Competing Pre-marital Investments

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Abstract

Pre-marital investments by spouses are largely viewed as public goods within the marriage. So individuals may underinvest. But individuals also use their investments to compete for spouses with higher investments. In a large marriage market, the higher equilibrium match quality obtained by increasing pre-marital investment exactly internalizes the external benefit of the investment so the competitive equilibrium is efficient. This model of competing investments in local public goods is a special case of Rosen's hedonic market model. In small marriage markets, the competition for spouses will raise incentives to invest in pre-marital investments as well as making these investments less predictable.

1 Introduction

In marriage, an individual derives utility from own pre-marital investment and the pre-marital invesment of his or her spouse. Much of this investment is human capital investment made by altruistic parents. Since pre-marital investment is a public good in marriage, parents may under invest in their children.

The under investment conjecture is premature. It ignores the competition for spouses in the marriage market. In this paper, we study pre-marital investments when children use these investments to compete for spouses. We are primarily interested in the implications of *assortative matching* equilibria which occur when wealthy individuals are matched with wealthy partners. Then altruistic parents take into account the additional utility their children will enjoy from wealthier partners, and this will increase their incentive to invest in their children on the margin.

Our first model considers the case where the number of families is very large. We study a *competitive* equilibrium in which all families on the same side of the market believe that they face the same non-stochastic *return* to their investment in their children. This return function adjusts until families beliefs are fulfilled in equilibrium.

Perhaps the most remarkable property of investment in the competitive equilibrium is the fact that the externalities associated with families' investments in their children are completely internalized by this return function. Any pair of families whose children match on the competitive equilibrium path, will make investments that are bilaterally pareto optimal. Despite the fact that neither family can directly compensate the other family for the investment that it makes in its child, the marriage market and the assortative matching that occurs there forces each family to compensate the other indirectly through the investment that it makes in its own child.

This efficiency result is an application of Rosen's hedonic pricing (Rosen 1974) approach to large matching problems. The market return function provides what is essentially a hedonic value for every investment level that a family might consider making. Families on the other side of the market need to provide these hedonic values in order to attract partners with specific investment levels. In equilibrium, each family's indifference curve (in the space of investments) will be tangent to this hedonic return function, and consequently, families whose children match will have indifference curves that are tangent to each other. The investments that families undertake will then be *bilaterally efficient* in the sense that there will not be another pair of investments that will make both of the matched families better off at the same time. Since the joint payoffs that we employ are supermodular ((Becker 1973),(Smith 1996)), assortative matching along with bilateral efficiency are sufficient to guarantee that the distribution of investments for the economy is efficient.

Matters are more complicated in small marriage markets. When the number of families and children is small, assortative matching among children will raise families incentive to invest and at the same time make families investments less predictable. If there are significant wealth disparities on the other side of the market, parents may find that they can increase the wealth of their child's partner significantly by raising their investment only slightly. This makes parental payoff functions discontinuous, which rules out pure strategy equilibria in some situations. In the mixed strategy equilibria that do prevail, parental investment is stochastic. Though rich families will invest more in their children on average than poor families do, there will be a positive probability that the poor families will invest more than rich families so that their children move up the wealth distribution. This creates endogenous intergenerational mobility.

Equilibria always exist for the families investment game when there is assortative matching. Since the equilibria involve mixed investment strategies, they can be complex. To get around this, we begin by looking at the simplest non-trivial market structure imaginable, in which there are only two families on each side of the market. For this case, we can give a complete characterization of equilibrium. We show that all families will invest more on average in their children than if there is no competition for spouses. While richer families will invest more on average than poorer families, the probability that parents and children switch places in the wealth distribution is non-zero. Furthermore, the investment levels that families do make will not generally be efficient.

The mixed strategy equilibria for this special case illustrate some of the properties of mixed strategy equilibrium for the more general case, but are otherwise quite special. Mixed strategy equilibria with arbitrary numbers of families are difficult to characterize, especially in the case where there are different numbers of families on each side of the market.

Our paper innovates primarily by making the level of investment that occurs on both sides of the market endogenous. Various authors have considered matching problems where investments occur prior to matching on one side of the market (for example (Shi 1997)). However, the impact of match quality on investment does not arise in these models. The paper that is most closely related to our own is (Cole, Malaith and Postlewaite 1998) who analyze a two sided matching investment problem. There are two important differences between our paper and theirs. First, they consider the case of transferable utility - partners can make agreements to transfer money or share surplus in a way that depends on investments. This approach is more appropriate to modelling firm worker matching problems where firms can make wages depend on human capital investments by firms. Incentives to invest are much stronger in such environments because the market provides an explicitly monetary reward for investment. No such reward exists in our paper. Furthermore, (Cole et al. 1998) use a *cooperative* matching process to pair traders and determine how surplus is shared in each match. For small marriage markets, our matching process is non-cooperative. (Siow and Zhu 1998) also study a two side matching investment problem with transferable utility and two wealth classes on each side of the market. They also study multigenerational equilibria. (Acemoglu 1997) studies a two side matching investment model with workers and firms. He obtains underinvestment because due to potential random matching, workers and firms are unable to fully capture the returns to their pre-employment investments.

While this paper focuses on the marriage market, our analysis applies to other partnerships in which the share of surplus in the partnership is not conditioned on the level of pre-partnership investment. Members of amateur sports teams and co-authors in economics usually do not divide surplus according to their levels of pre-partnership investments. In most of these markets, agents invest in pre-partnership human capital and then compete for partners. The results in this paper should be useful for thinking about those markets as well.

2 Preliminaries

Families begin with an endowment of wealth y which can be used partly as current consumption, and partly as an investment in children. Let w be the amount invested in the child. If the child subsequently matches with a partner whose wealth level is w' then utility for the parents is given by

$$V(y-w)+\frac{w+w'}{z}\ ,\ z>0$$

and utility of the child is

$$\frac{w+w'}{z}$$

If the family invests w and the child is not expected to match, then we assume that the child has utility w/z. The actual value of the childrens' utility when no match occurs is unimportant as long as both the child and family are at least weakly better off when a match occurs than if it doesn't.[1]

Assumption The function $V(\cdot)$ is monotonically increasing, strictly concave and satisfies

$$\lim_{x\to 0} V(x) = -\infty$$

The bilateral Nash, or non-competitive investment levels for each family are given by the solutions to

$$V'(y_i - w_i^*) = \frac{1}{z} \tag{1}$$

These are the investment levels that the families would make if they believed (for whatever reason) that their children's match partner is independent of parental investment. In the case where there are only two families, one on each side of the market, the investment of the family on the other side of the market would be fixed and equilibrium investment would satisfy 1. Investment would be inefficient in this case because 1 does not take account of the positive effect that the family's investment has on the family on the other side of the market.

After the families have made their investments, the children compete for partners in the marriage market. We will first study the investment and matching problem in a large marriage market. Then we will investigate properties of small marriage markets.

3 Large Marriage Markets

This section considers a large marriage market with a continuum of families on each side of the market. We refer to families with female children as families 'in F' and similarly, families with male children are families 'in M'. Let G and G' be measure of the set of families in M and F respectively.

[1] It is possible that children might strictly prefer not to match if the best available partner is too poor. This creates problems for our methodology, but these are not particularly relevant for the issues we wish to discuss.

Figure 1: Figure 1 - Matching Function

Interpret $G(B)$ to be the measure of the set of families whose endowments lie in the set B and similarly for G'. Let $g(w)$ represent the wealth of the wife that each family in M expects to match with from an investment of w in their son. Similarly, $g^{-1}(w')$ represents the wealth of the groom that each family in F expects to match with from an investment of w' in their daughter.

$g(w)$ is expected to be non-decreasing in w. Since it is costly for families to supply w beyond the Nash level, families in M will be willing to supply more w only if they can get a return for their investment. A non-decreasing $g(w)$ will imply assortative matching in equilibrium.

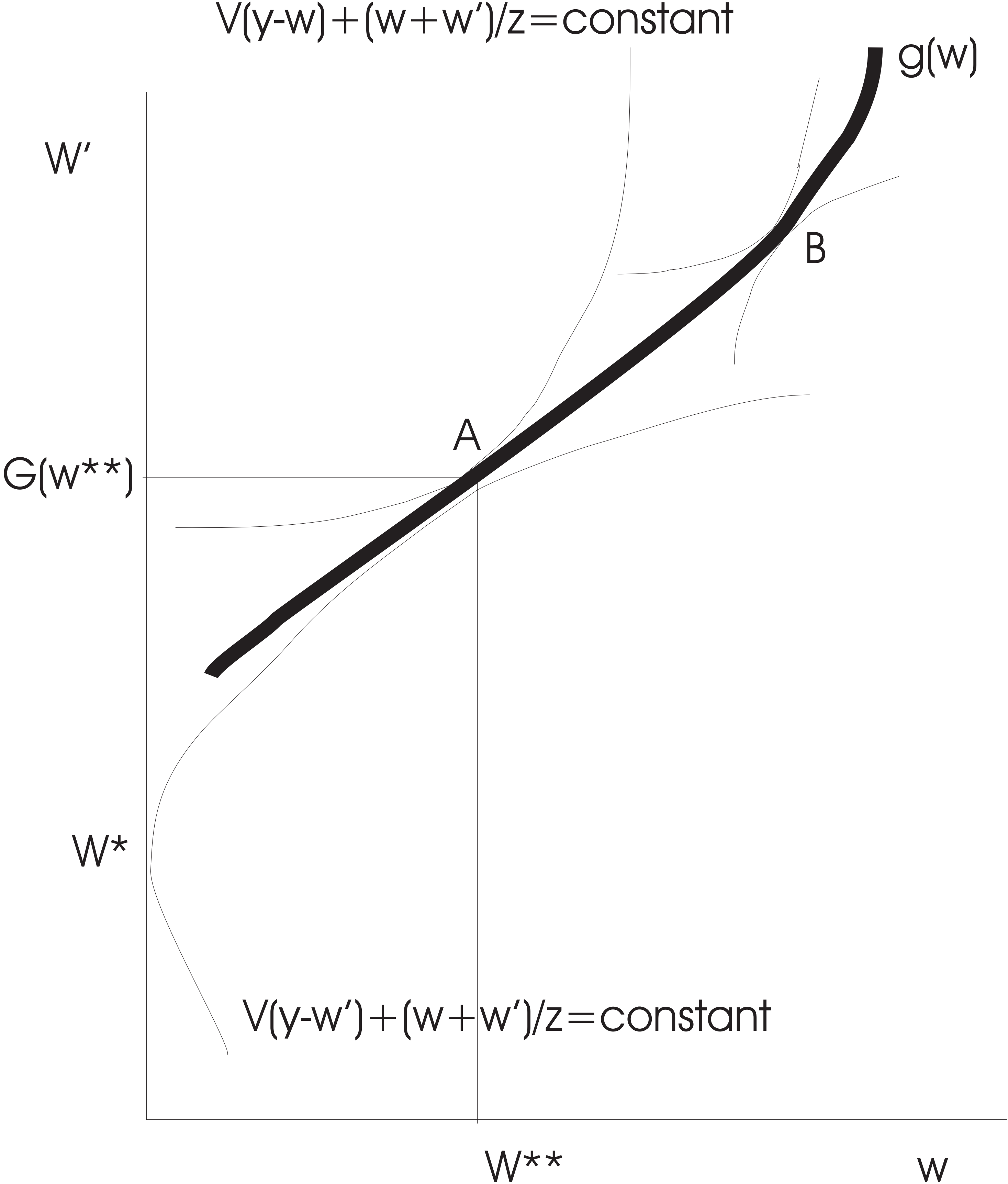
Definition 1 *The return function $g(w)$ is a* rational expectations equilibrium *if there is an interval $[\underline{w}, \overline{w}]$ such that for every $w \in [\underline{w}, \overline{w}]$ there exist income levels $y(w)$ and $y'(w)$ such that*

1. $G(y(\underline{w})) = 0$; $G'(y'(\underline{w})) = F - M$; $G(y(\overline{w})) = M$; $G'(y'(\overline{w})) = F$ and $M - G(y(w)) = F - G'(y'(w))$ for $w \in (\underline{w}, \overline{w})$;

2. $w \in \arg\max_x \left\{ u(y(w) - x) + \frac{x+g(x)}{z} \right\}$; and

$$g(w) \in \arg\max_x \left\{ u(y'(w) - x) + \frac{x + g^{-1}(x)}{z} \right\}$$

Figure 1 illustrates a rational expectations equilibrium. The investment levels for families in M are given along the horizontal axis, while investments for families in F are along the vertical axis. The dark curve illustrates the equilibrium matching function. The lighter curves that are convex upward are indifference curves for families in M, those that are convex downward are indifference curves for families in F. A family in M who invests w^{**} should expect their child to match with someone whose wealth is $g(w^{**})$. In equilibrium, if a family chooses investment w^{**} then their indifference curve should be tangent to the curve $g(w)$ at the point $(w^{**}, g(w^{**}))$. The reason is that the family thinks that $g(w)$ represents the market return function that they face. Similarly, any family in F who chooses to invest $g(w^{**})$ should expect



V(y-w)+(w+w')/z=constant
g(w)
W'
B
A
G(w**)
W*
V(y-w')+(w+w')/z=constant
W**
w

return w^{**}. In equilibrium this family must have an indifference curve tangent to the market trade-off function $g^{-1}(\cdot)$ at the point $(w^{**}, g(w^{**}))$.This implies that the indifference curves of the families of every pair of children who match in equilibrium will be tangent to each other and investment levels are bilaterally efficient. As the picture is drawn, the family from M is the one with the lowest endowment, while the family from F is the family with the lowest endowment who actually succeeds in matching.

To see this more formally, focus on the case where both G and G' are monotonic, and let $\alpha(w)$ satisfy $M - G(y) = F - G'(\alpha(y))$. By assortative matching, a family in M with income y should end up matching with a family from F whose income is $\alpha(y)$ provided that each families investment is an increasing function of its endowment. Suppose that the optimal investment level in the rational expectations equilibrium for each family in M is some function $w_m(y)$, while the corresponding function for families in F is $w_f(y)$. A necessary condition for optimality is that a family whose income is y should prefer to invest $w_m(y)$ to any investment level $w_m(y')$. Since each family's payoff is given by

$$V(y - w_m(y)) + \frac{w_m(y) + w_f(\alpha(y))}{z}$$

this gives the condition

$$w'_m(y) = \frac{w'_f(\alpha(y))\,\alpha'(y)}{z\left(V'(y - w_m(y)) - \frac{1}{z}\right)} \tag{2}$$

The corresponding condition for the family $\alpha(y)$ from F is that

$$w'_f(\alpha(y)) = \frac{w'_m(y)\,\frac{1}{\alpha'(y)}}{z\left(V'(y - w_f(\alpha(y))) - \frac{1}{z}\right)} \tag{3}$$

Substituting (3) into (2) gives the tangency of the indifference curves for family y in M and $\alpha(y)$ in F.

This bilateral efficiency is enough to ensure full pareto optimality of the equilibrium allocation. To show this, it is sufficient to demonstrate that pareto improvements cannot be achieved by rematching the families' children then adjusting the ex ante investment levels of families. Figure 1 can be used to illustrate this. Children are initially matched assortatively. Suppose that a pareto improvement can be obtained by rematching in such a way that family A in M is rematched with family B from F. To illustrate the argument, suppose that family A is the family in Figure 1 who chooses the point A while family B is the family from F who chooses the outcome B. If matching family B with family A results in a pareto improvement, then investment levels need to be adjusted so that both families end up on higher indifference curves than they attain in the initial assortative matching equilibrium. Since the allocations that A prefers to the allocation in the assortative matching equilibrium lie everywhere above the market return line $g(w)$ while the allocations that B prefers lie everywhere below it, this will not be possible.

4 Existence of a Rational Expectations Equilibrium

In some simple problems existence of a rational expectations equilibrium is immediate. For example, suppose that $G = G'$. Then set $g(w) = w$. Then each family chooses the (efficient) level of investment w^{**} that equates the marginal utility of consumption and 2. For a variety of reasons, this is not a good example of the rational expectations solution concept. We return to it momentarily.

A more illuminating example occurs when G and G' differ. Suppose as before that $F > M$ but that all the families in F have the same endowment. In figure 1, let the indifference curve touching the vertical axis be the common indifference curve for all these families. Define $g(x)$ to be equal to this indifference curve, and let the families in M choose the point on this indifference curve that they most desire. Families in F can be spread over the indifference curve in a manner that ensures the market clearing condition holds. The market return function will have a closed form solution provided that the indifference curve can be represented in closed form.

To see the solution for the more general case, suppose that G and G' are both strictly monotonic with differentiable inverse functions whose derivatives are bounded away from 0 and infinity. Let u be strictly concave and differentiable with marginal utility bounded above. Define for each y in the support of G $\gamma(y) = \{y' \in \operatorname{supp} G' : F - G'(y') = M - G(y)\}$. Since G and G' are both monotonic and differentiable, so is γ. Furthermore, the derivative of γ is bounded away from 0 and ∞. The first order condition for the optimal investment for a family of income y is given by

$$-u'(y-w)+\frac{1+g'(w)}{z}=0$$

Since u' is monotonic, it has an inverse. This implies that the income of the family in M who invests w must be equal to

$$u'^{-1}\left(\frac{1+g'(w)}{z}\right)+w$$

The family in F who invests $g(w)$ has an income level such that $g(w)$ satisfies

$$-u'(y'-g(w))+\frac{1+\frac{1}{g'(w)}}{z}=0$$

In equilibrium, this family matches with the family in M who invests w. If $g(\cdot)$ is a rational expectations solution, the measure of families who are wealthier than this family from M must be equal to the measure of families who are wealthier that y'. This requires that

$$y'=\gamma\left(u'^{-1}\left(\frac{1+g'(w)}{z}\right)+w\right)$$

This yields the ordinary differential equation

$$-u'\left(\gamma\left(u'^{-1}\left(\frac{1+g'(w)}{z}\right)+w\right)-g(w)\right)+\frac{1+\frac{1}{g'(w)}}{z}=0$$

Re-arranging gives

$$g(w)=\gamma\left(u'^{-1}\left(\frac{1+g'(w)}{z}+w\right)\right)-u'^{-1}\left(\frac{1+\frac{1}{g'(w)}}{z}\right)$$

Let $\phi(g',w)$ denote the expression on the right hand side of this equation. Since ϕ is monotonically decreasing in g' the inverse function $\phi^{-1}(\cdot,w)$ exists. So the market return function must satisfy

$$g'(w)=\phi^{-1}(g(w),w) \tag{4}$$

To see the initial condition, let y^* be the poorest family from F who successfully matches with some family from M in equilibrium (y^* satisfies $F-$

$G'(y^*) = M$). Let I^* be the highest indifference curve that the family with endowment y^* can attain when they do not match. Let $(\underline{w}, g(\underline{w}))$ be the point where this indifference curve is tangent to the indifference curve for a family in M who endowment is $\underline{y}$. The equilibrium market return function is then any solution to (4) with initial condition $(\underline{w}, g(\underline{w}))$. To ensure existence and uniqueness of the solution we need to impose additional restrictions on the problem to ensure that ϕ^{-1} satisfies the usual Lipshitz condition in g. We do not pursue these issues here.

5 A Hedonic Pricing Interpretation

The above model can be interpreted as a special case of Rosen's hedonic market model. To see this most easily, consider families in M as suppliers. Let y be the characteristic of a supplier. Let w be the level of output (pre-marital investment) that a supplier produces. Note that w also provides consumption value for the supplier and thus is not purely costly. $g(w)$ is the return that a son gets for supplying w. We may consider families in F as demanders. If a demander pays w', the daughter will match with a supplier whose output is $g^{-1}(w')$. Unlike Rosen, demanders value paying w'. However this does not cause any analytic difficulty because a demander, that is matched, will pay a higher w' than she is willing to pay if she is not matched. So as in the case of Rosen's firms, the demander will prefer to pay less w' for her matched supplier if she could.

With this interpretation, results that apply to Rosen's hedonic market model may apply here as well. For example, the difficulties of estimating hedonic pricing functions is well known (E.g. (Epple 1987) or (Kahn and Lang 1988)). These difficulties appear here as well (Botticini and Siow 1999).

6 Small Marriage Markets

In discussing small marriage markets, we have to first discuss the non-cooperative matching process among the children. The key property of this process is that it involves assortative matching. One non-cooperative procedure that will ensure this is the Gale-Shapley (Gale and Shapley 1962) algorithm. Male children publicly advertize their wealth levels. After seeing the wealth levels available, each female child chooses one and only one male as a potential partner and proposes a match to the male. The male can select one and only one of the proposals he receives and can form a match with the female who made the proposal.

To analyze this matching and investment process, we begin with equilibrium for the matching process. We can then proceed to the investment stage assuming that families foresee the impact that their investment will have on the outcome of the matching process.

7 The Matching Process

Without loss of generality, we can assume that there are n families in F and m families in M with $n > m$. The families in M have income levels $\{y_1, \ldots, y_m\}$ listed in descending order, so that family 1 is richest and family m is poorest. Similarly for F the endowments are $\{y_1, \ldots, y_n\}$ in descending order, so that family n in F is poorest. Denote

$$w_i^* = \arg\max \left\{ V\left(y_i - w\right) + \frac{w}{z} \right\}$$

for $i = 1, \ldots m$ as the *bilateral Nash investment level* for each family in M, as discussed above. Similarly, let $\left\{w_j^{\prime *}\right\}_{j=1,n}$ be the bilateral Nash investment levels for the various families in F. To simplify the argument, it will be assumed that $y_1 > y_2 > y_{3\ldots}$ for income levels in both M and F (which implies that the same property will hold for the Nash investment levels).

Fix an array $w = \{w_1, \ldots w_m\}$ of wealth levels for children in M and an array $w' = \{w'_1, \ldots w'_n\}$ of wealth levels for children in F. Alternatively, we will refer to the empirical distribution functions ϕ and ϕ' generated by these investments. Without loss of generality, it can be assumed that these wealth levels are listed in descending order, so that $w_j \geq w_{j+1}$ for all j, and $w'_k \geq w'_{k+1}$ for all k.

Males advertize their wealth levels, females make proposals, then males accept one of the ones they most prefer. This is a simple finite game played between the children, and as such it always has a Nash equilibrium. Some equilibria are problematic. For example, suppose there are two children on each side of the market and that all four children have the same wealth level. There is a mixed strategy equilibrium in which each female proposes to each male with equal probability, and each male accepts each female who proposes to him with equal probability. In this equilibrium, it can occur with positive probability that a male and female remain unmatched. This outcome does not seem reasonable. However, it is straightforward to show that the proposal game described above always has a pure strategy *assortative matching* equilibrium in which the child in M with investment w_i matches with probability 1 with the child from F who has investment w_i'. Call the children in the i^{th} position on each side of the market *designated partners*. Then a shorthand would be to say that there is a pure strategy equilibrium in which i matches with probability 1 with his or her designated partner.

Assortative matching is the property of this equilibrium and this matching process that will be used below. Since the arguments are the same on both sides, focus on families in M. Let ϕ' denote the empirical *distribution* of investment levels for families in F and let ϕ be the distribution of investments for all the families in M. Let $\tilde{g}_i(w_i; \phi, \phi')$ be the investment level of i's designated partner in F given the distribution ϕ' and given that i's investment level in ϕ has been replaced with w_i. We will also use the notation $\hat{\phi}(w)$ to denote the empirical distribution generated by the array of investments w. The function $\tilde{g}_i$ represents family i's return to investment. The assortative property of the matching process ensures that $\tilde{g}$ is non-decreasing. The matching process will not generally be anonymous since in the event of ties, equal investments by families in M might yield different investments on the part of the children with whom their children match. However if $w = (w_1, \dots w_m)$ has no two components in common (which is generic at least with respect to Lebesgue measure) then the matching process is anonymous in the sense that $\tilde{g}_i(w_i; \phi, \phi') = \tilde{g}_j(w_i; \phi, \phi')$.

In the sequel, we will be interested in situation in which the distribution ϕ' is itself random with some known distribution Φ. Let $\overline{\phi}$ be the distribution of the mean values of the order statistics associated with the random distributions ϕ'. Observe that for any ϕ

$$\mathbf{E}\tilde{g}_i(w_i; \phi, \phi') = \int \tilde{g}_i(w_i; \phi, \phi')\, d\Phi(\phi') = \tilde{g}_i\left(w_i; \phi, \overline{\phi}\right)$$

8 Investment and Equilibrium

The families choose investment levels in the first stage of the game, taking full account of the effect that their investment has on the continuation equilibrium in which their children match. Given distributions ϕ and ϕ' of investments for families in M and F respectively, the payoff to family i is given by

$$v_i(w, w') = V(y_i - w_i) + \frac{w_i + \tilde{g}_i(w_i; \phi, \phi')}{z}$$

This payoff function is discontinuous because if family i and family $i-1$ both have the same investment level, then family i can increase the wealth of their child's partner from w'_i to w'_{i-1} by raising investment slightly. Thus we cannot generally hope to find pure strategy equilibria for the investment game. However, the payoff functions have the properties required by the Reny (Reny 1999) existence theorem for equilibrium in mixed strategies.

Theorem 2 *The investment game has at least one equilibrium in mixed strategies.*

9 Minimal Competition Induces Additional Investments

The usefulness of the existence result in the previous section is limited by the fact that equilibrium strategies are quite complex. It is very difficult to give a useful characterization of equilibrium in the general case. To begin, we consider a very simple case where a complete characterization is possible. We follow by pointing out the properties of the equilibrium that do generalize.

We begin with the case where there are four families. One of the families from F and one from M have and initial endowment y_l while the other pair of families each have endowment y_h. Consider investment levels $w_l^f, w_h^f, w_l^m, w_h^m$, where, in an obvious notation the superscript f refers to children of families from F while the superscript m refers to children of families in M. In the case where the investments levels differ in the sense that for example $w_h^f > w_l^f$ the continuation equilibrium is straightforward. The wealthy child from F will propose to the wealthy child from M with probability 1 and this proposal will be accepted. Similarly for the less wealthy children. The poor F has no incentive to propose to the wealthy M because she expects the wealthy F to propose there with probability 1 and she knows that the wealthy male will always prefer her proposal.

If the wealth disparity of the families is large enough, it will never pay families to try to improve their children's match quality. In that case there will be an obvious equilibrium where families make non-cooperative investments and never mix. To avoid this we assume

Assumption

$$V\left(y_l^i - w_h^{*i}\right) + \frac{2w_h^{*i}}{z} > V\left(y_l^i - w_l^{*i}\right) + \frac{2w_l^{*i}}{z}$$

The assumption says that a poor family would be willing to raise investment to w_h^* if they believed that this would generate a match with a child from a wealthy family making the non-cooperative investment.

Then we have:

Theorem 3 *There exists a symmetric mixed strategy equilibrium in which the wealthy families both use mixture H_h while the poor families both use mixture H_l. These mixtures have the following properties:*

1. *H_l has an atom at w_l^* and H_h has an atom at w_h^* with $H_h(w_l^*) < H_l(w_h^*)$;*
2. *H_l and H_h both have smooth density functions on some interval $[w_h^*, w^{**}]$ with $H_l'(s) < H_h'(s)$ for all $s \in [w_h^*, w^{**}]$*

The proof of this theorem is constructive. The argument resembles the argument for a Bertrand competition with capacity constraints. If one family invests more than the other, that family's child will match with the wealthiest child on the other side of the market. In the mixed strategy equilibrium investments generate random returns since the investment level of the competing family is random. The major complication involved in this argument arises from the fact that family who invests most gets a random return equal to the first order statistic for investment levels on the other side of the market. The distribution of this order statistic is endogenous. This constitutes the main technical difficulty to be overcome in the proof.

The theorem illustrates nicely the inherent unpredictability of investment. Both kinds of families choose their bilateral Nash investment level with positive probability. However, they also use a strategy that involves investment at a level strictly above w_h^* with positive probability. It follows immediately that both families will invest, on average, strictly more than their bilateral Nash investment levels. From the position of the atoms, and the restriction on densities, it is immediate that the family with the highest endowment will invest more, on average, than the family with the low endowment. With randomization however, there is a strictly positive probability that the poor family will end up investing more than the wealthy family. Their child will then be at the top of the wealth distribution. We interpret this as endogenous intergenerational mobility in wealth.

10 Discussion

There is also a close connection between the models in this paper and the directed search models of the labor market (for example (Shi 1999) or (Moen 1997)). To see this suppose that there is a measurable set of firms who invest in physical capital and workers who invest in human capital. Firms have different technologies parameterized by some variable $y \in \mathbf{R}$ with the marginal product of capital increasing in y. Workers differ according to a parameter y' that determined the cost at which the worker can acquire human capital. This cost is assumed to be decreasing as y' increases. Each firm has a single job to be filled and each worker wishes to fill one job. The total output produced by the firm is some increasing function of the physical capital w invested by the firm and the human capital w' invested by the worker who fills the job. Physical capital is purchased by the firm at a fixed price r while human capital is acquired by the worker according to a convex and increasing cost $e(w')$. For the moment, assume that when a firm and worker match, each receives a fixed share of the profit that is created. So if a firm who invests physical capital w is matched with a worker with human capital w', the profit of the firm is

$$\alpha f(w, w'; y) - rw$$

while the profit of the worker is

$$(1-\alpha) f(w, w') - e(w'; y')$$

The functional form used in this example differs slightly from that used in the marriage market above, but otherwise the problems are identical. If we allow the firms to advertize their capital stocks after they make their investments so that workers can apply to the firm that they like, all the equilibria will involve assortative matching exactly as in the marriage problem.

The hedonic value of the firms investment w is given by some function $g(w)$ that gives the human capital that will be embodied in the worker that the firm expects to be able to attract. Conversely, any worker who wants a job at a firm with physical capital w will have to provide the level $g(w)$ of human capital to get the job. In equilibrium, this hedonic value will ensure that firms and workers will invest efficiently. This is similar to the result in (Moen 1997), though it generalizes that result by allowing firms and workers to differ, and by endogenizing the investments on both sides of the market. There are also some important differences. In the existing literature on directed search in labor markets, frictions generated by workers inability to coordinate their search decisions play an important role.[2] The hedonic value of any given wage that a firm offers to pay workers is then measured by the size of the queue of applicants that the firm attracts. The model here shows that when families or workers differ in equilibrium, the mixed strategy equilibria that support these frictions disappear - the matching equilibria that occur after wages are posted or capital stocks are chosen involve pure assortative matching - families use their own characteristics to coordinate their search decisions. Despite this, the hedonic interpretation in which the market responds to specific investments with a predictable return is supported.

The other major difference is that there are no side payments in the model studied here (in the labor market interpretation, firms do not offer wages but instead simply give workers an exogenously determined split of the profit). The case where workers and firms have multidimensional characteristics is certainly likely to support a hedonic interpretation, but so far models of this form have not been studied.

One of the predictions of the model studied here is certainly too strong - pure assortative matching. Clearly the model needs to be extended to allow for unobservable or match specific characteristics. The payoff to focussing on the case with perfect information is the simplicity of the model that it delivers. A synthesis of the directed search models of the kind discussed here and the random matching models that characterize the older literature is clearly an important topic for future research.

11 Appendix

11.1 Pure strategy equilibria in the matching game

[2] Frictions are generated by the fact that workers use mixed strategies when they choose which firms to apply to (Shi 1999), or (Peters 1999).

played among the children 4:

Males choose a selection rule that specifies for each subset of the set of females, the probability with which he will accept each different proposal from that set. Let $\mathcal{P}$ be the set of all subsets of females. A selection rule for male i is a mapping $\sigma^i : \mathcal{P} \rightarrow \mathbf{R}^M$ satisfying $\sigma_j^i(S) \geq 0$ for all $j = 1, \ldots M$; $\sum_{j=1}^{M} \sigma_j^i(S) \leq 1$; and $\sigma_{j'}^i = 0$ whenever $j' \notin S$. Each female can make a proposal to one and only one male. Let π_j^i be the probability that female j proposes to male i.Write $Q_j^i(\pi, \sigma)$ as the probability with which a proposal by female j to male i is accepted when males and females are using the strategies π and σ. Let (σ^*, π^*) be a Nash (continuation) equilibrium for the matching game. Equilibrium for this game exists by the finiteness of the underlying pure action space.

Equilibrium for this process involves *assortative* matching - a child who has wealth $w' > w$ will have a wealthier partner than a child whose wealth is w. The rationale is as follows. Consider a candidate non-assortative matching equilibrium in which female j is matched with male i and female $j' < j$ is matched with male $i' > i$ or unmatched. Since female j' has wealth larger than female j, she can make a marriage offer to male i and her offer will be accepted. Moreover she will have the incentive to do so. Thus the candidate non-assortative matching equilibrium cannot be an equilibrium (Lam 1988).

More formally, think of male i as the *designated partner* for female i and conversely. Then we have the following lemma

Lemma 4 *There exists a pure strategy Nash equilibrium for the choice process in which each child matches with his or her designated partner with probability 1. In particular, $\pi_j^M = 1$ for $j > M$; $\pi_j^j = 1$ for $j = 1, \ldots, M$; and for $i = 1, \ldots N$, σ^i is defined by*

$$\sigma_j^i(S) = \begin{cases} 1 & \text{if } j \in S;\ j = i;\ w'_{j'} \leq w'_i \text{ for all } j' \in S \text{ and } \frac{w_i + w'_i}{z} \geq \underline{w} \\ 1 & \text{if } j \in S;\ w'_j > w'_{j'} \text{ for all } j' \in S \text{ and } \frac{w_i + w'_j}{z} \geq \underline{w} \\ 1 & \text{if } j \in S;\ j \geq j' \text{ for all } j' \in S\ ; i \notin S \text{ and } \frac{w_i + w'_j}{z} \geq \underline{w} \\ 0 & \text{otherwise} \end{cases}$$

Remark 5 *Observe that each male i selects his 'designated partner' female i if she is available among the set S who propose, and if her wealth is at least as large as the wealth of all the others who propose. In particular, if there is some female other than i who proposes to male i but who has the same* *wealth as male i's designated partner, male i will always select his designated partner. So this equilibrium is not anonymous - the probability with which a male accepts a female proposal depends both on her wealth and her identity.*

The proof of this lemma is straightforward. The male's strategy is clearly optimal, he simply chooses from all the females who propose to him, the one with the highest wealth level. If there are multiple proposals in this category, he chooses his designated partner when she is available, or the highest index in the group if she is not. Females with very low wealth levels (indices $M+1$ through N) all propose to male M. Their proposals are all rejected on the equilibrium path, either because male M does not find it worthwhile to match, or because male M accepts the proposal from his designated partner female M. Female j proposes with probability 1 to her designated partner male j. Females with higher wealth (lower indices) propose to male j with probability 0, so that female j can expect her proposal to be accepted with probability 1. Thus she can never gain by deviating and proposing to a male whose wealth level is no higher than the wealth level of her designated partner. On the other hand, if she deviates by proposing to a male whose wealth is strictly higher than the wealth of her designated partner, her proposal will be rejected with probability1 since the seller to whom she proposes will either have a proposal from a female with more wealth, or will choose his designated partner. Since males and females share the same ex post payoff and the same payoff when they do not match, they will agree on whether or not to form a match ex post.■

11.2 Proof of Theorem 2:

To verify the existence of a mixed strategy equilibrium, we take the approach of Reny (Reny 1999) and show that the mixed extension of the first stage game conditional on the continuation equilibrium described in Lemma 4 is *reciprocally upper semi-continuous* and payoff secure. Observe first that the match value $\tilde{g}$ is weakly increasing in investment. This means that family i will never invest less than it's bilateral Nash investment level w_i^*. By Assumption 2, no family will invest more than it's endowment y_i. So without loss of generality, we can restrict each families strategy space to the convex interval $[w_i^*, y_i]$. Furthermore, since the matching function is determined by the equilibrium of the second stage, the return to investment is bounded above by the endowment of the richest family on the other side of the market.

The *first stage game* is the normal form game in which families have strategy spaces $[w_i^*, y_i]$ and payoffs

$$V(y_i - w_i) + \frac{w_i + \tilde{g}_i\left(w_i; \hat{\phi}(w_i, w_{-i}), \hat{\phi}(w')\right)}{z}.$$

This first stage 'game' is a compact game with metric strategy spaces.

Let $\triangle_i$ be the compact set of regular countably additive probability measures on (the Borel sets of) $[w_i^*, y_i]$. The mixed extension of the first stage game is the game in which the families' strategy spaces are given by $\triangle_i$ and for any vector of probability measures $\mu = \{\mu_1, \dots \mu_n, \mu_{n+1}, \dots \mu_{n+m}\}$ the payoffs are

$$u_i(\mu) \equiv \int V(y_i - w_i) + \frac{w_i + \tilde{g}_i\left(w_i; \hat{\phi}(w_i, w_{-i}), \hat{\phi}(w')\right)}{z} d\mu$$

The sum of payoffs in the first stage game is given by

$$\sum_{i=1}^{m} V(y_i - w_i) + \sum_{j=1}^{n} V\left(y_j - w_j'\right) +$$

$$\frac{\sum_{i=1}^{m}\left\{w_i + \tilde{g}_i\left(w_i; \hat{\phi}(w_i, w_{-i}), \hat{\phi}(w')\right)\right\} + \sum_{j=1}^{n}\left\{w_j + \tilde{g}_j\left(w_j; \hat{\phi}\left(w_j', w_{-j}'\right), \hat{\phi}(w)\right)\right\}}{z}$$

Since the $\tilde{g}$ are simply the wealth levels of families designated partners, this is equal to

$$\sum_{i=1}^{m} V(y_i - w_i) + \sum_{j=1}^{n} V\left(y_j - w_j'\right) +$$

$$\frac{\sum_{i=1}^{m} w_i + \max_{j_1,\dots j_m} \sum_{k=j_1}^{j_n} w_k' + \sum_{j=1}^{n} w_j' + \sum_{i=1}^{m} w_i}{z}$$

By assumption 2, this is continuous in (w, w'), and therefore upper semi-continuous. Therefore by (Reny 1999, Proposition 5), the mixed extension of the first stage game is reciprocally upper-semicontinuous.

The mixed extension of the first stage game is *payoff secure* if for every array of strategies μ and for any each player i has a strategy $\overline{\mu}_i$ such that

$$u_i\left(\overline{\mu}_i, \mu'_{-i}\right) \geq u_i(\mu) - \varepsilon$$

for all μ'_{-i} in some neighborhood of μ_{-i}. To show that this property holds, rewrite the payoff function using linearity as

$$u_i(w_i, \mu_{-i}) \equiv V(y_i - w_i) + \frac{w_i + \int \tilde{g}_i\left(w_i; \hat{\phi}(w_i, w_{-i}), \hat{\phi}(w')\right) d\mu}{z}$$

This function is linear, and therefore continuous in the strategies $\{\mu_{-i}, \mu'\}$. Using assortative matching, the payoff function family i faces can be further specialized to

$$= V(y_i - w_i) + \frac{w_i + \int \tilde{g}_i\left(w_i; \hat{\phi}(w_i, w_{-i}), \hat{\phi}(\overline{w}')\right) d\mu_1 \dots d\mu_m}{z}$$

where $\overline{w}'$ is the vector of expected values of the order statistics generated by draws from the distributions $\mu' = \{\mu'_1, \dots \mu'_n\}$ used by the families in F. Finally, this can be written as

$$V(y_i - w_i) + \frac{w_i + \sum_{k=1}^m q_k(w_i)\,\overline{w}'_k}{z}$$

where $q_k(w_i)$ is the probability with which the designated partner of family $i \in M$ is family $k \in F$. Let $w'' > w_i$. Then by assortative matching, in any state in which an investment of w_i earns return $\overline{w}_k$, raising investment to w'' will either leave i's designated partner unchanged, or raise it. Thus

$$V(y_i - w'') + \frac{w'' + \sum_{k=1}^m q_k(w'')\,\overline{w}'_k}{z} \geq V(y_i - w'') + \frac{w'' + \sum_{k=1}^m q_k(w_i)\,\overline{w}'_k}{z}$$

Then the continuity of V, for any $\varepsilon/2 > 0$, there is a w'' such that

$$V(y_i - w'') + \frac{w'' + \sum_{k=1}^m q_k(w'')\,\overline{w}'_k}{z} > V(y_i - w_i) + \frac{w_i + \sum_{k=1}^m q_k(w_i)\,\overline{w}'_k}{z} - \varepsilon/2$$

By the continuity of the payoff in the strategies $\{\mu_{-i}, \mu'\}$ of the other families, this inequality holds on some open neighborhood of μ_{-i}. To get payoff security in the mixed extension, choose w_i such that

$$V(y_i - w_i) + \frac{w_i + \sum_{k=1}^m q_k(w_i)\,\overline{w}'_k}{z}$$

is within $\varepsilon/2$ of the supremum of this payoff. This implies that family i can secure a payoff within ε of the supremum of this payoff function.

By (Reny 1999, Corollary 5.2), reciprocal upper semi-continuity and payoff security of the mixed extension together imply that the first stage game has a mixed strategy equilibrium.■

11.3 Proof of Theorem 3:

Restrict attention for the moment to families in M. The proof is constructive. Let $\underline{w}$ and $\overline{w}$ be the ex post expected level of investment of the poorest and wealthiest child in F. Let H_w and H_l denote the probability mixtures used by the wealthy and rich families in M respectively and suppose that these satisfy the hypothesis of the theorem. In other words, H_l and H_h have atoms at w_l^* and w_h^* respectively, and are otherwise smooth on some interval $[w_h^*, w^{**}]$.

Consider the wealthy family first. If it turns out that it's child has the highest investment level ex post, then he will match with the wealthiest child in F who will have expected investment $\overline{w}$. If he is poorest ex post, he will match with the poorest child in F, gaining a partner whose expected wealth is $\underline{w}$. The only tie we need to worry about occurs when the poor family invests exactly w_h^* and this is matched by the rich family. In this case we assume that the wealthy child from F proposes to the male from the poor family for sure.[3] Under these conditions, the expected payoff when the family invests $w \in [w_h^*, w^{**}]$ is

$$H_l(w)\left\{V(y_h - w) + \frac{w + \overline{w}}{z}\right\} + (1 - H_l(w))\left\{V(y_h - w) + \frac{w + \underline{w}}{z}\right\} \quad (5)$$

To support the equilibrium this must be constant along $[w_h^*, w^{**}]$ and equal to

$$V(y_h - w_h^*) + \frac{w_h^* + (1 - H_l(w_h^*))\underline{w} + H_l(w_h^*)\overline{w}}{z}$$

to induce the wealthy family in M to make the investment. If the function is constant, it's derivative should be almost everywhere 0, or

$$-V'(y_h - w) + \frac{1}{z} + H_l'(w)\frac{\overline{w} - \underline{w}}{z} = 0$$

[3]This is for notational convenience only - our results are unaffected by the tie breaking rule. If the rich child from F randomizes in some fashion, the payoff to playing w_h^* exactly will still be smaller than the limit of the payoffs associated with playing slightly more than w_h^* and this is all that is required to support our equilibrium.

which gives

$$H_l'(w) = \frac{V'(y_h - w) - \frac{1}{z}}{\frac{\overline{w}-\underline{w}}{z}}$$

The function H_l is then determined (up to a constant) by integrating

$$H_l(w) - H_l(w_h^*) = \frac{\int_{w_h^*}^{w} \left\{V'(y_h - s)\, ds - \frac{1}{z}\right\} ds}{\frac{\overline{w}-\underline{w}}{z}}$$

$$= \frac{V(y_h - w_h^*) - V(y_h - w) - \frac{w - w_h^*}{z}}{\frac{\overline{w}-\underline{w}}{z}} \tag{6}$$

where $H_l(w_h^*)$ is the probability with which the low wealth family invests w_l^*. This atom, and the value of w^{**} are determined below.

The poor family faces a similar problem. Let H_h be the distribution of investments by the wealthy family. The poor family's payoff is

$$H_h(w)\left\{V(y_l - w) + \frac{w + \overline{w}}{z}\right\} + (1 - H_h(w))\left\{V(y_l - w) + \frac{w + \underline{w}}{z}\right\}$$

and this should be constant on the interval $[w_h^*, w^{**}]$ and equal to

$$H_h(w_h^*)\left\{V(y_l - w_h^*) + \frac{w_h^* + \overline{w}}{z}\right\} + (1 - H_h(w_h^*))\left\{V(y_l - w_h^*) + \frac{w_h^* + \underline{w}}{z}\right\} \tag{7}$$

where $H_h(w_h^*)$ is the probability with which the wealthy family invests w_h^*.

The poor family will choose the investment level w_l^* on the equilibrium path, so the atom in $H_h(w_h^*)$ should be chosen to make the poor family indifferent between the investment levels w_l^* and w_h^* conditional on the assumption that if the poor family invests w_h^* it will match with the rich family on the other side of the market in the event of ties. To accomplish this, assign the atom $H_h(w_h^*)$ so that

$$H_h(w_h^*)\left\{V(y_l - w_h^*) + \frac{w_h^* + \overline{w}}{z}\right\} + (1 - H_h(w_h^*))\left\{V(y_l - w_h^*) + \frac{w_h^* + \underline{w}}{z}\right\} =$$

$$V(y_l - w_l^*) + \frac{w_l^* + \underline{w}}{z}$$

or

$$H_h\left(w_h^*\right)=\frac{V\left(y_l-w_l^*\right)+\frac{w_l^*}{z}-V\left(y_l-w_h^*\right)-\frac{w_h^*}{z}}{\frac{\overline{w}-\underline{w}}{z}} \tag{8}$$

Then reasoning as above, we have that

$$H_h\left(w\right)=H_h\left(w_h^*\right)+\frac{V\left(y_l-w_h^*\right)-V\left(y_l-w\right)-\frac{w-w_h^*}{z}}{\frac{\overline{w}-\underline{w}}{z}}$$

$$=\frac{V\left(y_l-w_l^*\right)-V\left(y_l-w\right)-\frac{w-w_h^*}{z}}{\frac{\overline{w}-\underline{w}}{z}} \tag{9}$$

Finally, there can be no atoms at the top of the distribution of investments because of the discontinuous increase in expected wealth that this creates. So the atom at $H_l\left(w_h^*\right)$ (i.e., the probability that the poor family chooses investment w_l^*) should be chosen so that the top of the supports of H_l and H_h coincide. The top of the support of H_h is given by the solution to

$$\frac{V\left(y_l-w_l^*\right)-V\left(y_l-w^{**}\right)-\frac{w^{**}-w_l^*}{z}}{\frac{\overline{w}-\underline{w}}{z}}=1 \tag{10}$$

This determines

$$H_l\left(w_h^*\right)=1-\frac{V\left(y_h-w_h^*\right)-V\left(y_h-w^{**}\right)-\frac{w^{**}-w_h^*}{z}}{\frac{\overline{w}-\underline{w}}{z}} \tag{11}$$

Note that since the density of the distribution H_h is uniformly higher on the interval $[w_h^*, w^{**}]$ than the density of H_l, it follows that the atom $H_l^*\left(w_h^*\right)$ is strictly larger than the atom $H_h\left(w_l^*\right)$ which verifies the two properties of the distributions mentioned in the theorem. One implication of this is that the mean investment of the wealthy family exceeds the mean investment of the poor family.

Conditional on the mean payoff levels, $\overline{w}$ and $\underline{w}$ it is straightforward to show that neither family can profitably deviate from this strategy. The poor family is indifferent between investing w_l^* and any investment level in the support $[w_h^*, w^{**}]$ by construction. Investment levels between w_l^* and w_h^* guarantee a match with a partner whose expected wealth is $\underline{w}$. Since this outcome is the same for every investment level on the interval $[w_l^*, w_h^*)$ the poor family's expected utility is strictly higher when they invest w_l^* than it is when they invest any amount in (w_l^*, w_h^*) by the strict concavity of V. Similarly, the quality of the poor family's match is independent of it's investment level if it tries to invest more than w^{**}, so investments above w^{**} are strictly dominated. The arguments supporting the rich families strategy are identical.

These distributions generate mean wealth levels for the wealthy and rich children in M as given by

$$\underline{w}' = H_h(w_h^*)\{H_l(w_h^*)w_l^* + (1 - H_l(w_h^*))w_h^*\}$$

$$+ \int_{w_h^*}^{w^{**}} H_h'(s)\left\{H_l(w_h^*)w_l^* + \int_{w_h^*}^{s} H_l'(t)\,tdt + [1 - H_l(s)]\,s\right\}ds \tag{12}$$

and

$$\overline{w}' = H_h(w_h^*)\left\{H_l(w_h^*)w_h^* + \int_{w_h^*}^{w^{**}} H_l'(s)\,sds\right\}$$

$$+ \int_{w_h^*}^{w^{**}} H_h'(s)\left\{\int_{s}^{w^{**}} H_l'(t)\,tdt + H_l(s)\,s\right\}ds \tag{13}$$

The problem faced by families in F is completely symmetric. So to verify the existence of equilibrium it remains to show that if families in F also use these distributions, then the mean wealth levels $\underline{w}'$ and $\overline{w}'$ that they generate are equal to the wealth levels $\overline{w}$ and $\underline{w}$ used to construct the strategies. In other words, this will constitute and equilibrium if there are values for $\underline{w}$ and $\overline{w}$ that constitute a fixed point in equations (12) and (13).

To prove that such values exist, we simplify the approach slightly by defining a single dimensional transformation. Let

$$\triangle \in \left[\frac{V(y_l - w_l^*) + \frac{w_l^*}{z} - V(y_l - w_h^*) - \frac{w_h^*}{z}}{z}, \infty\right]$$

represent the expected gain $\frac{\overline{w} - \underline{w}}{z}$ associated with matching with the wealthy child. For each value $\triangle$, equations 6,9, 10, and 11 completely determine the 'equilibrium'[4] mixed strategies for each of the families in M as described above. From 10 and Assumption 2, an upper bound w^{**} exists for each

$$\frac{\overline{w}-\underline{w}}{z} \in \left[\frac{V\left(y_l - w_l^*\right) + \frac{w_l^*}{z} - V\left(y_l - w_h^*\right) - \frac{w_h^*}{z}}{z}, \infty \right].$$

Furthermore, this upper bound varies continuously. Since

$$\frac{V\left(y_l - w_l^*\right) + \frac{w_l^*}{z} - V\left(y_l - w_h^*\right) - \frac{w_h^*}{z}}{z} > 0$$

by the strict concavity of V, it is apparent by inspection that the right hand sides of equations (12) and (13) both vary continuously with $\frac{\overline{w}-\underline{w}}{z}$. Thus the transformation

$$\frac{\overline{w}-\underline{w}}{z} \to \frac{\overline{w}'-\underline{w}'}{z}$$

defined by (12) and (13) is continuous. At the lower limit choose $\frac{\overline{w}_0-\underline{w}_0}{z}$ such that

$$\frac{V\left(y_l - w_l^*\right) + \frac{w_l^*}{z} - V\left(y_l - w_h^*\right) - \frac{w_h^*}{z}}{\frac{\overline{w}_0-\underline{w}_0}{z}} = 1$$

so that the wealthy family puts all of it's probability weight on w_h^* by (8), then

$$\frac{\overline{w}_0-\underline{w}_0}{z} \to \frac{\overline{w}'-\underline{w}'}{z} = \frac{w_h^*-w_l^*}{z}.$$

Observe that Assumption 9 says that

$$V\left(y_l - w_h^*\right) + \frac{2w_h^*}{z} > V\left(y_l - w_l^*\right) + \frac{2w_l^*}{z}$$

which implies that

$$\frac{w_h^*-w_l^*}{z} > V\left(y_l - w_l^*\right) + \frac{w_l^*}{z} - V\left(y_l - w_h^*\right) - \frac{w_h^*}{z}$$

[4]Bear in mind that these may not be full equilibrium strategies, since the strategies being used by the families in F may not support the expected gain $\triangle$.

so that at the lower end of the interval

$$\frac{\overline{w}_0 - \underline{w}_0}{z} < \frac{w_h^* - w_l^*}{z}$$

On the other hand, as $\frac{\overline{w}-\underline{w}}{z}$ gets large, the support of the mixed strategies of both families are constrained by Assumption 2 to lie in a bounded interval $[w_l^*, y_l]$. Thus the maximum value for the transformed means is $\frac{y_l - w_l^*}{z}$. This implies that for $\frac{\overline{w}-\underline{w}}{z}$ large enough

$$\frac{\overline{w} - \underline{w}}{z} \rightarrow \frac{\overline{w}' - \underline{w}'}{z} < \frac{\overline{w} - \underline{w}}{z}$$

Thus by the intermediate value theorem the transformation $\frac{\overline{w}-\underline{w}}{z} \rightarrow \frac{\overline{w}'-\underline{w}'}{z}$ has a fixed point which defines the full equilibrium strategies.■

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