An explicit bound on ε for non-emptiness of the ε-core of an arbitrary game with side payments.*

Alexander Kovalenkov†
Department of Economics, Universitat Autònoma de Barcelona, 08193, Bellaterra (Barcelona), Spain

Myrna Holtz Wooders‡
Department of Economics, University of Toronto, 150 St.George St., Toronto, M5S 3G7, Canada

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Abstract

We introduce the concept of a parameterized collection of games with side payments and determine a bound on ε so that all sufficiently large games in such a collection have non-empty ε-cores. A parameterized collection of games is described by (a) the number of approximate player types and the accuracy of this approximation, (b) the size of approximately effective small groups, and (c) the size of the bound on maximal per capita payoffs achievable in coalitions. An explicit relationship between the required size of ε, the parameters describing

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the collection, and the number of players in the game under consideration is established. Given the parameters, the larger the number of players in a game, the smaller the ε that can be chosen.

1 Introduction.

For an arbitrary game with side payments it is known that for some sufficiently large non-negative real number ε, the ε-core of the game is non-empty. In this paper we develop the notion of parameterized collections of games with side payments and obtain an explicit bound on ε so that all sufficiently large games in the collection have non-empty ε-cores. The required size of ε depends on the parameters describing the games: (a) the number of approximate player types and the accuracy of the approximation, (b) an upper bound on the size of near-effective groups and the closeness of the groups to being effective for the realization of all gains to collective activities and (c) the supremum of per capita payoffs achievable in coalitions. Given these parameters, the larger the number of players in a game, the smaller the ε that can be chosen so that the game has a non-empty ε-core. The bound on the number of players in the game is expressed in terms of the parameters describing the games; thus it is apparent how the bound on ε varies as the parameters are varied.

Our result is obtained in situations where the bound on the size of near-effective groups is expressed in terms of absolute sizes of groups of players. We also present a Corollary showing the application of our results to sequences of games where the relative sizes of effective coalitions become small.

The framework of parameterized collections of games with side payments is presented in Section 2. Section 3 contains the Theorem, a corollary and several examples. The Theorem is proven in Section 4. Section 5 relates our model and results to other works in the literature and Section 6 concludes the paper.

2 Parameterized collections of games with side payments.

Let (N, v) be a pair consisting of a finite set N, called the *player set*, and a function v, called the *characteristic function*, from subsets of N to the non-negative real numbers with $v(\emptyset) = 0$. The pair (N, v) is a *game with side payments*, also called a *TU game* or simply a game. By appropriate normalization we can assume that $v(\{i\}) = 0$ for any $i \in N$. Non-empty subsets of N are called *coalitions* or *groups*. A game (N, v) is *superadditive* if $v(S) \geq \sum_k v(S^k)$ for all groups $S \subset N$ and for all partitions $\left\{S^k\right\}$ of S.

<u>*$\delta-$substitute partitions.*</u> Let (N, v) be a game and let $\delta \geq 0$ be a non-negative real

number. A δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. Formally, given a partition $\{N[t] : t = 1,..,T\}$ of N, a permutation τ of N is *type-consistent* if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A *δ-substitute partition* of N is a partition $\{N[t] : t = 1,..,T\}$ of N with the property that, for any type-consistent permutation τ and any coalition S,

$$|v(S) - v(\tau(S))| \leq \delta |S| . \tag{1}$$

Note that in general a δ-substitute partition of N is not uniquely determined.

A special case of a δ-substitute partition is the case where $\delta = 0$. Then, if two players i and j are δ-substitutes (1) is satisfied if and only if for all coalitions $S \subset N$ where $i \notin S, j \notin S$ it holds that $v(S \cup \{i\}) = v(S \cup \{j\})$.

profiles. Let $\delta \geq 0$ be a non-negative real number and let (N, v) be a game. For a δ-substitute partition $\{N[t] : t = 1,..,T\}$ of N, the set $N[t]$ is interpreted as an *approximate type*. Players in the same element of a δ-substitute partition are *δ-substitutes* and, when $\delta = 0$, they are *exact substitutes*. A *profile* relative to $\{N[t]\}$ is a vector of non-negative integers $f \in Z^T_+$. Given $S \subset N$ the *profile of* S is a profile, say $s \in Z^T_+$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of the numbers of players of each approximate type in the group. The *norm* of a profile f, denoted by $\|f\|$, is simply the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

(δ,T)- type games. Let δ be a non-negative real number and let T be a positive integer. A game (N, v) is a (δ, T)*-type game* if there is a T-member δ-substitute partition $\{N[t] : t = 1,..,T\}$ of N (in other words, if there are T approximate types).

ε-effective B-bounded groups. Let ε be a given non-negative real number, and let B be a given integer. A game (N, v) has *ε-effective B-bounded groups* if for every group $S \subset N$ there is a partition $\left\{S^k\right\}$ of S into subgroups with $\left|S^k\right| \leq B$ for each k and $v(S) - \sum_k v(S^k) \leq \varepsilon |S|$. When $\varepsilon = 0$, 0-effective B-bounded groups are called *strictly effective B-bounded groups.*

per capita boundedness. Let C be a positive real number. A game (N, v) has a *per capita bound* of C if $\frac{v(S)}{|S|} \leq C$ for all coalitions $S \subset N$.

parametrized collections of games $\Gamma((\delta, T), C, (\varepsilon, B))$. Let T and B be positive integers and let C be a positive real number. Let $\Gamma((\delta, T), C, (\varepsilon, B)$ be the collection of all (δ, T)-type games that are superadditive and have both ε-effective B-bounded groups and a per capita bound of C.

Remark 1. Note that the condition of per capita boundedness is apparently non-restrictive. The following considerations lead us to view the assumption of ε-effective B-bounded groups as also remarkably mild. First, this condition is an analogue, for arbitrary games, of the concept of *small group effectiveness*, the condition that all or almost all gains to collective activities can be realized by groups of players bounded in absolute size. In games derived from pregames with a finite number of types and sufficiently many players of each type, small group effectiveness is *equivalent* to the apparently mild condition of per capita boundedness (Wooders 1994b, Section 5).[1] Moreover, if there are many players of each type that appears in the games, the models of economies such as those of Shapley and Shubik (1966) satisfy small group effectiveness. Thus, we view the condition of ε-effective B-bounded groups, the natural extension of small group effectiveness to arbitrary games, as non-restrictive.

Remark 2. There are several definitions of ε-effective B-bounded groups that would suffice for our purposes. For example, we could require that every payoff that can be ε-improved upon can be ε-improved upon by a group of players bounded in size by B. This condition is asymptotically equivalent to the condition of this paper (see Wooders 1994a, Proposition 3.8 for a related result for games derived from pregames), and is more difficult to work with. Moreover, the condition of ε-effective B-bounded groups is more directly verifiable from economic data.

3 Non-emptiness of symmetric ε-cores.

the core and ε-cores. Let (N, v) be a game and let ε be a non-negative real number. A payoff x is in the *ε-core* of (N, v) if and only if $\sum_{a \in N} x_a \leq v(N)$ and $\sum_{a \in S} x_a \geq v(S) - \varepsilon |S|$ for all $S \subset N$. Then $\varepsilon = 0$, the ε-core is the *core.*

the equal treatment epsilon core. Given non-negative real numbers ε and δ, define the *equal treatment ε-core* of a game (N, v) relative to a δ-substitute partition $\{N[t]\}$ of the player set as the set of payoffs x in the ε-core with the property that for each t and all i and j in $N[t]$, $x_i = x_j$.

3.1 The Theorem.

Our main result states a bound on the size of ε ensuring non-emptiness of the ε-core. In fact, it shows non-emptiness of the symmetric (equal treatment of approximate substitutes) ε-core as well.

It is convenient to define a constant $K(T, B) := \sum_{l=1}^{B} \frac{(T+l-1)!}{(T-1)!(l-1)!}$.

[1] This paper also provides a definition of a pregame with a finite number of player types and of small group effectiveness.

Theorem. Let $(N, v) \in \Gamma((\delta, T), C, (\varepsilon, B))$ and let ε^* be a positive real number. If

$$\varepsilon^* \geq \frac{K(T,B)C}{|N|} + \delta + \varepsilon$$

then the equal treatment ε^*-core of (N, v) is non-empty.

Notice that the required size of ε^* depends on the parameters describing the game, including T (the number of player types), C (the per capita bound), B (the bound on ε-effective B-bounded groups), ε (how close B-bounded groups are to being strictly effective) and δ (the closeness of the approximate types to exact substitutes). It is immediate that ε^* is increasing in T, B, C, δ and ε but decreasing $|N|$, the total number of players.

We present the proof of the Theorem in the next Section. The following example illustrates the statement of the Theorem.

Example 1. (Exact types and strictly effective small groups) Let us consider a game (N, v) with two types of players. Assume that any player alone can get only 0 units or less, that is $v(\{i\}) = 0$ for all $i \in N$. Let γ_{11}, $\gamma_{12} = \gamma_{21}$, and γ_{22} be some numbers from the interval $[0, 1]$. Suppose that any coalition of the two players of types i and j can get up to γ_{ij} units of payoff to divide. An arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than two players.

We leave it to the reader to check that $(N, v) \in \Gamma((0, 2), \frac{1}{2}, (0, 2))$. Since $K(T, B) = 2! + 3! = 8$, we have from the Theorem that for $\varepsilon^* \geq \frac{4}{|N|}$ the equal treatment ε^*-core of (N, v) is non-empty. Notice that this result holds uniformly for all possible numbers γ_{11}, $\gamma_{12} = \gamma_{21}$, and γ_{22}.

The following example illustrates how our result can apply to games derived from pregames with a compact metric space of player types. For brevity, our example is somewhat informal.

Example 2. (Approximate player types.) Consider a pregame[2] with two sorts of players, firms and workers. The set of possible types of workers is given by the points in the interval $[0, 1)$ and the set of possible types of firms is given by the points in the interval $[1, 2]$. Formally, let N be any finite player set and let ξ be an *attribute function*, that is, a function from N into $[0, 2]$. If $\xi(i) \in [0, 1)$ then i is a worker and if $\xi(i) \in [1, 2]$ then i is a firm.

Firms can profitably hire up to three workers and the payoff to a firm i and

[2]We refer the reader to Wooders and Zame (1984,1987) or Wooders (1992) for the definition of a pregame with a compact metric space of player types, as in this example.

a set of workers $W(i) \subset N$, containing no more than 3 members, is given by $v(\{i\} \bigcup W(i)) = \xi(i) + \sum_{j \in W(i)} \xi(j)$. Workers and firms can earn positive payoff only by cooperating so $v(\{i\}) = 0$ for all $i \in N$. For any coalition $S \subset N$ define $v(S)$ as the maximum payoff the group S could realize by splitting into coalitions containing either workers only, or 1 firm and no more than 3 workers. This completes the specification of the game.

We leave it to the reader to verify that for any positive integer m every game derived from the pregame is a $(\frac{1}{m}, 2m)$-type game and even a member of the class $\Gamma((\frac{1}{m}, 2m), 2, (0,4))$. Then the Theorem states that for any $\varepsilon^* \geq \frac{2K(2m,4)}{|N|} + \frac{1}{m}$ the equal treatment ε^*-core of (N, v) is non-empty.

This implies that for any $\varepsilon^0 > 0$ there is a positive integer $N(\varepsilon^0)$ such that for any $|N| \geq N(\varepsilon^0)$ the game (N, v) has a non-empty equal treatment ε^0-core. (For an explicit bound take an integer $m^0 \geq \frac{2}{\varepsilon^0}$ and define $N(\varepsilon^0) \geq \frac{4}{\varepsilon^0} K(2m^0, 4)$.)

For completeness, we present an example with ε-effective B-bounded groups where $\varepsilon \neq 0$.

Example 3. (Nearly effective groups bounded in size) Call a game (N, v) a k-quota game if any coalition $S \subset N$ of the size less than k can realize only 0 units (that is, $v(S) = 1$ if $|S| < k$), any coalition of the size k can realize 1 unit (that is, $v(S) = 0$ if $|S| = k$), and an arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than k players. Let Q be a collection, across all k, of all k-quota games with player set N.

We leave it to the reader to verify that for any positive integer m every game in the collection Q has $\frac{1}{m}$-effective $(m-1)$-bounded groups. Moreover the class Q is contained in the class $\Gamma((0,1), 1, (\frac{1}{m}, m-1))$. Then our Theorem states that for any $\varepsilon^* \geq \frac{K(1,m-1)}{|N|} + \frac{1}{m}$ and for any $(N, v) \in Q$ the equal treatment ε^*-core of (N, v) is non-empty. This implies that for any $\varepsilon^0 > 0$ there is a positive integer $N(\varepsilon^0)$ such that for any $|N| \geq N(\varepsilon^0)$ any game $(N, v) \in Q$ has a non-empty equal treatment ε^0-core. (For an explicit bound take an integer $m^0 \geq \frac{2}{\varepsilon^0}$ and define $N(\varepsilon^0) \geq \frac{2}{\varepsilon^0} K(1, m^0 - 1)$.)

3.2 Games with relatively small effective coalitions.

Bounding absolute sizes of near-effective groups seems to be a natural approach – marriage games and buyer-seller games (see for example, Shapley and Shubik (1972) and Kaneko (1982)), the college admission problem (see for example Roth and Sotomayor (1990)), and worker-firm sorting models (see for example Crawford and Knoer (1981)) all have bounds on absolute sizes of effective coalitions. In some economic situations, however, the size of effective groups may be most naturally stated in terms of relative sizes of coalitions. An example is voting games or weighted voting games, where the percentage of players in a coalition is crucial. An important example of models where relative sizes of coalitions are relevant is models with a continuum of players and coalitions of positive measure. Since relative sizes of effective coalitions are important in some models and may be important in others, we provide a corollary treating sequences of arbitrary games where relative sizes of effective coalitions become small.

Corollary. Consider a sequence of games (N_m, v_m) satisfying the properties that:

(a) $N_m \to \infty$ as $m \to \infty$ (games in the sequence became large),

(b) (N_m, v_m) is a (δ_m, T_m)-game, where $\frac{T_m}{|N_m|} \to 0$ and $\delta_m \to 0$ as $m \to \infty$ (players in the large games can be separated in relatively small number of the approximate types, and approximation is close),

(c) (N_m, v_m) has ε_m-effective B_m-bounded groups, where $\frac{B_m}{|N_m|} \to 0$ and $\varepsilon_m \to 0$ as $m \to \infty$ (almost all gain to the collective activity can be captured by groups small in the relative size),

(d) (N_m, v_m) has per capita bound of C_m , where $\frac{C_m}{|N_m|} \to 0$ as $m \to \infty$ (the gain in per capita terms to the collective activity can grow but slower than the number of players in the game),

(e) T_m, B_m,and C_m satisfy the condition that $\frac{K(T_m,B_m)C_m}{|N_m|} \to 0$ as $m \to \infty$ (the parameters describing the games can grow, but they are required to grow slowly relative to the size of the total player set).

<u>THEN:</u> for any $\varepsilon^* > 0$, there is $m(\varepsilon^*)$, such that for any $m \geq m(\varepsilon^*)$ the game (N_m, v_m) has a non-empty ε^*-core.

We leave the proof of the Corollary to the reader. The following example illustrates the statement of the Corollary.

Example 4. (Unbounded effective group sizes.) Consider the following sequence of games (N_m, v_m). Let us define for each m a size of effective groups B_m and let $B_m \to \infty$ as $m \to \infty$. Then let $|N_m| > |B_m|$ for each m. Assume that, for any coalition $S \subset N_m$, $v(S) = 0$ if $|S| < B_m$, $v(S) = B_m$ if $|S| = B_m$, and an arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than B_m players.

According to the notations of the Corollary we define $T_m = 1$, $\delta_m = \varepsilon_m = 0$, and $C_m = 1$. Note that $K(B, 1) = \sum_{l=1}^{B} l = \frac{B(B+1)}{2}$. Thus we have that if $\frac{|N_m|}{B_m(B_m+1)} \to \infty$ as $m \to \infty$ then, by the Corollary, for any $\varepsilon^* > 0$, there is $m(\varepsilon^*)$, such that for any $m \geq m(\varepsilon^*)$ the game (N_m, v_m) has a non-empty ε^*-core.

Remark 3. Within the context of a pregame, relating bounds on relative sizes of effective coalitions and absolute sizes of effective coalitions is more straightforward since, when there are sufficiently many players of each type, per capita boundedness and small group effectiveness are equivalent and, if some type becomes scarce as games derived from the pregame grow large, small group effectiveness plays the essential role. (The reader may find it informative to know that small group effectiveness implies a Lipschitz condition on the per capita payoff function; see Wooders 1992.) Moreover, small group effectiveness and per capita boundedness apply to the entire pregame structure and simultaneously to all games derived from that structure. Thus, for sequences of games derived from pregames with compact metric spaces of player types, the condition of small group effectiveness alone yields the conclusion of our Corollary. Our result above applies to *all* games satisfying conditions (a)-(e) rather than only to sequences of games derived from a particular pregame.

4 The Proof of the Theorem.

Let us first prove the following Lemma.

Lemma. Let $(N, v) \in \Gamma((0, T), C, (0, B))$. If $\varepsilon^* \geq \frac{K(T,B)C}{|N|}$ then the equal treatment ε^*-core of (N, v) is non-empty.

Proof of Lemma: In this proof we will use the notion of a totally balanced cover for a game. Let us first define balanced collections and balancing weights. Let (N, v) be a game, let $S \subset N$, and let β denote a collection of subsets of S. The collection β is a *balanced collection of subsets* of S if there is a collection of non-negative real numbers $(\omega_{S'})_{S' \in \beta}$, called *balancing weights*, such that for each $i \in N$,

$$\sum_{S' : i \in S',\, S' \in \beta} \omega_{S'} = 1.$$

Let (N, v) be a game and let v^b be the characteristic function defined for each subset S of N by

$$v^b(S) := \max_{\beta} \sum_{S' \in \beta} \omega_{S'} v(S'),$$

where the maximum is taken over all balanced collections β of S with corresponding balancing weights $(\omega_{S'})_{S' \in \beta}$. Then (N, v^b) is a game, called the *totally balanced cover* of (N, v).

Bondareva (1962) and Shapley (1967) have shown that a TU game has a non-empty core if and only if $v^b(N) = v(N)$. It follows easily from their results that a game has a non-empty ε-core if and only if $v^b(N) \leq v(N) + \varepsilon |N|$ (see Wooders 1980 Theorem 1 or 1994a Proposition 2.2).

To begin the proof of Lemma we place a bound on the difference $v^b(N) - v(N)$. Let $\left\{m^k\right\}_{k=1}^{K(T,B)}$ denote the collection of all profiles m^k relative to the partition $\{N[t]\}$ of N, where $\left\|m^k\right\| \leq B$ for each m^k in the collection. Let f denote the profile of N. Define a characteristic function $\bar{v}$ mapping profiles into R_+^T by

$$\bar{v}(m) := v(M) \text{ for any group } M \text{ with profile } m.$$

Since (N, v) satisfies boundedness of effective group sizes with bound B there is a balanced collection β of profiles of N where each $m \in \beta$ is in $\left\{m^k\right\}$ and for some collection of balanced weights (ω_k),

$$v^b(N) = \sum_k \omega_k \overline{v}(m^k).$$

From balancedness, it holds that $\sum_k \omega_k m^k = f$.

Since there is a finite number of distinct profiles in the set $\left\{m^k\right\}$, we can write each ω_k as an integer plus a fraction, say $\omega_k = l_k + q_k$, where $q_k \in [0, 1)$. Since the game (N, v) satisfies boundedness of effective group sizes and superadditivity it holds that

$$\sum_k l_k \bar{v}(m^k) \leq v(N).$$

Now

$$\begin{aligned} v^b(N) - v(N) &\leq \sum_k \omega_k \bar{v}(m^k) - \sum_k l_k \bar{v}(m^k) = \sum_k (\omega_k - l_k) \bar{v}(m^k) \\ &= \sum_k q_k \bar{v}(m^k) = \sum_k \bar{v}(m^k) = \sum_k \left\|m^k\right\| C. \end{aligned}$$

Let us denote by $k(T, l)$ the number of distinct profiles with norm l. It is easy to check that $k(T, l) = \frac{(T+l-1)!}{(T-1)!(l)!}$. Thus $v^b(N) - v(N) \leq \sum_k \left\|m^k\right\| C \leq \sum_{l=1}^B k(T, l) l C = \sum_{l=1}^B \frac{(T+l-1)!}{(T-1)!(l-1)!} C = K(T, B) C$. Hence, for any $\varepsilon^* \geq \frac{K(T,B)C}{|N|}$, the ε^*-core of (N, v) is non-empty.

Note that if some payoff x belongs to the ε^*-core then for any type-consistent permutation τ of N, a payoff y, defined by its components as $y_a := x_{\tau(a)}$, belongs to the ε^*-core (since all agents of one type are exact substitutes and the payoff sets are unaffected by any permutation of substitute players). Let us consider an ε^*-core payoff x. Then there is an equal treatment ε^*-core payoff z defined by its components $z_t := \frac{1}{|N[t]|} \sum_{a \in N[t]} x_a$ (since the ε^*-core of a TU game is convex). Therefore, the equal treatment ε^*-core of (N, v) is non-empty. ■

Now we will prove the statement of Theorem.

Proof of Theorem. Let $(N,v) \in \Gamma((\delta,T),C,(\varepsilon,B))$ and let ε^* be a positive real number. We first construct another game with strictly effective groups bounded in size by B. From the definition of effective groups, for any $S \subset N$ there exists a partition $\left\{S^k\right\}$ of S, $\left|S^k\right| \leq B$ for each k, such that $v(S) - \sum_k v(S^k) \leq \varepsilon |S|$. Let us define $w(S) := \max_{\{s^k\}} \sum v(S^k)$ where the maximum is taken over all partitions $\left\{S^k\right\}$ of S with $\left|S^k\right| \leq B$ for each k. Then $(N,w) \in \Gamma((\delta,T),C,(0,B))$ and $\varepsilon |S| \geq v(S) - w(S) \geq 0$ for any $S \subset N$.

Next we construct a related game by identifying all players of the same approximate type. First, for the game (N,w) let $\{N[t]\}$ be a δ-substitute partition of N. Given a group $S \subset N$ let s denote the profile of S. Define

$$w^*(S) := \max \{w(S') : S' \text{ has profile } s\}.$$

Define w^c as the superadditive cover of w^*, i.e. for any $S \subset N$,

$$w^c(S) := \max_{\{S^k\}} \sum_k w^*(S^k),$$

where the maximum is taken over all partitions of S. Then $(N,w^c) \in \Gamma((0,T),C,(0,B))$ and

$$\delta |S| \geq w^c(S) - w(S) \geq 0 \text{ for each } S \subset N.$$

By Lemma the game (N,w^c) has a non-empty equal treatment $\frac{K(T,B)C}{|N|}$-core. Let x belong to the equal treatment $\frac{K(T,B)C}{|N|}$-core of (N,w^c). Hence

$$\sum_{a\in N} x_a \leq w^c(N) \text{ and } \sum_{a\in S} x_a + \frac{K(T,B)C}{|N|}|S| \geq w^c(S).$$

Now define a payoff vector y by

$$y(\{i\}) := x(\{i\}) - \delta$$

for each $i \in N$. Then

$$\begin{aligned}\sum_{a\in N} y_a &= \sum_{a\in N} x_a - \delta |N| \\ &\leq w^c(N) - \delta |N| \leq w(N) \leq v(N)\end{aligned}$$

and for any group S it holds that

$$\begin{aligned}\sum_{a\in S} y_a + \left(\frac{K(T,B)C}{|N|} + \delta + \varepsilon\right)|S| &= \sum_{a\in S} x_a + \frac{K(T,B)C}{|N|}|S| + \varepsilon |S| \\ &\geq w^c(S) + \varepsilon |S| \geq w(S) + \varepsilon |S| \geq v(S).\end{aligned}$$

It follows that y is in the ε^*-core for any $\varepsilon^* \geq \frac{K(T,B)C}{|N|} + \delta + \varepsilon$. Since y has equal treatment property by construction, the equal treatment ε^*-core of (N,v) is non-empty. ■

5 Relationships to the literature.

There are numerous interesting economic situations where the core may be empty, for example, economies with indivisibilities and/or non-convexities, and economies with shared goods subject to crowding (which can be positive or negative) and exclusion. There are now, however, a number of results in the literature showing non-emptiness of approximate cores of large games and economies.[3] For example: Shapley and Shubik (1966) show that large exchange economies with quasi-linear preferences have non-empty approximate cores; Wooders (1980,1983) shows that large games with and without side payments and satisfying a mild condition of boundedness of per capita payoffs (simply finiteness of the supremum of the average payoff function) have non-empty approximate cores; and Kaneko and Wooders (1982) show that large games with bounded group sizes have non-empty approximate cores where the approximation is independent of the structure of payoffs. Further results generalizing the model for games with side payments to allow a compact metric space of possible player types appear in Wooders and Zame (1984) and Wooders (1992). These results, however, are all obtained in the context of pregames.

Our model and results provide extensions of prior results for games derived from pregames. Recall that a pregame specifies a topological space of player types and a payoff set (or number) for every possible coalition in any game induced by the pregame.[4] More precisely, given a compact metric space of player "types" or "attributes" (possibly finite), the payoff function of a pregame assigns a payoff set to every finite list of player types, repetitions allowed. Given any finite player set and an attribute function, assigning a type to each player in the player set, the payoff function of the game is determined by the payoff function of the pregame. Thus, the payoff set to any collection of players having a certain set of attributes is *independent* of the total player set in which it is embedded. The pregame structure itself has hidden consequences, such as the equivalence between per capita boundedness and small group effectiveness. No such consequences can be hidden within parameterized collections of games since there is no necessary relationship between any of the games in the collection. Moreover, as our examples indicate, our model and results incorpo-

[3] These results are obtained without "balancedness" assumptions. In finite games, it is immediate, from the results of Bondareva (1962), Shapley (1967) and Scarf (1967), that the assumption of balancedness implies games have non-empty approximate cores. This observation is due to Weber (1981).

[4] Note that this description covers the class of games induced by the economies in Shapley and Shubik (1966).

rate games derived from pregames and, in addition, unlike the previous papers, we provide an explicit bound on the size of ε required for a game to have a non-empty ε-core as a function of the parameters describing the games.

To study large games generally, without the structure and implicit assumptions imposed by a pregame, Kovalenkov and Wooders (1996a,b) introduce the concept of parameterized collections of games without side payments and show non-emptiness of approximate cores of arbitrary large games. The difference between the approach of arbitrary games and games derived from pregames may be viewed as analogous to the differences in approach of Kannai (1970) on one hand, who requires a topology on the space of preferences[5], and Anderson (1978) and Manelli (1991a,b) on the other hand, who consider arbitrary economies described by certain parameters.

While Kovalenkov and Wooders (1996a,b) show non-emptiness of approximate cores of arbitrary large games, no explicit bound on the required size of the games is provided and the dependence of the required size on the parameters is not explicitly demonstrated. The bounds in the current paper are used in Kovalenkov and Wooders (1996c) to obtain analogous bounds for games without side payments. Although a special case, games with side payments, however, are important. Moreover, in the side payments case, the results are especially clear and intuitive. Thus, we present our results for games with side payments in this separate paper.

6 Conclusions.

The framework of parametrized collections of games introduced in this paper promises to have a number of applications and extensions. For example, in Kovalenkov and Wooders (1996d) we extend the "law of scarcity" to show that, for arbitrary games with small effective groups, the abundance of players of a particular type and their core payoffs are inversely related.[6] In Kovalenkov, Wooders and Zhang (1994) we initiate the study of convergence of approximate cores of arbitrary games.[7] It may be that the approach in this paper will find application in the theory of lottery equilibria of games and economies, as in Garratt and Qin (1996), for example.

Perhaps most importantly, our framework and approach suggests that similar approaches might be fruitful in the study of specific economic models, for example,

[5]Of course Kannai (1970) was considering the convergence of approximate cores to the core of a continuum limit economy, the sort of situation where a topology on the space of player types seems natural.

[6]This was first shown in Scotchmer and Wooders (1988) for games derived from pregames satisfying Wooders' (1983) condition of "minimum efficient scale," – that is, all improvement upon feasible payoffs can be carried out by coalitions bounded in size.

[7]Convergence of approximate cores of large games (with the mild assumptions of superadditivity and per capita boundedness) was first shown in Wooders (1980). The same result also appears in Shubik and Wooders (1982) and Wooders (1994a, Proposition A.1.2).

economies with shared goods or even in economies with private goods only, where there may be as many different commodities as there are units of commodities. Indeed, in Kovalenkov and Wooders (1996a) we apply our results to a model of an economy where agents may be members of multiple clubs, each engaging in possibly distinct activities. But consider the familiar example of an exchange economy and core convergence theorems. (These theorems of course assume that the core is non-empty.) The conditions of Manelli (1991a,b), for example, to show competitiveness of large economies without monotone preferences, free disposal, or convexity of preferences, are especially suggestive of the parameters used in this paper and the conditions on these parameters. Thus it may be that the approach of arbitrary games will lead to further understanding of large economies.

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