Department of Economics
and
Institute for Policy Analysis
University of Toronto
150 St. George St.
Toronto, Canada
M5S 1A1

WORKING PAPER
NUMBER UT-ECIPA-PETERS-95-02

# Competition Among Sellers who offer auctions instead of Prices

Michael Peters
Department of Economics
University of Toronto

Sergei Severinov
Department of Economics
Stanford University

July 10, 1995


Department of Economics
University of Toronto
ISSN 0829-4909
Author's email: peters@epas.utoronto.ca
On-line Version: http://www.epas.utoronto.ca:5680/wpa/wpa.html

**Abstract**

In this paper we study a large market in which sellers compete by offering auctions to buyers instead of simple fixed price contracts. Two variants of the model are studied. One extends a model first analyzed by Wolinsky [12] in which buyers learn their valuations only after meeting sellers. The other variant extends the model of McAfee [4] in which buyers know their valuations before they choose among available auctions. The equilibrium array of auctions is characterized for each case and the efficiency properties of the equilibria are analyzed.

# 1 Introduction

The purpose of this paper is to try to endogenize the number of buyers who participate in an auction in a reasonable way[1]. We consider a market with many buyers and many sellers of a homogeneous good. In the first 'stage' of the market process sellers compete by offering auctions. In the second stage buyers select among them. Then, if a seller tries to increase the surplus that he extracts from buyers by, say, increasing his reserve price, the number of buyers that he can expect to participate in the auction will fall. The response of buyer 'demand' to a change in the seller's reserve price is determined by the requirement that the buyers' choices among auctions be a best reply to the choices of the other buyers.

We consider two possible cases that differ according to the amount of private information that buyers have at the time that they are forced to choose among the various auctions that are available. In the first case, buyers learn their valuations after they choose an auction and actually inspect the product that the seller has to offer. In the second case, buyers are assumed to know their own valuations before they choose among the various auctions. If the bidder has to study the commodity for some time to learn his valuation, the former assumption is reasonable. For example, a renovation contractor has to study the specifics of the job before he can submit a reasonable estimate of his cost of doing it. On the other hand, if the bidder is simply searching for some predefined set of attributes, the latter assumption is better. This might apply in certain segments of the housing market, where house buyers are simply looking for a certain number of bathrooms, and proximity to public transit.

The simple two stage process by which equilibrium auctions are determined is analytically intractable in many environments. The continuation equilibrium correspondence for the buyers' selection process is not generally well behaved. For example, it does not change continuously with variations in the auction that a deviating seller offers. This means that the profit functions that sellers face in the first stage of the game in which they design their auctions, are not continuous in their feasible actions. As a result, equilibrium (even in mixed strategies) for the process will typically not exist[2].

[1]The formal arguments in this paper all assume that the seller organizes the auction and chooses the reserve price. Naturally the arguments apply as well to procurement auctions. We ignore this distinction in the introduction and leave it up to the reader to make all the appropriate sign changes in the informal part of the paper.

[2]McAfee [4] shows that if there are two auctioneers competing for two different buyers who use identical random strategies in selecting among the auctions, then an equilibrium will not exist. His example relies on a non-convexity of the seller's profit functions in the first stage of the game. This problem could conceivably be resolved by allowing sellers to resort to mixed strategies. In general, mixed strategies will not resolve existence problems because the sellers profits are not continuous in the

The main contribution of this paper is a limit equilibrium concept that can be applied to markets such as these when there are infinitely many buyers and sellers. The equilibrium concept begins with a finite number of buyers and sellers and fixes the distribution of offers by the sellers in the first stage of the game. The continuation equilibrium for the buyers' choice problem is then explicitly derived and the induced profits of all the seller's are calculated. The payoffs of sellers in the infinite game are then derived by taking limits of the payoffs to sellers as the number of buyers and sellers gets large and the distribution of seller offers is held constant.

In the case where sellers compete by offering auctions, the resulting equilibrium concept is tractable and has desirable convergence properties.

The only other equilibrium concept that has been proposed for this problem in the literature is given in McAfee [4]. He assumes that there are finitely many buyers and sellers, but that sellers ignore the impact that changes in their mechanisms have on the payoffs associated with the continuation equilibrium in the buyers' subgame. He is able to prove that when sellers compete by offering arbitrary direct mechanisms, there is an equilibrium in which all sellers offer second price auctions with reserve prices equal to their costs.

Here we are more interested in how buyers respond to changes in the mechanisms that sellers offer. Our limit equilibrium gets around the analytical difficulties presented by the two stage game, just as McAfee's solution did. Yet our solution retains the restrictions imposed by subgame perfection in the finite version of the game. The disadvantage is that we need to restrict attention to the case where sellers compete by offering different auctions to buyers.

One by product of this is a verification of McAfee's conjecture that deviations by one seller will not affect the payoff that buyers can get with other sellers when the number of buyers and sellers is large. However, there is a more subtle point at issue. When a seller in McAfee selects his best reply to the mechanisms being offered by the others, he counter factually assumes that the payoffs that buyers get with other sellers are independent of the mechanism that he offers. This imposes a lower bound on the payoff that a deviating seller can offer buyers.

Since deviating sellers will not offer buyers more than this minimum, buyers will generally be indifferent between selecting the deviator and their best alternative. Hence McAfee's assumption does not tie down the probability with which buyers will select the deviator. McAfee makes the optimistic assumption that buyers will select the deviator with a probability that maximizes the deviator's profit (subject to the constraint that

auction that he offers.

buyers remain indifferent between the deviator's mechanism and the fixed alternative payoff).

Even though the impact of a deviation on buyers' payoffs does decline as the number of buyers and sellers gets large, it is not clear why sellers should be able to count on this optimistic response in the probability with which buyers choose a deviator. By deriving the continuation equilibrium for the buyers' choice problem explicitly in the finite case, then taking its limit, we ensure that the buyers' choice probability is endogenous (and correctly foreseen by the seller) right into the limit.

Furthermore, our analysis of the continuation equilibrium in the finite case makes it possible to demonstrate some of the convergence properties of our equilibrium concept. We show that when competition is restricted to reserve prices only, if exact equilibria exist when there are a large but finite number of buyers and sellers, then these exact equilibria must be close to the equilibria for the limit game that we describe here.

We believe that our solution concept will be somewhat easier to apply in more general problems than the ones analyzed by McAfee, and by the problems analyzed here. The usefulness of McAfee's equilibrium concept is to some extent tied to the clever technique that he devised for characterizing equilibrium. It is not obvious that this technique will work in related environments. Our direct characterization of equilibrium in the limit is more flexible in the sense that it does not rely on any non-conventional argument.

The explicit characterization of the payoff earned by a deviating seller provides additional insight into the nature of equilibrium. The fact that sellers must set reserve prices equal to their use values, or costs, in McAfee is initially quite surprising. We are able to show (as has often been conjectured) that this is due to the fact that seller's profits are discontinuous in the seller's reserve price in a way that resembles the discontinuity in a Bertrand pricing game. The seller does not capture the entire market by undercutting the reserve price offered elsewhere, but he does capture a measurable segment of the market.

Finally, the formulas we derive for limiting payoffs in the independent values environment are extremely simple and elegant. This allows us to derive some additional results. In particular we focus on the 'allocative efficiency' of equilibria in auction markets. Allocative efficiency as defined in this paper refers to the market's ability to attract the right number and types of buyers to the auction market and away from their best alternatives.

We show that the performance of auction markets varies with seller's ability to advertize their reserve prices, and with the knowledge that buyers have at the time that they select among the mechanisms available to them.

The auction market performs 'efficiently' when sellers can advertize their reserve prices provided that buyers learn their valuations after selecting among auctions. If sellers are unable to advertize their reserve prices, then these prices rise and an inefficiently small number of buyers are attracted to the auction market.

On the other hand, if buyers know their valuations before they select one of the sellers' auctions, reserve prices are driven down to sellers costs, as they are in McAfee. In this case, there is excessive entry in the sense that if all sellers could simultaneously raise their reserve prices, the increased profits that this would yield would more than offset the loss in profits that occurs because of the decline in the number of buyers who enter.

The first of the two models studied in this paper is a direct extension of simpler competitive models without asymmetric information that have recently been the focus of some attention [8],[7],[10],[1]. The analysis generalizes the results of Wolinsky [12] and McAfee and McMillan[6]. Wolinsky considers a model with many auctioneers where buyers are exogenously and randomly assigned across the different auctions that are available. McAfee and McMillan consider a model of optimal auctions in which the number of participants is exogenous. By allowing sellers to advertize their reserve prices before matching occurs in this paper, our paper basically endogenizes the assignment of buyers to sellers' auctions, and analyzes the allocational implications of this.

In the next section we outline the basic properties of the model. The following section examines the case where buyers do not learn their valuations until after they choose a particular auction. The case where buyers know their valuations before they choose is discussed in section 4. Finally we conclude with a discussion of some of the many possible extensions of the analysis given in the paper.

# 2 The Model

There are a large countable number of buyers and sellers who try to transact by participating in a *market*. The set of sellers participating in this market is fixed. We will use the notation $J$ to stand for the number of sellers. To define payoffs to buyers and sellers we will calculate the corresponding payoff when $J$ is finite, then compute the limit of this payoff as $J$ goes to infinity to define the payoff in the limit process that we are interested in.

The number of potential buyers is very large relative to the number of sellers $J$. The number of buyers who participate in the market on the other hand will be determined endogenously. When the number of buyers who participate in the market is known, we will use the notation $k$ to refer to the ratio of the number of participating buyers to the number of participating sellers.

Each seller possesses a single indivisible unit of a commodity (like a house) that he wishes to sell at a non–negative price. Notice that this implies that each seller's use value, or cost for the commodity has been normalized to zero. Each buyer wishes to acquire exactly one unit of this commodity. Buyers valuations for this commodity will be restricted to be between 0 and 1. These valuations are private information to the buyers. Since buyer and seller beliefs vary in the two models that we analyze in this paper, the description of beliefs is deferred momentarily. Buyers and sellers are risk neutral. A buyer with valuation $x$ who trades at a price $p$ gets surplus $x - p$ while the seller in the same circumstance gets the surplus $p$.

The trading process begins when buyers make their entry decisions. It is assumed that this decision is taken before the buyers learn anything about the auctions that sellers are offering, or about their valuations for the commodities offered by the different sellers. We view the opportunity cost of participation in the auction market to be the time that is taken to assess the various available alternatives. It is natural to assume that buyers also learn something more about the nature of the commodity being trading during this process. The relevant entry decision then involves a decision about whether to incur the time cost associated with reading the paper that lists the sellers reserve prices and describes the features of the articles that the sellers are offering. Our timing assumptions seem natural in this context.

Buyers, of course, will have to correctly anticipate both the number of other buyers who will enter, and the reserve prices that sellers will offer them once they enter. As mentioned above, we characterize the outcome of the entry process by the number $k$ which depicts the ratio of the number of buyers who choose to enter the auction market, to the number of sellers who are holding auctions[3]. Buyers who choose not to participate in the trading process at all earn a sure surplus $\beta$. This outside opportunity is foregone by any buyer who chooses to enter.

Once sellers have observed the number of buyers who have entered, they try to sell their output by conducting second price auctions[4]. They do this by publicly announcing the reserve prices that they plan to use in the conduct of these auctions.

After buyers see the various reserve prices on offer, they choose to participate in one and only one of the available auctions. Buyers who are outbid, and sellers who are not chosen by any buyers leave the market without trading[5].

[3]It would also be plausible to assume that the entry decision is taken after buyers see the auctions that sellers have to offer. This modification changes details but none of the qualitative results.

[4]The auctions do not have to be second price auctions. The arguments below apply to any mechanism that awards the good to the buyer who actually bids in the auction who has the highest valuation. We continue to refer to second price auctions for clarity.

[5]The process can be made dynamic by letting unsuccessful traders repeat the procedure in the

In this paper we will search for *symmetric* equilibria. To assume that sellers offer the same auction in equilibrium does not seem a strong symmetry assumption. However we will also impose strong symmetry assumptions on the buyers' selection decision. We will search for equilibria in which all the buyers use the same selection *rule*. Equilibrium with identical selection rules for buyers typically requires that if two sellers offer the same reserve price, then they will be chosen by all buyers with the same probability.

The symmetric continuation equilibria are not the only plausible ones. For instance, when all sellers are the same, there is a continuation equilibrium where buyers use pure selection strategies and sort themselves among the sellers. This continuation leads to many more trades than the the equilibrium that we consider here. We believe that the symmetric continuation equilibrium that we characterize in this paper captures the trading frictions that are often present in large markets in an effective way. Furthermore, competition among sellers plays a much more important role here than it does for continuation equilibria in which buyers use pure strategies. We will defer a more extensive discussion of some alternative continuation equilibria to the end of the paper.

# 3 Buyers learn their valuations after choosing an auction

We begin by extending Wolinsky's [12] model to allow for competition among sellers. The timing of events in this version of the model is as follows; buyers begin by making their entry decisions, choosing either to exercise their outside option for a sure payoff $\beta$, or entering the market. Sellers announce the reserve prices they plan to use in their auctions. Buyers then choose among the sellers' auctions, learn their valuations and submit their bids.

Let $kJ$ be the number of buyers who choose to enter. It is assumed that once a buyer selects a seller, his valuation is independently drawn from a continuously differentiable probability distribution function $F$ whose support is $[0, 1]$.

Sellers choose a *reserve price strategy* which is a rule that specifies a reserve price for each level of entry $k$ that sellers might observe. Each buyer chooses a selection rule which gives the probability with which he will select each of the sellers conditional on the level of entry $k$, and the array of reserve prices on offer[6].

following period. This is straightforward provided that traders believe that these future payoffs are independent of their current actions.

[6]In this, and the sequel we will simply ignore the buyers' bidding strategies. It is a weakly dominant strategy for buyers to bid their true valuations in second price auctions, so we just assume that they do so.

We will search for a *symmetric* equilibrium in which all sellers employ the same reservation price rule, and all buyers use the same selection rule.

To begin suppose that some fixed set of buyers has entered. Since we want all sellers to use the same reservation price rule, let $r$ denote the reserve price that is offered by non-deviating sellers, while $r'$ is the reserve price that is offered by a single deviating seller. We begin with some preliminary, and well known results from the theory of optimal auctions (for example Riley and Samuelson [9] or McAfee and McMillan [5]).

The seller's payoff from holding a second price auction with reserve price $r$, when the seller is matched with exactly $n$ buyers is just

$$\begin{gathered}\Phi_n(r) = \\ n\left\{\int_r^1 \{[vF'(v) + F(v) - 1]\, F^{n-1}(v)\} dv\right\}\end{gathered} \tag{1}$$

Let $P^n(r', r, k)$ denote the probability with which each seller believes that he will be selected by exactly $n$ buyers when he offers the reserve price $r'$ at the beginning of a period, when each of the other sellers offers the reserve price $r$ and $k$ buyers have entered for each seller in the market. In the *competitive matching equilibrium* discussed in this paper, sellers will understand that their ability to attract buyers depends on the reserve price that they offer. In equilibrium they must be able to foresee the relationship correctly.

Ex ante each seller expects a payoff

$$\sum_{i=0}^{kJ} P^i(r', r, k) \cdot \Phi_i(r'). \tag{2}$$

We are looking for an equilibrium in which buyers use the same selection rule. Since we are only interested in unilateral deviations by sellers, we can write this selection rule as a single function $\pi_J(r', r, k)$ which denotes the probability with which each buyer selects the deviating seller when $kJ$ buyers enter. If buyers all use the same selection rule, the probability with which each of these buyers picks any one of the non-deviating sellers must be the same in equilibrium. Along with the requirement that the sum of the choice probabilities is one, this implies that we might as well restrict attention to a single choice probabilities for all the non-deviators equal to

$$\frac{1 - \pi_J(r', r, k)}{J - 1}$$

For a particular seller $j$, suppose that each buyer is selecting seller $j$ with probability $\pi$. Then the probability that exactly $n$ buyers select seller $j$ is given by

$$\frac{kJ!}{n!(kJ-n)!}\pi^n(1-\pi)^{kJ-n}$$

Again, using standard results from the theory of auctions([9]), a buyer with valuation $x$ who participates in the seller's auction with $n$ other buyers when the reserve price is $r$ will receive the payoff

$$\int_r^x F^n(s)ds$$

Notice that this is monotonically declining in the number $n$ of buyers who participate. Ex ante the buyer does not know what his valuation will be for the object being offered by any particular seller. Nor does she know how many other buyers will select the same seller. Hence the buyer's ex ante expected payoff from choosing a seller with reserve price $r$ when other buyers are choosing that seller with probability $\pi$ is given by

$$\sum_{n=0}^{kJ-1} \frac{(kJ-1)!}{n!(kJ-1-n)!}\pi^n(1-\pi)^{kJ-1-n}\left\{\int_r^1 \int_r^x F^n(s)dsf(x)dx\right\}$$

$$= \sum_{n=0}^{kJ-1} \frac{(kJ-1)!}{n!(kJ-1-n)!}\pi^n(1-\pi)^{kJ-1-n}\left\{\int_r^1 (1-F(x))F^n(x)dx\right\}$$

$$\equiv \sum_{n=0}^{kJ-1} \frac{(kJ-1)!}{n!(kJ-1-n)!}\pi^n(1-\pi)^{kJ-1-n}V_n(r) \tag{3}$$

where $V_n(r) = \left\{\int_r^1 (1-F(x))F^n(x)dx\right\}$. Notice that the ex ante payoff $V_n(r)$ is a declining function of $n$ and that this function has a limit 0 as $n$ goes to infinity.

Finally let

$$v_J'(r',r,k) \equiv$$

$$\sum_{n=0}^{kJ-1} \frac{(kJ-1)!}{n!(kJ-1-n)!}\pi(r',r,k)^n(1-\pi(r',r,k))^{kJ-1-n}V_n(r') \tag{4}$$

denote the expected payoff that a buyer gets by choosing the deviator, with $v_J(r',r,k)$ as the corresponding payoff from choosing any of the non-deviating sellers.

## 3.1 Competitive Matching Equilibrium

We now introduce the limit equilibrium concept. The equilibrium notion requires that buyers' and sellers' actions be best replies for them given their beliefs. Furthermore, sellers expectations about the impact that deviations in reserve price have on the probability distribution of buyers who select them must be *rational.*

**Definition 1** *A* competitive matching equilibrium *is a reserve price strategy* $r^*: [0,\infty] \to [0,1]$*, a belief function* $P$ *for sellers, a participation ratio* $k^*$ *and a selection rule* $\pi$ *such that*

1. (Optimal selection by buyers) for every $J$, $r$, $r'$, and $k$,

$$\begin{gathered}\pi_J(r',r,k)=0\Rightarrow v'_J(r',r,k)\leq v_J(r',r,k)\\ \pi_J(r',r,k)=1\Rightarrow v'_J(r',r,k)\geq v_J(r',r,k)\\ \pi_J(r',r,k)\in(0,1)\Rightarrow v'_J(r',r,k)=v_J(r',r,k)\end{gathered}$$

2. (Rational Expectations) for each $k$ and every pair $(r',r)$,

$$P^n(r',r,k)=\lim_{J\to\infty}\frac{kJ!}{n!(kJ-n)!}\pi_J\left(r',r,k\right)^n\left(1-\pi_J\left(r',r,k\right)\right)^{kJ-n}$$

3. (Profit maximization) for all $r'\in[0,1]$ and every $k\in[0,\infty]$

$$\sum_{n=0}^{\infty}P^n(r^*\left(k\right),r^*\left(k\right),k)\cdot\Phi_n(r^*\left(k\right))\geq\sum_{n=0}^{\infty}P^n(r',r^*\left(k\right),k)\Phi_n(r')$$

4. (Free entry condition)

$$\lim_{J\to\infty}v'_J\left(r^*\left(k^*\right),r^*\left(k^*\right),k^*\right)=\beta$$

The conditions are written to work backwards through the various stages of the trading process. The first condition (Optimal selection by buyers) restricts the strategies that buyers use in selecting among the various auctions. This is the final stage of the trading process. At this stage buyers have already observed the number of other buyers who have entered, and the offers that were made by sellers. In the spirit of subgame perfection, the equilibrium condition is written to be uniform in the history $(r,r',k)$.

This condition stands out because it is also required to hold when there are only finitely many buyers and sellers. The difficulty here is that it is not helpful to talk about the probability that the deviator is selected when the number of buyers and sellers is infinite. In any equilibrium this probability will have to be infinitesimal if buyers are to get a payoff. We get around this by calculating the exact equilibrium for all finite numbers of buyers and sellers, substituting the results into the seller's profit functions, then taking limits so that we can avoid dealing with the limiting strategies directly.

The *Rational Expectations* condition simply requires that sellers' beliefs about the relationship between the reserve prices they set and the number of buyers that they attract will be correct when there are an infinite number of buyers and sellers[7].

The *Profit maximization* requires that no matter how many buyers enter the trading process, sellers respond with a reserve price that is a best reply.

Finally, since potential buyers can foresee the outcome of the competition among sellers, we require that buyers enter until their expected payoff from doing so yields them exactly the same payoff that they get from their outside alternative. This is what the *free entry condition* specifies.

[7]We could just as easily require that this condition hold uniformly in $J$ instead of simply using the limit. Nothing would be affected by this change. The limiting version seems simpler.

## 3.2 Properties of a Competitive Matching Equilibrium

First, it should be noted that there is always a trivial equilibrium for this process in which no buyers enter, and sellers all set their reserve prices equal to 1. We will ignore this equilibrium as it is simply a consequence of the large numbers assumption. In any finite version of the model a buyer would realize that if she were the only entrant, the tremendous competition for her business would allow her to purchase for sure at a very low price. This does not occur in the limit only because the payoffs in the continuation equilibrium are insensitive to any *unilateral* deviation.

The first result shows that for any fixed positive level of entry, the payoffs in the continuation subgame can be well defined as limits of corresponding finite versions of the subgame.

**Lemma 1** *Suppose there are $k$ buyers for every seller. Let $\{r'_J\}$ be a sequence of deviations converging to $r' < 1$. Let $\{r_J\}$ be any sequence of reserve prices for the non-deviators converging to $r_\infty < 1$. Suppose that $\pi$ satisfies the optimal selection condition for buyers. Then*

*1.*

$$\lim_{J\to\infty}\sum_{n=0}^{kJ-1}\frac{(kJ-1)!}{n!(kJ-1-n)!}\frac{1-\pi_J}{J-1}^n(1-\frac{1-\pi_J}{J-1})^{kJ-1-n}V_n(r_J)$$

$$=\sum_{n=0}^{\infty}\frac{k^n e^{-k}}{n!}V_n(r_\infty)$$

*and*

*2.*

$$\lim_{J\to\infty}\sum_{n=0}^{kJ}\frac{kJ!}{n!(kJ-n)!}\pi_J^n(1-\pi_J)^{kJ-n}\Phi_n(r')$$

$$=\sum_{n=0}^{\infty}\frac{\bar{k}^n e^{-\bar{k}}}{n!}\Phi_n(r')$$

*where $\bar{k}$ is equal to $0$ or the unique solution to*

$$\sum_{n=0}^{\infty}\frac{x^n e^{-x}}{n!}V_n(r')=\sum_{n=0}^{\infty}\frac{k^n e^{-k}}{n!}V_n(r_\infty)$$

*whichever is larger.*

This Lemma has a number of uses. First note that it verifies the McAfee assumption for this environment. McAfee's assumption was that a seller who deviates to some alternative mechanism has no effect on the payoff that buyers can get by choosing non-deviating sellers. Taking the sequence $r_J$ to be constant and equal to $r_\infty$, this is verified for the case where there are an infinite number of buyers and sellers by the first result in the Lemma. But observe that the second result puts very precise restrictions on the trading probability that the deviating seller can expect in this case. So even though this expectational assumption is true in the limit, the assumption that this leaves the choice probability as a free parameter for the seller does not appear to be valid.

Secondly, we can restate the Lemma in a slightly different way since it implies a special *fixed indifference curve* property that is extremely useful in characterizing equilibrium. Again taking $r_J$ to be a constant sequence, consider a subgame in which $k > 0$ buyers have entered for every seller in the market. Then if the non-deviators offer $r_\infty < 1$ while the deviators offer $r' < 1$, condition 3 in the Lemma says that if the Optimal Selection condition for buyers and the Rational Expectations condition are satisfied, then $P^n(r', r, k)$ must be given by

$$\frac{\bar{k}^n e^{-\bar{k}}}{n!}$$

where $\bar{k}$ is the unique solution to

$$\sum_{n=0}^{\infty} \frac{x^n e^{-x}}{n!} V_n(r') = \sum_{n=0}^{\infty} \frac{k^n e^{-k}}{n!} V_n(r) \tag{5}$$

If the seller simply offers the same reserve price $r$ that all the other sellers are offering, then $P^n(r, r, k) = \frac{ke^{-k}}{n!}$. In this same situation, buyers payoffs would be given by the right hand side of (5). The Lemma then says that when the seller deviates to $r'$, the profits that he earns are the same as the profits that he would earn if *all* the sellers in the economy offered the reserve price $r'$ while the ratio of buyers to sellers is adjusted to $\bar{k}$ so that buyers' expected payoff in position $\left(r', \bar{k}\right)$ is the same as their payoff in the initial position $(r, k)$. Each seller's profits adjust as if he had the ability to position the symmetric outcome anywhere that he wants on the buyers' indifference curve through the point $(r, k)$ in the space of symmetric outcomes.

Finally, the lemma provides a useful convergence result. Again taking $k$ to be fixed, suppose that as $J$ goes to infinity, there is a sequence of symmetric pure strategy subgame perfect continuation equilibria for the subgames that start after buyer entry. Let $r_J$ denote the corresponding sequence of equilibrium reserve prices for the subgames. Let $r_\infty$ be a limit point of this sequence. Then $r_\infty$ must satisfy the profit maximization condition given in the definition of a competitive matching equilibrium. In other words, there are no profitable deviations from $r_\infty$ in the limit game, so that symmetric sub-game perfect equilibria for large $J$ and fixed $k$ must be close to the continuation equilibrium in the CME of our limit game.

To see this, note that we can immediately rule out the possibility that $r_\infty$ is equal to 1. If it were then $r_J$ is close to 1 for large $J$ and sellers would earn arbitrarily small profits in the symmetric equilibrium since the probability that they will trade with each buyer will be arbitrarily small and the expected number of buyers for each firm in the symmetric outcome is bounded. Furthermore for large $J$, a seller who deviates and offers a zero reserve price will attract all buyers with high probability, increasing his profits. We conclude that $r_\infty < 1$.

Now suppose that $r_\infty$ does not satisfy the profit maximization condition. Then at the limit point $r_\infty$ there must be some strictly profitable deviation, say $r'$, in the limit game. Taking the sequence of deviations to be constant, the second result in Lemma 1 says that the payoffs of the single deviator, who plays $r'$ for all $J$ against the $r_J$ used by each of the other sellers, converge to his payoff in the limit game when he plays $r'$ against $r_\infty$. This payoff is strictly larger than the payoff to playing $r_\infty$ in the limit game. On the other hand, taking the sequence of deviations $r'_J = r_J$ and applying the second result shows that if the deviator simply plays $r_J$ for all $J$, his payoff will converge to the payoff that he gets by playing $r_\infty$ in the limit game. This implies that if there is a profitable deviation (to $r'$) in the limit game, then there must also be for some large $J$, a contradiction.

It is shown below that for each $k$ there is a unique CME reserve price. It follows that *if* there are symmetric pure strategy subgame perfect equilibrium when $J$ is large, then the reserve prices that these equilibria support must be close to the CME reserve price that we describe below. The complication in all of this is that we have not yet been able to prove that these symmetric pure strategy equilibria exist for large $J$.

It is possible by using the fixed indifference curve property to give a simple graphical characterization of equilibrium. The necessary tool is provided by the following Lemma.

**Lemma 2** *Let $\nu(x) = x - \frac{1-F(x)}{f(x)}$ be the virtual valuation function. The reserve pricing rule $r^*(\cdot)$ and the buyer seller ratio $k^*$ satisfy the profit maximization and free entry conditions of a CME if and only if*

1. for each $k$, the pair $\{r^*(k), k\}$ maximizes $k' \int_{r'}^1 \nu(x) e^{-k'(1-F(x))} f(x) dx$ subject to

$$\int_{r'}^1 (1-F(x)) e^{-k'(1-F(x))} dx = \int_{r^*(k)}^1 (1-F(x)) e^{-k(1-F(x))} dx \tag{6}$$

and

2. 

$$\int_{r^*(k^*)}^1 (1-F(x)) e^{-k^*(1-F(x))} dx = \beta \tag{7}$$

The first condition in this lemma is simply another way of writing the fixed indifference curve property. The maximand is a simplified version of the profits that all sellers would get in the symmetric outcome where they all offer the reserve price $r'$ while $k'$ buyers enter per seller. The constraint represents all of the symmetric outcomes that yield buyers the same level of utility as the outcome $(r^*(k), k)$. The fixed indifference curve property says that a seller who deviates to $r'$ while other sellers offer $r^*(k)$ gets the same profits as he would get in a symmetric outcome where all sellers offer $r'$ while the level of entry is adjusted to $k'$, where $k'$ satisfies (6). Thus the constraint represents all the feasible outcomes open to a deviating seller. If $r^*(k)$ is to be part of an equilibrium, then $\{r^*(k), k\}$ must be the symmetric outcome in this set that is most preferred by all sellers.

The second condition is the free entry condition.

The Lemma shows that the CME can be characterized by finding the solution to a maximization problem. The maximization problem is relatively simple. For example, it is straightforward to show that the solution to the maximization problem is unique. The solution and equilibrium for this problem are described in Figure 1. The concave curve is the locus of solutions to (7). That is to say, it is the locus of all symmetric outcomes that yield buyers an expected payoff equal to $\beta$, the value of their outside alternative. Any CME reserve price must lie on this locus in order to satisfy the free entry condition. There is, of course, a whole family of similarly shaped curves, one for each possible expected payoff. We will simply refer to them as indifference curves. Higher expected utility is attained by buyers as they move toward the origin since in this direction they pay lower reserve prices and face less competition.

The C-shaped curves are loci of solutions to

$$\bar{k}\int_{r'}^{1} \nu(x)e^{-\bar{k}(1-F(x))}f(x)dx = \text{constant}$$

which is the maximand in the first condition given in Lemma 2. They represent the locus of symmetric outcomes along which sellers' expected profits are constant. Profits rise moving to the right along any horizontal ray since sellers prefer to have many buyers competing in their auction. The ray through $0B$ represents the locus along which the indifference curves of buyers and the iso profits curves of sellers are tangent.

The point $(r^0, k^0)$ in Figure 1 cannot represent an equilibrium for the subgame in which the ratio of buyers to sellers is $k^0$. By the fixed indifference curve property, a seller who wishes to deviate from $(r^0, k^0)$ should try reserve price $r'$. If he does, his profits will change as if all the sellers in the market had adjusted their reserve prices to $r'$, while the level of entry by buyers adjusts to $k'$ so that buyers expected utility is the same as it was at the initial outcome. From Figure 1, it is apparent that the point $(r', k')$

16

Figure 1:

lies on a higher iso-profit curve $\Pi_3$ than does the initial outcome $(r^0, k^0)$. Thus there is a profitable deviation.

When the ratio of buyers to sellers is $k^0$, the only reserve price where there is no profitable deviation is $r^*(k^0)$ at which the pair $(r^*(k^0), k^0)$ lies on the locus of tangencies $(0B)$ between the two indifference curves.

This same argument generates a continuation equilibrium at the tangency (along $0B$) for all values of $k$, which leads to the conclusion that the CME reserve pricing rule $r^*(k)$ is given by this locus through $0B$ of tangencies of indifference curves and iso-profit curves.

Finally, since potential entrants will understand this relationship between reserve prices and the number of buyers who enter, the CME buyer seller ratio is given by the point where the pricing rule $r^*(k)$ crosses the indifference curve for buyers along which they get the expected payoff $\beta$. This is the point $(r', k')$ in Figure 1.

Despite the fact that sellers compete in price in this problem, the reserve price does not fall to zero in equilibrium. To see this, note that the seller's profit function can be written as

$$k\int_r^1 \nu(x)e^{-k(1-F(x))}f(x)dx = k\int_r^1 \left(x - \frac{1-F(x)}{f(x)}\right)e^{-k(1-F(x))}f(x)dx$$

$$= k\int_r^1 xe^{-k(1-F(x))}f(x)\,dx - k\int_r^1 (1-F(x))\,e^{-k(1-F(x))}dx \tag{8}$$

Consider the point $(0, k'')$ in Figure 1, which lies on the indifference curve along which buyers get the expected payoff $\beta$. The level set of the first term in (8) is flat at this point. The second term is equal to $k\beta$ at this point. Then, moving up the indifference curve along which buyers get payoff $\beta$, the second term must be declining, while the first term is increasing. By the fixed indifference curve property, this means there is a profitable deviation at the point $(0, k'')$.

The reserve price does not fall to zero because the usual discontinuity in Bertrand pricing problems is not apparent here. When a deviator cuts his reserve price relative to other sellers, the utility that buyers get from his auction rises. This causes them to raise the probability with which they select his auction a little bit until the expected payoff in his auction is the same as it is with the non-deviators. The equilibrium reserve price remains above the seller's cost by an amount that depends on how many buyers enter in equilibrium.

There is another focal point in Figure 1. If sellers cannot advertize their reserve prices at all, then buyers will not learn the reserve price they have to pay until after they have committed themselves to one of the sellers. In this case they might as well choose each seller ex ante with the same probability $1/J$. Sellers will receive a random number of buyers, but this number will be independent of their reserve price. As is well known[6], this optimal reserve price is equal to the valuation $x$ at which the buyers virtual valuation $x-\frac{1-F(x)}{f(x)}$ is equal to zero. This reserve price is given by $r^0$ in Figure 1. As the sellers iso profit lines must be vertical at this point, while the buyers' indifference curve is downward sloping, the fixed indifference curve property again implies that there is a profitable deviation from this reserve price when sellers can advertize their prices.

All of this leads to the conclusion that the equilibrium reserve price lies between the seller's cost and the 'monopoly' reserve price described above.

### 3.3 Allocational Efficiency

In the discussion above, note that as the reserve price that sellers are expected to offer buyers in equilibrium falls, the equilibrium ratio of buyers to sellers rises, so sellers profits vary across these different outcomes. An outcome $(r,k)$ will be said to be *allocationally efficient* if it has the property that there does not exist an alternative outcome $(r',k')$ that yields buyers an expected payoff equal to the expected payoff that they receive with the outcome $(r,k)$, while yielding sellers a strictly higher payoff than they get at $(r,k)$.

The CME must be allocationally efficient. For suppose it is not. Then there is an alternative outcome $(r',k')$ such that sellers' expected profits are strictly higher at $(r',k')$ than they are at $(r^*(k^*),k^*)$ while buyers get the same expected payoff at $(r',k')$ as they do at $(r^*(k^*),k^*)$. If this is so, then some seller should deviate from $r^*(k)$ to $r'$. By the fixed indifference curve property, his payoff after this deviation will be the same as his payoff at the outcome $(r',k')$. Thus there will be a profitable deviation, contradicting the definition of a CME.

It should be noted that allocational efficiency is a much weaker concept than any of the notions of ex ante, interim or ex post efficiency that are normally applied to the study of auctions [11, 3]. For example, allocational efficiency does not guarantee that all mutually beneficial trades are carried out. The reason is that some buyers and sellers who could profitably trade with one another will simply never be matched together. Allocational efficiency simply takes for granted the fact that buyers can do no better at coordinating their choices than they can by playing the symmetric equilibrium of the their selection problem

Thus, when buyers are ill informed about their valuations in the sense that they do not learn them until after they commit themselves to a particular seller, competition among sellers attracts exactly the right number of buyers to the auction market. In particular, the equilibrium outcome in which sellers advertize their reserve prices allocationally dominates the equilibrium in which there is no price advertizing, despite the fact that the advertizing lowers the reserve prices that sellers offer in equilibrium. The large surplus that sellers extract by exploiting the Diamond like [2] monopoly power in the no advertizing case deters entry by buyers. Sellers are actually better off with the lower reserve prices because of the higher trading probability that this affords them.

# 4 Buyers know their valuations before they choose among sellers

In this section we analyze the situation where the buyers know their valuations for the objects that the sellers have to offer before they choose among the various auctions. We will retain the assumption that buyers make their entry decisions before they learn their valuations for the unit of output. This permits the most straightforward comparison between the results in this section, and the results that apply when buyers learn their valuations after they make their choices[8].

To review the timing, buyers first choose simultaneously whether to participate in the market or pursue an outside alternative that yields a sure payoff $\beta$. The outcome of this stage of the game is a buyer seller ratio $k$. Buyers who choose to participate then learn their valuation for the homogeneous unit of output that is offered by all the sellers and how many other buyers have chosen to participate.

Once sellers have observed how many buyers have chosen to participate in the market, they simultaneously announce reserve prices. After seeing the reserve prices, buyers simultaneously select one and only one seller as a potential trading partner and submit a bid to that seller. After sellers have collected all the bids that are submitted to them they sell to the buyer who submitted the highest bid at a price equal to the second highest bid or the reserve price, whichever is higher.

A strategy for the seller is a rule that specifies the reserve price he will offer for each level of entry by buyers. A strategy for buyer $i$ is a rule that specifies the probability that the buyer will select each of the available sellers as a function of the vector of reserve prices offered by sellers at the beginning of the period and of the buyer's valuation.

Again, we will focus on equilibria that are symmetric in the sense that i) all sellers use the same (pure strategy) reserve price rule; and ii) all buyers use the same selection rule. As before, the power of this second restriction lies in the fact that if two sellers

[8]The model works very differently when buyers make their entry decision after they learn their valuation. Then, only certain types of buyers will enter, and the support of the distribution of types will also be endogenous. If there is an unlimited number of potential entrants, only those buyers with the highest types will enter in equilibrium. Since entry involves a costly commitment, the assumption here seems reasonable. The opportunity cost of entry is the time cost associated with learning about the commodity being exchanged.

offer identical reserve prices, they must be selected with the same probability by every buyer in any symmetric equilibrium.

Let us return to the convention that $r'$ denotes the reserve price offered by some deviating seller while $r$ is the reserve price offered by each of the non-deviating sellers. Let $v'_J(x,r',r,k)$ and $v_J(x,r',r,k)$ be the expected payoff that each buyer expects to receive if she selects the deviating seller or the non-deviating seller respectively, when her valuation is $x$ and there are $J$ sellers and $kJ$ buyers participating in the market. Finally let $\pi_J(x,r',r,k)$ denote the probability with which each buyer will select the deviating seller when her valuation is $x$.

Once the buyer learns her valuation for the object being sold, the opportunity cost $\beta$ is sunk. Hence we can treat the decision not to bid in an auction as equivalent to the decision to submit a non-serious bid. Then, since all the non-deviating sellers are identical we might as well set the probability with which each buyer selects a non-deviator to be equal to $\frac{1-\pi_J(x,r',r,k)}{J-1}$ as above.

The probability that a buyer wins the deviator's second price auction is equal to the probability that all the other buyers either have lower valuations or choose to go to some other seller. The probability[9] that any given buyer either has a valuation lower than $x$ or chooses some other seller is given by

$$z_J(x,r',r,k) = 1 - \int_x^1 \pi_J(s,r',r,k) f(s) ds \tag{9}$$

This gives the probability that a buyer of valuation $x$ wins the deviator's auction as $z_J^{kJ-1}(x,r',r,k)$.

Standard results from the theory of auctions yield[10]

$$v'_J(x,r',r,k) = \int_\rho^x z_J^{kJ-1}(s,r',r,k) ds + K_J(r',r,k) \tag{10}$$

where $\rho = \inf\{x : \pi_J(x,r',r,k) > 0\}$ and $K_J(r) = \max[0, v_J(\rho,r',r,k)]$. In words, $\rho$ is the lowest valuation at which any buyer will consider selecting the deviating seller and $K_J(r',r,k)$ is what the buyer of type $\rho$ can get by selecting a non-deviating seller instead of a deviating seller. The constant is derived from the idea that the buyer of type $\rho$ must be just indifferent between participating in the deviating sellers auction and choosing her best alternative.

Note that in the usual way, the structures defined above yield the reduced form for the buyer's profit which is just

$$v'_J(x,r',r,k) = Q_J(x,r',r,k)x - P_J(x,r',r,k) \tag{11}$$

[9] This approach to the problem was first used by McAfee [4].
[10] See for example Riley and Samuelson [9].

where $Q_J$ and $P_J$ represent the reduced form probability of trading and price for any buyer who selects the deviating seller.

The deviating seller's profit is given by

$$\Phi'_J(r', r, k) = kJ \int_0^1 P_J(x, r', r, k)\pi_J(x, r', r, k)f(x)dx$$

Integrating using the usual incentive compatibility conditions gives the payoff (following McAfee [4]) as

$$\Phi'_J(r', r, k) = kJ \int_\rho^1 \left[x z_J^{kJ-1}(x, r', r, k) - v'_J(x, r', r, k)\right] z'_J(x, r', r, k)dx \tag{12}$$

where $z'_J(x, r', r, k) = \pi_J(x, r', r, k)f(x)$ is the derivative of $z_J(x, r', r, k)$ with respect to $x$.

We are now ready to state the conditions for a CME in this version of the problem. Once again

**Definition 2** *A* competitive matching equilibrium *is a reserve pricing rule* $r^* : [0, \infty) \to [0, 1]$, *a participation ratio* $k^*$ *and a selection rule* $\pi$ *such that*

1. (Optimal selection by buyers) for every $k$, $J$, $r'$, and $r$, and for every type $x$,

$$\begin{gathered}\pi_J(x, r', r, k) = 0 \Rightarrow v'_J(x, r', r, k) \leq v_J(x, r', r, k) \\ \pi_J(x, r', r, k) = 1 \Rightarrow v'_J(x, r', r, k) \geq v_J(x, r', r, k) \\ \pi_J(x, r', r, k) \in (0, 1) \Rightarrow v'_J(x, r', r, k) = v_J(x, r', r, k)\end{gathered}$$

2. (Profit maximization) for all $k$ and $r' \in [0, 1]$

$$\lim_{J\to\infty} \Phi'_J(r^*(k), r^*(k), k) \geq \lim_{J\to\infty} \Phi'_J(r', r^*(k), k)$$

3. (Free entry condition)

$$\lim_{J\to\infty} \int_{r^*(k^*)}^1 v_J(x, r^*(k^*), r^*(k^*), k)f(x)dx = \beta$$

The conditions are almost identical to the conditions for a CME in the problem discussed in the first part of the paper except that the optimality of the buyers' selection strategy is now required to hold uniformly in the buyer's type, and the rationality of the sellers' beliefs has been imbedded directly into the definition of the sellers' profits. Finally, note that the buyers' payoff before deciding to enter the market is just the expectation of the payoff that the buyer will earn conditional on her type after she enters.

## 4.1 Continuation Equilibria in a CME

A CME explicitly requires that the sellers' set the same reserve prices. As a result, we will only check the case in which there is a single deviating seller who offers the reserve price $r'$, while all of the other (non–deviating) sellers offer the reserve price $r$.

For this case we have the following

**Lemma 3** *The selection rule $\pi$ satisfies the Optimal Selection criteria for buyers if and only if for all $k$, $J$, and $r$, i) when $r' > r$, then there is a $y' > r$ such that*

$$\pi_J(x, r', r, k) = \begin{cases} 1/J \; if \; x \geq y' \\ 0 \; otherwise \end{cases}$$

*and $y'$ is the solution to*

$$(y' - r')\left\{1 - \frac{1 - F(y')}{J}\right\}^{kJ-1} = \int_r^{y'} \left\{1 - \frac{F(y') - F(x)}{J-1} - \frac{1 - F(y')}{J}\right\}^{kJ-1} dx \qquad (13)$$

*ii) when $r' < r$, then there is a $y' > r$ such that*

$$\pi_J(x, r', r', k) = \begin{cases} 1/J \; if \; x \geq y' \\ 1 \; otherwise \end{cases}$$

*and $y'$ is the solution to*

$$(y' - r)\left\{1 - \frac{1 - F(y')}{J}\right\}^{kJ-1} = \int_{r'}^{y'} \left\{1 - [F(y') - F(x)] - \frac{1 - F(y')}{J}\right\}^{kJ-1} dx \qquad (14)$$

Suppose there is an upward deviation, so that $r' > r$. Some buyers with valuations between $r$ and $r'$ are no longer willing to pay the reserve price $r'$ because it is higher than their valuations. These buyer types should clearly select the deviating seller with probability 0. Buyers with valuations just slightly higher than $r'$ should also choose the deviator with zero probability since even when they win, the surplus that they earn from the deviator is arbitrarily small.

However once buyers decide it is sensible to select the deviator, they will choose the deviator with exactly the same probability with which they choose the non-deviators. This results from the fact that uniform equality of the payoffs implies uniform equality of the trading probabilities, which implies (differentiate twice) uniform equality of the choice probabilities. This means that once we choose the cutoff $y'$ for the deviator, this determines the trading probabilities and payoffs for kinds of seller.

The cutoff valuation is determined by the condition that the marginal buyer type $y'$ is just indifferent between taking the chance at being the only bidder at the deviator's auction and paying the reserve price (the left hand side of (13) and (14)), and being the high bidder at the non-deviator's auction (the right hand side of the two equations).

It is straightforward to show that the solution to these equations has a unique limit as $J$ goes to infinity. In the case where the seller deviates upwards this limit solution is given by the solution to

$$(y-r')e^{-k(1-F(y))} = \int_r^y e^{-k(1-F(x))}dx \tag{15}$$

while in the case of a downward deviation the limiting value for $y$ must be equal to $r$ since the integral on the right hand side of (14) is converging to zero.

We can now calculate the payoff functions in the limit game.

**Theorem 4** *Let $k$ be fixed and let $\{r'_J\}$ be a sequence of prices charged by the deviator with $\lim_{J\to\infty} r'_J = r' < 1$. Let $\{r_J\}$ be any sequence of reserve prices offered by the non-deviators and suppose that this sequence is converging to $r_\infty < 1$. Suppose that $\pi(\cdot)$ satisfies the optimal selection criteria for buyers. Let $y'$ denote the solution to*

$$(y-r')e^{-k(1-F(y))} = \int_{r_\infty}^y e^{-k(1-F(x))}dx$$

*Then*

1. $\lim_{J\to\infty} v_J(x, r'_J, r_J, k) = \int_{r_\infty}^x e^{-k(1-F(x))}dx$ *for each* $x \in [0,1]$

2. *if* $r' > r_\infty$

$$\lim_{J\to\infty} \Phi'_J(r'_J, r_J, k) =$$

$$= k\int_{y'}^1 \left\{xe^{-k(1-F(x))} - \int_{r_\infty}^x e^{-k(1-F(s))}ds\right\} f(x)dx$$

$$= k\int_{y'}^1 \left\{\nu(x)\, e^{-k(1-F(x))}\right\} f(x)dx - k\left[1-F(y')\right]\int_{r_\infty}^{y'} e^{-k(1-F(s))}ds$$

*where $\nu(x)$ is the virtual valuation function defined above*

3. *if* $r' < r_\infty$

$$\lim_{J\to\infty} \Phi'_J(r'_J, r_J, k) =$$

$$k\int_{r_\infty}^1 \left\{\nu(x)\, e^{-k(1-F(x))}\right\} f(x)dx + r_\infty e^{-k(1-F(r_\infty))}$$

The first result is again a verification of McAfee's assumption. When there are infinitely many buyers and sellers, the payoff that buyers get by going to non-deviating sellers is independent of the particular deviation that has occurred. Once again, from Lemma 3, the choice probability for the deviator is tightly determined.

The third result gives a notable property of the payoffs in the limiting CME. The deviating seller's payoff function is discontinuous at the point where his reserve price is equal to the one offered by his competitors. Furthermore once the seller has undercut his competitors his profits are thereafter independent of the reserve price that he sets. This discontinuity resembles the one that occurs in standard Bertrand games. There are some important differences however.

First, according to Lemma 3, if the seller cuts his reserve price from $r$ to $r'$, then every buyer whose valuation is between $r$ and $r'$ must choose the deviating seller with probability one. When there are a very large number of buyers, this means that the seller is virtually certain to receive a bid arbitrarily close to $r$, no matter what the seller does with his reserve price. This explains why profits are constant once the seller has undercut. The discontinuity is explained by the fact that with a very large number of buyers the seller is also almost sure to attract at least one bidder with a valuation arbitrarily close to $r$. Hence unlike the non–deviators, who get no bidders at all with a strictly positive probability, the deviating seller is sure to trade.

It is also worth attempting to interpret the payoff that the seller gets when he deviates upwards. This has been written in two equivalent ways in the Theorem. The first equality gives the deviating seller's profit in the limit game as the integral over the all possible valuations with which the seller might trade, of the difference between the surplus that a buyer generates (his trading probability times his valuation) and the payoff that the seller needs to offer the buyer to get him to come in the first place.

The second result says that in the limit both components of this difference are independent of the seller's deviation for all buyer types who continue to trade with the seller after the deviation occurs. If a seller deviates by raising his reserve price relative to that offered by the other sellers, then in the limit, all this can do is to prevent some buyer types from selecting him.

These results imply McAfee's remarkable conclusion. The unique CME occurs when all sellers offer a reserve price equal to their cost 0. If all sellers are offering a 0 reserve price, then it is not hard to see why no seller will deviate upward. Any buyer type who continues to choose the deviator after the increase in the reserve price will generate exactly the same profit for the seller as before according to the second part of Theorem 4. In words, the expected price that these high valuation buyers pay the seller is not affected by the change in the seller's reserve price. Thus when a seller raises his reserve price relative to other sellers and chases low valuation buyers away, he simply loses the expected payments that they would otherwise have made to him, without gaining revenue from other types of buyers. Thus there is no incentive for sellers to raise reserve prices to extract higher revenues from high valuation buyers when there is a lot of competition[11].

The discontinuity implied by the third part of Theorem 4 shows why there can be no other symmetric equilibrium.

We can make use of Theorem 4 and the uniqueness of the CME reserve price to show that if there are exact symmetric equilibria when the number of buyers and sellers is large, then the reserve prices that are supported by these equilibria must be close to the CME reserve price (i.e., 0). The argument mimics the argument presented above for the case where buyers learn their reserve prices after selecting an auction.

It may be worthwhile to note that this result that the reserve price falls to the seller's cost in equilibrium can be generalized readily to the case where the seller's cost lies anywhere within the support of the distribution function $F$ of buyers' valuations. For this case we simply ignore the buyers whose valuations are below the seller's costs. A glance at the seller's profit function illustrates that the sellers will have no interest in trying to attract these buyers.

The case where the seller's cost lies strictly below the support of the distribution of buyers' valuations is slightly more complicated. In this case there are many payoff equivalent equilibria in which all firms set reserve prices at or below the infinum of the support of $F$. For example, there is one equilibrium in which all sellers set reserve prices at the infinum $\bar{r}$ of the support of $F$. Upward deviations are unprofitable for exactly the same reasons as described above. The story is slightly different when a firm deviates by setting a reserve price that is strictly below the reserve price $\bar{r}$ set by all the other firms. In this case buyers who have valuations slightly above $\bar{r}$ will choose the deviating seller with probability 1, while other types will randomize between all sellers equally.

Buyers with these low valuations do not benefit much from the lower reserve price when $J$ gets large. Buyers with the lowest valuations in this interval are almost surely outbid by buyers with higher valuations, while the buyers with the higher valuations are almost sure to find a second highest bidder who bids more than the reserve price. Thus this interval of buyer types who select the deviator shrinks as $J$ gets large.

This implies that sellers will not gain from cutting the reserve price when $J$ is very large. On the other hand, the fact that buyers with valuations in this small interval are selecting the deviator with probability one means that the seller is almost sure to receive two bids above $\bar{r}$ when $J$ is large. This means that the seller does not incur any cost by cutting his reserve price. In the limit, profits are unaffected by changes in the reserve price, as long as the reserve price remains below the support of the distribution of buyer

[11]It is straightforward to verify this argument formally. Simply differentiate the profit function given in Theorem 4 with respect to the cutoff valuation $y$. When the non-deviators set a zero reserve price, this derivative is non-positive.

valuations.

## 4.2 Allocational Efficiency once again

An outcome is allocationally efficient if it maximizes the expected profit that each seller earns conditional on each buyer getting the same expected payoff that he gets from his best alternative. At the time that buyers are forced to choose between visiting the auction market and moving to their alternative, they do not yet know their valuations for the commodity. If they expect sellers to offer the reserve price $r$, and expect $k$ buyers to enter for each seller, then their expected payoffs will be (from Theorem 4)

$$\lim_{J\to\infty}\int_r^1 v_J(x,r,r,k)f(x)dx =$$

$$\lim_{J\to\infty}\int_r^1\int_r^x z_J^{kJ-1}(s,r,r,k)dsf(x)dx$$

$$= \lim_{J\to\infty}\int_r^1 (1-F(x))\, z_j^{kJ-1}(x,r,r,k)dx$$

$$\lim_{J\to\infty}\int_r^1 (1-F(x))\left[1-\frac{1-F(x)}{J}\right]^{kJ-1} dx$$

$$= \int_r^1 (1-F(x))e^{-k(1-F(x))}dx \tag{16}$$

Referring back to equation (7) in the first section of the paper, it is apparent that the buyers' expected payoff is the same as it was in the case where buyers learned their valuations after choosing a seller. Buyers have the same distribution over valuations in the two cases considered. Then in every symmetric situation, buyers behave in exactly the same way conditional on their valuations, whether they learn their valuations before they select among sellers or not. When there is a deviation from the symmetric outcome, there will be a sorting effect in the model where buyers know their valuations that will not be present when they don't. This is why the equilibria for the two models differ.

Again referring to Theorem 4, if all sellers set the reserve price $r$ each of them will earn profit

$$k\int_r^1\left\{\left[x-\frac{1-F(x)}{f(x)}\right]e^{-k(1-F(x))}\right\}f(x)dx$$

$$= k\int_r^1\left\{\nu\left(x\right)e^{-k(1-F(x))}\right\}f(x)dx$$

Referring back to the first condition of Lemma 2, note that this is exactly the same as the profit earned by sellers in a symmetric outcome when buyers learn their valuation after selecting a seller.

This means that when buyers and sellers compare any two symmetric outcomes $(r, k)$ and $(r', k')$, their rankings of these two outcomes will be independent of whether or not buyers learn their valuations before they need to select among sellers' auctions. We concluded in the first section that sellers profits were higher in the equilibrium where all sellers advertized their reserve prices than they were when sales occurred whenever a positive bid was received. Thus this same result applies here. Sellers profits would be strictly higher with free entry if they could jointly raise the reserve price that they offered above zero. The additional buyers that are attracted to the auction market by the prospect of lower reserve prices do not compensate for the revenues that sellers lose by setting such low prices.

Our results suggest that there are two things that sellers who could collude might like to do about this. First, as Figure 1 is drawn, sellers could increase their (joint) profits if they could simply agree not to advertize their reserve prices. If they could agree to this, then they would set the reserve price $r^0$ that is optimal when the seller expects buyers to choose among sellers with equal probability. This may be one of the reasons why real estate companies have prospective house sellers set fairly standardized list prices that differ from their costs.

Yet our results indicate that it is not the price competition per se that is harmful for sellers. Sellers are jointly better off if they do compete in reserve price, provided buyers are not too well informed about the commodities being exchanged.

# 5 Extensions

As mentioned in the introduction, the models presented in this paper are highly stylized. We have chosen to frame the discussion around competition in reserve prices because this class of mechanisms is so well known. It is difficult to think of markets in which sellers compete in reserve prices in the manner described in this paper. Trying to devise an explanation for this fact is one of the central objectives of this line of research. To do this it is clearly necessary to allow sellers to offer mechanisms taken from a much broader class.

McAfee [4] deals with this problem and shows that it will be an *equilibrium* for sellers who can choose to offer any direct mechanism to offer mechanisms having the property that the seller chooses to trade with the bidder who has the highest valuation among all bidders who select the seller's mechanism. Presumably a result like this can be proved here, for the environment is much simpler than McAfee's. However, this is in some sense the wrong result since auctions tend not to be used in competitive markets like those discussed by McAfee. Hopefully the relatively simple limit equilibrium discussed in this paper will make it possible to discover the transactions costs that are missing from McAfee's story (and from this story).

The symmetry assumptions that we have made play a central role in this paper. The key assumption is the restriction on the continuation equilibrium for the buyers selection problem. Buyers are required to randomize equally over all the sellers when they offer the same mechanism. There are other plausible continuation equilibria. For example when $k$ buyers enter per seller we could assign $k$ buyers who would choose the first seller with probability 1, $k$ more buyers who would choose the second seller with probability 1 and so on.

Then, if there is a deviation, there are a couple of options. First of all, if the deviation is small, the situation in which all buyers continue to use the same pure selection strategies that they started with will be an equilibrium. The unlucky buyers who are choosing the deviator will be made worse off, but there will be nothing for them to do about it. If they select one of the non-deviating sellers they will add a new buyer to that sellers share. This will have a measurable effect of the surplus that is attainable from the non-deviating seller even when the number of buyers and sellers is infinitely large.

A version of McAfee's assumption holds for this kind of continuation equilibrium. In the symmetric situation where sellers all offer the reserve price $r$, any buyer who decides to try another seller will receive the payoff in an auction with reserve price $r$ in which he will be competing with $k$ other buyers instead of the $k-1$ buyers that he faces in the auction to which he is assigned. If his seller deviates by offering a higher reserve price, then his payoff will fall, but his best alternative will still be to compete against $k$ other buyers in an auction with reserve price $r$.

There are many alternative equilibria for the continuation. A referee has suggested the following variant of the story told above. Buyers are allocated across sellers as above when all sellers offer the same reserve price. If there is a deviation, all buyers are thrown back into a pool and re-allocated across the non-deviating sellers. The first $k$ buyers are assumed to choose the first non-deviator with probability $1-\epsilon$ and the deviator with probability $\epsilon$. The buyers indexed $k+1$ through $2k$ are assumed to choose the second non-deviator with probability $1-\epsilon$, and the deviator with probability $\epsilon$, and so on. The selection probability $\epsilon$ is chosen to equalize the payoff that buyers get from the deviator and the non-deviator who they choose with large probability[12]. This continuation allows more trades to occur on the equilibrium path and still permits competition to exert some influence on outcomes. McAfee's assumption is likely to hold for this continuation.

Perhaps a simpler variant of this idea is to allow buyers to sort among sellers as above

[12] This story needs to be modified slightly to get an exact equilibrium when the number of buyers and sellers is finite, but the argument conveys the idea.

when all sellers offer the same mechanism, then to revert to the symmetric equilibrium that we have described whenever there is a deviation. This continuation seems to increase the cost of a deviation since a deviation will create new matching inefficiencies that will be costly for sellers.

We prefer the purely symmetric continuation for two reasons. First it has the realistic property that sellers receive a random number of bidders along the equilibrium path. Secondly it has a more decentralized flavor in that it does not require any kind of coordination of buyers among sellers.

The models discussed in this paper are both static. Buyers who do not win an auction should have the opportunity to bid in another one, especially since our equilibrium requires that there be some sellers who do not trade.

It is straightforward to extend the model to allow rationed buyers and sellers to attempt to trade again in the following period, provided that it is reasonable to assume that the payoff that buyers and sellers can get next period is independent of their current actions. This seems plausible when there are infinitely many buyers and sellers but is not so realistic in a small market. If this assumption is plausible, the analysis is essentially unchanged except that buyers who do not trade get the discounted value of next periods payoff instead of 0.

# 6 Proofs of Theorems

## 6.1 Proof of Lemma 1

**Proof.** Define $\gamma_J$ such that

$$\pi_J = \gamma_J / J$$

The payoff that a buyer gets from selecting the deviator is given by substituting $r'_J$ and $\gamma_J/J$ into equation (3). The expected number of participants in the deviator's auction is $k\gamma_J$.

The expected number of participants in a non–deviators auction is $k\frac{1-\pi_J}{J-1}J = k\frac{1-\frac{\gamma_J}{J}}{J-1}J$. This latter expression can only have bounded limit points since $\pi_J$ must lie between 0 and 1. Substituting $r_J$ and $\frac{1-\frac{\gamma_J}{J}}{J-1}$ into (3) illustrates that the payoff to going to the non–deviator will have a strictly positive limit point since $r_J < 1$ for large enough $J$.

Now suppose that some subsequence of $\{\gamma_J\}$ is unbounded. Then from (3), the payoff offered by the deviator will be converging to zero along this subsequence. But this contradicts the definition of $\pi_J$ which is chosen to equate the payoff to the deviator and non-deviator. We conclude that the sequence $k\gamma_J$ is bounded above.

Let $\gamma_\infty$ be any limit point of the sequence $\{\gamma_J\}$. Then along the corresponding subsequence

$$\lim_{J\to\infty} \frac{kJ!}{n!(kJ-n)!}\pi_J^n(1-\pi_J)^{kJ-n}$$

$$= \lim_{J\to\infty} \frac{kJ!}{n!(kJ-n)!}\left(\frac{\gamma_J}{J}\right)^n\left(1-\frac{\gamma_J}{J}\right)^{kJ-n}$$

$$= \lim_{J\to\infty} \frac{(k\gamma_J)^n}{n!}\left(1-\frac{\gamma_J}{J}\right)^{kJ-n}$$

$$= \frac{(k\gamma_\infty)^n}{n!}e^{-k\gamma_\infty}$$

while for the non- deviator

$$\lim_{J\to\infty} \frac{kJ!}{n!(kJ-n)!}\left[\frac{1-\pi_J}{J-1}\right]^n\left(1-\frac{1-\pi}{J-1}\right)^{kJ-n}$$

$$\lim_{J\to\infty} \frac{kJ!}{n!(kJ-n)!}\left[\frac{1-\frac{\gamma_J}{J}}{J-1}\right]^n\left(1-\frac{1-\frac{\gamma_J}{J}}{J-1}\right)^{kJ-n}$$

$$= \lim_{J\to\infty} \frac{\left(k\left(1-\frac{\gamma_J}{J}\right)\right)^n}{n!}\left(1-\frac{1-\frac{\gamma_J}{J}}{J-1}\right)^{kJ-n}$$

$$= \frac{k^n}{n!}e^{-k}$$

since $\gamma_J$ has a finite limit along every converging subsequence. By the continuity of the functions $V_n$, the payoff at the non-deviators converges to

$$\sum_{n=0}^{\infty} \frac{k^n}{n!}e^{-k}V_n(r_\infty)$$

along each subsequence $\{\gamma_J\}$.

Since $\pi_J$ is chosen to equate the payoff that the buyer gets at the deviator and non-deviator whenever each is chosen with positive probability, it must be that

$$\sum_{n=0}^{\infty} \frac{(k\gamma_\infty)^n}{n!}e^{-k\gamma_\infty}V_n(r') = \sum_{n=0}^{\infty} \frac{(k)^n}{n!}e^{-k}V_n(r_\infty) \tag{17}$$

Since every limit point of the sequence $\{\gamma_J\}$ must satisfy this equation, and because the left hand side of this expression is monotonically declining in $\gamma$ (since $V_n$ is monotonically declining in $n$), we conclude that every limit point of the sequence $\{\gamma_J\}$ is the same, and is given by the solution to this last equation.

Substituting this limit into the deviating seller's payoff function gives

$$\lim_{J\to\infty} \sum_{n=0}^{kJ} \frac{kJ!}{n!(kJ-n)!}\pi_J^n(1-\pi_J)^{kJ-n}\Phi_n'(r_J')$$

$$= \sum_{n=0}^{\infty} \frac{(k\gamma_\infty)^n}{n!} e^{-k\gamma_\infty} \Phi_n'(r')$$
$$= \sum_{n=0}^{\infty} \frac{\left(\bar{k}'\right)^n}{n!} e^{-\bar{k}'} \Phi_n'(r')$$
where $\bar{k}$ is given by the solution to (17). ■

## 6.2 Proof of Lemma 2

**Proof.** First suppose that $r^*(k)$ is part of a CME. In any subgame consisting of $k$ buyers for each seller, if the Optimal Selection condition and the Rational Expectations condition hold, then for any $r'$, the deviating seller's payoff can be written

$$\sum_{n=0}^{\infty} P^n(r', r^*(k), k)\Phi_n'(r')$$
$$= \sum_{n=0}^{\infty} \frac{\bar{k}^n e^{-\bar{k}}}{n!} \Phi_n'(r')$$
$$= \sum_{n=0}^{\infty} \frac{\bar{k}^n e^{-\bar{k}}}{n!} \left[n \int_{r'}^1 \nu(x) F^{n-1}(x) f(x) dx\right]$$
$$= \int_{r'}^1 \nu(x) \sum_{n=0}^{\infty} \frac{\bar{k}^n e^{-\bar{k}}}{n!} n F^{n-1}(x) f(x) dx$$
$$= \bar{k} \int_{r'}^1 \nu(x) e^{-\bar{k}(1-F(x))} f(x) dx$$
where $\nu(x) \equiv x - \frac{1-F(x)}{f(x)}$ is the virtual valuation function and $\bar{k}$ satisfies
$$\sum_{n=0}^{\infty} \frac{\bar{k}^n e^{-x}}{n!} V_n(r') = \sum_{n=0}^{\infty} \frac{k^n e^{-k}}{n!} V_n(r^*(k))$$
By the definition of $V_n$, the buyer's payoff can be written
$$\sum_{n=0}^{\infty} \frac{\bar{k}^n e^{-\bar{k}}}{n!} \int_{r'}^1 (1 - F(x)) F^n(x) dx$$
$$= \int_{r'}^1 (1 - F(x)) e^{-\bar{k}(1-F(x))} dx$$
Then since $r^*(k)$ is part of a CME, it must be that
$$\bar{k} \int_{r'}^1 \nu(x) e^{-\bar{k}(1-F(x))} f(x) dx \leq k \int_{r^*(k)}^1 \nu(x) e^{-k(1-F(x))} f(x) dx$$
for each pair $\left(r', \bar{k}\right)$ satisfying
$$\int_{r'}^1 (1 - F(x)) e^{-\bar{k}(1-F(x))} dx = \int_{r^*(k)}^1 (1 - F(x)) e^{-k(1-F(x))} dx$$

This proves necessity of the conditions.

The proof of sufficiency is straightforward given the simplified formulas provided above. ■

## 6.3 Proof of Lemma 3

We will prove this theorem by showing that it is a special case of a more general result. We characterize the continuation equilibrium in a problem where there are a finite number $J$ of sellers, $k$ buyers have entered for each seller, and the sellers have offered an arbitrary array of reserve prices ordered without loss of generality as $r_1 \leq r_2 \leq \ldots r_J$. Let $v_j(x, r)$ denote the expected payoff for a buyer with valuation $x$ who chooses seller $j$ when the vector of offers is given by $r$.

**Theorem 5** *For each array of reserve prices $r_1 \leq r_2 \leq \ldots r_J$, every* symmetric *continuation equilibrium for the buyers search problem has the property that there exists a non-decreasing sequence $y_0, y_1, \ldots y_J, y_{J+1}$ with $y_0 = 0$, $y_1 = r_1$ and $y_n > r_n$ and $y_{J+1} = 1$ such that i) buyers with valuations in the interval $[y_0, y_1]$ choose not to bid; and ii) whenever $y_{n-1} \neq y_n$, each buyer with a valuation in the interval $[y_{n-1}, y_n]$ selects each seller $1 \leq j < n$ with equal probability $1/(n-1)$ and all other sellers with probability $0$.*

**Theorem 6** *The numbers $\{y_2, \ldots, y_J\}$ must satisfy the following system of equations*

$$(y_j - r_j)\left\{1 - \sum_{n=j+1}^{J} \frac{1}{n}\left[F(y_{n+1}) - F(y_n)\right]\right\}^{kJ-1} =$$

$$= \sum_{m=1}^{j-1} \int_{y_m}^{y_{m+1}} \left\{1 - \frac{\left[F(y_{m+1}) - F(s)\right]}{m} - \sum_{n=m+1}^{J} \frac{1}{n}\left[F(y_{n+1}) - F(y_n)\right]\right\}^{kJ-1} ds$$

*for $j = 2, \ldots J$. Furthermore, any set of solutions to this system of equations will support a symmetric continuation equilibrium.*

### 6.4 Proof of Theorem 5

**Proof.** A necessary and sufficient condition for a selection strategy

$$\pi(x,r) = \{\pi_1(x,r),\ldots,\pi_J(x,r)\}$$

to be a symmetric continuation equilibrium for the buyers' search problem is that for all $j$,

$$\pi_j(x,r) = \left\{ \begin{array}{c} 0\, if\ \exists\, i : v_i(x,r) > v_j(x,r) \\ 1\, if\ v_j(x,r) > v_i(x,r)\,\forall i \neq j \\ \in [0,1]\ otherwise \end{array} \right\} \tag{18}$$

Let the strategy $\pi(x,r)$ be a continuation equilibrium for some $r$. Let $E_j = \{x : \pi_j(x,r) > 0\}$ and let $E_{ji} = E_j \cap E_i$.

**Lemma 7** *For each $j$, $v_j(x,r)$ is a continuous function with derivative equal to $z_j^{kJ-1}(x,r)$. Furthermore $z_j^{kJ-1}(x,r)$ is an increasing function if $\pi_j(x,r) > 0$.*

**Proof.** These conclusions follow immediately from (10). They are standard consequences of incentive compatibility. ∎

**Lemma 8** *Suppose that the set $E_j$ has strictly positive measure. Then the set of all its cluster points is a connected interval. Futhermore $\pi_j(x,r) = \pi_i(x,r)$ a.e. $x \in E_{ji}$.*

**Proof.** To see this note that when $E_j$ has positive measure, the set of its cluster points is a union of closed intervals of positive measure. If this set is not connected, there is an interval $[a,b]$ and an $\epsilon > 0$ such that $\pi_j(x_1,r) > 0$ $\forall x_1 \in [a-\epsilon,a]$; $\pi_j(x_2,r) > 0$ $\forall x_2 \in [b,b+\epsilon]$; while $\pi_j(x,r) = 0\, a.e.\ x \in [a,b]$.

From (9) and Lemma 7 the fact that $\pi_j(x,r) = 0$ means that $v_j(x,r)$ has a constant (positive) derivative along the interior of the interval $[a,b]$. Furthermore since $\pi_j(x,r) > 0\ a.e.\ x \in [a-\epsilon,a]$, $v_j(a,r) \geq 0$.

Let $v_{-j}(x,r) \equiv \max_{i\neq j} v_i(x,r)$. The function $v_{-j}(x,r)$ is continuous and increasing, thus it is differentiable almost everywhere. By (18),

$$v_{-j}(x,r) = \sum_{i=1}^{J} \pi_i(x,r)\, v_i(x,r)$$

Then the derivative of $v_{-j}(x,r)$ when it exists is given by

$$\sum_{i=1}^{J} \pi_i(x,r)\, z_i^{kJ-1}(x,r) + \sum_{i=1}^{J} \frac{d\pi_i(x,r)}{dx} v_i(x,r)$$

$$= \sum_{i=1}^{J} \pi_i (x, r)\, z_i^{kJ-1} (x, r) + v_{-j} (x, r) \sum_{i=1}^{J} \frac{d\pi_i (x, r)}{dx}$$
$$= \sum_{i=1}^{J} \pi_i (x, r)\, z_i^{kJ-1} (x, r)$$

By Lemma 7 this function is increasing. But this implies that $v_{-j}(b, r) > v_j(b, r)$. Continuity of the profits functions along with (18) then require that $\pi_j(x, r) = 0$ on every neighborhood of $x = b$. This contradiction establishes the connectedness of this set of cluster points.

Now if there is an interval along which a.e. $v_j(x, r) = v_i(x, r)$, the first and second derivatives of these functions must also be equal a.e. along this interval. Differentiating both sides of this equation using the expression (10) establishes that $\pi_j(x, r) = \pi_i(x, r)$ a.e. $x \in E_{ji}$. ∎

**Lemma 9** *If $x \in E_j$ then a.e. $y \geq x$, $y \in E_j$.*

**Proof.** Suppose not. By Lemma 8, the cluster points of $E_j$ form a connected interval. Thus if the set $\{y \geq x : \pi_j (y) = 0\}$ has positive measure, its cluster points must form a connected set containing the point 1. Then choose $b$ and $\epsilon$ (we allow the possibility that $\epsilon = 0$ to account for the possibility that $E_j$ is a single point) such that $\pi_j(x, r) > 0$ for every $x \in [b - \epsilon, b]$ while $\pi_j(x, r) = 0$ *a.e.* $x > b$. For any $y > b$

$$v_j(y, r) = v_j(b, r) + \int_b^y z_j^{kJ-1}(s, r)ds \geq v_i(b, r) + \int_b^y z_j^{kJ-1}(s, r)ds$$
$$> v_i(b, r) + \int_b^y z_i^{kJ-1}(s, r)ds = v_i(y, r)$$

The first inequality follows from the fact that $\pi_j(b, r) > 0$, while the second inequality follows because $z_j(y, r) = 1$ for all $y > b$ as long as $\pi_j(y, r) = 0\, \forall y > b$. This strict inequality violates (18). ∎

These results imply that $E_j$ is an interval containing 1 for every firm. Where these intervals overlap, buyers will select more than one firm with the same probability.

**Lemma 10** *$E_j \subset E_i$ if and only if $r_i \leq r_j$.*

**Proof.** Let $b_t = \inf\{x : x \in E_t\}$. Suppose $E_j \subset E_i$. Consider a buyer of type $b_i$. Such a buyer is more likely to trade with seller $j$ than with seller $i$. To induce her to choose seller $i$ she must expect to pay a lower price with seller $i$. But if she wins the auction at either seller, she will end up paying the seller's reserve price (if anyone else comes, they will have a valuation on the interior of either $E_j$ or $E_i$). Hence we conclude that $r_i \leq r_j$. The converse argument is established in a similar fashion. Suppose $r_i \leq r_j$ but that $E_i$ is strictly contained in $E_j$. Use the same reasoning to establish a contradiction. Theorem 5 follows from Lemmas 7 to 10. ∎

## 6.5 Proof of Theorem 6

**Proof.** A buyer who has a valuation equal to $y_j$ must be just indifferent between the seller $j$ offering a reserve price $r_j$, and any of the sellers 1 through $j-1$. Using Theorem 5, the payoff with seller $j$ to a buyer of type $y_j$ is given by

$$(y_j - r_j) z_j^{kJ-1}(y_j, r) = (y_j - r_j)\left\{1 - \int_{y_j}^{1} \pi_j(x, r) f(x) dx\right\}^{kJ-1}$$

$$= (y_j - r_j)\left\{1 - \sum_{n=j+1}^{J} \int_{y_n}^{y_{n+1}} \frac{1}{n} f(x) dx\right\}^{kJ-1}$$

$$= (y_j - r_j)\left\{1 - \sum_{n=j+1}^{J} \frac{1}{n}\left[F(y_{n+1}) - F(y_n)\right]\right\}^{kJ-1}$$

On the other hand the payoff that the buyer of type $y_j$ gets from each of the sellers 1 through $j-1$ is equal to the payoff he gets by choosing seller 1. Again using Theorem 5 this gives

$$\int_{y_1}^{y_j} z_1^{kJ-1}(s, r) ds = \sum_{m=1}^{j-1} \int_{y_m}^{y_{m+1}} z_1^{kJ-1}(s, r) ds$$

$$= \sum_{m=1}^{j-1} \int_{y_m}^{y_{m+1}} \left\{1 - \int_{s}^{1} \pi_j(x, r) f(x) dx\right\}^{kJ-1} ds$$

$$= \sum_{m=1}^{j-1} \int_{y_m}^{y_{m+1}} \left\{1 - \frac{[F(y_{m+1}) - F(s)]}{m} - \sum_{n=m+1}^{J} \frac{1}{n}\left[F(y_{n+1}) - F(y_n)\right]\right\}^{kJ-1} ds$$

Thus in any equilibrium, these two expressions must be equal for a buyer of valuation $y_j$.

Suppose that these two expression are equal for $y_j$. A buyer with a valuation above $y_j$ will be indifferent between sellers 1 through $j$ by construction, and then will be happy to choose each with probability $1/j$. The payoff at seller $j$ is linear for valuations below $y_j$. The payoff at seller $j-1$ is convex. Since the probability of trading at seller $j$ and $j-1$ is the same for a buyer of type $y_j$, buyers with lower valuations must strictly prefer seller $j-1$ so they will be happy to select seller $j$ with probability 0. ∎

## 6.6 Proof of Theorem 4

**Proof.** Consider first the case where $r' > r_\infty$. Then $r'_J > r_J$ for large enough $J$. Then by Lemma 3, the deviator's profits are given by

$$\Phi'_J(r'_J, r_J, k) = kJ \int_{y_J}^{1} \left[x z_J^{kJ-1}(x, r'_J, r_J, k) - v'_J(x, r'_J, r_J, k)\right] z'_J(x, r'_J, r_J, k) dx$$

$$= kJ \int_{y_J}^{1} \left[ x z_J^{kJ-1}(x, r'_J, r_J, k) - v_J(x, r'_J, r_J, k) \right] z'_J(x, r'_J, r_J, k) dx$$

$$= k \int_{y_J}^{1} \left\{ x \left\{ 1 - \frac{1-F(x)}{J} \right\}^{kJ-1} \right.$$

$$- \int_{r_J}^{y_J} \left[ 1 - \frac{F(y)-F(s)}{J-1} - \frac{1-F(y)}{J} \right]^{kJ-1} ds$$

$$\left. - \int_{y_J}^{x} \left[ 1 - \frac{1-F(s)}{J} \right]^{kJ-1} ds \right\} f(x) dx$$

where $y_J$ is the solution to

$$(y - r') \left\{ 1 - \frac{1-F(y)}{J} \right\}^{kJ-1} = \int_{r_J}^{y} \left\{ 1 - \frac{F(y)-F(x)}{J-1} - \frac{1-F(y)}{J} \right\}^{kJ-1} dx$$

to show that $y_J$ is unique, and therefore continuous in $(r'_J, r_J)$ simply note that the equation will be solved if and only if

$$y - r'_J = \int_{r_J}^{y} \left\{ \frac{1 - \frac{F(y)-F(x)}{J-1} - \frac{1-F(y)}{J}}{1 - \frac{1-F(y)}{J}} \right\}^{kJ-1} dx$$

It is then readily shown that the derivative of the right hand side of this expression exceeds 1.

Now using the bounded convergence theorem take limits (point wise in $x$) of the deviator's profits to get the formula

$$= k \int_{y'}^{1} \left\{ x e^{-k(1-F(x))} - \int_{r_\infty}^{x} e^{-k(1-F(s))} ds \right\} f(x) dx$$

$$= k \int_{y'}^{1} \left\{ \left[ x - \frac{1-F(x)}{f(x)} \right] e^{-k(1-F(x))} \right\} f(x) dx - k \left[ 1 - F(y) \right] \int_{r_\infty}^{y'} e^{-k(1-F(s))} ds$$

The last equality follows from integration by parts.

Note that the limit of $v_J$ is calculated explicitly as a part of this argument and is as given in the theorem.

A slightly more complicated calculation serves in the case where $r_\infty > r'$.

$$\Phi'_J(r'_J, r_J, k) = kJ \int_{r'_J}^{1} \left[ x z_J^{kJ-1}(x, r'_J, r_J, k) - v'_J(x, r'_J, r_J, k) \right] z'_J(x, r'_J, r_J, k) dx$$

$$= kJ \int_{r'_J}^{y_J} \left[ x z_J^{kJ-1}(x, r'_J, r_J, k) - v'_J(x, r'_J, r_J, k) \right] z'_J(x, r'_J, r_J) dx$$

$$+ kJ \int_{y_J}^{1} \left[ x z_J^{kJ-1}(x, r'_J, r_J, k) - v'_J(x, r'_J, r_J, k) \right] z'_J(x, r'_J, r_J, k) dx$$

where now $y_J$ is the solution to

$$(y - r_J)\left\{1 - \frac{1-F(y)}{J}\right\}^{kJ-1} = \int_{r'_J}^{y}\left\{1 - \frac{F(y)-F(x)}{J-1} - \frac{1-F(y)}{J}\right\}^{kJ-1} dx$$

Note that $y_J$ must converge to $r_J$ as $J$ goes to infinity as the right hand side of this expression is converging to 0.

Substituting for $z$ according to Lemma 3 now gives this equal to

$$= kJ\int_{r'_J}^{y_J}\left[x\left[1 - [F(y) - F(x)] - \frac{1-F(y_J)}{J}\right]^{kJ-1} - \int_{r'_J}^{x} z_J^{kJ-1}(s, r'_J, r, k)ds\right] f(x)dx$$

$$+kJ\int_{y_J}^{1}\left[x\left\{1 - \frac{1-F(x)}{J}\right\}^{kJ-1} - \int_{r'_J}^{x} z_J^{kJ-1}(s, r'_J, r, k)ds\right]\frac{f(x)}{J}dx$$

$$= kJ\int_{r'_J}^{y_J}\left\{x\left[1 - [F(y_J) - F(x)] - \frac{1-F(y_J)}{J}\right]^{kJ-1}\right.$$

$$\left. - \int_{r'_J}^{x}\left[1 - [F(y_J) - F(s)] - \frac{1-F(y_J)}{J}\right]^{kJ-1} ds\right\} f(x)dx$$

$$+k\int_{y_J}^{1}\left\{x\left[1 - \frac{1-F(x)}{J}\right]^{kJ-1}\right.$$

$$- \int_{r'_J}^{y_J}\left[1 - [F(y_J) - F(s)] - \frac{1-F(y_J)}{J}\right]^{kJ-1} ds$$

$$\left. - \int_{y_J}^{x}\left[1 - \frac{1-F(s)}{J}\right]^{kJ-1} ds\right\} f(x)dx$$

The limit of the first term in this series is given by

$$\lim_{J\to\infty} kJ\int_{r'_J}^{y_J}\left\{x\left[1 - [F(y_J) - F(x)] - \frac{1-F(y_J)}{J}\right]^{kJ-1}\right\} f(x)dx$$

$$= \lim_{J\to\infty}\int_{r'_J}^{y_J} kJ\left\{x\left[1 - [F(y_J) - F(x)] - \frac{1-F(y_J)}{J}\right]^{kJ-1}\right\} f(x)dx$$

$$= \lim_{J\to\infty}\left\{x\left[1 - [F(y_J) - F(x)] - \frac{1-F(y_J)}{J}\right]^{kJ}\Bigg|_{r'_J}^{y_J} - \right.$$

$$\left.\int_{r'_J}^{y_J}\left[1 - [F(y_J) - F(x)] - \frac{1-F(y_J)}{J}\right]^{kJ} dx\right\}$$

$$= r_\infty e^{-k(1-F(r_\infty))}$$

by the bounded convergence theorem. The limit of the second term is

$$\lim_{J\to\infty} kJ \int_{r'_J}^{y_J} \left\{ \int_{r'_J}^{x} \left[1-[F(y)-F(s)]-\frac{1-F(y_J)}{J}\right]^{kJ-1} ds \right\} f(x)dx$$

$$= \lim_{J\to\infty} kJ \int_{r'_J}^{y_J} [F(y_J)-F(x)] \left[1-[F(y_J)-F(x)]-\frac{1-F(y_J)}{J}\right]^{kJ-1} dx$$

$$= \lim_{J\to\infty} \left\{ -[F(y_J)-F(r'_J)] \left[1-[F(y_J)-F(r'_J)]-\frac{1-F(y_J)}{J}\right]^{kJ} - \right.$$

$$\left. \int_{r'_J}^{y_J} \left[1-[F(y_J)-F(x)]-\frac{1-F(y_J)}{J}\right]^{kJ} \right\} d\left(\frac{F(y_J)-F(x)}{f(x)}\right) = 0$$

provided of course that $f(x)$ is uniformly non-zero.

The limits of the other terms are taken using the bounded convergence theorem as in the case of the upward deviation.

The payoff to a buyer who goes to the non-deviating seller in this case is given by

$$\int_{r'_J}^{x} z_J^{kJ-1}(s, r'_J, r, k)ds$$

$$= \int_{r'_J}^{r_J} F^{kJ-1}(s)\, ds + \int_{r_J}^{x} \left[1-\frac{1-F(s)}{J}\right]^{kJ-1} ds$$

which has limit $\int_{r_\infty}^{x} e^{-k[1-F(s)]}ds$. ∎

# References


1. R. Denekere and J. Peck, Competition over price and service rate when demand is stochastic, Northwestern University Discussion paper - to appear in Rand Journal, (1992).

2. P. Diamond, A model of price adjustment, *J. Econ. Theory* **3** (1971), 156–168.

3. B. Holmstrom and R. Myerson, Efficient and durable decision rules with incomplete information, *Econometrica* **51** (1983), 1799–1819.

4. P. McAfee, Mechanism design by competing sellers, *Econometrica* **61** (1993), 1281–1312.

5. P. McAfee and J. McMillan, Auctions and competitive bidding, *J. Econ. Literature* **25** (1987), 699–755.

6. P. McAfee and J. McMillan, Auctions with a stochastic number of bidders, *J. Econ. Theory* **43** (1987), 1–20.

7. J. Peck, Competition in transactions mechanisms: the emergence of competition, Ohio State University mimeo, (1994).

8. M. Peters, Equilibrium mechanisms in a decentralized market, manuscript University of Toronto to appear in Journal of Economic Theory, (1992).

9. J. Riley and W. Samuelson, Optimal auctions, *Amer. Econ. Rev.* **71** (1981), 381–392.

10. R. Wilson, Efficient and competitive rationing, *Econometrica* **57** (1989), 1–40.

11. R. Wilson, Incentive efficiency of double auctions, *Econometrica* **53** (1985), 1101–1115.

12. A. Wolinsky, Dynamic markets with competitive bidding, *Rev. Econ. Stud.* **55** (1988), 71–84.