k\int_s^x\frac{dF}{H^n}}$ is bounded above by 1, the integral $\int_0^x e^{-k\int_s^x\frac{dF}{H}}ds$ converges by the dominated convergence theorem. Finally, as the distribution function G is continuous almost everywhere, we have that

$$G\left(x-\int_0^x e^{-k\int_s^x\frac{dF}{H^n}}ds\right)\to_{a.e.} G\left(x-\int_0^x e^{-k\int_s^x\frac{dF}{H}}ds\right)$$

This almost everywhere convergence implies weak convergence of the functional described by the left hand side of this equation. This implies that the transformation defined by the equilibrium condition is a continuous (in the weak convergence topology) mapping from the compact space Ξ into itself. By the Fan-Glicksberg fixed point theorem the transformation has a fixed point. ■

10.5 Proof of Lemma 8

Proof. From the definition given in equation (5)

$$\bar{v}_1(x,\mu',J) =$$

$$\int_{y_1}^{x}\left[1-\int_s^1 \frac{1-\bar{\pi}_J(y,\mu)}{n_J(y,H)}f(y)dy\right]^{kJ-1} ds$$

$$\geq \int_{y_1}^{x}\left[1-\int_s^1 \frac{1}{n_J(y,H)}f(y)dy\right]^{kJ-1} ds = v_J^*(x,H)$$

The inequality follows from the fact that $\pi_J(x,\mu)\geq 0$. ■

10.6 Proof of Theorem 9

Proof.

$$\lim_{J\to\infty}\Phi_J^* =$$

$$\lim_{J\to\infty} w+kJ\int_\omega^1\left\{(x-w)\left[1-\int_x^1\frac{1}{n_J(s,H)}f(s)ds\right]^{kJ-1}\right.$$

$$\left.-\int_0^x\left[1-\int_y^1\frac{1}{n_J(s,H)}f(s)ds\right]^{kJ-1}dy\right\}\frac{1}{n_J(x,H)}f(x)dx =$$

$$\lim_{J\to\infty} w+k\int_\omega^1\left\{(x-w)\left[1-\int_x^1\frac{1}{\bar{H}_J(s)J}f(s)ds\right]^{kJ-1}\right.$$

$$\left.-\int_0^x\left[1-\int_y^1\frac{1}{\bar{H}_J(s)J}f(s)ds\right]^{kJ-1}dy\right\}\frac{1}{\bar{H}_J(s)}f(x)dx =$$

$$w+k\int_y^1\left[(x-w)e^{-k\int_x^1\frac{dF}{H}}-\int_y^x e^{-k\int_s^1\frac{dF}{H}}ds\right]\frac{1}{H(x)}f(x)dx$$

This last expression is equal to the limit payoff given in Lemma 4 for the deviating seller who offers an auction. ■

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