The Tiebout Hypothesis: On the Existence of Pareto Efficient Competitive Equilibrium*

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Abstract

We introduce a model of an local public goods economy with a continuum of agents and jurisdictions with finite population. Under an especially mild condition of boundedness of per capita payoffs we show nonemptiness of the core of the economy. We then demonstrate that the equal treatment core coincides with the set of competitive allocations. Under an additional condition ensuring that all outcomes in the core have the equal treatment property, existence of equilibrium follows from nonemptiness of the core and equivalence of the core

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to the set of equilibrium outcomes. Our approach provides a new technique for showing existence of equilibrium in economies with a continuum of agents.

1 Introduction

In his seminal paper, Tiebout (1956) suggests that if public goods are subject to congestion, the benefits of sharing costs over a large number of agents will eventually be offset by the negative effects of crowding. Balancing the effects of cost-sharing and congestion make it advantageous for agents to be partitioned into a system of disjoint jurisdictions. Tiebout speculates that these jurisdictions would offer competing bundles of local public goods and tax liabilities and that agents would move to jurisdictions whose membership, public goods and taxes most closely approximate their ideal combinations. Tiebout concludes that if public goods are local, agents will reveal their preferences through their locational choice and the free rider problem will disappear.

In effect, Tiebout hypothesizes that in a local public goods economy, competitive forces will lead to efficient equilibrium outcomes. Tiebout's paper is quite informal, however, and while he makes a plausible case for the preference revelation properties of equilibrium, he does not deal with the question of existence. Subsequent more rigorous investigations have shown that the nonexistence problem is far from trivial. Perhaps the most famous of these is the paper by Bewley (1981) which shows in a series of examples that competitive equilibrium may not exist and, when it does, it may not be efficient. While many of Bewley's counterexamples are open to criticism (in particular, Bewley ignores the essential role of small group effectiveness), his basic point still holds. It is possible to write down reasonable examples of economies for which competitive equilibrium does not exist. Bewley's criticism goes to the very foundation of the Tiebout hypothesis. If he is correct, then the relevance of the vast literature that Tiebout's paper generated is in serious doubt. There is limited value in studying the properties of an equilibrium concept that seldom exists. Not surprisingly, the question of existence has occupied the attention of many authors. We divide their contributions into four branches and give a brief discussion below.

The first branch treats a model with a continuum of agents who divide themselves into an exogenously fixed, finite number of jurisdictions each of which subsequently chooses public goods levels according to a voting rule. All of these papers consider Nash equilibrium and differ mainly in their various treatments of land and taxation rules. Papers using income taxes with majority voting over public good levels include Westhoff (1977), who considers a model without land, and Dunz (1989), who adds indivisible land. Greenberg and Shitovitz (1988) treat a related model in which land is divisible and the d-majority voting rule developed in Greenberg (1979) is used to decide within-jurisdiction allocations. Agents are immobile in this model, however, and so are not allowed to move to their most preferred jurisdiction. Konishi (1996) extends the model by allowing mobility of agents across jurisdictional boundaries. All these papers show the existence of equilibria.[1] Income taxes, however, have been criticized as unrealistic since the vast majority of local public goods in the United States are funded by property taxes. Rose-Ackerman (1979) shows that, unfortunately, when land is divisible and property taxes are used to fund public good provision, consumers' preferences may not be single-peaked and so majority voting equilibrium may not exist. Epple et al. (1984, 1993) succeed in proving existence in Rose-Ackerman's economy for a special case which imposes a restrictive condition on preferences, the so called "single-crossing property". In contrast, Nechyba (1996) shows existence of equilibrium in a fairly general model with indivisible land and property taxes.

This literature succeeds in addressing the very difficult question of the existence of equilibrium in a variety of interesting and realistic institutional environments. There is one important respect, however, in which it falls short of confirming Tiebout's hypothesis. While the equilibria these authors describe can be shown to exist, they are not, in general, Pareto efficient. The most important reason for this is that the set of Nash equilibria can be quite large. It is easy for agents to become trapped in suboptimal states for which coalitional deviations would yield a significant Pareto improvement, but which are stable against Nash's unilateral deviations. In many of these models it may even be the case that none of the Nash equilibria are Pareto

[1]A closely related literature uses a similar model but employs Foley's (1979) public competitive equilibrium rather than voting to choose within-jurisdiction public goods levels. This approach was initiated by Richter (1975) and Greenberg (1977) in models which do not allow agents to migrate between jurisdictions. The migration restriction is relaxed in Richter (1982) and Greenberg (1983).

optimal. This is because the institutions of taxation (especially) and voting constrain agents' actions. For example, if equal sharing of costs is imposed, poor agents may simply not be able afford to join coalitions with rich agents who have high demands for public goods. Thus, even if crowding is not a significant factor, in equilibrium agents will segregate by income. This is despite the fact that they would all be better off pooling their resources and living in one large jurisdiction with unequal sharing of costs. At best, Nash equilibria are Pareto optimal over all allocations which respect the institutional constraint, but are not in the unconstrained core.[2] This contrasts with most notions of economic equilibrium, and casts some doubt on the stability of these institutions. We hasten to add that this should be interpreted as a criticism of the soundness of Tiebout's conjecture in real economic settings and not as a theoretical failing on the part of the above mentioned authors. All of the papers in this literature also share the feature that equilibrium involves an uncountable number of agents being packed into a finite set of jurisdictions. We discuss this in more detail below.

A second branch of the literature addresses some of the issues discussed above by considering the existence of the core in finite economies. Obviously, the core is efficient and only finite numbers of agents will be in each jurisdiction in a finite economy. Guesnerie and Oddou (1981) explore such a model without land and using proportional income taxes. They show that if the population is less than or equal to three agents, the core will exist. Weber and Zamir (1985) then show the complementary proposition: in general the core will not exist if there are more than three agents. Subsequently, Greenberg and Weber (1986) show that for any finitely-sized economy, the core will exist, but only under fairly severe restrictions on preferences. Thus, this literature comes to a largely negative conclusion on the question of existence of equilibrium.

A third branch of the literature was initiated by Ellickson (1979), who treats local public goods as indivisible commodities. Ellickson defines a notion of Nash equilibrium in which all agents who choose the same type of indivisible commodities can be thought of as being in the same jurisdiction. His model has a finite number of agents and multiple, divisible private commodities. In addition to showing that equilibrium exists, Ellickson shows

[2] Konishi, Le Breton, and Weber provide an example that shows that it may even the case the no Nash equilibria satisfy constrained Pareto optimality.

that it can be decentralized by a set of average cost prices. These results, however depend on a strong assumption on the technology. Specifically, Ellickson assumes that eventually the cost of providing any type of (indivisible) public good is linear in the number of agents. Thus, jurisdictions may be too small to exhaust returns to scale, but they can never be to large. Put another way, congestion takes place in production, but for sufficiently large jurisdictions, there is neither per capita harm nor benefit from adding agents to the jurisdictions. In this sense, his indivisible public goods are closer in spirit to public services as described in Bewley (1981), for example, than to nonrival public goods. Under weaker conditions, Vohra (1987) treats a similar model and shows the existence of approximate equilibrium which are second best. In a related paper, Vohra (1984) treats a continuum version of this model. He shows the existence of a type of exact equilibrium in which the continuum of agents choose over a finite set of indivisible commodities (in interpretation, jurisdictions.) He is able to relax some of the restrictive assumptions of congestion imposed by Ellickson (1979) and Vohra (1984). The main problem with these approaches is that while equilibria do exist, a first welfare theorem does not hold. It may be possible to support many quite inefficient allocations in general.[3]

To summarize these first three literatures, except under the very special conditions of Greenberg and Weber (1986), there is no general proof of the existence of the core, or of Nash or price taking equilibria which are always efficient over the set of all feasible allocations. We are left with a choice between an efficient equilibrium notion which may not exist, or an equilibrium which does exist, but which may not be efficient. In either case, Tiebout's main conjecture remains unproven.

A problem shared by all the continuum models of Tiebout economies discussed above is that, by construction, they require that almost all agents live in jurisdictions with uncountably infinite populations. It is far from clear that this is the right continuum limit of a finite economy. What does it mean to have an infinitely large city? Do we really think that as the population increases we would see cities grow without bound? At best, only a measure zero of agents could live in small (meaning finite) towns. Thus, these authors

[3]Also see Cole and Prescott (1997) for a more recent contribution in this spirit. This paper shows the existence of valuation equilibrium for lotteries over club memberships and the equivalence of valuation equilibrium outcomes to the core.

describe a state of the world in which cities are infinitely large, and essentially all small towns disappear.

The final branch of the literature we discuss suggests a possible solution to these problems and motivates the approach taken in the current paper. Wooders (1980b) considers a finite economy in which crowding or congestion limits the size of efficient jurisdictions to be small compared to the population.[4] She notes that in such an economy, what is driving the nonexistence of the core is that in general, the total number of agents is not an exact multiple of the efficiently sized jurisdictions. For example, if optional jurisdictions consist of five agents, then when the population is 11, or 101, or 1001, there are "left-over" agents. These left-overs induce a cycle which results in the nonexistence of the core. Note, however, that as the population grows the proportion of left-overs goes to zero. Thus, it is immediate that there exist approximate equilibria where a small percentage of players are ignored. Wooders proposes a notion of ϵ-core which has a natural interpretation in the context of local public goods economies. Informally, she modifies the notion of what it means to improve on a feasible state by requiring that it be possible to make the defecting agents better off while paying a small cost of jurisdictional formation. In other words, agents who contemplate defecting must each pay a transaction cost of $\epsilon \geq 0$, which may represent moving or setup costs, and must still be better off in the new jurisdiction. Wooders demonstrates that for any for $\epsilon > 0$, for all sufficiently large economies the ϵ-core exists and is equivalent to a similarly defined ϵ-admission price equilibrium.[5] A feature of this model is that the sizes of optimal jurisdictions do not change as the economy grows without bound. The model, following Wooders (1978), is restricted to anonymous crowding. This restriction makes the analysis of the core especially straightforward since it implies that all improvement can be carried out by taste-homogeneous coalitions. Wooders (1985,1997)[6] obtain similar results for an economy with multiple private and public goods and *differentiated* crowding: that is, preferences and/or production possibilities may depend on the composition as well as

[4]Similar models appear in McGuire (1974), Berglas and Pines (1981) and a number of subsequent papers.

[5]In an admission price equilibrium, each agent pays what amounts to a head tax to pay for public goods. This contrasts to the income and property taxes above.

[6]Related results, for the special case of one private good and one public good are presented in Scotchmer and Wooders (1986).

the size of jurisdictions.

Wooders' approach suggests three things. First, since the ε-core and ε-equilibrium exist for arbitrarily small epsilon if the economy is sufficiently large, it follows that if the economy were infinitely large the exact core and equilibrium should exist. Second, the continuum limit would have agents living in finite jurisdictions rather than in jurisdictions of positive measure. Third, the techniques of proof of internalizing externalities within jurisdictions and then using the Minkowski separating hyperplane theorem to show existence of equilibrium used in Wooders (1985,1997) does not depend on the finiteness of the economy and thus may extend to continuum economies.

The approach we take in this paper is to define a continuum economy with local public goods by adapting the f-core notion of Kaneko and Wooders (1986,1989) and Hammond, Kaneko and Wooders (1989). Using this, we show in a remarkably general model the existence of Pareto efficient Tiebout equilibria with a continuum of agents. Following Kaneko and Wooders (1986), we require that admissible partitions of individuals into jurisdictions consist only of jurisdictions with finite membership and satisfy measurement-consistency. This last condition ensures that the partition is consistent with the proportions of agents of each type given by the measure on the total agent set and so is feasible given the population. We introduce a particularly elegant statement of this condition. Our notion of the core requires that no finite coalition can improve. Since the number of private goods may be greater than one, improving coalitions may form multiple jurisdictions. In the context of a local public goods economy, the idea that improvements by finite coalitions should block feasible allocations is compelling. Tiebout's argument is based on the assumption that small groups are sufficient to realize all or almost all gains from cooperation. Restricting improvements to coalitions of positive measure (as in Aumann 1964) would therefore seem to require that these basic building blocks of the economy ignore their best interests. We note, however, that in this environment our notion of the core and Aumann's turn out to be equivalent.

To summarize, in this paper we introduce a model of an economy with multiple public and private goods and a continuum of agents. Following Conley and Wooders (1997b), we make a distinction between the unobservable taste types of agents and their observable crowding types. The crowding type of an agent determines his effects on other agents and/or on production possibilities. We define a competitive equilibrium concept in which admission prices for jurisdictions depend only on observable crowding types of agents. Under extraordinarily mild conditions we show that the core is nonempty. We also show that the set of core outcomes with the equal treatment property coincides with the set of equilibrium outcomes. Conditions are described under which all outcomes in the core have the equal treatment property and we show that they imply that core coincides with the equilibrium outcomes. Thus, from nonemptiness of the core and the equivalence of the core with the competitive outcomes, we obtain existence of competitive equilibria which are always Pareto efficient.

2 The Model

Let (N, β, μ) be a measure space where N is a Borel subset of a complete separable metric space, let β be a σ-algebra of all Borel subsets of N and let μ be Lesbegue measure with $0 < \mu < +\infty$. The set N is the *set of agents* in an economy and $i \in N$ will typically denote an agent. Each agent is one of C different sorts of crowding types,[7] denoted $c \in \{1, \ldots, C\} := \mathcal{C}$, and one of T different sorts of taste types, denoted $t \in \{1, \ldots, T\} := \mathcal{T}$. An element of $\mathcal{C} \times \mathcal{T}$ is typically represented by a pair (c, t) and called a *type*. The crowding and taste types of agents are given by a pair of *attribute functions*, denoted, respectively, by κ and τ. Formally $\kappa : N \mapsto \mathcal{C}$ and $\tau : N \mapsto \mathcal{T}$. For each (c, t), and for any measurable subset S of N, define

$$S_{ct} = \{i \in S : \kappa(i) = c \text{ and } \tau(i) = t, \}$$

$$S_c = \{i \in S : \kappa(i) = c\} \text{ and}$$

$$S_t = \{i \in S : \tau(i) = t\}.$$

A *jurisdiction* G is a finite subset of N. Let $\mathcal{F}$ be a set of *admissible jurisdictions*. The set $\mathcal{F}$ is required to satisfy the property that for each $i \in N, \{i\} \in \mathcal{F}$.

Remark 1 It may be the case that $\mathcal{F}$ contains only singleton sets, that is, $\mathcal{F} = \{\{i\} \in N\}$. In this case, the economy has, in effect, only private goods.

[7] Note that a crowding type may be a point in a finite or infinite dimensional commodity space.

We describe a jurisdiction by the numbers of agents of each type in the jurisdiction. Let $\mathcal{Z}$ denote the nonnegative integers and let $\mathcal{Z}^{CT}$ denote the CT-fold Cartesian product of $\mathcal{Z}$. Given an admissible jurisdiction $G \in \mathcal{F}$, the profile of G, denoted by $pro(G)$, is a vector in $\mathcal{Z}^{CT}$ defined by its components

$$pro(G)_{ct} = |G_{ct}|$$

where $|\cdot|$ denotes the cardinality of a set. The vector $pro(G)$ describes the jurisdiction G by the number of agents of each type in the jurisdiction.

In a similar vein, we will be interested in the taste and crowding profiles of jurisdictions. Given an admissible jurisdiction $S \in \mathcal{F}$, the *crowding profile* of an admissible jurisdiction G, denoted by $\mathcal{C}pro(G)$, is a vector in $\mathcal{Z}^{C}$ defined by its components

$$\mathcal{C}pro(G)_c = |G_c|\,.$$

Symmetrically, the *taste profile* of G is a vector in $\mathcal{Z}^{T}$, defined by its coordinates

$$\mathcal{T}pro(G)_t = |G_t|\,.$$

One of the most crucial concepts in our work is that of an admissible jurisdiction structure. Since agents consume public goods jointly with other members of finite jurisdictions, a feasible state of the economy must specify a partition of the set of agents into finite jurisdictions that is consistent with the measure on the total player set. Thus, an admissible jurisdiction structure is a measurement-consistent partition in the sense of Kaneko and Wooders (1986). We provide here a simpler definition.

Let S be a measurable subset of N and let π denote a partition of S into admissible jurisdictions. An *index set of the partition* π is a measurable set $n \subset S$ such that

$$\text{for each } G \in \pi,\ |G \cap n| = 1;$$

that is, n contains one and only one member of each jurisdiction G.

Let S be a measurable subset of N. A partition π is a *jurisdiction structure* of S if

1. There exists an index set for π and,

2. for all index sets n and n' of π,

$$\mu(n) = \mu(n').$$

Remark 2 As pointed out by Kaneko and Wooders (1986), to allow finite coalitions with a continuum of agents requires that the magnitudes of finite coalitions and of partitions of players into finite coalitions be consistent with the proportions given by the measure on the continuum. Consider a marriage game where the (Lebesgue) measure of the set of males and the measure of the set of females both equal one. Since there are sets of measure zero with the cardinality of the continuum it is possible to define a one-to-one mapping from the set of males to a subset of females of measure zero. Thus, there are partitions of the set of players into marriages and singletons so that all the males are married but virtually all the females (all the females except for a set of measure zero) are single. Our definition of admissible partitions avoids this problem. An index set consisting of the set of married males plus the set of unmarried females would have measure two but the index set consisting of married females plus the set of unmarried females would have measure one. In fact, from Lemma A.2 of Kaneko and Wooders (1986), it follows that when coalitions are constrained to be finite, our definition of jurisdiction structures coincides with the Kaneko-Wooders definition of measurement-consistent partitions.[8]

Given a jurisdiction structure π and $i \in N$, let π_i denote the jurisdiction in π containing player i.

We consider an economy with L private goods and a topological space of public projects, denoted by $\mathcal{X}^{public}$, containing a distinguished element denoted by $\underline{0}$. A bundle of private goods is denoted by $x \in \mathbf{R}^L$ and a public project is denoted by $y \in \mathcal{X}^{public}$. An *endowment function* is given by a measurable and integrable function ω^0 from N to $\mathbf{R}^L$ such that for all agents i and j with $\tau(i) = \tau(j)$ it holds that $\omega^0(i) = \omega^0(j)$.

[8] In the language of Kaneko and Wooders (1986), the set of admissible index sets of a "measurement-consistent" partition includes (a) all the first members of k-member coalitions for $k = 1, 2, ...$, (b) all the singleton coalitions plus all the second members of k-member coalitions, $k = 1, 2, ...$, etc. See Kaneko and Wooders (1986).

Each agent of type $(c,t) \in \mathcal{C} \times \mathcal{T}$ has a consumption set

$$\mathcal{X}_{ct} = \mathcal{X}^{private} \times \mathcal{X}^{public} \times \mathcal{Z}_{ct},$$

where $\mathcal{X}^{private} \subset \mathbf{R}^L$ and $\mathcal{Z}_{ct} \subset \mathcal{Z}^{CT}$ is the set of profiles $pro(S)$ with $pro(S)_{ct} \neq 0$. The following assumptions are required:

(A.1)

(a) $\mathcal{X}_{ct}$ is closed in $\mathbf{R}^L \times \mathcal{X}^{public} \times \mathcal{Z}_{ct}$ and

(b) for any nonempty subset $X' \subset \mathcal{X}^{private}$ the vector a, defined by its components $a_\ell = \inf\{a'_\ell : a' \in X'\}$ for $\ell = 1, ..., L$ is in $\mathcal{X}^{private}$.

(A.2) $(\omega^0(i), \underline{0}, pro(\{i\})) \in \mathcal{X}_{ct}$ for all $i \in N_{ct}$.

Condition (A.1)(b) ensures that the consumption set for private goods has a lattice structure. Condition (A.2) dictates that, for each agent, producing "zero" public projects while consuming his endowment is in his agent consumption set. Note that neither $\mathcal{X}^{private}$ nor $\mathcal{X}^{public}$ depend on agents' types. An agent's preferences, however, are only defined over those jurisdiction profiles containing agents of his crowding type; if an agent is of type (c,t) and $(x, y, pro(G))$ is in $\mathcal{X}_{ct}$, then $pro(G)_{ct}$ is not equal to zero. Also, note that we have not required any divisibility of commodities.

The preferences of an agent of type t are described by a utility function u_t mapping $\mathcal{X}_{ct}$ into $\mathbf{R}_+$. In interpretation,

$$u_t(x, y, pro(G)) < u_t(x', y', pro(G'))$$

means that an agent of taste type t, residing in a jurisdiction with profile $pro(G')$, enjoys the consumption bundle x' of private goods and y' of public goods more than he would enjoy the consumption bundle x of private goods and y of public goods while residing in a jurisdiction with profile $pro(G)$.

Remark 3 Following Mas-Colell (1980) for economies with public projects and Manning (1992) for local public projects, we do not require a linear structure on the space $\mathcal{X}^{public}$. This is costless – the proofs of our results are the same as in the case where $\mathcal{X}^{public}$ is contained in some finite dimensional Euclidean space.

For each taste type $t \in \mathcal{T}$ we make the following assumptions:

(A.3) u_t is continuous.

(A.4) *Taste anonymity in consumption* (TAC): For all $x \in \mathcal{X}^{private}$, $y \in \mathcal{X}^{public}$ and all $G, G' \in \mathcal{F}$ such that $(x, y, pro(G)), (x, y, pro(G')) \in \mathcal{X}_{ct}$ and $\mathcal{C}pro(G) = \mathcal{C}pro(G')$ it holds that $u_t(x, y, pro(G)) = u_t(x, y, pro(G'))$.

TAC dictates that preferences depend only crowding characteristics of agents in the same jurisdiction, and not on their preferences.

Production is also subject to crowding. The production technology is given by a mapping P from the set of profiles to nonempty, closed subsets of $\mathbf{R}^L \times \mathcal{X}^{public}$ containing $(0, \underline{0})$. Thus, for each jurisdiction G,

$$\begin{aligned} P(pro(G)) &\subset \mathbf{R}^L \times \mathcal{X}^{public} \text{ and} \\ (0, \underline{0}) &\in P(pro(G)) \end{aligned}$$

and $P(pro(G))$ represents the technology for all jurisdictions with profile $pro(G)$.

It is required that a finite amount of inputs can produce only some bounded amount of output. That is,

(A.5) Given a jurisdiction $G \in \mathcal{F}$ and an input vector $z' \in \mathbf{R}^L$, the set $\{y : (z', y) \in P(pro(G))\}$ is compact.

For our price system to be Pareto-optimal, we require taste anonymity in production as well as in consumption.

(A.6) *Taste anonymity in production* (TAP): For all $G, G' \in \mathcal{F}$ such that $\mathcal{C}pro(G) = \mathcal{C}pro(G')$ it holds that $P(pro(G)) = P(pro(G'))$.

TAP, analogously to TAC, dictates that production possibilities do not depend on tastes. Note that we have not required the standard condition of production economies of positive semi-independence. Since we will (later) assume per capita boundedness, this condition is not required.

To define feasible states of the economy, following the f-core approach of Hammond, Kaneko and Wooders[9], we require that any feasible state is

[9]Kaneko and Wooders (1986,1989,1996) and Hammond, Kaneko, and Wooders (1989).

the limit of "f-feasible" states – states of the economy that are feasible by trade only within finite coalitions.[10] The members of a finite coalition may divide into many jurisdictions, each providing public goods for their membership, but feasibility requires that trade of private goods occurs only among members of a coalition. Thus, we must define admissible coalitions structures relative to a given jurisdiction structure π. A coalition structure will be denoted by ξ.

A pair (ξ, π) is an *admissible coalition-jurisdiction structure* if, for each coalition $W \in \xi$, there is a collection of jurisdictions $\{\pi^k\}$ such that

$$W = \cup_k \pi^k$$

– that is, ξ is a coarsening of π. Since π is a jurisdiction structure, it follows that there exists an index set for ξ, say m, and that for every index set m and m' of ξ, $\mu(m) = \mu(m')$. The set of coalitions in an admissible coalition-jurisdiction structure is measurement-consistent in the sense of Kaneko and Wooders (1986).

Let S be a measurable subset of N and let (ξ, π) be an admissible coalition-jurisdiction structure of S. A *feasible state for S relative to* (ξ, π) is a list $((\xi, \pi), X, Y, Z)$ where (ξ, π) is an admissible coalition-jurisdiction structure of S, $X : S \to \mathbf{R}^L$ is a private goods consumption mapping, $Y : S \to \mathcal{X}^{public}$ is a public goods consumption mapping, and $Z : S \to \mathbf{R}^L$ is an input accounting device, such that:

1. For almost all $i, j \in S$ if $\pi_i = \pi_j$ then $Y(i) = Y(j)$.

2. For almost all $i \in S$, $(X(i), Y(i), pro(\pi_i)) \in \mathcal{X}_{ct}$.

3. The public projects consumption and production mappings are feasible:

[10] In an earlier version of this paper, we required instead the usual Aumann-feasibility condition on allocations of private goods. This meant that to obtain our equal-treatment result (Theorem 4) it was necessary to include the proof of the equivalence of Aumann-feasibility and f-core feasibility. The proof of this equivalence, however, is essentially the same as that of Kaneko and Wooders (1986, Lemma 3.2), which demonstrates the result for a private goods exchange economy. Thus, we've chosen to shorten the paper and make it completely self-contained by using the f-core definition of feasibility. (Hammond, Kaneko and Wooders 1989 and Hammond 1992 also relate to the equivalence of Aumann-feasibility and f-feasibility in the context of exchange economies.)

(a) For almost all $i \in N$ it holds that $(\sum_{j\in\pi_i} Z(j), Y(j)) \in P(pro(\pi_i))$; and

(b) The distribution of private projects is feasible. That is, for each coalition $W \in \xi$ it holds that

$$\sum_{i\in W} (\omega^0(i) - X(i) - Z(i)) \geq 0.$$

Remark 4 Note that in the above definition we "assigned" to each individual i inputs of private goods $Z(i)$ used to produce public goods in the jurisdiction containing that individual – inputs are indexed by individuals. Thus, total input in π_i of private goods into production of public projects is given by $\sum_{j\in\pi_i} Z(j)$. This accounting device is simply for convenience. An alternative approach would be to define another variable, say $\widehat{Z}$, so that $\widehat{Z}(\pi_i)$ is the total input used in the jurisdiction containing agent i. For convenience, however, we follow the convention of 3.(b), distributing inputs used in a jurisdiction over the members of a jurisdiction and summing over the entire population in the jurisdiction. Also note that S may be finite.

Given a measurable subset $S \subset N$ and given a jurisdiction structure π, define $F_S(\pi)$ by

$$\begin{aligned} F_S(\pi) = \{(X,Y,Z) :\ & \text{there is an admissible coalition-jurisdiction structure } (\xi,\pi) \\ & \text{and a feasible state } ((\xi,\pi),X',Y',Z') \text{ of the economy relative to } (\xi,\pi) \\ & \text{such that } X = X',\ Y' = Y, \text{ and } Z = Z'\}. \end{aligned}$$

Define the set $F_S^*(\pi)$ by

$$\begin{aligned} F_S^*(\pi) = \{(X^*,Y^*,Z^*) :\ & \text{for some sequence } \{(X^\nu,Y^\nu,Z^\nu)\} \text{ in } F_S(\pi), \\ & \{(X^\nu,Y^\nu,Z^\nu)\} \text{ converges in measure to } (X^*,Y^*,Z^*)\}. \end{aligned}$$

A *feasible state for* S is a list (π,X,Y,Z) where π is a jurisdiction structure of S and $(X,Y,Z) \in F_S^*(\pi)$.

Let S be a measurable subset of N. A feasible state (π,X,Y,Z) for S has the *equal treatment property* if there is a subset S^0 of S of full measure $(\mu(S^0) = \mu(S))$ such that

$$u_{\tau(i)}(X(i),Y(i),pro(\pi_i)) = u_{\tau(j)}(X(j),Y(j),pro(\pi_j))$$

whenever $\tau(i) = \tau(j)$ and $\kappa(i) = \kappa(j)$. When S is finite, the equality must hold for all agents $i \in S$.

Let (π, X, Y, Z) be a state of the economy N. A measurable subset $S \subset N$ of the total population of agents *improves upon* (π, X, Y, Z) with a feasible state of the economy for S, say (π', X', Y', Z'), if for every $i \in S$,

$$u_{\tau(i)}(X'(i), Y'(i), pro('\pi_i)) > u_{\tau(i)}(X(i), Y(i), pro(\pi_i)).$$

For our results, we will typically restrict improving coalitions to be finite.

The *f-core,* or simply the *core*, of the economy consists of those feasible states of the economy (π, X, Y, Z) with the property that, for some subset of agents $N^0 \subset N$ of full measure, there is no finite coalition $S \subset N^0$ that can improve upon (π, X, Y, Z).

The *equal-treatment core* consists of those states of the economy (π, X, Y, Z) in the core with the property that for almost all $i, j \in N$, if $(\kappa(i), \tau(i)) = (\kappa(j), \tau(j))$ then

$$u_i(X(i), Y(i), pro(\pi_i)) = u_i(X(i), Y(i), pro(\pi_i)).$$

The core has the *equal treatment property* if the core is equivalent to the equal-treatment core.

Our first Theorem requires per capita boundedness of utility of finite coalitions in the neighborhood of the population proportions given by the measure, as defined in Kaneko and Wooders (1986).

(A.7) The economy is *per capita bounded* if there are positive numbers δ and K, with $0 < \delta < 1$, such that for every finite subset $S \subset N$ and for any feasible state (π, X, Y, Z) of S with the equal treatment property,

$$(1+\delta)\tfrac{\mu(N_{ct})}{\mu(N)} \geq \tfrac{|S_{ct}|}{|S|} \geq (1-\delta)\tfrac{\mu(N_{ct})}{\mu(N)} \text{ for all } (c,t) \in \mathcal{C} \times \mathcal{T}$$

$$\Rightarrow u_{\tau(i)}(X(i), Y(i), pro(\pi_i)) \leq K \text{ for all } i \in S.$$

Theorem 1. *Nonemptiness of the core.* Under assumptions (A.1)-(A.3), (A.5), and (A.7), the equal-treatment core of the economy is nonempty.[11]

[11]The anonymity assumptions (A.4) and (A.6) are not required for this result.

Proof of Theorem 1. The proof of this result is an application of the main result of Kaneko and Wooders (1986). See the Appendix for details. ■

An *equilibrium price system for crowding type* c is a mapping

$$\psi_c : \mathcal{X}^{public} \times \mathcal{Z}^C \to \mathbf{R}.$$

The value $\psi_c(y, \mathcal{C}pro(S))$ of the mapping ψ_c at $(y, \mathcal{C}pro(S))$ is interpreted as the amount of money that an agent of crowding type c would be required to pay to join a jurisdiction with crowding profile $\mathcal{C}pro(S)$ and consume a vector y of public goods. A *price system* ψ *for public goods* is a collection of price systems, one for each crowding type.

In the following definition, note that, as in the definition of a feasible state, for the purposes of adding up the total input of private goods into public good production, we distribute the input of private good in a jurisdiction among the members of the jurisdiction.

An *equilibrium* is a feasible state (π, X, Y, Z) of the economy N, a price system $p \in \mathbf{R}^L_+$ for private goods, and a price system ψ for public goods such that:

1. For almost all $i \in N, (X(i), Y(i), \mathcal{C}pro(\pi_i)) \in \mathcal{X}_{\kappa(i)\tau(i)}$ and

$$p \cdot X(i) + \psi_{\kappa(i)}(Y(i), \mathcal{C}pro(\pi_i))) = p \cdot \omega^0(i).$$

2. For almost all $i \in N$, for all jurisdictions $G \in \mathcal{F}$ such that $i \in G$, for all possible bundles of private goods $x \in \mathcal{X}^{private}$ and public projects $y \in \mathcal{X}^{public}$, if

$$u_{\tau(i)}(x, y, pro(G)) > u_{\tau(i)}(X(i), Y(i), pro(\pi_i))$$

then

$$p \cdot x + \psi_{\kappa(i)}(y, \mathcal{C}pro(G))) > p \cdot \omega^0(i).$$

3. For almost all $i \in N$

$$\sum_{j \in \pi_i} \psi_{\kappa(j)}(Y(j), \mathcal{C}pro(\pi_i)) - p \cdot \sum_{j \in \pi_i} Z(j) \quad = \quad 0.$$

4. For every $G \in \mathcal{F}$, there does not exist $(z, y) \in P(pro(G))$ such that

$$\sum_{j \in G} \psi_{\kappa(j)}(y, \mathcal{C}pro(G)) - p \cdot z > 0.$$

Theorem 2. *An equilibrium state of the economy is in the core.* Assume (A.4) and (A.7). If a feasible state of the economy (π, X, Y, Z) and price systems p and ψ constitute an equilibrium, then (π, X, Y, Z) is in the core.

Proof. See the Appendix.

Remark 5 Theorem 2 is proven using the game-theoretic notion of the core, where all members of an improving coalition must be better off. In general, this leads to a larger core than the notion frequently used in economics, where all members of an improving coalition must be at least as well off and one must be strictly better off. The Theorem is also easily proven for the this notion of the core.

The following Theorem depends on the assumption that crowding is taste-anonymous, both in production and consumption. Without these assumptions, our equilibrium concept will not necessarily satisfy a First Welfare Theorem. Let S be a subset of N of positive measure and let (π, X, Y, Z) be a feasible state for S. The state (π, X, Y, Z) for S is *Pareto dominated* by another feasible state (π', X', Y', Z') for S if, for almost every $i \in S$,

$$u_{\tau(i)}(X'(i), Y'(i), pro(\pi'_i)) \geq u_{\tau(i)}(X(i), Y(i), pro(\pi_i))$$

and there exists a set of agents $S' \subset S$ with positive measure such that for every $i \in S'$

$$u_{\tau(i)}(X'(i), Y'(i), pro(\pi'_i)) > u_{\tau(i)}(X(i), Y(i), pro(\pi_i)).$$

A feasible state of the economy is *Pareto optimal* if it is not Pareto-dominated by any other feasible state.

Theorem 3. Assume (A.4) and (A.6). Let (π, X, Y, Z) be an equilibrium state of the economy. Then (π, X, Y, Z) is Pareto-optimal.

Proof of Theorem 3. Theorem 3 is immediate from Theorem 2. ■

Next we demonstrate an equal treatment theorem, extending the equal-treatment property of the core of replicated exchange economies and games with strictly effective small groups to continuum economies with local public goods. This depends on our assumption that agents of the same taste type have the same endowment[12] and on two additional assumptions.

(A.8) *Divisibility and monotonicity.* Private goods are infinitely divisible and utility functions are strictly monotonic increasing in private goods.

(A.9) *Desirability of the endowment of divisible private goods.* For each agent i, it holds that:

$$u_{\tau(i)}(\omega^0(i), \underline{0}, pro(\{i\})) > u_{\tau(i)}(x, y, pro(S))$$

where x is any private goods bundle containing none of the divisible private goods.

Assumptions related to (A.9) are frequently used in the literature on economies with local public goods to ensure the equal treatment property of the core, cf., Wooders (1978, Theorem 5) and Wooders (1985,1997). Exactly the same assumptions appear in Hammond, Kaneko and Wooders (1989), Kaneko and Wooders (1989) and Ellickson et al. (1997). They ensure that if one agent in a coalition can be made better off, then all agents in the coalition can be made better off – informally, there is always some way to transfer "utility," although not necessarily at a one-to-one rate. If such a condition does not hold, then it may be that no states of the economy in the core have the equal treatment property.[13] Other assumptions would suffice to ensure this

[12]Debreu and Scarf (1963) treat replicated exchange economies. The equal-treatment property of the core for replicated games with strictly effective small groups is shown in Wooders (1983) while Wooders (1980,1994) shows that cores of large games have the equal treatment property. (These results also appear in Shubik and Wooders 1982.)

[13]Consider a two-sided matching model where the members of a matched pair can each realize one unit of utility, but there is no transferability of utility. For example, commodities might all be indivisible. In this case, if there are more members on one side of the market than the other, no core payoffs will have the equal treatment property.

property. For example, as discussed in Kaneko and Wooders (1994), it could simply be assumed that all individuals have strictly increasing preferences for a divisible commodity, all individuals are endowed with that commodity, and the boundary constraint is never binding. With this assumption, we can also demonstrate that the core has the equal-treatment property.

Remark 6 The above assumptions, ensuring that the core has the equal treatment property, also ensure the equivalence of the game-theoretic notion of the core, requiring that all agents in an improving coalition be strictly better-off (as in Scarf 1967, for example) and the economic notion, that at least one agent in an improving coalition be strictly better off (as in Debreu and Scarf 1963). This equivalence has consequences for economic results, cf. Page and Wooders (1996)).

Theorem 4. *The equal treatment property of the core.* Assume (A.1)-(A.5), and (A.8)-(A.9). Let (π, X, Y, Z) be a core state of the economy. Then there is a subset $N^0 \subset N$, $\mu(N^0) = \mu(N)$, such that for every pair of agents $i, j \in N^0$ satisfying $\tau(i) = \tau(j)$ and $\kappa(i) = \kappa(j)$ it holds that

$$u_{\tau(i)}(X(i), Y(i), pro(\pi_i)) = u_{\tau(j)}(X(j), Y(j), pro(\pi_j)).$$

Proof of Theorem 4. See the Appendix. ■

Theorem 4 is used in our proof of the equivalence of the core and the equilibrium states of the economy. Note that the following result does not require that all states of the economy in the core have the equal-treatment property and does not require (A.9).

Theorem 5. *Equivalence of the equal-treatment core and the equilibrium states of the economy.* Let (π, X, Y, Z) be an equal-treatment core state of the economy satisfying (A.3)-(A.6), and (A.8). Then there is a price system p for private goods and a price system ψ for public goods such that (π, X, Y, Z), p and ψ constitute an equilibrium.

Proof of Theorem 5. See the Appendix. ■

Theorem 6. *Existence of equilibrium.* Assume that the economy satisfies (A.1)-(A.7), and (A.8). Then there exists an equilibrium for the economy.

Proof of Theorem 6. From Theorem 1 the equal-treatment core is nonempty. From Theorem 5 every state of the economy in the equal-treatment core is an equilibrium state. Thus, an equilibrium exists. ■

The following Theorem concludes our results.

Theorem 7. *Core-equilibrium equivalence.* Assume that the economy satisfies (A.1)-(A.9). Then the equilibrium exists and the set of equilibrium states of the economy is equivalent to the core.

Proof of Theorem 7. This is immediate from Theorem 1, Theorem 4, and Theorem 5. ■

3 Some further remarks on the literature

Before concluding, we contrast our work to some other approaches to continuum economies with small effective groups.

Remark 7 The f-core approach was initially developed for games in characteristic form. These models encompasses games derived from a variety of economies, including economies with local public goods as the current paper illustrates. The level of abstraction of games in characteristic form which permits the framework to encompass diverse economic situations, however, does not permit the analysis of economic equilibrium in specific environments. In Hammond et al. (1989) and Kaneko and Wooders (1986), the authors extend the f-core approach to pure exchange economies with wide-spread externalities – externalities relative to which individual agents are negligible – and show equivalence of the f-core to the competitive allocations (and non-equivalence of the Aumann-core).[14] In contrast, this paper is first application of the f-core framework to a Tiebout economy, an environment in which finite membership coalitions are particularly compelling.

[14]Since the authors do not consider competitive equilibrium, their work is less related, but we note that Legros and Newman (1996) apply the f-core nonemptiness to study certain sorts of partnerships.

Remark 8 A feature of our model is that there are only a finite number of types of consumers. This allows us to use the results of Kaneko and Wooders (1986). It is apparent that, using the results of Kaneko and Wooders (1996), the same results as those of this paper can be obtained with a measure space of consumer types. The only major difference would be that we would then require a bound on jurisdiction sizes; otherwise the assumptions of the economic model could remain unchanged, and essentially the same proof could be used. At the cost of stronger restrictions on the economic model, by resorting to further unpublished research due to Kaneko and Wooders, we could allow a measure space of consumer types and unbounded jurisdiction sizes but of course are the cost of further restrictions of other sorts on the economic model. Specifically, some sort of limited "side paymentness" would be required.

Remark 9 There are some relationships between this paper and Ellickson et al. (1997). Recall that Ellickson (1979) treats local public goods as indivisible private commodities. Ellickson et al. (1997) adopt the approach of Ellickson (1979) to study a Buchanan economy. By fixing a finite menu of admissible sorts of club types and allowing only a finite number of distinct public projects, Ellickson et al. are able to adopt techniques from finite-dimensional private goods exchange economies to prove existence and equivalence. In Ellickson et. al. agents are permitted to join several clubs as opposed to requiring that agents join one and only one jurisdiction as would be appropriate in a local public goods context. In this respect, their paper is more general than the current work, although extending our model to include multiple memberships does not appear to present any great difficulties. Like us, (and in contrast to Ellickson 1979) Ellickson et al. treat the group of agents consuming a public project as part of the description of the project. Ellickson et al. show that the core and equilibrium exist and are equivalent. They assert, and illustrate by an example, that a second welfare theorem does not hold for their model. In contrast, preliminary results indicate that a second welfare theorem does hold for the model described in this paper.

There are two major differences between the approach of Ellickson et al. and ours. The first relates to the treatment of crowding. Ellickson et al. describe agents using a single attribute which determines their tastes, endowments and external crowding effects. In other words, tastes and crowding effects are never treated separately and so this model is in the tradition of standard differentiated crowding economies (as in Wooders 1985,1997, for example) and not in the crowding types tradition of the current paper. This feature means that the equilibrium prices that Ellickson et al. define are not anonymous since they must discriminate over agents based on an index which identifies them by tastes.

The second major difference is that Ellickson et al. allow only a finite menu of possible club types. Thus, some types of clubs with finite membership which would be feasible given the population are not allowed to form either in the core or in equilibrium. As a result, the equilibrium Ellickson et al. describe is Pareto efficient, but only over the exogenously fixed, finite menu of admissible clubs. Their equilibrium is not necessarily Pareto efficient over all feasible, finite clubs. In contrast, we allow an uncountably infinite set of public projects; standard continuously divisible Euclidean public goods are a special case. We allow clubs to be arbitrarily large (but finite) so club sizes are unbounded. Thus, our equilibrium notion is Pareto efficient over the set of all clubs with finite membership.

The differences between these two approaches makes a substantial difference in the techniques of proof. The restrictions on the model of Ellickson et al. plus an extension of the notion of measurement-consistency allow them to use techniques similar to those of finite-dimensional exchange economies with some indivisible commodities, as in Hammond, Kaneko and Wooders (1989), for example. Our method of proof is completely different. The proof that the core is nonempty is based on nonemptiness of the f-core of Kaneko and Wooders (1986), itself rooted in nonemptiness of approximate cores of large games of Wooders (1983). Our equivalence and existence result are outgrowths of equivalence results for large finite economies. Rather than treating clubs as indivisible commodities, we treat clubs as groups that internalize externalities.

Remark 10 We conclude by noting an important difference between the approach of this paper, allowing public projects with minimal assumptions on production, and the approach of Wooders (1985,1997).[15] Recall that Wooders' model required that production sets for public goods be closed convex

[15] Wooders (1985,1997) use the same proof of convergence, except the later paper shows that the prices for public goods derived in the proofs of the earlier papers are Lindahl and also determine admission prices. Conley and Wooders (1994) discusses the differences between Lindahl pricing and admission pricing.

cones and that pricing was differentiated – that is, prices for public goods were based on agents' types where "'type" included taste type. Our model does not require these restrictions. In the course of our proof, following Wooders's earlier papers, we define preferred sets of net trades of private goods for jurisdictions, Wooders obtained existence of equilibrium prices for private goods by separating the preferred sets of jurisdictions from the origin. We also use such a separating hyperplane argument, but we separate only preferred sets for jurisdictions in the core from the origin. From the prices for private goods thus determined, we are able to construction prices for public goods for all jurisdictions. To the best of our knowledge, there is no precedent for this technique.

4 Conclusions

The point of this paper has been to confirm Tiebout's hypothesis that when public goods are local, markets are able to decentralize the efficient outcomes. Traditionally, there has been a trade-off in the literature. On the one hand, it is has been widely demonstrated that, without special assumptions, both the core and competitive equilibrium may fail to exist in finite economies. Defining equilibrium notions that can be shown to exist, on the other hand, typically involves restricting agents' alternatives to a subset of all feasible allocations (for example, requiring that agents always share the cost of public goods equally or restricting the menu of admissible clubs to an arbitrary, finite subset of the feasible set). Thus, although these equilibrium exist, at best they are only to be Pareto optimal within the constraints imposed and not over the whole feasible set. It is often the case, however, that many other equilibria also exist which do not even satisfy this constrained optimality

The intuition for our results comes the ϵ-equilibrium Tiebout literature. This literature suggests that the failure of existence is caused by the presence of a group of "left-over" agents who cannot find placement in optimal jurisdictions. In the continuum limit, the problem of left-over agents disappears. Extending the intuition of ε-cores to economies with a continuum of agents creates both technical and intuitive problems. Much of the literature supposes that agents end up in an finite number of infinitely large jurisdictions in equilibrium. This reflects neither everyday observation, nor is it the limiting case of the ϵ-equilibrium.

These considerations motivate our use of an f-core approach in an environment with local public goods and production of both public and private goods. This economy allows the multiple private goods to be traded freely across jurisdictional boundaries, but requires that crowding and public projects be consumed only within jurisdictions. The space of public projects we consider is quite abstract and does not require any linear structure, but includes standard infinitely divisible public goods as a special case. The most innovative part of this paper is showing the equivalence of the equal-treatment core and the competitive outcomes, thereby obtaining existence of equilibrium in a novel way.

Our main result is that under fairly standard conditions on production and preferences, the core is nonempty and is equivalent to the set of anonymous admission price equilibrium outcomes. Thus, Tiebout's hypothesis is confirmed in the sense that save for at most negligible fraction of agents, competitive equilibria exist and are first best.

There are several ways in which the research of this paper might be furthered. In particular, what prevents the proving a second welfare theorem in the case of a finite Tiebout economy is the general failure of existence of competitive equilibrium. Thus, we speculate that it should be possible to prove a second welfare theorem in the generality of our model. We have also treated crowding characteristics as exogenously given (for example gender, race or intelligence might be externality producing characteristics which are exogenous to agents). It would be interesting to extend this model to endogenously chosen externality producing characteristics like skills or being a smoker as in Conley and Wooders (1997a), for example. Finally, it should be possible to prove results similar to those given in this paper when agents are allowed to join more than one club at a time. The modeling challenge, as we perceive it, is to maintain measurement-consistency while allowing agents to join an arbitrary number of clubs, each of which may be able to produce an arbitrary level of public goods.

5 Appendix

For the convenience of the reader, we first list the assumptions:

(A.1)

(a) $\mathcal{X}_{ct}$ is closed in $\mathbf{R}^L \times \mathcal{X}^{public} \times \mathcal{Z}_{ct}$ and

(b) for any nonempty subset $X' \subset \mathcal{X}^{private}$ the vector a, defined by its components $a_\ell = \inf\{a'_\ell : a' \in X'\}$ for $\ell = 1, ..., L$ is in $\mathcal{X}^{private}$.

(A.2) $(\omega^0(i), \underline{0}, pro(\{i\})) \in \mathcal{X}_{ct}$ for all $i \in N_{ct}$.

(A.3) u_t is continuous.

(A.4) *Taste anonymity in consumption* (TAC): For all $x \in \mathcal{X}^{private}$, $y \in \mathcal{X}^{public}$ and all $G, G' \in \mathcal{F}$ such that $(x, y, pro(G)), (x, y, pro(G')) \in \mathcal{X}_{ct}$ and $\mathcal{C}pro(G) = \mathcal{C}pro(G')$ it holds that $u_t(x, y, pro(G)) = u_t(x, y, pro(G'))$.

(A.5) Given a jurisdiction $G \in \mathcal{F}$ and an input vector $z' \in \mathbf{R}^L$, the set $\{y : (z', y) \in P(pro(G))\}$ is compact.

For our price system to be Pareto-optimal, we require taste anonymity in production as well as in consumption.

(A.6) *Taste anonymity in production* (TAP): For all $G, G' \in \mathcal{F}$ such that $\mathcal{C}pro(G) = \mathcal{C}pro(G')$ it holds that $P(pro(G)) = P(pro(G'))$.

(A.7) The economy is *per capita bounded* if there are positive numbers δ and K, with $0 < \delta < 1$, such that for every finite subset $S \subset N$ and for any feasible state (π, X, Y, Z) of S with the equal treatment property,

$$(1+\delta)\tfrac{\mu(N_{ct})}{\mu(N)} \geq \tfrac{|S_{ct}|}{|S|} \geq (1-\delta)\tfrac{\mu(N_{ct})}{\mu(N)} \text{ for all } (c,t) \in \mathcal{C} \times \mathcal{T}$$

$$\Rightarrow u_{\tau(i)}(X(i), Y(i), pro(\pi_i)) \leq K \text{ for all } i \in S.$$

(A.8) *Divisibility and monotonicity.* Private goods are infinitely divisible and utility functions are strictly monotonic increasing in private goods.

(A.9) *Desirability of the endowment of divisible private goods.* For each agent i, it holds that:

$$u_{\tau(i)}(\omega^0(i), \underline{0}, pro(\{i\})) > u_{\tau(i)}(x, y, pro(S))$$

where x is any private goods bundle containing none of the divisible private goods.

Theorem 1. *Nonemptiness of the core.* Under assumptions (A.1)-(A.3), (A.5), and (A.7), the equal-treatment core of the economy is nonempty

Proof of Theorem 1. The proof of this Theorem follows by showing that the conditions required for Kaneko and Wooders (1986,Theorem 1), demonstrating the nonemptiness of the f-core of a continuum game, are satisfied by the game induced by the economy. First, we will review the model and statement of Theorem 1 of Kaneko and Wooders (1986).

Given the measure space of players N, a *characteristic function game* V *without side payments* is a correspondence on $\mathcal{F}$ which assigns to each coalition $S \in \mathcal{F}$ a subset $V(S)$ with the following properties:

1. $V(S)$ is a nonempty closed subset of $\mathbf{R}^S$ for all $S \in \mathcal{F}$;
2. $V(S) \times V(W) \subset V(S \cup W)$ for any $S, W \in \mathcal{F}$ with $S \cap W = \emptyset$;
3. $\inf_{i \in N} \sup V(\{i\}) > -\infty$;
4. for any $S \in \mathcal{F}$, $V(S)/ \cup [interior V(\{i\}) \times R^{S-\{i\}}]$ is nonempty and bounded.

A characteristic function game can be generated by an economy in the usual way. Specifically, given a finite coalition S, define

$V(S) = \{\overline{u} \in \mathbf{R}^S :$ for some feasible state of the economy for S, say (π, X, Y, Z), it holds that for each $i \in S$, $u_{\tau(i)}(X(i), Y(i), \mathcal{C}pro(\pi_i)) \geq \overline{u}_i\}$.

Condition 1. is satisfied for the game derived from the economy from closedness of the consumption sets $\mathcal{X}_{ct}$,(A.1), continuity of utility functions (A.3), and closedness of the production possibility sets. Condition 2. is immediate since one possibility open to a coalition (or a jurisdiction) consisting of $S \cup W$, $(S \cap W = \emptyset)$ is to form a partition into disjoint coalitions S and W. Condition 3. is also immediate since there are only a finite number of types and since the supremum of a finite set of real numbers is a real number. Condition 4 is simply that the set of feasible and individually rational payoffs is bounded above. This follows from the assumptions that a finite amount of private goods can produce only a bounded amount of public and private goods (A.5). Kaneko and Wooders also require that collection of finite games satisfies per capita boundedness, which we have imposed directly.

The nonemptiness Theorem stated in Kaneko and Wooders (1986) does not mention the equal-treatment property. Their result is proven, however, by showing existence of an equal-treatment f-core payoff that is the limit of equal-treatment payoffs in finite approximating games. We will briefly sketch the result.

Wooders (1983) shows that sequences of games with types satisfying per capita boundedness have nonempty approximate cores. This result is based on the result that, in large games, when all "improvement" can be carried out by coalitions bounded in size and the bound is small relative to the economy, provided payoff sets do not contain segments parallel to the axes – called strong comprehensiveness or nonlevelness– then *all* payoffs in the core of a (finite) game have the equal treatment property. Moreover, even without strong comprehensiveness, under these conditions the core, when nonempty, contains an equal-treatment payoff. Building on these results, Shubik and Wooders (1983) establish that for any sequence of games satisfying per capita boundedness, eventually there are equal treatment payoffs in approximate cores. Kaneko and Wooders (1986) use this result to show that there is a sequence of vectors $\{u^\nu\}$, where $u \in \mathbf{R}^{CT}$ (taking CT as the number of types of players) represents an equal treatment payoff in the core of a finite game with proportions of players in the finite games converging to the proportions in the continuum limit game. ¿From per capita boundedness, the sequence $\{u^\nu\}$ has a converging subsequence, converging to, say u^*. The function $f : N \rightarrow \mathbf{R}$ defined by $f(i) = u^*_{ct}$ when i is of type (c,t) is in the core of a continuum limit game. ■

Theorem 2. *An equilibrium state of the economy is in the core.* Assume (A.4) and (A.7). If a feasible state of the economy (π, X, Y, Z) and price systems p and ψ constitute an equilibrium, then (π, X, Y, Z) is in the core.

Proof of Theorem 2. Suppose that a feasible state of the economy (π, X, Y, Z), a price system $p \in \mathbf{R}^L$ for private goods and a price system ψ for public goods constitute an equilibrium. Note that the state of the economy (π, X, Y, Z) has the equal treatment property – that is, almost all agents of the same type are treated equally. Suppose the equilibrium is not in the core. Then there is at least one finite coalition, say M, that can improve upon (π, X, Y, Z) for its members. Moreover, M can be chosen so that for each type (c,t) represented in M it holds that $\mu(N_{ct}) > 0$, and that all agents of the same type in

M are treated equally. From Kaneko and Wooders (1986, Lemma A.1) and the fact that (π, X, Y, Z) has the equal-treatment property it follows that there is a coalition S of positive measure ($\mu(S) > 0$) containing the same proportion of agents of each type as M, a feasible state of the economy for S, say (π', X', Y', Z'), and a positive real number δ such that for almost all agents $i \in S$ it holds that

$$u_{\tau(i)}(X'(i), Y'(i), pro(\pi_i')) > u_{\tau(i)}(X(i), Y(i), pro(\pi_i)) \text{ and}$$

$$p \cdot X'(i) + \psi_{\kappa(j)}(Y'(i), \mathcal{C}pro(\pi_i')) \quad > \quad p \cdot \omega^0(i) + \delta.$$

[Since, for each type (c, t) represented in M, all players of type (c, t) in M have the same equilibrium utility levels and since the (π, X, Y, Z) has the equal treatment property, we may select a collection of disjoint subsets, say $\{M_i\}_{i \in M}$, so that all players in M_i are the same type as i and so that $\mu(M_i) = \mu(M_{i'})$ for each $i, i' \in M$. From Kaneko and Wooders (1986, Lemma A.2), there is a one-to-one measure-preserving isomorphism (mod 0) between any two sets M_i and $M_{i'}$ in the collection. Let $S = \cup_{i \in M} M_i$. Let π' be a partition consisting of one agent from each set M_i and singleton sets, so that the measure of the set of agents in coalitions with the same profiles as M is $\mu(S)$. Those coalitions with the same profile as M can all improve upon the state of the economy.]

¿From the definition of a feasible state of the economy, there is a sequence of admissible coalition-jurisdiction structures, say $\{(\xi^\nu, \pi')\}_\nu$, and, for each ν, a feasible state $((\xi^\nu, \pi'), X^\nu, Y^\nu, Z^\nu) \in F(\pi')$ relative to (ξ^ν, π') such that $\{(X^\nu, Y^\nu, Z^\nu)\}_\nu$ converges in measure to $(X'(i), Y'(i), Z'(i))$. It follows from continuity of preferences that for some subset S^0 of positive measure of S, for all v sufficiently large, for each coalition $W \subset S^0$, $W \in \xi^\nu$,

$$p \cdot \sum_W X^\nu(i) + \sum_W \psi_{\kappa(j)}(Y^\nu(i), \mathcal{C}pro(\pi_i') > p \cdot \sum_W \omega^0(i) + \frac{\delta}{2}$$

and

$$u_{\tau(i)}(X^\nu(i), Y^\nu(i), pro(\pi_i') > u_{\tau(i)}(X(i), Y(i), pro(\pi_i))$$

for all $i \in S^0$. From the condition of equilibrium that profits are nonpositive, for almost all $i \in S$,

$$\sum_{j \in \pi'_i} \psi_{\kappa(j)}(Y'(j), \mathcal{C}pro(G)) - p \cdot \sum_{j \in \pi'_i} Z'(j) \leq 0$$

It follows that for all ν sufficiently large, for all $i \in S^0$,

$$\sum_{j \in \pi'_i} \psi_{\kappa(j)}(Y^{\nu}(j), \mathcal{C}pro(j)) - p \cdot \sum_{j \in \pi'_i} Z^{\nu}(j) \leq \frac{\delta}{2}.$$

>From the above expressions it now follows that for all ν sufficiently large and for all coalitions $W \subset S$, $W \in \xi^{\nu}$,

$$p \cdot \sum_W \omega^0(i) + \tfrac{\delta}{2} < p \cdot \sum_W X^{\nu}(i) + \sum_W \psi_{\kappa(i)}(Y^{\nu}(i), \mathcal{C}pro(\pi'_i))$$

$$\leq p \cdot \sum_W X^{\nu}(i) + p \cdot \sum_W Z^{\nu}(i) + \tfrac{\delta}{2}.$$

But, from feasibility, for almost all $W \in \xi^{\nu}$ it holds that

$$\sum_W \omega^0(i) \geq \sum_W X^{\nu}(i) + \sum_W Z^{\nu}(i)$$

and therefore

$$\frac{\delta}{2} + p \cdot \sum_W \omega^0(i) \geq p \cdot \sum_W X^{\nu}(i) + p \cdot \sum_W Z^{\nu}(i) + \frac{\delta}{2} > p \cdot \sum_W \omega^0(i) + \frac{\delta}{2}.$$

the desired contradiction. ■

Theorem 4. *The equal treatment property of the core.* Assume (A.1)-(A.4), (A.5), and (A.8)-(A.9). Let (π, X, Y, Z) be a core state of the economy. Then there is a subset $N^0 \subset N$, $\mu(N^0) = \mu(N)$, such that for every pair of agents $i, j \in N^0$ satisfying $\tau(i) = \tau(j)$ and $\kappa(i) = \kappa(j)$ it holds that

$$u_{\tau(i)}(X(i), Y(i), pro(\pi_i)) = u_{\tau(j)}(X(j), Y(j), pro(\pi_j)).$$

Proof of Theorem 4. The proof of this Theorem is an adaptation of the equal treatment result of Wooders (1983,Theorem 3) for games, itself an adaptation of an argument due to Debreu and Scarf (1963) for economies. In

Wooders' proof, if all gains to improvement can be exhausted by coalitions smaller than the number of players of each type and if two agents in different jurisdictions are treated unequally, the less-well treated agent could "replace" the better off agent, and make all players in the new coalition better off. Intuitively, in the model of the current paper, since all gains to coalition formation are exhausted by finite coalitions, "in the limit" the same ideas can be used in the proof of the current theorem.

Let (π, X, Y, Z) be a core state of the economy. Suppose that there is a type (c', t'), a positive real number $\gamma > 0$ and two measurable subsets of players $S^1_{c't'}$ and $S^2_{c't'}$ such that

1. for $k = 1, 2,$ $\mu(S^k_{c't'}) > 0$ and $S^k_{c't'} \subset N_{c't'}$ and
2. if $i \in S^1_{c't'}$ and $j \in S^2_{c't'}$ then

$$u_{\tau(i)}(X(i), Y(i), pro(\pi_i)) + \gamma < u_{\tau(j)}(X(j), Y(j), pro(\pi_j)).$$

¿From the definition of feasible states of the economy there is a sequence of coalition-jurisdiction structures $\{(\xi^\nu, \pi)\}_\nu$ and, for each ν, a feasible state $((\xi^\nu, \pi), X^\nu, Y^v, Z^\nu)$ relative to (ξ^ν, π), such that the sequence $\{(X^\nu, Y^v, Z^\nu)\}$ converges in measure to (X, Y, Z). Since $S^1_{c't'}$ and $S^2_{c't'}$ have positive measure, for each ν we may select two disjoint subsets of positive measure, say S^ν_1 and S^ν_2, such that for disjoint subcollections $\xi^{1\nu}$ and $\xi^{2\nu}$ of ξ^ν, it holds that:

1. each member of $\xi^{1\nu}$ contains a member of $S^1_{c't'}$,
2. each member of $\xi^{2\nu}$ contains a member of $S^2_{c't'}$,
3. $S^\nu_1 = \cup_{i \in W,\, W \in \xi^{1\nu}} W$,
4. $S^\nu_2 = \cup_{i \in W,\, W \in \xi^{2\nu}} W$, and
5. for some $\delta > 0$, $\mu(S^1_{c't'} \cap S^\nu_1) = \mu(S^2_{c't'} \cap S^\nu_2) > \delta$.

Define $W^\nu = S^1_{c't'} \cap S^\nu_1$ and $B^\nu = S^2_{c't'} \cap S^\nu_2$ (for "W" is for "worse-off" and B is for "better off.") Define $S^{*\nu} = S^\nu_2 \cup W^\nu \backslash B^\nu$. Now from Kaneko and Wooders (1986, Lemma A.1) there is a measure preserving isomorphism

φ from W^ν to B^ν. Using such a mapping and its inverse, we can "switch places" between members of B^ν and members of W^ν in a one-to-one, measure-preserving way (mod 0). This induces a coalition-jurisdiction structure of $S^{*\nu}$ – the same as $\xi^{\nu 2}$ except that agents in B^ν are replaced by agents in W^ν. Denote the resulting jurisdiction structure by π^ν,the coalition structure of $S^{*\nu}$ by $\xi^{*\nu}$, and the state of the economy by $((\xi^{*\nu}, \pi^\nu), X^{*\nu}, Y^{*v}, Z^{*\nu})$.

From (A.8) and (A.9) it follows that in the state of the economy (π, X, Y, Z) almost all agents are allocated some (divisible) private good. Thus, for all sufficiently large ν, it follows that in the state of the economy $((\xi^{*\nu}, \pi^\nu), X^{*\nu}, Y^{*v}, Z^{*\nu})$ almost all agents are allocated some positive amount of divisible private goods. Therefore almost all agents in $S^{*\nu}$ are allocated some positive amounts of the divisible private goods. It follows that there is a feasible state of the economy for $S^{*\nu}$ in which all members of $S^{*\nu}$ are better off than in $((\xi^\nu, \pi), X^\nu, Y^v, Z^\nu)$ – the agents in W^ν can each make a small transfer of divisible private goods to all other members of the coalition in which they are contained. Moreover, from continuity and the fact that for (X, Y, Z) is the limit, with respect to convergence in measure, of $\{(X^\nu, Y^v, Z^\nu)\}_\nu$, all sufficiently large ν the members of $S^{*\nu}$ can be made better off by some amount that is bounded away from zero. Since this can be done for each v, we have a contradiction to the supposition that (π, X, Y, Z) is a core state of the economy. ■

Sketch of the proof of Theorem 5. In Step 1, we obtain the existence of a price system, say p^ν, for private goods for a finite approximating economy. To obtain this result, we follow a technique arising from Debreu-Scarf (1963), of separating preferred sets from the origin. Instead of considering the preferred sets of individual consumers as in Debreu-Scarf (1963) and Foley (1970), we consider preferred sets for jurisdictions, as in Wooders (1985,1997) and treat the preferred sets of private goods for jurisdictions, for all jurisdictions in π, with numbers of members bounded by an arbitrary positive integer for an arbitrary positive integer ν. A point in the preferred set of a jurisdiction is an amount of private goods sufficiently large so that, with this amount of resources, it is possible for all members of the jurisdiction to be better off than they are in the state of the economy in the core. We then let ν become large, thus obtaining a sequence $\{p^\nu\}$ of prices. Since, without loss of generality, we normalize prices to lie in the simplex, the sequence $\{p^\nu\}$ has a converging subsequence. Let p denote the limit of this subsequence. In

Step 2, it is verified that p satisfies the conditions of the Theorem. Finally, in Step 3, using the same techniques as in Conley and Wooders (1997) for a one-private-good case, the price system ψ is constructed from the price system p. Step 4 completes the proof by showing that all the properties of a competitive equilibrium are satisfied.

Theorem 5. *Equivalence of the equal-treatment core and the equilibrium states of the economy.* Let (π, X, Y, Z) be an equal-treatment core state of the economy satisfying (A.3)-(A.6), and (A.8). Then there is a price system p for private goods and a price system ψ for public goods such that (π, X, Y, Z), p and ψ constitute an equilibrium.

Proof of Theorem 5.

Preliminaries. Let (π, X, Y, Z) be an equal-treatment state of the economy in the core. From the equal treatment property, in (π, X, Y, Z) almost all agents of the same type receive the same utility levels. For an agent of type (c, t) let $u(c, t)$ denote this utility level.

Let N^0 be a subset of N with the property that $\mu(N^0) = \mu(N)$ and, if $i \in N^0$ and $(\kappa(i), \tau(i)) = (c, t)$ then there is an infinite number of agents in N^0 of type (c, t).

Step 1. Given a positive integer ν, let $\mathcal{G}^\nu$ denote the collection of all jurisdictions G in π and also contained in N^0 with $|G| < \nu$. For each $G \in \mathcal{G}^\nu$ let Ω^ν_G denote the set of private goods bundles b in $\mathbf{R}^L$ with the properties that, for each $i \in G$, there is an $x^i_G \in \mathcal{X}^{private}$ such that:

$$u_{\tau(i)}(x^i_G, Y(i), pro(G)) > u_{\tau(i)}(X(i), Y(i), pro(G)) = u(c, t) \text{ and}$$

$$b = \sum_{i \in G} (x^i_G - Z(i) - w^0(i)).$$

The set Ω^ν_G is a subset of $\mathbf{R}^L$, called the *preferred set for* G. For jurisdiction G, Ω^ν_G describes the set of required aggregate net trades (or transfers) of private goods with the property that there is some production of public goods and some distribution of private goods so that, for each member of the jurisdiction, the given allocation is preferred to the allocation which he is assigned in the initially given state of the economy (π, X, Y, Z). (Note that Ω^{ν}_G may be empty.)

Let Ω^{ν} denote the convex hull of the union $\cup_{G\in\mathcal{G}^{\nu}}\Omega^{\nu}_G$. We next show that $0 \notin \Omega^{\nu}$. First, suppose that $0 \in \Omega^{\nu}$. Then, since Ω^{ν} is the convex hull of the sets $\{\Omega^{\nu}_G\}$, for each $G \in \mathcal{G}^{\nu}$ there exists a convex combination λ^{ν}_G of weights ($0 \leq \lambda^{\nu}_G \leq 1$ and $\sum_G \lambda^{\nu}_G = 1$), and for all G with $\lambda^{\nu}_G > 0$, x^i_G for each $i \in G$, such that:

(a) $0 = \sum_G[\lambda^{\nu}_G(\sum_{i\in G}\ (x^i_G - Z(i) - \omega^0(i)))]$, and

(c) $u_{\tau(i)}(x^{i\nu}_G, Y(i), pro(G)) > u_{\tau(i)}(X(i), Y(i), pro(G))$.

From continuity of preferences, (A.3), and (A.8), the sets Ω^{ν}_G are open. Therefore, if any of the weights λ^{ν}_G are irrational, we can perturb the allocations of private goods x^i_G so that (a) and (b) are satisfied with rational weights. Thus, we suppose, without loss of generality, that the weights λ^{ν}_G are all rational numbers.

It holds that for some $\zeta^{\nu} > 0$,

$$\zeta^{\nu} \sum_{G\in\mathcal{G}^{\nu}} \lambda^{\nu}_G pro(G) \quad \leq \quad (\mu(N^0_{11}), ..., \mu(N^0_{CT})).$$

For each (c,t), let N'_{ct} be a subset of N with $\mu(N'_{ct}) = \zeta^{\nu}\sum_{G\in\mathcal{G}^{\nu}}\lambda^{\nu}_G\,|G_{ct}|$ and such that in the state of the economy (π, X, Y, Z) all members of N'_{ct} are treated equally. Let W denote a finite set of agents of agents with the same proportions of agents of each type (c,t) as $\cup_{ct}N'_{ct}$ and sufficiently large so that for some nonnegative integer r,

$$pro(W) = r\sum_{G\in\mathcal{G}^{\nu}}\zeta^{\nu}\lambda^{\nu}_G pro_{ct}(G).$$

¿From Kaneko and Wooders (1986, Lemma A.2), the construction of W and from (a) above, for the coalition W, a state of the economy described by (a)-(b) above is feasible. The set $\cup_{ct}N'_{ct}$ can be partitioned into coalitions W' so that, for each coalition W', the proportion of jurisdictions in the coalition with profile $pro(G)$ is $\frac{\mu(N'_{ct})}{\sum_{c't'}\mu(N'_{c't'})}$ and so that for each coalition W', $pro(W') = pro(W)$. The equation $\sum_G[\lambda^{\nu}_G(\sum_{i\in G}\ (x^i_G - Z(i) - \omega^0(i)))] = 0$ implies that $r\zeta^{\nu}\sum_G[\lambda^{\nu}_G(\sum_{i\in G}\ (x^i_G - Z(i) - \omega^0(i)))] = 0$. Therefore each coalition W' can improve upon the state of the economy (π, X, Y, Z) for its members and we have a contradiction to the supposition that (π, X, Y, Z) is in the core. Thus, $0 \notin \Omega^\nu$.

¿From the fact that Ω^ν is convex and $0 \notin \Omega^\nu$ it follows that there exists a price system p^ν for private goods which separates the preferred sets for jurisdictions G in π with $|G| < \nu$ from their affordable net trades of private commodities. ¿From (A.8) it follows that $p^\nu_\ell > 0$ for each private good ℓ.

Step 2. A private-goods price system for the continuum economy.

We assume without loss of generality that, for each ν, p^ν is in the unit simplex. Thus, we may suppose, by passing to a subsequence if necessary, that the price system $\{p^\nu\}$ converges, say to p.

Let π^0 denote the jurisdiction structure π restricted to agents contained in N^0. We now show that p satisfies the properties of an equilibrium price system for private goods that (a) for almost all agents $i \in N^0$ it holds that the jurisdiction π_i can afford its bundle of private goods and (b) except possibly for a set of measure zero, no jurisdiction G of consumers could be better off given prices p. Observe that for each $G \in \pi^0$ there exists a ν_0 such that $\nu_0 > |G|$. For each $\nu \geq \nu_0$, from the definition of Ω^ν and since Ω^ν is open, it holds that for any $i \in G$, $\sum_{j \in \pi_i^0} [X(j) + Z(j) - \sum_{j \in \pi_i^0} \omega^0(j)]$ is in the closure of Ω^ν_G. Thus, it follows that

$$p \cdot \sum_{j \in \pi_i^0} (X(j) + Z(j)) \geq p \cdot \sum_{j \in \pi_i^0} \omega^0(j).$$

¿From the separating hyperplane property of p, it follows that for almost all agents $i \in N^0$,

$$\sum_{j \in \pi_i^0} (X(j) + Z(j)) \leq \sum_{j \in \pi_i^0} \omega^0(j).$$

Thus, for almost all agents $i \in N^0$,

$$p \cdot \sum_{j \in \pi_i^0} (X(j) + Z(j)) = p \cdot \sum_{j \in \pi_i^0} \omega^0(j).$$

Thus, for almost all agents $i \in N^0$, the jurisdiction π_i can afford the allocation for its members given by (π, X, Y, Z), so (a) is true. Claim (b) follows from the construction of p. For suppose (b) does not hold. Then if follows that, for a sufficiently large ν, there is a jurisdiction G for which (b) does not hold, a contradiction to the separating hyperplane property of p^ν.

Step 4. A public-goods price system. We must now construct prices for public goods. Here we follow the techniques of Conley and Wooders (1997). Although this paper only has one private good, once prices for private goods are given the problem becomes quite similar to the problem in the one-private-good case. The following Lemma is that analogue of Conley and Wooders (1997, Theorem 2).

Lemma 1. Let (π, X, Y, Z) be a state of the economy in the core. Let p be as determined above. Then for any $c \in \mathcal{C}$ and any pair of agents $i \in \pi_i$, $j \in \pi_j, \kappa(i) = \kappa(j) = c$, and $\pi_i = \pi_j$ it holds that

$$p \cdot (\omega^0(i) - X(i)) = p \cdot (\omega^0(j) - X(j)).$$

Proof of Lemma 1. Suppose not. Suppose that

$$p \cdot (\omega^0(i) - X(i)) > p \cdot (\omega^0(j) - X(j)).$$

¿From Theorem 4 and from the fact that jurisdictions are finite (and thus, of measure zero), there is an agent $i' \notin \pi_i$ such that $\kappa(i') = \kappa(i)$, $\tau(i') = \tau(i)$ and $u_{\tau(i')}(X(i'), Y(i'), \pi_{i'}) = u_{\tau(i)}(X(i), Y(i), \pi_i)$. Now consider the jurisdiction $G = \pi_i \cup \{i'\} \backslash \{j\}$ and the allocation for G with consumption of public goods equal to $Y(i)$ for each agent and with consumptions of private goods of $X(i'')$ for all $i'' \in G$, $i'' \neq i, i'$, and for i and i', the consumption of private goods of $X(i)$. Note that the jurisdiction G can afford this allocation of private goods and the required input of private goods into production. From (A.7)-(A.8), and the strict monotonicity of preferences for divisible private goods, the membership of G can afford a bundle of private goods for private consumption and for public good production that would make all members of G better off than they are in the initial state of the economy (π, X, Y, Z). This contradicts the fact that p separates preferred sets from affordable bundles of private goods.

QED

For each agent i, define

$$\psi_{\kappa(i)}(Y(i), \mathcal{C}pro(\pi_i)) = p \cdot (\omega^0(i) - X(i)).$$

It remains to specify prices for public goods for jurisdictions and levels of public goods that do not appear in the core state.

Consider an arbitrary jurisdiction $G \in \mathcal{F}$ and an arbitrary $c \in \mathcal{C}$ with the property that $\mathcal{C}pro(G)_c \neq 0$. Let $(z, y) \in P(pro(G))$ and suppose that there does not appear a jurisdiction G' with $pro(G') = pro(G)$ offering public goods level y in the core state of the economy. Now take an arbitrary agent $i \in G$ of type c (or any agent i in any jurisdiction G' with the same profile) and consider how much he would be willing to pay to join this jurisdiction. There are three possibilities. First, it may be that this jurisdiction is so attractive to agent i that even when he pays the most it is feasible for him to pay, (that is: $p \cdot \omega^0_{\tau(i)}$) he is still better off than he is in the core state. Second, it may be that this jurisdiction and public good package is so unattractive that if the agent were a member of this jurisdiction, no amount of income could make him as well off as he was in the core state. Third, it may be that there is an amount of income $(p \cdot \omega^0(i))$ which is feasible for the agent to pay, and which leaves him exactly indifferent between this jurisdiction and the one he is assigned in the core state. This creates a partition of agents which we define formally thus:

$$A_c(y, pro(G)) := \{i \in N_c : (a, y, pro(G)) \succ_{\tau(i)} (X(i), Y(i), \pi_i)\},$$ [16]

$$U_c(y, pro(G)) := \{i \in N_c : \forall x \in \mathcal{X}^{private}, (x, y, pro(G)) \prec_{\tau(i)} (X(i), Y(i), \pi_i)\}, \text{ and}$$

$$I_c(y, pro(G)) := \{i \in N_c : \exists x \in \mathcal{X}^{private} \text{ such that } (x, y, pro(G)) \sim_{\tau(i)} (X(i), Y(i), pro(\pi_i)\}.$$

For agents in the set $I_c(y, pro(G))$, the "willingness to pay" to join the jurisdiction G is well-defined. Define the maximum willingness to pay over all such agents (that is, the maximum over all taste types of crowding type c represented in the jurisdiction G) as follows:

$$MaxWTP_c(y, pro(G)) = \max_{i \in I_c(y, pro(G))} \sup_{x \in \mathcal{X}^{private}} \{p \cdot \omega^0(i) - p \cdot x : (x, y, pro(G)) \sim_{\tau(i)} (X(i), Y(i), pro(\pi_i))\}$$

[16]Recall that a is the component-wise infimum of the consumption set which is an element of $\mathcal{X}^{private}$ by assumption (A.1). The bundle a has minimal value in the consumption set under any set of nonnegative private good prices.

Given any $\varepsilon > 0$, we can think of $MaxWTP_c(y, pro(G)) + \varepsilon$ as discouraging agents in $I_c(y, pro(G))$ from choosing the package $(y, pro(G))$.

Agents in the set $A_c(y, pro(G))$ can spend their entire income on admission to a jurisdiction with profile $pro(G)$ offering public goods y and still be better off than in the core state of the economy. For these agents define

$$Max\psi_c(y, pro(G)) = \max_{i \in A_c(y, pro(G))} \{p \cdot \omega^0(i)\}.$$

Given any $\varepsilon > 0$, we can think of $Max\psi_c(y, pro(G)) + \varepsilon$ as a prohibitive price, barring agents in $A_c(y, pro(G))$ from affording the package $(y, pro(G))$.

To complete the price system, choose $\epsilon > 0$ and let the admission price for any jurisdiction with profile $pro(G)$ offering public goods level y be defined as follows:

$$\psi_c(y, \mathcal{C}pro(G)) = \left\{ \begin{array}{l} (a) \ \ \frac{-1}{\varepsilon} \text{ if } \cup_t G_{ct} \subset U_c(y, pro(G)). \\ \\ (b) \ \ Max\psi_c(y, pro(G)) + \epsilon \text{ if } A_c(y, pro(G)) \neq \emptyset \text{ and,} \\ \quad \text{if } \ (I_c(y, pro(G)) \neq \emptyset, \text{ then} \\ \qquad\qquad Max\psi_c(y, pro(G))) \geq \ MaxWTP_c(y, pro(G)). \\ \\ (c) \ \ MaxWTP_c(y, pro(G)) \text{ if } I_c(y, pro(G)) \neq \emptyset \text{ and,} \\ \quad \text{if } A_c(y, pro(G)) \neq \emptyset \text{ , then} \\ \qquad\qquad MaxWTP_c(y, pro(G)) > Max\psi_c(y, pro(G))). \end{array} \right\}$$

First, suppose that case (a) obtains. Then except possibly for a set of measure zero, *no* amount of income is sufficient to induce any agent of crowding type c to switch to a jurisdiction with profile $pro(G)$ offering y. Therefore, for all $\epsilon > 0$, all agents are strictly worse off if they join a jurisdiction offering $(y, \mathcal{C}pro(G))$ at the price $-\frac{1}{\varepsilon}$ then they are at the core state.

Suppose instead that case (b) obtains. By the argument above, any agents who happen to be in the set $U_c(y, pro(G))$ are worse off in the new jurisdiction under these prices then they are in their core jurisdictions. By construction, agents in $I_c(y, pro(G))$ are no better off, and possible worse off if they choose the new jurisdiction under these prices, and any agent who happens to be in $A_c(y, pro(G))$ finds the new jurisdiction unaffordable.

Finally, suppose that case (c), obtains. Again, agents in $U_c(y, pro(G))$, if there are any, would be made worse off by joining a jurisdiction offering $(y, \mathcal{C}pro(G))$. By construction, agents in $I_c(y, pro(G))$ are equally well, or worse off in the new jurisdiction under these prices as they are in their core jurisdictions and agents in $A_c(y, pro(G))$ the jurisdiction is again unaffordable. Thus, condition 1. of the definition of the Tiebout equilibrium is satisfied.

It only remains to show that there exists $\epsilon > 0$ such that for any z satisfying $(z, y) \in P(\mathcal{C}pro(G))$, profits are nonpositive. First note that if for any $c \in \mathcal{C}$, case (a) holds, we can choose ϵ arbitrarily close to zero, which makes the admission price for type c an arbitrarily large negative number. Obviously then, for small enough ϵ

$$\sum_{\mathcal{C}pro(G)_c \neq 0} n_c \psi_c(y, \mathcal{C}pro(G)) \leq p \cdot z \text{ for any } z \text{ such that } (z, y) \in P(\mathcal{C}pro(G)) \ .$$

Next suppose that for at least one crowding type $c \in \mathcal{C}$ case (b) holds but for no crowding type does case (a) hold. Also suppose that for all $\epsilon > 0$,

$$\sum_{\mathcal{C}pro(G)_c \neq 0} n_c \psi_c(y, \mathcal{C}pro(G)) > p \cdot z \text{ for some } z \text{ such that } (z, y) \in P(\mathcal{C}pro(G)).$$

Then, for $\epsilon = 0$,

$$\sum_{\mathcal{C}pro(G)_c \neq 0} n_c \psi_c(y, \mathcal{C}pro(G)) \geq p \cdot z \text{ for some } z \text{ such that } (z, y) \in P(\mathcal{C}pro(G)).$$

In this case, (i) there is enough revenue to cover the costs of producing y, (ii) there exist agents in $A_c(y, G)$ who are strictly better off when they join this jurisdiction, and for whom the jurisdiction is affordable and (iii) if $I_c(y, pro(G))$ is nonempty, then there exist agents who are exactly as well off when they join this jurisdiction at the posted prices. Thus, given these prices, the members of the jurisdiction G could (collectively) afford and prefer the jurisdiction G offering the project y. This is a contradiction to the hypothesis that (π, X, Y, Z) is a core state.

Finally, suppose that every crowding type $c \in \mathcal{C}$ case (b) holds but for no crowding type does case (a) or (b) hold (note this exhausts the possibilities. Also suppose

$$\sum_{\mathcal{C}pro(G)_c \neq 0} n_c \psi_c(y, \mathcal{C}pro(G)) > p \cdot z \text{ for some } z \text{ such that } (z, y) \in P(\mathcal{C}pro(G)).$$

By construction, all the agents who join the jurisdiction at these prices are exactly as well off as in the core state. Since the admission prices they pay more than cover the cost of the public good, an improving allocation is feasible by redistributing the surplus to the agents in this jurisdiction. This contradicts the hypothesis that (X, Y, A, S) is a core state. Therefore, condition 3. of the definition of the Tiebout equilibrium is satisfied at these prices.

Step 4. $(\pi, X, Y, Z), p$ and ψ satisfy the requirements for a competitive equilibrium.

First, from the above definition of $\psi_c(y, pro(G))$ it is immediate that the budget constraint of each agent is satisfied. From the construction of prices for public goods, it follows that no agent i can afford a bundle preferred to $(X(i), Y(i), pro(\pi_i))$ for if he could, then, from desirability of divisible commodities and sufficiency of endowments ((A.4) and (A.7)), there would be an affordable bundle of private goods for the jurisdiction π_i that would enable all members of π_i to be better off than they are in the initially given core state of the economy (π, X, Y, Z). This contradicts the separating hyperplane property of the price system p.

We have already shown zero profits for those jurisdictions in π (condition 3.) To prove the impossibility of positive profits (condition 4.), suppose that for some jurisdiction $G \in \mathcal{F}$ and some $(z, y) \in P(pro(G))$, it holds that

$$\sum_{i \in G} \psi_{\kappa(i)}(y, pro(G)) - p \cdot z > 0.$$

But this implies that

$$\sum_{i \in G} (p \cdot \omega^0(i) - p \cdot x^i) - p \cdot z > 0$$

where, for each $i \in G$ it holds that $u_{\tau(i)}(x^i, y, G) = u_{\tau(i)}(X(i), Y(i), \pi_i)$. However, the above expression implies that there is are amounts of divisible private goods for each consumer, say $\delta^i \in \mathbf{R}^L$ with all components of δ^i equal to zero except for those representing divisible commodities, for consumer i so that $\sum_{i \in G}(p \cdot \omega^0(i) - p \cdot x^i - p \cdot \delta^i) - p \cdot z = 0$. This contradicts the separating hyperplane property of the price system p. ■

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