Department of Economics
and
Institute for Policy Analysis
University of Toronto
150 St. George St.
Toronto, Canada
M5S 1A1

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# On the Equivalence of Walrasian and Non-Walrasian Equilibria in Contract Markets: The case of Complete Contracts

Michael Peters*
Department of Economics
University of Toronto
150 St. George St.
Toronto, Canada
M5S 1A1

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Author's email: peters@chass.utoronto.ca
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**Abstract**

This paper explores two models of an economy in which contracts are exchanged. In the first version contracts are exchanged on a competitive market in which traders expectations concerning conditions that prevail within specific markets adjust until markets 'clear'. In the second model contract designers compete directly against one another by offering alternate contracts. It is shown that Walrasian allocations correspond with the equilibrium allocations in the model with direct competition when the number of traders is made large. Furthermore, the expectational assumptions that drive the Walrasian analysis coincide with off the equilibrium path conjectures in the problem with direct competition.

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# 1 Introduction

In a recent series of papers, Gale (1992),(1994) has proposed a Walrasian theory for contract markets. In this theory each imaginable contract is traded on a separate market. Since contracts normally specify trading prices explicitly, the usual Walrasian notion in which prices adjust to clear markets must be adapted. In Gale's model traders form expectations about the conditions that will prevail in different markets, and the requirement that traders expectations be rational in all active markets replaces the standard market clearing condition.

One of the key advantages of these contract market models is that contract designers can compete with one another for trading partners in a non-trivial way. This has numerous applications. There is evidence, for example, that an increase in the minimum wage affects the rate at which workers apply to firms affected by the change (Holzer, Katz, and Kreuger 1991), as well as the rate at which higher wage offers are accepted. This phenomena is difficult to explain in a standard search model where workers sample wage offers randomly. It is an immediate consequence when workers can focus their search activity as they can in a Walrasian model. The Walrasian approach provides a simple resolution of the Diamond paradox (Diamond 1971) in which wage setting firms can act as monopsonists in a competitive search market. (Moen 1994) uses the Walrasian approach to explain why Walrasian wage distributions must be efficient. Older applications include (Carlton 1978) who used the approach to explain consumers choices among rationing schemes at electric utilities[1]

These applications all assume that contracts are *complete* in the sense that the value of the contract to any trader who accepts it is determined en-

[1]Competition among contract designers who can offer different rationing schemes as well as prices is the concern of papers by (Peters 1985) ,(Wilson 1989), and (Denekere and Peck 1992).

tirely by the terms of the contract. Gale was primarily interested in whether or not 'Walrasian' contract markets would be efficient in the presence of adverse selection[2]. This paper restricts attention to the complete contracts case of the older literature. The primary contribution of this paper is a decentralization theorem that shows that in an economy consisting of an infinitely large number of traders, the allocations generated by Gale's Walrasian equilibrium coincide exactly with the equilibrium allocations of a non-Walrasian model which makes more conventional assumptions about traders' beliefs in off equilibrium situations.

The Walrasian model is conceptually simpler than any of the models that involve direct competition. Indeed this is the primary motivation suggested by Gale for the study of Walrasian models[3]. Yet even without adverse selection, the Walrasian theory requires additional, and very strong informational assumptions to generate results. The reason is that the theory only restricts traders' expectations in what Gale calls *active* markets. These are simply markets in which some transactions occur in equilibrium. This assumption is in the spirit of subjective equilibrium (Kalai 1993) since traders will never observe anything inconsistent with their beliefs if the condition is satisfied. Yet when expectations elsewhere are unrestricted many equilibria can be supported. If traders jointly believe that they will not be able to find trading partners in some market, then they will rationally avoid this market and their belief will be fulfilled. This makes the set of active markets arbitrary.

This problem can be avoided by imposing two restrictions on traders' beliefs. First, traders must have common beliefs about inactive markets.

[2]Adverse selection here means that when a contract is exchanged, the value of the contract to the trader on one side of the exchange depends on the private characteristics of the trader on the other side of the exchange.

[3]The analytical difficulties associated with modelling the competitive process directly are well illustrated in (Wilson 1989) who discusses competitive rationing in the last part of his paper. More recently some progress has been made on the competitive rationing process with direct competition among sellers (Peters 1994, Peters and Severinov 1993, McAfee 1993). The strategic considerations that arise are complicated.

Second, they must believe that there are many buyers and sellers in all markets. If buyers, for example, believe that they have zero probability of finding a partner in a specific market, this should be because they believe that too many buyers go to the market, not because no sellers do. Though it is simple enough to present these assumptions, it is not obvious why they should be true.

In the non-Walrasian model of direct competition presented in this paper, inactive markets correspond to deviations that sellers consider, but reject on the equilibrium path. For sellers to be able to reject these deviations, they must be able to correctly anticipate the effect that such deviations will have on their payoff. The decentralization theorem proved in this paper, in addition to showing the equivalence of the equilibrium allocations in the two models, shows that these correct conjectures in the game with direct competition coincide with one of the sets of beliefs about conditions in inactive markets that support the Walrasian equilibrium.

This paper is organized in the following way. The first section gives an abstract description of the queuing problem in a fairly general economic environment. The second section describes the Walrasian equilibrium for the contract market, gives conditions under which an equilibrium can be shown to exist. The third section describes the non-Walrasian model and proves the equivalence of it's equilibrium allocations to that of the Walrasian model. Finally we discuss some possible issues associated with extending the model to the more general case with incomplete contracts.

# 2 The Economy

The economy consists of an ocean of buyers and sellers. Sellers are characterized by their type $s \in S$ where $S$ is an interval $[s_0, s_1]$ in $\mathbf{R}^+$, while buyers are characterized by their type $b \in B \subset \mathbf{R}^+$ where again, $B$ is assumed to be an interval $[b_0, b_1]$. Let the distribution functions $G$ and $F$ characterize the distributions of seller and buyer types respectively. It will be assumed that each of these functions can be written as the sum of a jump function with a *finite* support, and an absolutely continuous function[4]. The measures of $S$ and $B$ are assumed to be $N$ and $M$ respectively. We reserve the notation $\bar{k} = M/N$ to denote the ratio of the measure of buyers to the measure of sellers in the entire economy.

Let $G(s)$ be the measure of the set of sellers whose types are less than or equal to $s$ for each $s \in S$. If $\Sigma$ is a subset of $S$, we will occasionally use the notation $G(\Sigma) = \int K_{\Sigma}(s)dG(s)$ to denote[5] the measure of the set of sellers whose types belong to the set $\Sigma$.

The names buyer and seller are arbitrary in this context. The important assumption is that a mutually beneficial exchange requires exactly one member from each group in the sense that a pair of buyers cannot profitably trade with one another. A *contract* between a buyer and seller is a vector $x \in \mathbf{R}^l$ for $l > 0$. We will assume that the set of feasible contracts, $\Xi$, is closed and bounded in $\mathbf{R}^l$.

Buyers and sellers maximize expected utility. Payoffs for a buyer or seller respectively who finds a trader on the other side of the market willing to exchange the contract $x$ are given by $u(x, b)$ and $v(x, s)$ where $b$ and $s$ are the buyer's and seller's types. The functions $u$ and $v$ are assumed to be jointly continuous, and strictly quasi-convex. The payoffs for buyers or sellers who do not succeed in finding a trading partner are given by $\bar{u}$ for buyers and $\bar{v}$ for sellers.

It will be assumed throughout the paper that there exists a contract in $\Xi$ which yields all buyers and sellers an ex post payoff that is equal to their reservation utility. The availability of this contract makes it possible

[4]Any distribution function can always be decomposed into a jump function with a countable support and a *continuous* function. So the key assumptions here are finiteness of the support and absolute continuity.

[5]Here $K_S(x)$ is the indicator function for the set $S$.

to allow for entry and exit. For example, if firms are required by law to pay a minimum wage that makes it impossible for some firm types to make profits paying that wage, then there might be a wage contract that pays the minimum wage but offers 0 hours of work. Firms who cannot make a profit at the minimum wage can then choose this contract in equilibrium, which will be equivalent to exiting the process completely. By the same token, if many buyers and sellers choose this contract in 'equilibrium', then they represent a pool of potential entrants into the process. The equilibrium will then be immune to entry in this sense.

## 2.1 The Matching Technology

The set of markets in the Walrasian story below coincides with the set of feasible contracts. Every seller who offers a contract in a particular market must offer the same contract as every other seller in that market. The probability $q$ that a buyer can find a seller to trade with in a specific market depends on the level of congestion in that market. The level of congestion is measured by the ratio $k$ of the measure of buyers who choose the market to the measure of the set of sellers who choose the same market. Let $q : \mathbf{R} \to [0,1]$ denote this relationship. It is customary to assume that $q$ satisfies

1. $q$ is continuous and non-increasing with $q(0) = 1$; $q(\infty) = 0$;
2. $q(k)k$ is non-decreasing with limits 1 and 0 respectively as $k$ goes to infinity and 0.

The number $q(k)$ denotes the probability that a buyer locates a trading partner within any measurable set of markets, when the ratio of the measure of the set of buyers who choose those markets to the measure of the set of sellers who choose those markets is $k$. Since $m$ is the measure of buyers assigned to markets in $\bar{X}$, the total number of trades within $\bar{X}$ will then be $q(k)m$. This means that the probability that a seller who chooses a market in $\bar{X}$ trades is equal to $q(k)\frac{m}{n}$ or $q(k)k$, which explains the restrictions on this function.

Traders can now rank the different trading opportunities. A buyer's expected payoff in market $x$ (i.e., the payoff to choosing the location where $x$ is traded) is given by

$$U(x, b, k) = q(k)\, u(x, b) + [1 - q(k)]\, \bar{u} \tag{1}$$

while a seller's expected payoff in the same location is

$$V(x, s, k) = q(k)\, kv(x, s) + [1 - q(k)\, k]\, \bar{v} \tag{2}$$

Though the description of Walrasian equilibrium given in this paper applies to a broad class of matching technologies, in this paper interest will be focussed on two particular technologies. In any finite version of the model, a seller with a contract may be matched with many buyers. If anonymity is desirable, then the only reasonable way for him to select among the buyers is to choose one of them at random. This simple randomization scheme leads to the somewhat standard matching technology for the market in which traders on the short side of the market always trade, while every trader on the long side of the market is equally likely to trade. This means that $q(k) = 1$ if $k < 1$ while $q(k) = 1/k$ otherwise. This technology will be referred to henceforth as the *classical matching technology.*

It will become apparent below that traders who select randomly among their available options at an individual level do not necessarily generate a market that behaves according to this classical matching technology. The reason is that there is an intervening selection rule that describes how buyers allocate themselves among the sellers in the first place. The selection rule that is used below to describe buyers search behavior will endogenously generate a matching technology for the market that behaves according to

$$q(k) = \frac{1 - e^{-k}}{k}$$

This technology will be referred to as the *exponential matching technology.*

## 2.2 Allocations

Let $\Psi^s$ and $\Psi^b$ be distribution functions whose domains are $S \times \Xi$ and $B \times \Xi$ respectively. These distributions are called *allocations* if their corresponding marginal distributions on $S$ and $B$ coincide with $G$ and $F$ respectively, and if the marginal distributions $\sigma^s$ and $\sigma^b$ on the space of contracts $\Xi$ are both integrable functions that can be written as the sum of a jump function with a finite support, and a function that is absolutely continuous with respect to Lebesque measure. The jump function deals with contracts that are allocated a non zero measure of traders, like the contract that yields traders their reservation utility. The function also makes it possible to deal with situations in which there are a finite number of types of buyers and sellers, an assumption that is used in examples, and throughout most of the literature.

The measure of the sets of buyers and sellers who are assigned to particular contracts will play an important role in what follows. Let $X^i_{jump}$ denote the finite set of points at which the distribution $\sigma^i$ has a 'jump'[6]. Let $\sigma^i_{jump}(x)$ denote the *jumpsize* at $x$ (note that $\sigma^i_{jump}(x) = 0$ if $x \notin X^i_{jump}$). The distributions $\sigma^i$ for $i = s, b$ can be written as

$$\sigma^i(x) = \sum_{y \in X^i_{jump}; y \leq x} \sigma^i_{jump}(y) + \int_{x^0}^{x} \tilde{\mu}^i(y)\, dy$$

for some integrable functions $\tilde{\mu}^i$. Let $X_{jump}$ denote the (finite) union of the supports of $\sigma^s_j$ and $\sigma^b_j$. The *generalized derivative of* $\sigma^i$ at $x$ is defined to be

$$\mu^i(x) \equiv \begin{cases} \sigma^i_{jump}(x) & if\ x \in X_{jump} \\ \tilde{\mu}^i(x) & otherwise \end{cases}$$

If there is a jump in the distribution $\sigma^i$ at $x$, then we will interpret the jumpsize as the measure of the set of sellers assigned to $x$. If there is no jump

[6] Formally, let $\Sigma^n$ be a declining sequence of measurable sets in $\Xi$ whose limit superior is equal to $x$. Then the jumpsize of the distribution $\sigma^i$ at $x$ is defined to be the supremum over all such sequences of $\lim_{n\to\infty} \int K_{\Sigma^n} d\sigma^i$.

in the distribution, then it is more reasonable to use the density, represented by the Radon Nikodym derivative, as the assignment.

As a slight abuse of notation we will refer to the marginal distributions $\sigma^s$ and $\sigma^b$ as allocations and take it for granted that these are derived from a joint distribution whose marginal on the space of buyers and sellers is $F$ and $G$ respectively.

# 3 Walrasian theory

In each particular contract market $x$, traders conjecture what the value of $k$ will be, say $k(x)$. This conjecture determines traders' rankings among markets through (1) and (2). Traders then allocate themselves to the market that yields them the highest expected utility. These allocations generate densities of say, $\mu^s(x)$ sellers and $\mu^b(x)$ buyers in market $x$.

In equilibrium traders' conjectures about $k$ should be correct. This leads to the condition $\frac{\mu^b(x)}{\mu^s(x)} = k(x)$, or, in a more convenient form, to the *generalized market clearing condition* $\mu^b(x) = k(x)\mu^s(x)$. This condition reduces to the usual demand equal to supply condition in market clearing models when the simple rationing rule is in effect.

The power of the contract market approach comes from two key assumptions. First, traders must all share the same beliefs about the amount of congestion in each market. Secondly, traders must be able to formulate a well defined expectation for every possible market. These are basic competitive assumptions. The first is analogous to the assumption that all traders agree on the price at which specific commodities can be exchanged. The second assumption is analogous to the complete markets assumption in standard competitive theory.

Notice that the assumption that there is a well defined $k$ in every market means that the conjecture by buyers that there is no point going to a market because there will not be any sellers there is ruled out. Buyers believe that there are sellers in every market. If the chance of completing a trade in a market is small, it is because there are too many buyers in the market, not too few sellers. Gale (1994) calls this property *orderliness* and attributes it to Hahn (1978).

Now given the relationship $k(x)$, define the sets

$$S \supseteq \phi^s(x,k) = \{s' \in S : \exists x' \leq x; V(x', s', k(x')) \geq V(y, s', k(y)) \forall y \in \Xi\}$$

and

$$B \supseteq \phi^b(x,k) = \{b' \in B : \exists x' \leq x; U(x', b', k(x')) \geq U(y, b', k(y)) \forall y \in \Xi\}$$

Here $\phi^s(x,k)$ is just the set of seller types who can find a contract below $x$ that is at least as good as any other available contract given the relationship that they believe prevails between $x$ and $k$[7].

An allocation $\sigma = \left\{\sigma^s, \sigma^b\right\}$ and a trading rule $k(x)$ constitute a *Walrasian equilibrium for the contract market* if

- ($\sigma^s$ is an aggregate supply function) $\sigma^s(\Xi) = N$ and for almost all $x \in \Xi$,
$$\sigma^s(x) \leq \int K_{\phi^s(x,k)} dG[s];$$
- ($\sigma^b$ is an aggregate demand function) $\sigma^b(\Xi) = M$ and for almost all $x \in \Xi$,
$$\sigma^b(x) \leq \int K_{\phi^b(x,k)} dF[b];$$
- (generalized market clearing condition) for almost all $x$ in the union of the supports of the distributions $\sigma^s$ and $\sigma^b$,
$$\mu^b(x) = \mu^s(x) k(x)$$

[7]For the moment, we will ignore the problem of whether or not these sets are measurable. The assumptions below will guarantee that they are intervals which will resolve this problem.

where 'almost all' should be understood to mean relative to the measures induced by the distribution functions $\sigma^s$ and $\sigma^b$.

The first condition requires, along with the obvious adding up requirement, that the measure of the set of sellers allocated to contracts below $x$, is no larger than the measure of the set of sellers whose optimal choice of contracts is below $x$ given beliefs $k$. The second condition is the same thing for buyers. These are both analogous to the requirement that the aggregate demand and supply functions be the sum of individual demand and supply functions in the usual Walrasian story. The third condition is the generalized market clearing condition described above. It requires that buyers' and sellers' beliefs about the trading probability in each market are correct in each market which is assigned some buyers or sellers[8].

## 3.1 An Example

To illustrate some of the properties of equilibrium, and some of the problems that can arise, it is useful to refer to a simple example. In this example, workers attempt to find jobs with firms who offer vacancies in the various contract markets. To keep matters simple, assume that a contract specifies the wage that a worker receives for carrying out a specific job for the firm. The set of feasible contracts is the set of wages in some interval $[\underline{w}, \bar{w}]$. The disutility of effort for the worker is given by $b$, the worker's type, while the profit for the firm who fills a vacancy is given by $s$. Utility is given by $u(w, b) = w - b$ while $v(w, s) = s - w$. The payoff to a worker who does not find a job can be normalized to zero so that the expected payoff to a worker who searches for a wage $w$ and faces a ratio of workers to firms equal to $k$ is $U(w, b, k) = q(k)(w - b)$. Similarly for firms the expected payoff is $V(w, s, k) = q(k)\, k\,(s - w)$.

Various possibilities are illustrated in Figures 1 and 2. In the first figure,

[8](Gale 1994) refers to this condition as *fullfilled expectations in active markets.*

Figure 1:

all buyers have the same disutility for labour and all the firms have the same productivity. The worker's trade-off between wage and congestion is represented by a family of indifference curves of which the curve $\bar{U}$ is a typical member. The preferences of firms are given by a family of indifference curves of which $\bar{V}$ is a typical member.

The curve $k(w)$ represents the relationship between congestion and wages that traders think prevails in the market. Traders are free to choose any point on this relationship, so both sides will choose the wage-congestion pair at which their indifference curves are tangent to this relationship.

In this example with homogenous tastes, the equilibrium is very simple. With strictly convex indifference curves of the kind shown, all traders on one side of the transaction will choose to search in the same market. Thus the rational expectation condition requires that all traders select the same market, and that the level of congestion expected in this market be equal to the ratio of the measure of searching workers to searching firms.

Furthermore, since traders on both sides of the market are supposed to perceive themselves as selecting along a common opportunity locus, the indifference curves of workers and firms must be tangent at the equilibrium choice when this choice is on the interior of the set of feasible contracts. The line $M^*M^*$ represents the locus of tangencies between worker firm indifference curves. The equilibrium is found by finding the intersection of this locus with the vertical line through the market congestion ratio (given as $k^*$ in the picture).

The case where all traders have the same tastes is not very interesting, but it does illustrate the power of the assumption that all traders perceive the same trade-off between the wage and the level of congestion. The equilibrium in this simple example is unique, and clearly has desirable efficiency properties (Moen (1994)).

A similar example is illustrated in Figure 1 for the case in which there is an active market on the boundary of the set of feasible contracts. This might occur if there were a minimum wage. This possibility is illustrated by the point $(w_1, k^*)$ in the figure. Here the indifference curves need not be tangent since the contract cannot offer lower wages that the minimum $w_1$. The feasible locus generated by the expectation function must pass between the indifference curves of workers and firms to the right of the contract $(\bar{w}, k^*)$.[9]

Figure 2 shows a more complicated example. There are two types of workers and two types of firms. The first set of workers and firms is the same set as in Figure 1. Now however, the families of indifference curves and

[9]If it happens that production is unprofitable at the minimum wage, then some null contract has to be added to the feasible set of contracts. Firms would all choose this null contract. Workers could also be induced to choose this null contract as well by setting $k$ equal to infinity for all the existing wage contracts.

the locus of their tangencies are given by $\bar{U}_1$, $\bar{V}_1$, and the line $M_1^*M_1^*$. The second set of workers have the flatter family of indifference curves with typical member $\bar{V}_2$. This new type of worker has a higher disutility of leisure. They are matched up in equilibrium with a new kind of firm that has a steeper set of iso-profit lines with typical member $\bar{U}_2$ in the figure. The locus of tangencies between these new sets of indifference curves is given by the line $M_2^*M_2^*$ in figure 2.

As before the expected relationship between the wage on offer and the level of congestion is given by the line $k(w)$. This expectation is shared equally by all traders in the market. Workers and firms of type 1 congregate in the market $w_1$ while workers and firms of type 2 congregate in the market $w_2$. Provided that the actual ratio of the measure of workers of type 1 to firms of type 1 in the economy is $k_1$ while the actual ratio of the measure of workers of type 2 to firms of type 2 is $k_2$, then the outcome shown is an equilibrium.

As in the first example, the distribution of types across contracts involves mass points at the two contracts $w_1$ and $w_2$. In order for the rational expectations condition to hold, the level of congestion in each market must be equal to the ratio of the measures of workers to firms who participate in each market. Problems arise if the ratio of workers to firms of type 2 is given by $k_3$. In this case, the point on the locus of tangencies $M_2^*M_2^*$ where $k = k_3^*$ generates a contract that is not incentive compatible when paired with $(w_1, k_1)$ (since firms of type 1 prefer the contract in the market involving traders of type 2).

The difficulty with the solution concept is that a necessary condition for equilibrium is that in active contract markets on the interior of the contract space, no more that one type from each side of the market can participate (again assuming that marginal rates of substitution are monotone in type)[10].

[10]As a referee has pointed out, this need not be true on the boundary of the feasible set of contracts. For example, in the equilibrium under the minimum wage given in Figure

Figure 2:

In any active market, the wage–congestion pair must lie on a locus that passes between the indifference curves of the worker and firm since equilibrium requires them to maximize along the equilibrium locus. Any new type of worker must necessarily have an indifference curve that cuts through this feasible locus. Then this new type of worker must be attracted to some other market.

To check for equilibrium in this case, it is necessary to match the type 1 workers together with the type 2 firms, and the type 2 workers together with the type 1 firms. This generates two new congestion levels and two new wage contracts that equate the marginal rates of substitution for workers and firms in each pair. If this pair of wage-congestion combinations is not incentive compatible, then there will be no equilibrium at all.

## 3.2 An Existence Theorem

If there is preference diversity on both sides of the market, then it will be possible to find distributions of traders such that equilibrium does not exist. As is apparent from the diagram, there will also be distributions for which equilibrium exists even when the distributions are perturbed slightly. The only reasonable sufficient condition for existence that I have been able to establish so far is homogeneity of tastes on one side of the market[11].

**Assumption** 1. Buyers all have identical tastes, i.e., $\exists b^* : F(b) = 0 \forall b < b^*; F(b) = 1\ otherwise.$

2. The function $V(x, s, k) = kq(k)v(x, s) + (1 - kq(k))\bar{v}$ is strictly quasi-convex in $(x, k)$ and satisfies the single crossing property: for any $(x', k') \neq (x, k); k' \geq k, s' > s$ then $V(x', s', k') \geq V(x, s', k) \Rightarrow V(x', s, k') > V(x, s, k)$.

1, it would be possible to modify the preferences of some of the workers by making their indifference curves steeper. Since the wage cannot fall below the minimum, these new workers would mix with the old workers without creating any difficulty.

[11]This assumption is also common in the literature (Gale 1994, Moen 1994).

3. The function $U(x, b^*, k) = q(k)u(x, b^*) + (1 - q(k))\bar{u}$ is strictly quasi convex in $(x, k)$.

If the space of contracts were single dimensional, like the set of possible trading prices, then this assumption makes the indifference curves for buyers and sellers in $(p, k)$ space convex downward and upward respectively. Furthermore, an increase in $s$ would make the indifference curve flatter. Note that these assumptions involve a joint restriction on preferences and the matching technology.

**Theorem 1** *Under Assumptions 1, 2 and 3, there exists a Walrasian equilibrium for the contractual market.*

The proof of this and the other theorems in the paper is given in the appendix. In outline the proof is straightforward. It exploits the fact that all buyers have the same preferences. Pick $u^0$ and for each $x \in \Xi$, let $k(x) = \{k : U(x, b^*, k) = u^0\}$ if such a solution exists and $k(x) = 0$ otherwise. Now let sellers allocate themselves on the surface $\{x, k(x)\}$ as they desire. If buyers believe that the relationship between contracts and matching probabilities is given by $k(x)$ then they will not care which contract they choose. Thus it makes sense to allocate buyers across the various markets until the ratio of the density of buyers allocated to market $x$ to the density of sellers who choose market $x$ is just equal to $k(x)$. Of course this might not be feasible, so we manipulate $u^0$ until the integral of all the buyers allocated across all the markets is equal to the measure of the set of buyers. This integral turns out to be a continuous function of $u^0$ under our assumptions which is sufficient to make it possible to prove existence.

# 4 A Decentralized Model of Contractual Markets

As may be apparent from the preceding discussion, there are two restrictions on beliefs that do most of the work in contract market equilibrium. The first restriction is that all traders agree about the value for $k$ that is expected to prevail in inactive markets. Perhaps more important, traders don't really believe in inactive markets at all. They persist in expecting to trade with positive probability in markets which provide no incentive for traders to visit. The purpose of this section is to show the degree to which these assumptions make sense.

In this section an explicit game is considered in which an economy consists of a finite number of sellers who compete directly with one another by offering different contracts to a finite number of buyers. The game involves two stages. In the first, sellers simultaneously and publicly offer contracts to buyers. In the second stage, buyers choose among these contracts.

In a game like this, there is no such thing as active or inactive markets. However, in any equilibrium, sellers can consider deviating from the offer they are supposed to make. If they do, they will be able to calculate the expected number of buyers who will respond to such an offer by using their information about equilibrium strategies. Of course, buyers should also understand exactly how many other buyers will respond to such a deviation, again from their knowledge of equilibrium strategies.

It is shown that, in the limit, the equilibrium allocation of this two stage game corresponds with the Walrasian equilibrium allocation of the economy to which the finite game converges. Furthermore, one of the sets of beliefs about congestion in inactive markets that supports the Walrasian allocation, coincides with the actual levels of congestion that prevail in the equilibrium of the limit game when a seller makes an off the equilibrium path contract offer.

### 4.1 The finite game

In the finite games that we describe, sellers' preferences vary, while all buyers are identical[12]. It is assumed that the distribution of seller's types in the limit economy is given by the distribution function $G(s)$. The theorem is proved by considering a series of finite approximations to this limit distribution.

To see how these approximations are constructed, begin by indexing the games by the number $J$ of participating sellers. Let $\bar{k}$ be the ratio of the measure of the set of buyers to the measure of the set of sellers in the Walrasian economy. The number of buyers in the finite game with $J$ sellers will be assumed to be $\bar{k}J$.

Second, let $s^j \equiv \{s : G(s) = nj/J\}$ for $j = 1, \ldots, J$. For each of the $J$ sellers in the game, it is assumed that $j$'s preferences are given by $\tilde{V}_j(x, k) \equiv V(x, k, s^j)$. This assignment of preferences approximates the preferences in the distribution $G$ in the sense that the distribution function $G_J$ of preferences that applies in the finite version of the game consisting of $J$ sellers and $kJ$ buyers, converges point-wise, and hence weakly to the distribution $G$. Notice that $G$ does not have to be an absolutely continuous distribution function in this formulation. In the interesting case in which all sellers are identical, the distribution $G$ is degenerate at some point. The finite approximation in that case simply involves a finite game with $J$ identical sellers.

The game form that governs the interaction between buyers and sellers may now be described. The sellers begin the procedure by simultaneously and publicly offering contracts to buyers. Contract offers must come from the space $\Xi$ which describes feasible markets in the Walrasian version of the model. The buyers, after seeing the contracts offered by each of the sellers, simultaneously choose one and only one of the sellers as a potential trading

[12]This is to resolve the difficulties with existence mentioned above. However the identity of buyers plays a key role in the proof, and I am not sure how to extend the theorem to the case buyers are different, even when it is known that a Walrasian equilibrium exists in the limit economy.

partner. If many buyers choose the same seller, then the seller selects one of the buyers at random and executes the contract $x$ that he offered with that buyer. The other buyers whom were not selected by the seller are out of luck, and are assumed to end the procedure without trading.

### 4.2 Equilibrium in the Decentralized Model

Let $\tilde{x} = \{x', x_{-j}\}$ denote the vector of contract offers made by each of the sellers, where $x'$ denotes the offer by the seller indexed $j$, and $x_{-j}$ denotes the vector of offers by each of the other sellers. Attention in this section will be (somewhat arbitrarily) restricted to equilibria in which all buyers behave symmetrically. So let $\pi_J^j(\tilde{x})$ denote the (common) probability with which *each* of the buyers selects seller $j$. Of course, since this choice is made after the buyers observe the vector of seller offers, the choice should be conditioned on $x$. Let $\pi_J(\cdot)$ denote the vector of such probability functions, assumed to satisfy $\sum_{i=1}^{J} \pi_J^i(\tilde{x}) = 1$ for all $\tilde{x} \in \Xi$.

If many buyers select a particular seller $j$, then one of them is chosen at random by seller $j$ as a trading partner.[13] The probability with which a buyer who wants to trade with seller $j$ will also be chosen as seller $j$'s trading partner is given by

$$\sum_{l=0}^{kJ-1} \left( \begin{array}{c} kJ-1 \\ l \end{array} \right) \left(\pi_J^j(\tilde{x})\right)^l \left(1 - \pi_J^j(\tilde{x})\right)^{kJ-1-l} \frac{1}{l+1}$$

$$= \frac{1}{kJ\pi_J^j(\tilde{x})} \left[1 - \left(1 - \pi_J^j(\tilde{x})\right)^{kJ}\right] \equiv \tilde{q}_J^j(\tilde{x})$$

The probability with which the seller trades is just

$$\left[1 - \left(1 - \pi_J^j(\tilde{x})\right)^{kJ}\right] \equiv \tilde{r}_J^j(\tilde{x})$$

[13] The assumption that each of the buyers is equally likely to be selected by seller $j$ might be motivated by the assumption that the buyers arrive randomly at different times and the seller simply trades with the first buyer who arrives.

The matching technology is said to be *incentive consistent* if for all $\tilde{x}$, $\pi_J^j(\tilde{x}) > 0$ implies that

$$\sum_{j=1}^{J} \pi_J^j \left\{ \tilde{q}_J^j(\tilde{x}) u(x_j) + (1 - \tilde{q}_J^j(\tilde{x}))\bar{u} \right\} \geq \tilde{q}_J^i(\tilde{x}) u(x_i) + (1 - \tilde{q}_J^i(\tilde{x}))\bar{u} \qquad (3)$$

for all $i = 1, \ldots, J$. We will also say that a vector of choice probabilities $\left\{\pi_J^j\right\}$ is incentive consistent at $x$ if 3 holds for $\tilde{x} = x$, and

$$\frac{1}{kJ\pi_J^j}\left[1 - \left(1 - \pi_J^j\right)^{kJ}\right] \equiv \tilde{q}_J^j(x)$$

for each $j = 1, \ldots, J$.

Notice that a matching technology is incentive consistent if the assignment probabilities that it specifies constitute a continuation equilibrium for the subgame that is played when buyers simultaneously choose among the sellers. It is, however, only one of many potential continuation equilibria.

The pair $\{x^*, \pi_J^*\}$ is an *equilibrium* if $\pi_J^*$ is incentive consistent and if for each $j$, and each possible deviation $x_j$,

$$\tilde{r}_J^j(x_j^*, x_{-j}^*) v_j(x_j^*) + (1 - \tilde{r}_J^j(x_j^*, x_{-j}^*))\bar{v} \geq$$
$$\tilde{r}_J^j(x_j, x_{-j}^*) v_j(x_j) + (1 - \tilde{r}_J^j(x_j, x_{-j}^*))\bar{v}$$

This much is straightforward. The purpose of this section is to try to extend these definitions to the limit where there are an infinite number of buyers and sellers. The limit is of interest because the strategic considerations that normally make Walrasian reasoning inappropriate, disappear when there are an infinite number of buyers and sellers.

# 5 Analysis of the Equilibrium of the Decentralized Model

The convexity assumptions that we have made ensure that the relation

$$x(\tilde{u}, s) \equiv \arg\max_x \{v(x, s) : u(x) = \tilde{u}\}$$

is a continuous function. This means that once we decide what utility payoff a seller wishes to offer to buyers to attract them, we can use the function $x(\tilde{u}, s)$ to work back to the contract. It is then possible to think of the seller offering a level of utility instead of a contract.

To begin we show a simple result that relates competition in contracts to competition in levels of utility. Let $u_0 = \min_{x \in \Xi} u(x)$, while $u_1 = \max_{x \in \Xi} u(x)$.

**Lemma 2** *Suppose that buyer and seller preferences satisfy assumptions 2 and 3 on the space $\Xi \times \mathbf{R}$. Then these preferences satisfy the same assumptions on the space $[u_0, u_1] \times \mathbf{R}$.*

## 5.1 The Infinite Game

By Lemma 2, it is possible to think of a seller making a one dimensional utility offer instead of a many dimensional contract offer. When the seller makes this offer in the limit game, he makes it against a fixed distribution of utility offers made by the other sellers. This distribution $H$ will generate an expected payoff for buyers that is subsequently independent of the utility or contract offer that is made by any single deviating seller. To motivate this interpretation a finite approximation of the limit game is described in which the distribution of utility offers made by all but one of the sellers, is approximately equal to the limit distribution $H$ that is of interest. The one additional seller is a deviator who makes an arbitrary utility offer. The payoff to the deviator from making this arbitrary offer in the limit game is then given by the limit of his payoffs from this arbitrary offer as the sequence of finite approximations converges to $H$.

To approximate the distribution $H$, we begin by finding each of the $J$ tiles of the distribution $H$. We then locate one seller at each of these $J$ tiles. That is, we have one seller offer a contract that is better in utility terms than the fraction $1/J$ of all offers in the distribution $H$, one seller offers a contract that is better than $2/J$ of the offers in $H$, and so on. Then for any deviation $u'$, we remove the seller from the largest tile that does not exceed $u'$ and replace him with the deviator offering $u'$[14].

The payoff that this deviator gets can then be calculated for each of the finite approximations to $H$ that are indexed by $J$. This sequence of payoffs to the deviator has a limit, that defines the payoff for the deviating seller in the limit game.

Since this procedure yields a payoff for each possible utility offer when it is made against a distribution $H$ of offers by the other sellers, it is possible to calculate a distribution of best replies to $H$. The fixed point of this mapping from distributions into distributions will constitute an equilibrium for the limit game.

To be specific, for any $H$ and any deviation $u'$, let $\tilde{u}_j(H,J) = \{u : H(u) = jN/J\}$ (recall that $N$ is the measure of the set of sellers) for $j = 1, \ldots, J$. Let $i$ be the largest index for which $\tilde{u}_j(H,J) \leq u'$. Let $\tilde{u}(H,J)$ denote the array of offers

$$\tilde{u}(H,J) \equiv \{\tilde{u}_1(H,J), \ldots \tilde{u}_{i-1}(H,J), u', \ldots, \tilde{u}_J(H,J)\}$$

The empirical distribution function that this implies is given by

$$H_J(u,u') = \frac{\#\{\tilde{u} \in \{\tilde{u}_1(H,J), \ldots \tilde{u}_{i-1}(H,J), u', \ldots, \tilde{u}_J(H,J)\}; \tilde{u} \leq u\}}{J} N$$

Notice that this distribution is a simple function that converges point-wise, and hence weakly to the distribution $H$ as $J \to \infty$ no matter what $u'$ is. Our first result is the following

**Theorem 3** *For any distribution $H$ of utilities offered by sellers, if the matching technology is incentive consistent, then there is a utility level $u^*(H)$ for buyers such that for any contract offer $x'$, any $j$ and any $s$, the limiting*

[14]This awkward procedure ensures that the sequence of approximating distributions to $H$ approaches it from below in an appropriate sense.

*payoff to seller $j$ is given by*

$$\lim_{J\to\infty}\left\{\left(1-\left(1-\pi_J^j\left(x',\tilde{x}_{-j}\left(H,J\right)\right)\right)^{kJ}\right)v(x',s)+\right.$$
$$\left.\left[1-\left(1-\left(1-\pi_J^j\left(x',\tilde{x}_{-j}\left(H,J\right)\right)\right)^{kJ}\right)\right]\bar{v}\right\}$$
$$=\left\{\begin{array}{c}\bar{v}\ if\ u\left(x'\right)<u^*(H)\\ \left(1-e^{-k'}\right)v(x',s)+e^{-k'}\bar{v}\ otherwise\end{array}\right\}$$

*where $k'$ is the unique solution to the equation*

$$\frac{1-e^{-k}}{k}u(x')+\left[1-\frac{1-e^{-k}}{k}\right]\bar{u}=u^*(H) \tag{4}$$

This is the key result in the paper. As the seller considers possible deviations to alternative contract offers he finds that the continuation equilibrium causes his trading probability to change exactly as if he were moving along a Walrasian like feasible set in the space of contracts and congestion levels. Furthermore, the feasible set corresponds to a buyer indifference curve.

Note further that the theorem applies to any contract offer that the seller considers making. Since we have not yet calculated the equilibrium distribution $H$, there is no sense in which this payoff applies only in active markets.

With the help of the key relation $H\to u^*\left(H\right)$ described by this theorem, it is now possible to give a complete description of equilibrium in the limit game. To keep down the notation, which is already dense enough, we will not give the most general definition. Define the relations

$$\hat{u}\left(s,u\right):s\to\underset{u'}{\operatorname{argmax}}\left\{\left[1-e^{-k}\right]v\left(x\left(u',s\right),s\right)+e^{-k}\bar{v}:\bar{u}+\frac{1-e^{-k}}{k}\left[u'-\bar{u}\right]\geq u\right\}$$

and

$$\hat{x}\left(s,u\right):s\to\underset{x}{\operatorname{argmax}}\left\{\left[1-e^{-k}\right]v\left(x,s\right)+e^{-k}\bar{v}:\bar{u}+\frac{1-e^{-k}}{k}\left[u\left(x\right)-\bar{u}\right]\geq u\right\}$$

It is readily shown that these are continuous functions. Furthermore, from the single crossing property of seller preferences, the function $\hat{x}$ has a continuous inverse.

Since $\hat{u}$ and $\hat{x}$ are continuous, they are measurable, so for each sub-interval $[u_0, u']$, the set $\{s : \hat{u}(s, u^*(H)) \in [u_0, u']\}$ is measurable. Then we have the following definition

**Definition 1** *The distribution $H$ of utilities is an* equilibrium for the limit game *if for all $u'$, $H(u') = G(\{s : \hat{u}(s, u^*(H)) \in [u_0, u']\})$.*

The definition says that a distribution $H$ is an equilibrium if the measure of the set of sellers who offer contracts yielding utility less than or equal to $u'$ for buyers is equal to the measure of the set of sellers whose best reply to the distribution $H$ is to offer a contract yielding buyers a level of utility no larger than $u'$.

The allocation that is supported by the equilibrium in the infinite game is just the induced allocation of buyers and sellers to contracts. Again, using the measurability of the function $\hat{x}(s, u^*(H^*))$, define for each $x \in \Xi$

$$\sigma^s(x) = G(\{s : \hat{x}(s, u^*(H^*)) \leq x\})$$

The continuity and invertability of $\hat{x}$, and the fact that $G$ is a distribution function, guarantee that $\sigma^s$ is continuous from above, and therefore is a distribution function. Define the transformation $T$ from the subsets of $\Xi$ to the subsets of $[u_0, u_1]$ as

$$\Sigma \to \hat{u}\left[\hat{x}^{-1}(\Sigma, u^*(H^*)), u^*(H^*)\right]$$

. The transformation $T$ is measurable as the composition of two measurable transformations. Now define

$$\sigma^b(x) = \int_{x_0}^{x} k^* \cdot T d\sigma^s(y)$$

where $k^*(u)$ is equal to the solution to the equation

$$\bar{u} + \frac{1 - e^{-k}}{k}[u - \bar{u}] = u^*(H^*) \tag{5}$$

if a solution exists and $k^*(u) = 0$ otherwise. The distribution functions $\sigma^s$ and $\sigma^b$ are referred to as the *allocations supported by the equilibrium in the infinite game.*

**Theorem 4** *Under the assumptions 1,2, and 3 used to prove Theorem 1, an allocation can be supported as an equilibrium allocation for the limit game if and only if it can be supported as a Walrasian allocation for an economy with the exponential matching technology.*

Note that there is no need to prove existence of a decentralized equilibrium. Simply apply Theorem 1 to the Walrasian economy with the exponential matching technology to verify the existence of a Walrasian equilibrium. Then use Theorem 4 to verify that the corresponding equilibrium allocation is also an equilibrium allocation for the decentralized problem.

The rationalization for the use of the Walrasian equilibrium concept in an economy with the exponential matching technology, then arises from the fact that the payoff that all traders expect to get when they go to inactive markets is equal to the payoff that they would actually get in the non-Walrasian equilibrium for the problem.

## 5.2 The example again

It is possible to illustrate the theorem using the simple labour market example used earlier in the paper. In that example, all sellers were identical (which satisfies the assumptions of Theorem 4). Now, however, the indifference curve $\bar{U}$ can be thought of as the locus of solutions to the equation

$$\frac{\left(1 - e^{-k}\right)}{k} v(w) + \left[1 - \frac{\left(1 - e^{-k}\right)}{k}\right] \bar{u} = \text{constant}$$

Theorem 3 gives a formula that makes it possible to calculate the payoff that a deviating seller gets in the limit game. Suppose that sellers are all offering the wage contract $w_2$ and that the ratio of workers to firms in the economy is given by $k^{**}$.

The theorem says that there is a level of expected utility for buyers, such that for any deviation that some firm tries, the firm's payoff will be the same as it would have been if all the other firms had deviated as well, while the ratio of workers to firms had been adjusted to maintain the workers' expected payoff. Evidently, the firm's feasible set is just the workers' indifference curve through the outcome $(w_2, k^{**})$. A necessary condition for equilibrium will then be that workers' and firms' indifference curves be tangent at $k^{**}$. This is exactly the condition for Walrasian equilibrium.

Once this tangency has been located, the indifference curve for the worker then coincides with one of the expectations functions that will support the Walrasian outcome.

# 6 Conclusions

The paper shows that the Walrasian theory developed by Gale approximates the outcome of a large game in which sellers compete directly against one another by offering contracts. The strong assumptions used in the Walrasian theory to narrow the set of equilibria are shown to coincide with off equilibrium path conjectures in the large game.

There are three important areas where further work is required. First, though the Walrasian equilibrium concept has desirable properties for a relatively large class of matching technologies, this paper only shows that it can be decentralized for the exponential matching technology. It is conventional in applications to choose an ad hoc specification for the matching technology that has desirable analytical properties. It seems more natural to restrict attention to matching technologies that approximate well defined assignment processes in finite games with direct competition. One of the indirect contributions of this paper is in fact to provide a matching technology that has this property.

A referee has pointed out that there may be other matching technologies that are reasonable according to this definition. For example, rather than having buyers choose sellers, buyers might choose contracts, then have the opportunity of matching with any seller who has chosen the same contract. The ex ante probability with which buyers trade in any market is then determined by the probability with which buyers and sellers choose that market[15]. The method in this paper can then be applied to try to determine whether the limit equilibria in this kind of game are Walrasian with respect to whatever the limiting matching technology is.

A second issue concerns the fact that the equivalence established in this paper is between the Walrasian equilibrium and the exact equilibrium of the infinite version of the game with direct competition. It is also of interest to know whether the Walrasian equilibrium is equal to the limit of exact equilibria of large but finite games. If it is known that these exact equilibria exist for all large finite games, then this result would follow from the result in this paper, and the usual upper hemicontinuity of the Nash correspondence.

The real difficulty with this argument is that I have not yet been able to show that exact equilibrium exist for the large finite approximations to the infinite economy. The game that is played in the non-Walrasian version of the model involves two stages. The continuation equilibrium for the second stage involves an incentive consistency condition which will be met by matching process that need not be continuous in the contract offers that are made in the first stage of the game. Continuity of the first stage payoffs is required to verify existence for all the well known fixed point arguments.

A third problem involves extensions of the result provided above to situations in which there is diversity on both sides of the market. This assumption

[15]So if $k$ is the ratio of the measure of the number of buyers who choose the market to the measure of the number of sellers who choose the market, then the trading probability is $\mathbb{E}\min[1, k]$.

is common in the literature. Though it is apparent from the example given above that the Walrasian equilibrium will exist for cases in which both sides of the market have diverse preferences, the homogeneity of buyer preferences is central in both the existence and the decentralization theorems proved in this paper. At the moment I do not have anything to offer on this issue.

Finally, the theorem presented here has no immediate extension to problems involving adverse selection. If contract offers convey information about the types of the sellers offering them, there will be multiple equilibria in both the Walrasian and non-Walrasian versions of the model. Whether there will be a correspondence between these two sets of equilibria is an open question.

# 7 Appendix-Proofs of Theorems

For the proof of Theorem , we will make use of the relations

$$\hat{u}(s,u) : s \rightarrow \operatorname*{argmax}_{u'} \left\{ \left[1-e^{-k}\right] v\left(x\left(u',s\right),s\right) + e^{-k}\bar{v} : \bar{u} + \frac{1-e^{-k}}{k}\left[u'-\bar{u}\right] \geq u \right\}$$

and

$$\hat{x}(s,u) : s \rightarrow \operatorname*{argmax}_{x} \left\{ \left[1-e^{-k}\right] v\left(x,s\right) + e^{-k}\bar{v} : \bar{u} + \frac{1-e^{-k}}{k}\left[u\left(x\right)-\bar{u}\right] \geq u \right\}$$

described earlier in the paper. These relations are continuous functions and $\hat{x}$ has a continuous inverse function.

## 7.1 Proof of Theorem 1

**Proof.** The relation

$$\tilde{k}(x,u) = \begin{cases} k : U\left(x,k,b^*\right) = u \; if \; u\left(x,b^*\right) > u \\ 0 \; otherwise \end{cases}$$

is readily shown to be a continuous function. In addition, this selection has the property that for any sequence $u^n \rightarrow u^0$, the function $\tilde{k}(\cdot,u^n)$ converges point-wise to the function $\tilde{k}(\cdot,u^0)$.

Now recall that the set

$$\phi^s(x, \tilde{k}(\cdot, u)) = \\ \left\{s' \in S : \exists x' \leq x; V(x', s', \tilde{k}(x, u)) \geq V(y, s', \tilde{k}(x, u)) \forall y \in \Xi\right\}$$

Since $\tilde{k}$ is continuous, this set is measurable for each $x$. So define the distribution function

$$\tilde{G}(x, u) = \int K_{\phi^s(x, \tilde{k}(\cdot, u))}(s) dG(s)$$

Let $u^*$ be a solution to the equation

$$M = \int \tilde{k}(x, u) d\tilde{G}(x, u) \tag{6}$$

**Lemma 5** *The expectations function $\tilde{k}(x, u^*)$ along with the allocations*

$$\sigma^{*s}(x) = \tilde{G}(x, u^*)$$

*and*

$$\sigma^{*b}(x) = \int_{x_0}^{x} \tilde{k}(x', u^*) d\tilde{G}(x', u^*)$$

*constitute a Walrasian equilibrium for the contract market.*

**Proof.** First, to verify that $\sigma^{*s}$ is an aggregate supply function, all we really need to show is that it is an allocation, in the sense that it can be written as the sum of jump function with a finite support and an absolutely continuous function. The other properties of the aggregate supply function are fulfilled by $\sigma^{*s}$ by construction.

Let $S_1$ be the set of jump points of the distribution $G$ and let $G_j(s)$ denote the jumpsize of the distribution $G$ at point $s$. Now writing

$$\sigma^{*s}(x) = \tilde{G}(x, u^*) = \int K_{\phi^s(x, \tilde{k}(\cdot, u^*))}(s) dG(s)$$

$$= \sum_{s \in S_1} K_{\phi^s(x, \tilde{k}(\cdot, u^*))} G_j(s) + \int_S K_{\phi^s(x, \tilde{k}(\cdot, u^*))} g(s) ds$$

$$= \sum_{x \in X_1} G_j \left( \hat{x}^{-1}(x) \right) + \int_{\Xi} g \left( \hat{x}^{-1}(x) \right) h(x)\, dx \tag{7}$$

where $X_1 = \{x : x = \hat{x}(s, u^*) \ \mathit{some}\ s \in S_1\} \cap \left\{ x : \tilde{k}(x, u^*) > 0 \right\}$ and $h$ is some measurable function. The first line follows from the fact that $G$ can be decomposed, while the second line follows from a standard change of variable theorem Burrill (1972).

By the construction of $\tilde{k}$, buyers are indifferent among all the contracts for which $\tilde{k}(x, u^*) > 0$. Hence for each $x$, $\phi^b(x, \tilde{k}(\cdot, u))$ is either empty, or equal to $b^*$ (the common type for all the buyers). Since

$$\int_{x_0}^{x} \tilde{k}(x', u^*)\, d\tilde{G}(x', u^*)$$

is a non-decreasing function of $x$, and $M = \int_{x_0}^{x_1} \tilde{k}(x, u^*) d\tilde{G}(x, u^*)$ by the choice of $u^*$, the integral is always less than or equal to the measure of the set $\phi^b(x, \tilde{k}(\cdot, u))$.

Finally, to verify that $\sigma^{*b}$ is an allocation, we have that

$$\sigma^{*b}(x) = \int_{x_0}^{x} \tilde{k}(x', u^*)\, d\tilde{G}(x', u^*)$$

$$= \sum_{X_1} \tilde{k}(x', u^*)\, G_j \left( x^{-1}(x) \right) + \int_{x_0}^{x} \tilde{k}(x', u^*)\, g \left( \hat{x}^{-1}(x) \right) h(x)\, dx \tag{8}$$

Comparing this last expression with (7) shows that the rational expectations condition must hold for almost all $x$. ■

This argument implies that an equilibrium will exist whenever the equation (6) has a solution. So we will now show that the expression $\int \tilde{k}(x, u) d\tilde{G}(x, u)$ is continuous in $u$, provided that $u > \bar{u}$.

Now consider the set $\Phi^n(k, x) = \left\{ y \in \Xi : \tilde{k}(y, u^n) \leq k \right\}$ and the distribution functions $F^n(k) = \int K_{\Phi^n(k,x)}(s) d\tilde{G}(s, u^n)$ and $F^*(k) = \int K_{\Phi^*(k,x)}(s) d\tilde{G}(s, u^*)$. The value $F^n(k)$ is the measure of the set of sellers who are allocated to contracts by the distribution $\tilde{G}$, for which the ratio of the measure of buyers to sellers allocated to the same market is less than or equal to $k$. Then changing variables in the integration, $\int \tilde{k}(s,u^n)d\tilde{G}(s,u^n) = \int kdF^n(k)$. The key point is that the integral on the left hand side of this equation converges as $u^n \to u$ if $F^n$ converges weakly to $F^*$.[16]

To show this latter point, we show that $F^n \to F$ at each continuity point of $F$. To see this, note that

$$\begin{aligned} F^n(k) &= \int K_{\left\{x:\tilde{k}(x,u^n)\leq k\right\}}(s)\, d\tilde{G}(s,u^n) \\ &= \int K_{\left\{s:\hat{k}(s,u^n)\leq k)\right\}}(s')\, dG(s') \\ &= 1 - G(\tilde{k}^{-1}(k,u^n)) \end{aligned}$$

The final equation follows from the single crossing property of seller preferences. Since $\tilde{k}$ is continuous, $F^0$ is continuous at $k$ if and only if $G$ is continuous at $\tilde{k}^{-1}(k,u^0)$.

Now let $k^0$ be any point at which $G(\tilde{k}^{-1}(k^0,u^0))$ is continuous. Let $s$ be any type of seller for whom $\hat{k}(s,u) < k^0$. The function $G(\tilde{k}^{-1}(k^0,u^0))$ represents the measure of the set of such sellers. Now for any such $s$, $\exists n^*$ large enough that $\tilde{k}(s,u^n) < k^0$ for all $n > n^*$. This follows from the joint continuity of $\hat{k}$ in $s$ and $u$. Since $\tilde{k}$ is a decreasing function, this statement can be strengthened to read, for and $s : \tilde{k}(s,u) < k^0$, there is an $n^*$ large enough that $\tilde{k}(s',u^n) < k^0$ for all $n > n^*$ and $s' > s$.

Now for any $\epsilon > 0$, $\exists s^\epsilon : \tilde{k}(s^\epsilon,u^0) < k^0$ and $F^0(k^0) \geq 1 - G(s^\epsilon) \geq F(k^0) - \epsilon$. By the reasoning of the last paragraph, there exists an $n^*$ large enough that $F^n(k^0) \geq 1 - G(s^\epsilon)$ for all $n > n^*$. Since this is true for all positive $\epsilon$, we have $\lim_{n\to\infty} F^n(k^0) \geq F(k^0)$. The reverse inequality is proved in exactly the same way.

This implies that the distribution function $F^n(\cdot)$ converges weakly to $F^*(\cdot)$ as $u^n \to u^*$. But then

$$\int \tilde{k}(s,u^n)d\tilde{G}(s,u^n) = \int kdF^n(k) \to \int kdF^*(k) = \int \tilde{k}(s,u^*)d\tilde{G}(s,u^*)$$

[16](Hildenbrand 1974) p. 52

Finally, to prove the existence of equilibrium, note that when $u = \bar{u}$, the constraint set in the maximization problem used to define $F^n$ consists of all contracts satisfying $u(x, b^*) = \bar{u}$, along with all possible values of $k$. Since sellers payoffs are increasing in $k$, we must have $\hat{k}(s, \bar{u}) = \infty$ for all $s \in S$.

On the other hand let $u > \max_x u(x, b^*) - \epsilon$. Then for all feasible $x$, there must exist a $k$ such that

$$q(k)u(x, b^*) + (1 - q(k))\bar{u} > \max_x u(x, b^*) - \epsilon$$

which gives

$$q(k) > 1 - \frac{\epsilon}{\max_x u(x, b^*) - \bar{u}}$$

By the continuity of $q$, for each $\delta > 0$, there is an $\epsilon > 0$, such that if $u > \max_x u(x, b^*) - \epsilon$, then all feasible values for $k$ will be less than $\delta$.

The existence of a solution to

$$\int \tilde{k}(s, u^n) d\tilde{G}(s, u^n) = m$$

then follows from the mean value theorem. ■

## 7.2 Proof of Lemma 2

**Proof.** The convexity of the buyer's preferences is immediate. For the quasi-convexity of the seller's preferences, let $(u_1, k_1)$ and $(u_2, k_2)$ denote a pair of outcomes in $[u_0, u_1] \times \mathbf{R}$ to which a seller of type $s$ is indifferent. Then

$$\begin{aligned}
&q(\lambda k_1 + (1-\lambda) k_2)\, v(x(\lambda u_1 + (1-\lambda) u_2, s), s) + [1 - q(\lambda k_1 + (1-\lambda) k_2)]\, \bar{v} \\
&\geq q(\lambda k_1 + (1-\lambda) k_2)\, v(\lambda x(u_1, s) + (1-\lambda) x(u_2, s), s) + [1 - q(\lambda k_1 + (1-\lambda) k_2)]\, \bar{v} \\
&\geq \min_{i=1,2} [q(k_i)\, v(x(u_i, s), s) + [1 - q(k_i)]\, \bar{v}] \\
&= q(k_1)\, v(x(u_1, s), s) + [1 - q(k_1)]\, \bar{v}
\end{aligned}$$

The first inequality follows by the properties of maximization, the second by the quasi-convexity of the seller's preferences on $\Xi \times \mathbf{R}$.

The single crossing property requires that if $(u', k') \neq (u, k); k' \geq k, s' \geq s$ then $\tilde{V}(x(u', s'), k', s') \geq \tilde{V}(x(u, s'), k, s') \Rightarrow \tilde{V}(x(u', s), k', s) > \tilde{V}(x(u, s), k, s)$. From the single crossing property which applies on the space of contracts, we know that

$$\tilde{V}(x(u', s'), k', s') \geq \tilde{V}(x(u, s'), k, s') \Rightarrow \tilde{V}(x(u', s'), k', s) > \tilde{V}(x(u, s'), k, s)$$

Then because $x(u, s)$ is the optimal contract offer for type $s$, we have

$$\tilde{V}(x(u', s'), k', s') \geq \tilde{V}(x(u, s'), k, s') \Rightarrow \tilde{V}(x(u', s'), k', s') \geq \tilde{V}(x(u, s), k, s')$$

Then by the single crossing property on the space of contracts, we have

$$\tilde{V}(x(u', s), k', s) \geq \tilde{V}(x(u', s'), k', s) > \tilde{V}(x(u, s), k, s)$$

■

## 7.3 Proof of Theorem 3

To study the properties of incentive consistent matching technologies, it will be useful to approach the problem in a slightly different way. It is straightforward to show that $\frac{1}{kJ\pi}\left[1-(1-\pi)^{kJ}\right](u-\bar{u})$ is a continuous and monotonically declining function of $\pi$ whenever $u > \bar{u}$. The limit of the function as $\pi$ goes to zero is $u - \bar{u}$, while its limit when $\pi$ goes to one is $(u_j - \bar{u})/kJ$. Define the inverse function

$$g_J(u, u_j) \equiv \begin{cases} 0 \; if \; u > u_j \\ 1 \; if \; u < u_j/kJ \\ \pi : \frac{1}{kJ\pi}\left[1-(1-\pi)^{kJ}\right](u_j-\bar{u}) = u - \bar{u} \; otherwise \end{cases} \tag{9}$$

The inverse function describes the assignment probability which will yield buyers any given level of expected utility. This inverse function is continuous and monotonically decreasing on the interval $[u_j/kJ, u_j]$. Now define the function

$$G_J(u, u_1, \ldots u_J) = \sum_{j=1}^{J} g_J(u, u_j)$$

This function[17] is continuous, as the sum of continuous functions. Furthermore, it is monotonically decreasing on the interval

$$\left[\min_j \frac{u_j}{kJ}, \max_j u_j\right]$$

**Lemma 6** *For any array $\tilde{u} = \{\tilde{u}_1, \ldots, \tilde{u}_J\}$ of utility offers by the sellers, the vector of choice strategies $[\pi_1, \ldots, \pi_J]$ is incentive consistent for $\tilde{u}$ if and only if there is a utility level $u^*$ such that*

$$G_J(u^*, \tilde{u}) = \sum_{j=1}^{J} g_J(u^*, \tilde{u}_j) = 1$$

*and*

$$g_J(u^*, \tilde{u}_j) = \pi_j$$

*for each $j$.*

**Proof.** First suppose that $\pi = [\pi_1, \ldots, \pi_J]$ is incentive consistent for $\tilde{u}$. Then by standard properties of the symmetric mixed strategy equilibrium, it must be that for each positive $\pi_j$,

$$\frac{1}{kJ\pi_J^j}\left[1-\left(1-\pi_J^j\right)^{kJ}\right]\tilde{u}_j + \left\{1 - \frac{1}{kJ\pi_J^j}\left[1-\left(1-\pi_J^j\right)^{kJ}\right]\right\}\bar{u} \geq$$

$$\frac{1}{kJ\pi_J^i}\left[1-\left(1-\pi_J^i\right)^{kJ}\right]\tilde{u}_i + \left\{1 - \frac{1}{kJ\pi_J^i}\left[1-\left(1-\pi_J^i\right)^{kJ}\right]\right\}\bar{u}$$

for each $i = 1, \ldots, J$. This implies that for each $j$ for which $\pi_j > 0$, the left hand side of this equation has the same value $u^* < \tilde{u}_j$. Furthermore, for each $j$ for which $\pi_j = 0$, we must have $\tilde{u}_j \leq u^*$. If $\pi_j = 1$ for some $j$, then $u^* = \tilde{u}_j/kJ$ and the result is immediate. If there are two or more $j$ for which $\pi_j > 0$, then $u^* > \tilde{u}_j/kJ$ for all $j$. This implies that the functions $g_J(u^*, \tilde{u}_j)$ are all monotonic at $u^*$, implying that $g_J(u^*, \tilde{u}_j) = \pi_j$.

[17] This function is exactly analogous to the sum of the marginal cost functions that is used to allocate production in a multiplant firm.

Alternatively, let $u^*$ be any solution to $G(u, \tilde{u}) = 1$. Then by definition, there exists a vector $\pi = [\pi_1, \ldots, \pi_J]$ such that

$$G_J(u^*, \tilde{u}_1, \ldots \tilde{u}_J) = \sum_{j=1}^{J} g_J(u^*, \tilde{u}_j) = \sum_{j=1}^{J} \pi_j = 1$$

and

$$\sum_{j=1}^{J} \pi_j \left\{ \frac{1}{kJ\pi_J^j} \left[1 - \left(1 - \pi_J^j\right)^{kJ}\right] \tilde{u}_j + \left\{1 - \frac{1}{kJ\pi_J^j} \left[1 - \left(1 - \pi_J^j\right)^{kJ}\right]\right\} \bar{u} \right\} \leq u^*$$

$$\leq \max_j \frac{1}{kJ\pi_J^j} \left[1 - \left(1 - \pi_J^j\right)^{kJ}\right] \tilde{u}_j + \left\{1 - \frac{1}{kJ\pi_J^j} \left[1 - \left(1 - \pi_J^j\right)^{kJ}\right]\right\} \bar{u}$$

so that the corresponding probabilities are incentive consistent for $\tilde{u}$. ■

Now consider again the vector $\{u', \tilde{u}_1(H, J), \ldots \tilde{u}_{J-1}(H, J)\}$ that is derived from the distribution $H$ of offers along with the deviation $u'$ by one of the sellers. Now for any $J$ and $u$,

$$g_J(u, u') + \sum_{i=1}^{J-1} g_J(u, \tilde{u}_i(H, J))$$

$$= \frac{J}{J} g_J(u, u') + \frac{J \sum_{i=1}^{J-1} g_J(u, \tilde{u}_i(H, J))}{J} = \int b_J(\tilde{u}, u) dH_J(\tilde{u}, u')$$

where for each $u$ in the interval $\left[\frac{u_{\max}}{kJ}, u_{\max}\right]$, $b_J(\tilde{u}, u)$ is a map from $[\bar{u}, u_{\max}]$ defined as

$$b_J(\tilde{u}, u) = \begin{cases} 0 \; if \; \tilde{u} < u \\ b : \frac{1-(1-b/J)^{kJ}}{kb} (\tilde{u} - \bar{u}) = (u - \bar{u}) \; otherwise \end{cases}$$

Notice that for each $i = 1, \ldots, J$, $b_J(\tilde{u}_i(H, J), u) = J g_J(u, \tilde{u}_i(H, J))$.

The function $b_J$ is relatively simple. In particular, it converges point-wise as $J$ gets large to the inverse function of $\frac{1-e^{-bk}}{bk}$. Furthermore, $H_J(\cdot, u')$ is converging point-wise to the distribution function $H$ as $J$ gets large for any possible deviation $u'$, as has been mentioned above.

Define

$$b(\tilde{u}, u) = \begin{cases} 0 \; if \; \tilde{u} < u \\ b : \frac{1-e^{-bk}}{bk} = (\tilde{u} - \bar{u}) \; otherwise \end{cases}$$

**Lemma 7** *For each $u > \bar{u}$, and each possible deviation $u'$,* $\lim_{J\to\infty} \int b_J(\tilde{u}, u)\, dH_J(\tilde{u}, u') = \int b(\tilde{u}, u)\, dH(\tilde{u})$.

**Proof.** If $u > \bar{u}$, there is some $J^*$ large enough that $u > \frac{u_{\max} - \bar{u}}{kJ^*}$. We will restrict attention to the class of functions $\{b_J\}$ for $J > J^*$. Since this sequence of functions converges point-wise to $b(\tilde{u}, u)$ on $[\bar{u}, u_{\max}]$, it converges in distribution in the sense of (Hildenbrand 1974)[18].

Second, note that the function

$$\frac{1 - (1 - b/J)^{kJ}}{bk}$$

is decreasing in $J$. To see this note that the derivative of this function with respect to $J$ is given by

$$-\left(1 - \frac{b}{J}\right)^{kJ} \left(k \ln(1 - \frac{b}{J}) + bkJ^{-1}\left(1 - \frac{b}{J}\right)^{-1}\right) b^{-1}k^{-1}$$

The sign of this expression is determined by the sign of the expression

$$\ln(1 - x) + x\,(1 - x)^{-1}$$

for $x \leq 1$ which is readily shown to be positive.

Now when $J = J^*$, $\sup_{\tilde{u}} b_{J^*}(\tilde{u}, u) = b_{J^*}(\tilde{u}_{\max}, u) < J^*$. This implies that for all $J > J^*$, $\int b_J(\tilde{u}, u)\, dH_J(\tilde{u}, u') < J^*$ which makes the family of functions $\{b_J\}$ uniformly integrable.

By the construction of the approximation, $H_J(u) \leq H(u)$. Then for all $x$, and all $J \geq J^*$,

$$H\{\tilde{u} : b(\tilde{u}) \geq x\} \geq H\{\tilde{u} : b_J(\tilde{u}) \geq x\} \geq H_J\{\tilde{u} : b_J(\tilde{u}) \geq x\}$$

[18] A sequence of functions $\{f^J\}$ from and interval $[a, b]$ in $\mathbf{R}$ into $\mathbf{R}$ converges in distribution to $f : [a, b] \to \mathbf{R}$ if for every $\epsilon$, the Lebesque measure of the set $\{\mu : |f^J(\mu) - f(\mu)| > \epsilon\}$ converges to zero as $J \to \infty$.

The first inequality follows because of the fact that $b_J(\tilde{u})$ is a decreasing function, while the sequence $b_J(x)$ is a declining sequence for all $x$. The second inequality follows as above from the construction of the partition.

The convergence of the integral then follows from the Generalized Lebesque theorem given in Hildenbrand (1974)(p 52). ■

**Lemma 8** *The sequence of solutions* $\{u_J\}$ *to the equation*

$$\sum_{i=1}^{J} g_J\left(u, \tilde{u}_i\left(H, J\right)\right) = 1$$

*has a limit* $u^*$ *that is given by the solution to the equation*

$$\int b\left(\tilde{u}, u\right) dH\left(\tilde{u}\right) = 1$$

.

**Proof.** Suppose the sequence $\{u_J\}$ has a limit point $u_\infty > u^*$ for example. Let $\delta = |u_\infty - u^*|$. Since the integral $\sum_{i=1}^{J} g_J\left(u, \tilde{u}_i\left(H, J\right)\right)$ converges pointwise in $u$ by Lemma 7, and is continuous, it converges uniformly on the interval $[u^* - \delta, u^* + \delta]$. Hence for any $\epsilon$ there is a $J^*$ large enough that $\left|\sum_{i=1}^{J} g_J\left(u, \tilde{u}_i\left(H, J\right)\right) - \int b\left(\tilde{u}, u\right) dH\left(\tilde{u}\right)\right| < \epsilon$ for all $J > J^*$ and for all $u \in [u^* - \delta, u^* + \delta]$. Let $\epsilon = \left|\int b\left(\tilde{u}, u^*\right) dH\left(\tilde{u}\right) - 1\right| > 0$. Then this implies we can choose $J^*$ large enough that $\sum_{i=1}^{J} g_J\left(u, \tilde{u}_i\left(H, J\right)\right) < 1$ for all $J > J^*$, a contradiction. ■

As the approximation to $H$ becomes finer and finer as $J$ goes to infinity, the payoff that buyers receive in the equilibrium continuation must converge to a payoff $u^*$ that is independent of the utility level offered by the deviating seller. For any distribution $H$, let $u^*(H)$ denote this limit payoff. Let $u'$ be the utility level that is offered by the deviating seller. If the deviating seller is chosen with positive probability, then the payoff that buyers receive if they select the deviator is converging to

$$u^*(H) = \lim_{J \to \infty} \frac{1 - \left(1 - {\pi'}_J\right)^{kJ}}{kJ\pi'}\left(u' - \bar{u}\right) + \bar{u}$$

where $\pi'$ is the probability with which all buyers select the deviator in the continuation equilibrium. On the other hand, the probability with which the deviating seller trades is given by the probability that at least one of the buyers selects him. This implies that the limiting value of the probability with which the deviating seller trades is given by

$$\lim_{J\to\infty}\left[1-(1-\pi')^{kJ}\right]$$

**Lemma 9** *For any deviation $u' < u^*(H)$,*

$$\lim_{J\to\infty}\left[1-(1-\pi')^{kJ}\right]=\begin{cases} 0 \; if \; u' < u^*(H) \\ 1-e^{-k'} \; otherwise \end{cases}$$

*where $k'$ is the solution to the equation*

$$\frac{1-e^{-k}}{k}\left(u'-\bar{u}\right)=u^*\left(H\right)-\bar{u}$$

**Proof.** First, for $u' < u^*\left(H\right)$ there must be some large $J^*$ such that the equilibrium payoff in the continuation strictly exceeds $u'$. Since $\frac{1-(1-\pi'_J)^{kJ}}{kJ\pi'_J}$ is bounded above by 1, then the payoff to a buyer who visits the deviator is smaller than the equilibrium payoff in the continuation no matter what the value of $\pi'_J$ is. Hence in this case, the Lemma is immediate.

So suppose instead that $u' \geq u^*$. Then we must have $\lim_{J\to\infty}\pi'_J = 0$, for otherwise the buyers' trading probability with the deviator goes to zero, the payoff with the deviator goes to $\bar{u}$, and incentive consistency is violated. Let $\tilde{J}_J$ and $\tilde{k}_J$ solve $\pi'_J = 1/\tilde{J}_J$ and

$$\frac{1-(1-\pi'_J)^{kJ}}{kJ\pi'_J}=\frac{1-\left(1-\frac{1}{\tilde{J}_J}\right)^{\tilde{k}_J\tilde{J}_J}}{\tilde{k}_J}$$

Since $\lim_{J\to\infty}\tilde{J}_J = \infty$ and

$$\lim_{\tilde{J}_J\to\infty}\frac{1-\left(1-\frac{1}{\tilde{J}_J}\right)^{\tilde{k}_J\tilde{J}_J}}{\tilde{k}_J}\left(u'-\bar{u}\right)=\left(u^*-\bar{u}\right)$$

the result follows. ■

This argument completes the proof of the Theorem.

## 7.4 Proof of Theorem 4

**Proof.** Let $\sigma^s$ and $\sigma^b$ denote an allocation that can be supported as an equilibrium allocation for the limit game in an economy with the classical matching technology. We will now construct the corresponding Walrasian allocation for the economy with the exponential matching technology.

First note that the equilibrium for the limit game generates an equilibrium distribution $H^*$ and equilibrium utility level $u^*(H^*)$. Let us begin by assigning to each contract $x$ in $\Xi$ the expectation $k^*(u(x))$ defined as the solution to the equation

$$\bar{u} + \frac{1-e^{-k}}{k}\left[u(x) - \bar{u}\right] = u^*(H^*)$$

Then evidently

$$\sigma^s(x) = G(\{s : \hat{x}(s, H^*) \leq x\})$$

$$= G\left\{\left\{s : \operatorname{argmax}\left[\left[1-e^{-k}\right] v(x,s) + e^{-k}\bar{v} : \bar{u} + \frac{1-e^{-k}}{k}\left[u(x) - \bar{u}\right] \geq u^*(H^*)\right] \leq x\right\}\right\}$$

$$\equiv G\{s' \in S : \exists x' < x ; V(x', s', k^*(u(x'))) \geq V(y, s', k^*(u(y))) \forall y \in \Xi\}$$

Under the assumption 2, the set $\{s : \hat{x}(s, H^*) \leq x\}$ is union of intervals, so for some such union $S^x$, $\sigma^s$ can be written as $\sigma^s(x) = G(S^x)$. Now let $S^1$ denote the set of points at which the distribution function $G$ jumps. Then

$$\sigma^s(x) = \sum_{y \in S^x \cup S^1} G^j(s) + \int_{S^x} g(s)\, ds$$

This equation uses the assumption that $G$ can be decomposed into a jump function and an absolutely continuous function. Now using a standard change of variable theorem for measurable transformations ((Burrill 1972)), this can be written as

$$= \sum_{x \in \hat{x}\{[s', s''] \cup S_1\}} G^j\left(\hat{x}^{-1}(x)\right) + \int_{\hat{x}(s')}^{\hat{x}(s'')} g\left(\hat{x}^{-1}(y)\right) e(y)\, dy \tag{10}$$

for some measurable function $e(y)$. This verifies the decomposition of $\sigma^s$. The distribution function $\sigma^s$ is then an aggregate supply function for the Walrasian economy with the exponential matching technology.

It remains to show that the induced allocation $\sigma^b$ is an aggregate demand function in the sense of the definition of Walrasian equilibrium, and that the generalized market clearing condition holds.

Now for any interval $[u_0, u']$, in equilibrium

$$H^*(u') = G(\{s : \hat{u}(s) \leq u'\}) = G(\{s : u(\hat{x}(s)) \leq u'\})$$
$$= \sigma^s\left(T^{-1}([u_0, u'])\right).$$

Now $H^*(u) = 0$ if $u < u^*(H^*)$; $H^*(u^*(H^*)) = G(\underline{s})$ where $\underline{s} = \sup\{s : \hat{u}(s) \leq u^*(H^*)\}$. Otherwise, since $\hat{u}$ is strictly increasing on the rest of it's domain, $H^*(u') = G(s')$ for some $s'$. It follows that $H^*(u)$ can be written as

$$H^*(u) = G(s') = G(\underline{s}) + \sum_{S^1 \cap (\underline{s}, s']} G^j(s) + \int_{\underline{s}}^{s'} g(y)\, dy$$

$$= G\left(\hat{u}^{-1}(u^*(H^*))\right) + \sum_{u \in \hat{u}(\{S^1 \cap (\underline{s}, s']\})} G^j\left(\hat{u}^{-1}(u)\right) + \int_{u^*}^{\hat{u}(s')} g\left(\hat{u}^{-1}(u)\right) f(u)\, du \tag{11}$$

for some measurable function $f(u)$[19]. This verifies that $H^*$ can be written as the sum of a jump function with a finite support and an absolutely continuous function.

It follows from a standard change of variables (Burrill 1972, Theorem 8.2) that

$$\sigma^b\left(T^{-1}([u_0, u'])\right) = \int_{T^{-1}[u_0, u']} k^* \cdot T d\sigma^s(y) = \int_{u_0}^{u'} k^*(u)\, dH^*(u)$$

[19]For example, if $\hat{u}$ is differentiable, then $f(u) = \left.\frac{1}{\hat{u}'(s)}\right|_{\hat{u}(s)=u}$. See (Burrill 1972, Theorem 7-8b).

and using the decomposition of $H^*$ given in (11), this equals

$$\int_{u_0}^{u'} k^*(u)\, d\left[H_j^*(u) + \int_{u^*}^{u} h(u')\, du'\right]$$

where $H_j^*(u) = G\left(\hat{u}^{-1}(u^*(H^*))\right) + \sum_{u\in\hat{u}(\{S^1\cap(\underline{s},s']\})} G^j\left(\hat{u}^{-1}(u)\right)$ and $h(u) = g\left(\hat{u}^{-1}(u)\right)$. This gives

$$\sigma^b\left(T^{-1}([u_o,u'])\right) = \sum_{u\in\hat{u}(\{S^1\cap(\underline{s},s']\})} k^*(u)\, G^j\left(\hat{u}^{-1}(u)\right) + \int_{u_0}^{u'} k^*(u)\, g\left(\hat{u}^{-1}(u)\right) f(u)\, du$$

because of the fact that $k^*(u^*) = 0$. The right hand side of this equation equals

$$\sum_{S^1\cap(\underline{s},s']} k^*(\hat{u}(s))\, G^j(s) + \int_{\underline{s}}^{s'} k^*(\hat{u}(y))\, g(y)\, dy$$

Finally using (10), this is equal to

$$= \sum_{x\in\hat{x}\{[s',s'']\cup S_1\}} k^*(u(x))\, G^j\left(\hat{x}^{-1}(x)\right) + \int_{\hat{x}(s')}^{\hat{x}(s'')} k^*(u(y))\, g\left(\hat{x}^{-1}(y)\right) e(y)\, dy \tag{12}$$

Now $\sigma^b$ can be written

$$\int_{u_0}^{u'} \bar{k}b(u,u^*)\, dH^*(u)$$

where $\bar{k}b(u,u^*) \equiv k^*(u)$. Since this integral is increasing in $u'$, and since

$$\int_{u_0}^{u^1} \bar{k}b(u,u^*)\, dH^*(u) = \bar{k}N \equiv M$$

by Lemma 7, $\sigma^b(\Xi) = M$. Finally, since buyers are identical, and expectations are chosen so that buyers are indifferent among all contracts,

$$F\left\{b' \in B : \exists x' < x; U(x', k^*(u(x'))) \geq U(y, k^*(u(y)))\forall y \in \Xi\right\}$$

is equal either to 0 or $M$. Hence the allocation of buyers is an aggregate demand function.

To verify the rational expectation condition simply take the generalized derivatives of (12) and (10).

The converse argument is straightforward. The Walrasian equilibrium with identical buyers generates an equilibrium level of utility which buyers get in all active markets. This is the utility level $u^*$ for the equilibrium in the decentralized model. The distribution of payoffs implied by the Walrasian equilibrium is the equilibrium distribution $H$. The details are straightforward and are omitted. ■

# References


C. W. Burrill, *Measure, Integration and Probability*, McGraw Hill (1972).

D. Carlton, Market behaviour with demand uncertainty and price inflexibility, *Amer. Econ. Rev.* **68** (1978), 571–587.

R. Denekere and J. Peck, Competition over price and service rate when demand is stochastic, Northwestern University Discussion paper - to appear in Rand Journal, (1992).

P. Diamond, A model of price adjustment, *J. Econ. Theory* **3** (1971), 156–168.

D. Gale, Equilibria and pareto optima of markets with adverse selection, Boston University Discussion paper, no. 46, (1994).

D. Gale, A walrasian theory of markets with adverse selection, *Rev. Econ. Stud.* **59** (1992), 229–255.

F. Hahn, On non-walrasian equilibria, *Rev. Econ. Stud.* **45** (1978), 1–18.

W. Hildenbrand, *Core and Equilibria of a Large Economy*, Princeton University Press (1974).

H. Holzer, L. Katz, and A. Kreuger, Job queues and wages, *Quart. J. Econ.* **151** (1991), 739–768.

E. Kalai, Subjective games and equilibria: i, presented at the 1993 Nobel Symposium on Game Theory, (1993).

P. McAfee, Mechanism design by competing sellers, *Econometrica* **61** (1993), 1281–1312.

E. Moen, A search model with wage announcement, LSE working paper, (1994).

M. Peters, A competitive distribution of auctions, University of Toronto manuscript, (1994).

M. Peters, Immobility, rationing, and price competition, *Rev. Econ. Stud.* **52** (1985), 593–604.

M. Peters and S. Severinov, Limit equilibrium concepts for problems with competing mechanism designers, unpublished University of Toronto, (1993).

R. Wilson, Efficient and competitive rationing, *Econometrica* **57** (1989), 1–40.